Equivalents of the Axiom of Choice

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Abstract

This development [1] proves the equivalence of seven formulations of the well-ordering theorem and twenty formulations of the axiom of choice. It formalizes the first two chapters of the monograph *Equivalents of the Axiom of Choice* by Rubin and Rubin [2]. Some of this mmaterial involves extremely complex techniques.

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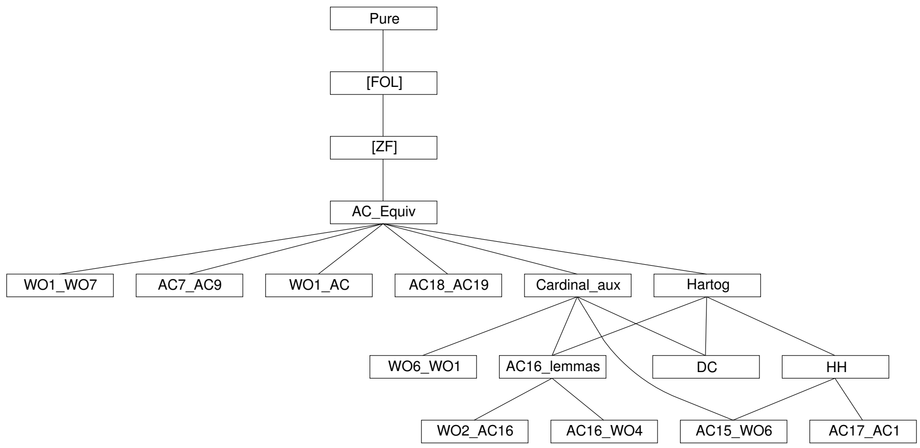

theory *AC_Equiv*
imports *Main*
begin

definition

```
"WO1 == ∀A. ∃R. well_ord(A,R)"
```

definition

```
"WO2 == ∀A. ∃a. Ord(a) & A≈a"
```

definition

```
"WO3 == ∀A. ∃a. Ord(a) & (∃b. b ⊆ a & A≈b)"
```

definition

```
"WO4(m) == ∀A. ∃a f. Ord(a) & domain(f)=a &
                    (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m)"
```

definition

```
"WO5 == ∃m ∈ nat. 1≤m & WO4(m)"
```

definition

```
"WO6 == ∀A. ∃m ∈ nat. 1≤m & (∃a f. Ord(a) & domain(f)=a
                          & (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m))"
```

definition

```
"WO7 == ∀A. Finite(A) <-> (∀R. well_ord(A,R) --> well_ord(A,converse(R)))"
```

definition

```
"WO8 == ∀A. (∃f. f ∈ (Π X ∈ A. X)) --> (∃R. well_ord(A,R))"
```

definition

pairwise_disjoint :: "i => o" **where**

```
"pairwise_disjoint(A) == ∀A1 ∈ A. ∀A2 ∈ A. A1 Int A2 ≠ 0 --> A1=A2"
```

definition

sets_of_size_between :: "[i, i, i] => o" **where**

```
"sets_of_size_between(A,m,n) == ∀B ∈ A. m ≲ B & B ≲ n"
```

definition

```
"AC0 == ∀A. ∃f. f ∈ (Π X ∈ Pow(A)-{0}. X)"
```

definition
```
"AC1 == ∀A. 0∉A --> (∃f. f ∈ (Π X ∈ A. X))"
```

definition
```
"AC2 == ∀A. 0∉A & pairwise_disjoint(A)
                --> (∃C. ∀B ∈ A. ∃y. B Int C = {y})"
```

definition
```
"AC3 == ∀A B. ∀f ∈ A->B. ∃g. g ∈ (Π x ∈ {a ∈ A. f`a≠0}. f`x)"
```

definition
```
"AC4 == ∀R A B. (R ⊆ A*B --> (∃f. f ∈ (Π x ∈ domain(R). R``{x})))"
```

definition
```
"AC5 == ∀A B. ∀f ∈ A->B. ∃g ∈ range(f)->A. ∀x ∈ domain(g). f`(g`x)
= x"
```

definition
```
"AC6 == ∀A. 0∉A --> (Π B ∈ A. B)≠0"
```

definition
```
"AC7 == ∀A. 0∉A & (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) --> (Π B ∈ A. B)
≠ 0"
```

definition
```
"AC8 == ∀A. (∀B ∈ A. ∃B1 B2. B=<B1,B2> & B1≈B2)
                --> (∃f. ∀B ∈ A. f`B ∈ bij(fst(B),snd(B)))"
```

definition
```
"AC9 == ∀A. (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) -->
                (∃f. ∀B1 ∈ A. ∀B2 ∈ A. f`<B1,B2> ∈ bij(B1,B2))"
```

definition
```
"AC10(n) ==  ∀A. (∀B ∈ A. ~Finite(B)) -->
                (∃f. ∀B ∈ A. (pairwise_disjoint(f`B) &
                sets_of_size_between(f`B, 2, succ(n)) & Union(f`B)=B))"
```

definition
```
"AC11 == ∃n ∈ nat. 1≤n & AC10(n)"
```

definition
```
"AC12 == ∀A. (∀B ∈ A. ~Finite(B)) -->
              (∃n ∈ nat. 1≤n & (∃f. ∀B ∈ A. (pairwise_disjoint(f`B)
&
                  sets_of_size_between(f`B, 2, succ(n)) & Union(f`B)=B))))"
```

definition
```
"AC13(m) == ∀A. 0∉A --> (∃f. ∀B ∈ A. f`B≠0 & f`B ⊆ B & f`B ≲
m)"
```

definition
```
"AC14 == ∃m ∈ nat. 1≤m & AC13(m)"
```

definition
```
"AC15 == ∀A. 0∉A -->
              (∃m ∈ nat. 1≤m & (∃f. ∀B ∈ A. f‘B≠0 & f‘B ⊆ B &
f‘B ≲ m))"
```

definition
```
"AC16(n, k)  ==
   ∀A. ~Finite(A) -->
       (∃T. T ⊆ {X ∈ Pow(A). X≈succ(n)} &
       (∀X ∈ {X ∈ Pow(A). X≈succ(k)}. ∃! Y. Y ∈ T & X ⊆ Y))"
```

definition
```
"AC17 == ∀A. ∀g ∈ (Pow(A)-{0} -> A) -> Pow(A)-{0}.
               ∃f ∈ Pow(A)-{0} -> A. f‘(g‘f) ∈ g‘f"
```

locale *AC18* =
  assumes *AC18*: *"A≠0 & (∀a ∈ A. B(a) ≠ 0) -->*
```
    ((⋂a ∈ A. ⋃b ∈ B(a). X(a,b)) =
      (⋃f ∈ Π a ∈ A. B(a). ⋂a ∈ A. X(a, f‘a)))"
```
  — AC18 cannot be expressed within the object-logic

definition
```
"AC19 == ∀A. A≠0 & 0∉A --> ((⋂a ∈ A. ⋃b ∈ a. b) =
               (⋃f ∈ (Π B ∈ A. B). ⋂a ∈ A. f‘a))"
```

lemma *rvimage_id: "rvimage(A,id(A),r) = r Int A*A"*
apply *(unfold rvimage_def)*
apply *(rule equalityI, safe)*
apply *(drule_tac P = "%a. <id (A) ‘xb,a>:r"* **in** *id_conv [THEN subst], assumption)*
apply *(drule_tac P = "%a. <a,ya>:r"* **in** *id_conv [THEN subst], (assumption+))*
apply *(fast intro: id_conv [THEN ssubst])*
done

lemma *ordertype_Int:*
  *"well_ord(A,r) ==> ordertype(A, r Int A*A) = ordertype(A,r)"*
apply *(rule_tac P = "%a. ordertype (A,a) =ordertype (A,r) "* **in** *rvimage_id [THEN subst])*

```
apply (erule id_bij [THEN bij_ordertype_vimage])
done

lemma lam_sing_bij: "(λx ∈ A. {x}) ∈ bij(A, {{x}. x ∈ A})"
apply (rule_tac d = "%z. THE x. z={x}" in lam_bijective)
apply (auto simp add: the_equality)
done

lemma inj_strengthen_type:
     "[| f ∈ inj(A, B);  !!a. a ∈ A ==> f`a ∈ C |] ==> f ∈ inj(A,C)"
by (unfold inj_def, blast intro: Pi_type)

lemma nat_not_Finite: "~ Finite(nat)"
by (unfold Finite_def, blast dest: eqpoll_imp_lepoll ltI lt_not_lepoll)

lemmas le_imp_lepoll = le_imp_subset [THEN subset_imp_lepoll]
```

```
lemma ex1_two_eq: "[| ∃! x. P(x); P(x); P(y) |] ==> x=y"
by blast
```

```
lemma surj_image_eq: "f ∈ surj(A, B) ==> f``A = B"
apply (unfold surj_def)
apply (erule CollectE)
apply (rule image_fun [THEN ssubst], assumption, rule subset_refl)
apply (blast dest: apply_type)
done
```

```
lemma first_in_B:
     "[| well_ord(Union(A),r); 0∉A; B ∈ A |] ==> (THE b. first(b,B,r))
∈ B"
by (blast dest!: well_ord_imp_ex1_first
                    [THEN theI, THEN first_def [THEN def_imp_iff, THEN
iffD1]])

lemma ex_choice_fun: "[| well_ord(Union(A), R); 0∉A |] ==> ∃f. f:(Π
X ∈ A. X)"
```

by *(fast elim!: first_in_B intro!: lam_type)*

lemma *ex_choice_fun_Pow: "well_ord(A, R) ==> ∃f. f:(Π X ∈ Pow(A)-{0}. X)"*
by *(fast elim!: well_ord_subset [THEN ex_choice_fun])*

lemma *lepoll_m_imp_domain_lepoll_m:*
"[| m ∈ nat; u ≲ m |] ==> domain(u) ≲ m"
apply *(unfold lepoll_def)*
apply *(erule exE)*
apply *(rule_tac x = "λx ∈ domain(u). LEAST i. ∃y. <x,y> ∈ u & f'<x,y> = i"*
in *exI)*
apply *(rule_tac d = "%y. fst (converse(f) ' y) " **in** lam_injective)*
apply *(fast intro: LeastI2 nat_into_Ord [THEN Ord_in_Ord]*
inj_is_fun [THEN apply_type])
apply *(erule domainE)*
apply *(frule inj_is_fun [THEN apply_type], assumption)*
apply *(rule LeastI2)*
apply *(auto elim!: nat_into_Ord [THEN Ord_in_Ord])*
done

lemma *rel_domain_ex1:*
"[| succ(m) ≲ domain(r); r ≲ succ(m); m ∈ nat |] ==> function(r)"
apply *(unfold function_def, safe)*
apply *(rule ccontr)*
apply *(fast elim!: lepoll_trans [THEN succ_lepoll_natE]*
lepoll_m_imp_domain_lepoll_m [OF _ Diff_sing_lepoll]
elim: domain_Diff_eq [OF _ not_sym, THEN subst])
done

lemma *rel_is_fun:*
"[| succ(m) ≲ domain(r); r ≲ succ(m); m ∈ nat;
*r ⊆ A*B; A=domain(r) |] ==> r ∈ A->B"*
by *(simp add: Pi_iff rel_domain_ex1)*

end

theory *Cardinal_aux* **imports** *AC_Equiv* **begin**

```
lemma Diff_lepoll: "[| A ≲ succ(m); B ⊆ A; B≠0 |] ==> A-B ≲ m"
apply (rule not_emptyE, assumption)
apply (blast intro: lepoll_trans [OF subset_imp_lepoll Diff_sing_lepoll])
done
```

```
lemma lepoll_imp_ex_le_eqpoll:
     "[| A ≲ i; Ord(i) |] ==> ∃j. j le i & A ≈ j"
by (blast intro!: lepoll_cardinal_le well_ord_Memrel
                  well_ord_cardinal_eqpoll [THEN eqpoll_sym]
          dest: lepoll_well_ord)
```

```
lemma lesspoll_imp_ex_lt_eqpoll:
     "[| A ≺ i; Ord(i) |] ==> ∃j. j<i & A ≈ j"
by (unfold lesspoll_def, blast dest!: lepoll_imp_ex_le_eqpoll elim!: leE)

lemma Inf_Ord_imp_InfCard_cardinal: "[| ~Finite(i); Ord(i) |] ==> InfCard(|i|)"
apply (unfold InfCard_def)
apply (rule conjI)
apply (rule Card_cardinal)
apply (rule Card_nat
            [THEN Card_def [THEN def_imp_iff, THEN iffD1, THEN ssubst]])
  — rewriting would loop!
apply (rule well_ord_Memrel [THEN well_ord_lepoll_imp_Card_le], assumption)

apply (rule nat_le_infinite_Ord [THEN le_imp_lepoll], assumption+)
done
```

An alternative and more general proof goes like this: A and B are both well-ordered (because they are injected into an ordinal), either A lepoll B or B lepoll A. Also both are equipollent to their cardinalities, so (if A and B are infinite) then A Un B lepoll —A—+—B— = max(—A—,—B—) lepoll i. In fact, the correctly strengthened version of this theorem appears below.

```
lemma Un_lepoll_Inf_Ord_weak:
     "[|A ≈ i; B ≈ i; ¬ Finite(i); Ord(i)|] ==> A ∪ B ≲ i"
apply (rule Un_lepoll_sum [THEN lepoll_trans])
apply (rule lepoll_imp_sum_lepoll_prod [THEN lepoll_trans])
apply (erule eqpoll_trans [THEN eqpoll_imp_lepoll])
apply (erule eqpoll_sym)
apply (rule subset_imp_lepoll [THEN lepoll_trans, THEN lepoll_trans])
```

```
apply (rule nat_2I [THEN OrdmemD], rule Ord_nat)
apply (rule nat_le_infinite_Ord [THEN le_imp_lepoll], assumption+)
apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll])
apply (erule prod_eqpoll_cong [THEN eqpoll_imp_lepoll, THEN lepoll_trans],
       assumption)
apply (rule eqpoll_imp_lepoll)
apply (rule well_ord_Memrel [THEN well_ord_InfCard_square_eq], assumption)

apply (rule Inf_Ord_imp_InfCard_cardinal, assumption+)
done

lemma Un_eqpoll_Inf_Ord:
     "[| A ≈ i; B ≈ i; ~Finite(i); Ord(i) |] ==> A Un B ≈ i"
apply (rule eqpollI)
apply (blast intro: Un_lepoll_Inf_Ord_weak)
apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_trans])
apply (rule Un_upper1 [THEN subset_imp_lepoll])
done

schematic_lemma paired_bij: "?f ∈ bij({{y,z}. y ∈ x}, x)"
apply (rule RepFun_bijective)
apply (simp add: doubleton_eq_iff, blast)
done

lemma paired_eqpoll: "{{y,z}. y ∈ x} ≈ x"
by (unfold eqpoll_def, fast intro!: paired_bij)

lemma ex_eqpoll_disjoint: "∃B. B ≈ A & B Int C = 0"
by (fast intro!: paired_eqpoll equals0I elim: mem_asym)


lemma Un_lepoll_Inf_Ord:
     "[| A ≲ i; B ≲ i; ~Finite(i); Ord(i) |] ==> A Un B ≲ i"
apply (rule_tac A1 = i and C1 = i in ex_eqpoll_disjoint [THEN exE])
apply (erule conjE)
apply (drule lepoll_trans)
apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll])
apply (rule Un_lepoll_Un [THEN lepoll_trans], (assumption+))
apply (blast intro: eqpoll_refl Un_eqpoll_Inf_Ord eqpoll_imp_lepoll)
done

lemma Least_in_Ord: "[| P(i); i ∈ j; Ord(j) |] ==> (LEAST i. P(i)) ∈
j"
apply (erule Least_le [THEN leE])
apply (erule Ord_in_Ord, assumption)
apply (erule ltE)
apply (fast dest: OrdmemD)
apply (erule subst_elem, assumption)
```

```
done

lemma Diff_first_lepoll:
     "[| well_ord(x,r); y ⊆ x; y ≲ succ(n); n ∈ nat |]
      ==> y - {THE b. first(b,y,r)} ≲ n"
apply (case_tac "y=0", simp add: empty_lepollI)
apply (fast intro!: Diff_sing_lepoll the_first_in)
done

lemma UN_subset_split:
     "(⋃x ∈ X. P(x)) ⊆ (⋃x ∈ X. P(x)-Q(x)) Un (⋃x ∈ X. Q(x))"
by blast

lemma UN_sing_lepoll: "Ord(a) ==> (⋃x ∈ a. {P(x)}) ≲ a"
apply (unfold lepoll_def)
apply (rule_tac x = "λz ∈ (⋃x ∈ a. {P (x) }) . (LEAST i. P (i) =z)
" in exI)
apply (rule_tac d = "%z. P (z) " in lam_injective)
apply (fast intro!: Least_in_Ord)
apply (fast intro: LeastI elim!: Ord_in_Ord)
done

lemma UN_fun_lepoll_lemma [rule_format]:
     "[| well_ord(T, R); ~Finite(a); Ord(a); n ∈ nat |]
      ==> ∀f. (∀b ∈ a. f`b ≲ n & f`b ⊆ T) --> (⋃b ∈ a. f`b) ≲ a"
apply (induct_tac "n")
apply (rule allI)
apply (rule impI)
apply (rule_tac b = "⋃b ∈ a. f`b" in subst)
apply (rule_tac [2] empty_lepollI)
apply (rule equals0I [symmetric], clarify)
apply (fast dest: lepoll_0_is_0 [THEN subst])
apply (rule allI)
apply (rule impI)
apply (erule_tac x = "λx ∈ a. f`x - {THE b. first (b,f`x,R) }" in allE)
apply (erule impE, simp)
apply (fast intro!: Diff_first_lepoll, simp)
apply (rule UN_subset_split [THEN subset_imp_lepoll, THEN lepoll_trans])
apply (fast intro: Un_lepoll_Inf_Ord UN_sing_lepoll)
done

lemma UN_fun_lepoll:
     "[| ∀b ∈ a. f`b ≲ n & f`b ⊆ T; well_ord(T, R);
         ~Finite(a); Ord(a); n ∈ nat |] ==> (⋃b ∈ a. f`b) ≲ a"
by (blast intro: UN_fun_lepoll_lemma)

lemma UN_lepoll:
     "[| ∀b ∈ a. F(b) ≲ n & F(b) ⊆ T; well_ord(T, R);
         ~Finite(a); Ord(a); n ∈ nat |]
```

```
      ==> (⋃b ∈ a. F(b)) ≲ a"
apply (rule rev_mp)
apply (rule_tac f="λb ∈ a. F (b)" in UN_fun_lepoll)
apply auto
done

lemma UN_eq_UN_Diffs:
     "Ord(a) ==> (⋃b ∈ a. F(b)) = (⋃b ∈ a. F(b) - (⋃c ∈ b. F(c)))"
apply (rule equalityI)
 prefer 2 apply fast
apply (rule subsetI)
apply (erule UN_E)
apply (rule UN_I)
 apply (rule_tac P = "%z. x ∈ F (z) " in Least_in_Ord, (assumption+))
apply (rule DiffI, best intro: Ord_in_Ord LeastI, clarify)
apply (erule_tac P = "%z. x ∈ F (z) " and i = c in less_LeastE)
apply (blast intro: Ord_Least ltI)
done

lemma lepoll_imp_eqpoll_subset:
     "a ≲ X ==> ∃Y. Y ⊆ X & a ≈ Y"
apply (unfold lepoll_def eqpoll_def, clarify)
apply (blast intro: restrict_bij
             dest: inj_is_fun [THEN fun_is_rel, THEN image_subset])
done




lemma Diff_lesspoll_eqpoll_Card_lemma:
     "[| A≈a; ~Finite(a); Card(a); B ≺ a; A-B ≺ a |] ==> P"
apply (elim lesspoll_imp_ex_lt_eqpoll [THEN exE] Card_is_Ord conjE)
apply (frule_tac j=xa in Un_upper1_le [OF lt_Ord lt_Ord], assumption)
apply (frule_tac j=xa in Un_upper2_le [OF lt_Ord lt_Ord], assumption)
apply (drule Un_least_lt, assumption)
apply (drule eqpoll_imp_lepoll [THEN lepoll_trans],
       rule le_imp_lepoll, assumption)+
apply (case_tac "Finite(x Un xa)")
```

finite case

```
 apply (drule Finite_Un [OF lepoll_Finite lepoll_Finite], assumption+)

 apply (drule subset_Un_Diff [THEN subset_imp_lepoll, THEN lepoll_Finite])
 apply (fast dest: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_Finite])
```

infinite case

```
apply (drule Un_lepoll_Inf_Ord, (assumption+))
apply (blast intro: le_Ord2)
```

```
apply (drule lesspoll_trans1
            [OF subset_Un_Diff [THEN subset_imp_lepoll, THEN lepoll_trans]

                lt_Card_imp_lesspoll], assumption+)
apply (simp add: lesspoll_def)
done

lemma Diff_lesspoll_eqpoll_Card:
     "[| A ≈ a; ~Finite(a); Card(a); B ≺ a |] ==> A - B ≈ a"
apply (rule ccontr)
apply (rule Diff_lesspoll_eqpoll_Card_lemma, (assumption+))
apply (blast intro: lesspoll_def [THEN def_imp_iff, THEN iffD2]
                    subset_imp_lepoll eqpoll_imp_lepoll lepoll_trans)
done

end
```

```
theory WO6_WO1
imports Cardinal_aux
begin


definition
  NN  :: "i => i"  where
    "NN(y) == {m ∈ nat. ∃a. ∃f. Ord(a) & domain(f)=a  &
                        (⋃b<a. f`b) = y & (∀b<a. f`b ≲ m)}"

definition
  uu  :: "[i, i, i, i] => i"  where
    "uu(f, beta, gamma, delta) == (f`beta * f`gamma) Int f`delta"



definition
  vv1 :: "[i, i, i] => i"  where
     "vv1(f,m,b) ==
           let g = LEAST g. (∃d. Ord(d) & (domain(uu(f,b,g,d)) ≠ 0 &

                                domain(uu(f,b,g,d)) ≲ m));
               d = LEAST d. domain(uu(f,b,g,d)) ≠ 0 &
                            domain(uu(f,b,g,d)) ≲ m
           in  if f`b ≠ 0 then domain(uu(f,b,g,d)) else 0"

definition
  ww1 :: "[i, i, i] => i"  where
     "ww1(f,m,b) == f`b - vv1(f,m,b)"
```

definition
```
  gg1 :: "[i, i, i] => i"  where
     "gg1(f,a,m) == λb ∈ a++a. if b<a then vv1(f,m,b) else ww1(f,m,b--a)"
```

definition
```
  vv2 :: "[i, i, i, i] => i"  where
     "vv2(f,b,g,s) ==
           if f`g ≠ 0 then {uu(f, b, g, LEAST d. uu(f,b,g,d) ≠ 0)`s}
else 0"
```

definition
```
  ww2 :: "[i, i, i, i] => i"  where
     "ww2(f,b,g,s) == f`g - vv2(f,b,g,s)"
```

definition
```
  gg2 :: "[i, i, i, i] => i"  where
     "gg2(f,a,b,s) ==
              λg ∈ a++a. if g<a then vv2(f,b,g,s) else ww2(f,b,g--a,s)"
```

```
lemma WO2_WO3: "WO2 ==> WO3"
by (unfold WO2_def WO3_def, fast)
```

```
lemma WO3_WO1: "WO3 ==> WO1"
apply (unfold eqpoll_def WO1_def WO3_def)
apply (intro allI)
apply (drule_tac x=A in spec)
apply (blast intro: bij_is_inj well_ord_rvimage
                    well_ord_Memrel [THEN well_ord_subset])
done
```

```
lemma WO1_WO2: "WO1 ==> WO2"
apply (unfold eqpoll_def WO1_def WO2_def)
apply (blast intro!: Ord_ordertype ordermap_bij)
done
```

```
lemma lam_sets: "f ∈ A->B ==> (λx ∈ A. {f`x}): A -> {{b}. b ∈ B}"
by (fast intro!: lam_type apply_type)
```

```
lemma surj_imp_eq': "f ∈ surj(A,B) ==> (⋃a ∈ A. {f`a}) = B"
apply (unfold surj_def)
```

```
apply (fast elim!: apply_type)
done

lemma surj_imp_eq: "[| f ∈ surj(A,B); Ord(A) |] ==> (⋃a<A. {f`a}) =
B"
by (fast dest!: surj_imp_eq' intro!: ltI elim!: ltE)

lemma WO1_WO4: "WO1 ==> WO4(1)"
apply (unfold WO1_def WO4_def)
apply (rule allI)
apply (erule_tac x = A in allE)
apply (erule exE)
apply (intro exI conjI)
apply (erule Ord_ordertype)
apply (erule ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun, THEN
lam_sets, THEN domain_of_fun])
apply (simp_all add: singleton_eqpoll_1 eqpoll_imp_lepoll Ord_ordertype
       ordermap_bij [THEN bij_converse_bij, THEN bij_is_surj, THEN surj_imp_eq]
       ltD)
done



lemma WO4_mono: "[| m≤n; WO4(m) |] ==> WO4(n)"
apply (unfold WO4_def)
apply (blast dest!: spec intro: lepoll_trans [OF _ le_imp_lepoll])
done



lemma WO4_WO5: "[| m ∈ nat; 1≤m; WO4(m) |] ==> WO5"
by (unfold WO4_def WO5_def, blast)



lemma WO5_WO6: "WO5 ==> WO6"
by (unfold WO4_def WO5_def WO6_def, blast)



lemma lt_oadd_odiff_disj:
      "[| k < i++j;  Ord(i);  Ord(j) |]
       ==> k < i  |  (~ k<i & k = i ++ (k--i) & (k--i)<j)"
apply (rule_tac i = k and j = i in Ord_linear2)
prefer 4
   apply (drule odiff_lt_mono2, assumption)
   apply (simp add: oadd_odiff_inverse odiff_oadd_inverse)
apply (auto elim!: lt_Ord)
```

done

lemma *domain_uu_subset: "domain(uu(f,b,g,d)) ⊆ f`b"*
by *(unfold uu_def, blast)*

lemma *quant_domain_uu_lepoll_m:*
"∀b<a. f`b ≲ m ==> ∀b<a. ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≲ m"
by *(blast intro: domain_uu_subset [THEN subset_imp_lepoll] lepoll_trans)*

lemma *uu_subset1: "uu(f,b,g,d) ⊆ f`b * f`g"*
by *(unfold uu_def, blast)*

lemma *uu_subset2: "uu(f,b,g,d) ⊆ f`d"*
by *(unfold uu_def, blast)*

lemma *uu_lepoll_m: "[| ∀b<a. f`b ≲ m; d<a |] ==> uu(f,b,g,d) ≲ m"*
by *(blast intro: uu_subset2 [THEN subset_imp_lepoll] lepoll_trans)*

lemma *cases:*
```
"∀b<a. ∀g<a. ∀d<a. u(f,b,g,d) ≲ m
 ==> (∀b<a. f`b ≠ 0 -->
              (∃g<a. ∃d<a. u(f,b,g,d) ≠ 0 & u(f,b,g,d) ≺ m))
   | (∃b<a. f`b ≠ 0 & (∀g<a. ∀d<a. u(f,b,g,d) ≠ 0 -->
                                   u(f,b,g,d) ≈ m))"
```
apply *(unfold lesspoll_def)*
apply *(blast del: equalityI)*
done

lemma *UN_oadd: "Ord(a) ==> (⋃b<a++a. C(b)) = (⋃b<a. C(b) Un C(a++b))"*
by *(blast intro: ltI lt_oadd1 oadd_lt_mono2 dest!: lt_oadd_disj)*

```
lemma vv1_subset: "vv1(f,m,b) ⊆ f`b"
by (simp add: vv1_def Let_def domain_uu_subset)
```

```
lemma UN_gg1_eq:
  "[| Ord(a);  m ∈ nat |] ==> (⋃b<a++a. gg1(f,a,m)`b) = (⋃b<a. f`b)"
by (simp add: gg1_def UN_oadd lt_oadd1 oadd_le_self [THEN le_imp_not_lt]

              lt_Ord odiff_oadd_inverse ltD vv1_subset [THEN Diff_partition]
              ww1_def)

lemma domain_gg1: "domain(gg1(f,a,m)) = a++a"
by (simp add: lam_funtype [THEN domain_of_fun] gg1_def)
```

```
lemma nested_LeastI:
     "[| P(a, b);  Ord(a);  Ord(b);
         Least_a = (LEAST a. ∃x. Ord(x) & P(a, x)) |]
      ==> P(Least_a, LEAST b. P(Least_a, b))"
apply (erule ssubst)
apply (rule_tac Q = "%z. P (z, LEAST b. P (z, b))" in LeastI2)
apply (fast elim!: LeastI)+
done

lemmas nested_Least_instance =
       nested_LeastI [of "%g d. domain(uu(f,b,g,d)) ≠ 0 &
                                domain(uu(f,b,g,d)) ≲ m",
                      standard]

lemma gg1_lepoll_m:
     "[| Ord(a);  m ∈ nat;
         ∀b<a. f`b ≠0 -->
             (∃g<a. ∃d<a. domain(uu(f,b,g,d)) ≠ 0  &
                          domain(uu(f,b,g,d)) ≲ m);
         ∀b<a. f`b ≲ succ(m);  b<a++a |]
      ==> gg1(f,a,m)`b ≲ m"
apply (simp add: gg1_def empty_lepollI)
apply (safe dest!: lt_oadd_odiff_disj)

 apply (simp add: vv1_def Let_def empty_lepollI)
 apply (fast intro: nested_Least_instance [THEN conjunct2]
             elim!: lt_Ord)
```

```
apply (simp add: ww1_def empty_lepollI)
apply (case_tac "f` (b--a) = 0", simp add: empty_lepollI)
apply (rule Diff_lepoll, blast)
apply (rule vv1_subset)
apply (drule ospec [THEN mp], assumption+)
apply (elim oexE conjE)
apply (simp add: vv1_def Let_def lt_Ord nested_Least_instance [THEN conjunct1])
done
```

```
lemma ex_d_uu_not_empty:
     "[| b<a;  g<a;  f`b≠0;  f`g≠0;
         y*y ⊆ y;  (⋃b<a. f`b)=y |]
      ==> ∃d<a. uu(f,b,g,d) ≠ 0"
by (unfold uu_def, blast)

lemma uu_not_empty:
     "[| b<a; g<a; f`b≠0; f`g≠0;  y*y ⊆ y; (⋃b<a. f`b)=y |]
      ==> uu(f,b,g,LEAST d. (uu(f,b,g,d) ≠ 0)) ≠ 0"
apply (drule ex_d_uu_not_empty, assumption+)
apply (fast elim!: LeastI lt_Ord)
done

lemma not_empty_rel_imp_domain: "[| r ⊆ A*B; r≠0 |] ==> domain(r)≠0"
by blast

lemma Least_uu_not_empty_lt_a:
     "[| b<a; g<a; f`b≠0; f`g≠0; y*y ⊆ y; (⋃b<a. f`b)=y |]
      ==> (LEAST d. uu(f,b,g,d) ≠ 0) < a"
apply (erule ex_d_uu_not_empty [THEN oexE], assumption+)
apply (blast intro: Least_le [THEN lt_trans1] lt_Ord)
done

lemma subset_Diff_sing: "[| B ⊆ A; a∉B |] ==> B ⊆ A-{a}"
by blast


lemma supset_lepoll_imp_eq:
     "[| A ≲ m; m ≲ B; B ⊆ A; m ∈ nat |] ==> A=B"
apply (erule natE)
apply (fast dest!: lepoll_0_is_0 intro!: equalityI)
apply (safe intro!: equalityI)
```

```
apply (rule ccontr)
apply (rule succ_lepoll_natE)
 apply (erule lepoll_trans)
 apply (rule lepoll_trans)
  apply (erule subset_Diff_sing [THEN subset_imp_lepoll], assumption)
 apply (rule Diff_sing_lepoll, assumption+)
done

lemma uu_Least_is_fun:
     "[| ∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 -->
          domain(uu(f, b, g, d)) ≈ succ(m);
          ∀b<a. f`b ≲ succ(m);  y*y ⊆ y;
          (⋃b<a. f`b)=y;  b<a;  g<a;  d<a;
          f`b≠0;  f`g≠0;  m ∈ nat;  s ∈ f`b |]
      ==> uu(f, b, g, LEAST d. uu(f,b,g,d)≠0) ∈ f`b -> f`g"
apply (drule_tac x2=g in ospec [THEN ospec, THEN mp])
   apply (rule_tac [3] not_empty_rel_imp_domain [OF uu_subset1 uu_not_empty])
        apply (rule_tac [2] Least_uu_not_empty_lt_a, assumption+)
apply (rule rel_is_fun)
    apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll])
   apply (erule uu_lepoll_m)
   apply (rule Least_uu_not_empty_lt_a, assumption+)
apply (rule uu_subset1)
apply (rule supset_lepoll_imp_eq [OF _ eqpoll_sym [THEN eqpoll_imp_lepoll]])
apply (fast intro!: domain_uu_subset)+
done

lemma vv2_subset:
     "[| ∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 -->
                       domain(uu(f, b, g, d)) ≈ succ(m);
         ∀b<a. f`b ≲ succ(m); y*y ⊆ y;
         (⋃b<a. f`b)=y;  b<a;  g<a;  m ∈ nat;  s ∈ f`b |]
      ==> vv2(f,b,g,s) ⊆ f`g"
apply (simp add: vv2_def)
apply (blast intro: uu_Least_is_fun [THEN apply_type])
done




lemma UN_gg2_eq:
     "[| ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 -->
             domain(uu(f,b,g,d)) ≈ succ(m);
         ∀b<a. f`b ≲ succ(m); y*y ⊆ y;
         (⋃b<a. f`b)=y;  Ord(a);  m ∈ nat;  s ∈ f`b;  b<a |]
      ==> (⋃g<a++a. gg2(f,a,b,s) ` g) = y"
apply (unfold gg2_def)
apply (drule sym)
apply (simp add: ltD UN_oadd  oadd_le_self [THEN le_imp_not_lt]
```

```
                   lt_Ord odiff_oadd_inverse ww2_def
                   vv2_subset [THEN Diff_partition])
done

lemma domain_gg2: "domain(gg2(f,a,b,s)) = a++a"
by (simp add: lam_funtype [THEN domain_of_fun] gg2_def)




lemma vv2_lepoll: "[| m ∈ nat; m≠0 |] ==> vv2(f,b,g,s) ≲ m"
apply (unfold vv2_def)
apply (simp add: empty_lepollI)
apply (fast dest!: le_imp_subset [THEN subset_imp_lepoll, THEN lepoll_0_is_0]

       intro!: singleton_eqpoll_1 [THEN eqpoll_imp_lepoll, THEN lepoll_trans]
               not_lt_imp_le [THEN le_imp_subset, THEN subset_imp_lepoll]
               nat_into_Ord nat_1I)
done

lemma ww2_lepoll:
    "[| ∀b<a. f`b ≲ succ(m);  g<a;  m ∈ nat;  vv2(f,b,g,d) ⊆ f`g |]

     ==> ww2(f,b,g,d) ≲ m"
apply (unfold ww2_def)
apply (case_tac "f`g = 0")
apply (simp add: empty_lepollI)
apply (drule ospec, assumption)
apply (rule Diff_lepoll, assumption+)
apply (simp add: vv2_def not_emptyI)
done

lemma gg2_lepoll_m:
     "[| ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 -->
                      domain(uu(f,b,g,d)) ≈ succ(m);
         ∀b<a. f`b ≲ succ(m);  y*y ⊆ y;
         (⋃b<a. f`b)=y;  b<a;  s ∈ f`b;  m ∈ nat;  m≠ 0;  g<a++a |]

      ==> gg2(f,a,b,s) ` g ≲ m"
apply (simp add: gg2_def empty_lepollI)
apply (safe elim!: lt_Ord2 dest!: lt_oadd_odiff_disj)
 apply (simp add: vv2_lepoll)
apply (simp add: ww2_lepoll vv2_subset)
done
```

```
lemma lemma_ii: "[| succ(m) ∈ NN(y); y*y ⊆ y; m ∈ nat; m≠0 |] ==>
m ∈ NN(y)"
apply (unfold NN_def)
apply (elim CollectE exE conjE)
apply (rule quant_domain_uu_lepoll_m [THEN cases, THEN disjE], assumption)

apply (simp add: lesspoll_succ_iff)
apply (rule_tac x = "a++a" in exI)
apply (fast intro!: Ord_oadd domain_gg1 UN_gg1_eq gg1_lepoll_m)

apply (elim oexE conjE)
apply (rule_tac A = "f`?B" in not_emptyE, assumption)
apply (rule CollectI)
apply (erule succ_natD)
apply (rule_tac x = "a++a" in exI)
apply (rule_tac x = "gg2 (f,a,b,x) " in exI)
apply (simp add: Ord_oadd domain_gg2 UN_gg2_eq gg2_lepoll_m)
done
```

```
lemma z_n_subset_z_succ_n:
     "∀n ∈ nat. rec(n, x, %k r. r Un r*r) ⊆ rec(succ(n), x, %k r. r
Un r*r)"
by (fast intro: rec_succ [THEN ssubst])

lemma le_subsets:
     "[| ∀n ∈ nat. f(n)<=f(succ(n)); n≤m; n ∈ nat; m ∈ nat |]
      ==> f(n)<=f(m)"
apply (erule_tac P = "n≤m" in rev_mp)
apply (rule_tac P = "%z. n≤z --> f (n) ⊆ f (z) " in nat_induct)
apply (auto simp add: le_iff)
done

lemma le_imp_rec_subset:
     "[| n≤m; m ∈ nat |]
      ==> rec(n, x, %k r. r Un r*r) ⊆ rec(m, x, %k r. r Un r*r)"
apply (rule z_n_subset_z_succ_n [THEN le_subsets])
apply (blast intro: lt_nat_in_nat)+
done
```

```
lemma lemma_iv: "∃y. x Un y*y ⊆ y"
apply (rule_tac x = "⋃n ∈ nat. rec (n, x, %k r. r Un r*r) " in exI)
apply safe
apply (rule nat_0I [THEN UN_I], simp)
apply (rule_tac a = "succ (n Un na) " in UN_I)
apply (erule Un_nat_type [THEN nat_succI], assumption)
apply (auto intro: le_imp_rec_subset [THEN subsetD]
            intro!: Un_upper1_le Un_upper2_le Un_nat_type
            elim!: nat_into_Ord)
done
```

```
lemma WO6_imp_NN_not_empty: "WO6 ==> NN(y) ≠ 0"
by (unfold WO6_def NN_def, clarify, blast)
```

```
lemma lemma1:
     "[| (⋃b<a. f`b)=y; x ∈ y; ∀b<a. f`b ≲ 1; Ord(a) |] ==> ∃c<a. f`c
= {x}"
by (fast elim!: lepoll_1_is_sing)

lemma lemma2:
     "[| (⋃b<a. f`b)=y; x ∈ y; ∀b<a. f`b ≲ 1; Ord(a) |]
      ==> f` (LEAST i. f`i = {x}) = {x}"
apply (drule lemma1, assumption+)
apply (fast elim!: lt_Ord intro: LeastI)
done

lemma NN_imp_ex_inj: "1 ∈ NN(y) ==> ∃a f. Ord(a) & f ∈ inj(y, a)"
apply (unfold NN_def)
apply (elim CollectE exE conjE)
apply (rule_tac x = a in exI)
```

```
apply (rule_tac x = "λx ∈ y. LEAST i. f`i = {x}" in exI)
apply (rule conjI, assumption)
apply (rule_tac d = "%i. THE x. x ∈ f`i" in lam_injective)
apply (drule lemma1, assumption+)
apply (fast elim!: Least_le [THEN lt_trans1, THEN ltD] lt_Ord)
apply (rule lemma2 [THEN ssubst], assumption+, blast)
done

lemma y_well_ord: "[| y*y ⊆ y; 1 ∈ NN(y) |] ==> ∃r. well_ord(y, r)"
apply (drule NN_imp_ex_inj)
apply (fast elim!: well_ord_rvimage [OF _ well_ord_Memrel])
done


lemma rev_induct_lemma [rule_format]:
     "[| n ∈ nat; !!m. [| m ∈ nat; m≠0; P(succ(m)) |] ==> P(m) |]
      ==> n≠0 --> P(n) --> P(1)"
by (erule nat_induct, blast+)

lemma rev_induct:
     "[| n ∈ nat;  P(n);  n≠0;
         !!m. [| m ∈ nat; m≠0; P(succ(m)) |] ==> P(m) |]
      ==> P(1)"
by (rule rev_induct_lemma, blast+)

lemma NN_into_nat: "n ∈ NN(y) ==> n ∈ nat"
by (simp add: NN_def)

lemma lemma3: "[| n ∈ NN(y); y*y ⊆ y; n≠0 |] ==> 1 ∈ NN(y)"
apply (rule rev_induct [OF NN_into_nat], assumption+)
apply (rule lemma_ii, assumption+)
done


lemma NN_y_0: "0 ∈ NN(y) ==> y=0"
apply (unfold NN_def)
apply (fast intro!: equalityI dest!: lepoll_0_is_0 elim: subst)
done

lemma WO6_imp_WO1: "WO6 ==> WO1"
apply (unfold WO1_def)
apply (rule allI)
```

apply *(case_tac "A=0")*
apply *(fast intro!: well_ord_Memrel nat_0I [THEN nat_into_Ord])*
apply *(rule_tac x1 = A* **in** *lemma_iv [THEN revcut_rl])*
apply *(erule exE)*
apply *(drule WO6_imp_NN_not_empty)*
apply *(erule Un_subset_iff [THEN iffD1, THEN conjE])*
apply *(erule_tac A = "NN (y) "* **in** *not_emptyE)*
apply *(frule y_well_ord)*
apply *(fast intro!: lemma3 dest!: NN_y_0 elim!: not_emptyE)*
apply *(fast elim: well_ord_subset)*
done

end

theory *WO1_WO7*
imports *AC_Equiv*
begin

definition
"LEMMA ==
$\forall$X. ~Finite(X) --> ($\exists$R. well_ord(X,R) & ~well_ord(X,converse(R)))"

lemma *WO7_iff_LEMMA: "WO7 <-> LEMMA"*
apply *(unfold WO7_def LEMMA_def)*
apply *(blast intro: Finite_well_ord_converse)*
done

lemma *LEMMA_imp_WO1: "LEMMA ==> WO1"*
apply *(unfold WO1_def LEMMA_def Finite_def eqpoll_def)*
apply *(blast intro!: well_ord_rvimage [OF bij_is_inj nat_implies_well_ord])*
done

```
lemma converse_Memrel_not_wf_on:
    "[| Ord(a); ~Finite(a) |] ==> ~wf[a](converse(Memrel(a)))"
apply (unfold wf_on_def wf_def)
apply (drule nat_le_infinite_Ord [THEN le_imp_subset], assumption)
apply (rule notI)
apply (erule_tac x = nat in allE, blast)
done

lemma converse_Memrel_not_well_ord:
    "[| Ord(a); ~Finite(a) |] ==> ~well_ord(a,converse(Memrel(a)))"
apply (unfold well_ord_def)
apply (blast dest: converse_Memrel_not_wf_on)
done

lemma well_ord_rvimage_ordertype:
     "well_ord(A,r) ⟹
       rvimage (ordertype(A,r), converse(ordermap(A,r)),r) =
       Memrel(ordertype(A,r))"
by (blast intro: ordertype_ord_iso [THEN ord_iso_sym] ord_iso_rvimage_eq
            Memrel_type [THEN subset_Int_iff [THEN iffD1]] trans)

lemma well_ord_converse_Memrel:
     "[| well_ord(A,r); well_ord(A,converse(r)) |]
      ==> well_ord(ordertype(A,r), converse(Memrel(ordertype(A,r))))"

apply (subst well_ord_rvimage_ordertype [symmetric], assumption)
apply (rule rvimage_converse [THEN subst])
apply (blast intro: ordertype_ord_iso ord_iso_sym ord_iso_is_bij
                 bij_is_inj well_ord_rvimage)
done

lemma WO1_imp_LEMMA: "WO1 ==> LEMMA"
apply (unfold WO1_def LEMMA_def, clarify)
apply (blast dest: well_ord_converse_Memrel
                 Ord_ordertype [THEN converse_Memrel_not_well_ord]
            intro: ordertype_ord_iso ord_iso_is_bij bij_is_inj lepoll_Finite
                 lepoll_def [THEN def_imp_iff, THEN iffD2] )
done

lemma WO1_iff_WO7: "WO1 <-> WO7"
apply (simp add: WO7_iff_LEMMA)
apply (blast intro: LEMMA_imp_WO1 WO1_imp_LEMMA)
done
```

```
lemma WO1_WO8: "WO1 ==> WO8"
by (unfold WO1_def WO8_def, fast)



lemma WO8_WO1: "WO8 ==> WO1"
apply (unfold WO1_def WO8_def)
apply (rule allI)
apply (erule_tac x = "{{x}. x ∈ A}" in allE)
apply (erule impE)
 apply (rule_tac x = "λa ∈ {{x}. x ∈ A}. THE x. a={x}" in exI)
 apply (force intro!: lam_type simp add: singleton_eq_iff the_equality)
apply (blast intro: lam_sing_bij bij_is_inj well_ord_rvimage)
done

end



theory AC7_AC9
imports AC_Equiv
begin



lemma Sigma_fun_space_not0: "[| 0∉A; B ∈ A |] ==> (nat->Union(A)) *
B ≠ 0"
by (blast dest!: Sigma_empty_iff [THEN iffD1] Union_empty_iff [THEN iffD1])

lemma inj_lemma:
        "C ∈ A ==> (λg ∈ (nat->Union(A))*C.
                (λn ∈ nat. if(n=0, snd(g), fst(g)`(n #- 1))))
                ∈ inj((nat->Union(A))*C, (nat->Union(A)) ) "
apply (unfold inj_def)
apply (rule CollectI)
apply (fast intro!: lam_type if_type apply_type fst_type snd_type, auto)

apply (rule fun_extension, assumption+)
```

```
apply (drule lam_eqE [OF _ nat_succI], assumption, simp)
apply (drule lam_eqE [OF _ nat_0I], simp)
done

lemma Sigma_fun_space_eqpoll:
     "[| C ∈ A; 0∉A |] ==> (nat->Union(A)) * C ≈ (nat->Union(A))"
apply (rule eqpollI)
apply (simp add: lepoll_def)
apply (fast intro!: inj_lemma)
apply (fast elim!: prod_lepoll_self not_sym [THEN not_emptyE] subst_elem

            elim: swap)
done
```

```
lemma AC6_AC7: "AC6 ==> AC7"
by (unfold AC6_def AC7_def, blast)
```

```
lemma lemma1_1: "y ∈ (Π B ∈ A. Y*B) ==> (λB ∈ A. snd(y‘B)) ∈ (Π B
∈ A. B)"
by (fast intro!: lam_type snd_type apply_type)

lemma lemma1_2:
     "y ∈ (Π B ∈ {Y*C. C ∈ A}. B) ==> (λB ∈ A. y‘(Y*B)) ∈ (Π B ∈ A.
Y*B)"
apply (fast intro!: lam_type apply_type)
done

lemma AC7_AC6_lemma1:
     "(Π B ∈ {(nat->Union(A))*C. C ∈ A}. B) ≠ 0 ==> (Π B ∈ A. B) ≠
0"
by (fast intro!: equals0I lemma1_1 lemma1_2)

lemma AC7_AC6_lemma2: "0 ∉ A ==> 0 ∉ {(nat -> Union(A)) * C. C ∈ A}"
by (blast dest: Sigma_fun_space_not0)

lemma AC7_AC6: "AC7 ==> AC6"
apply (unfold AC6_def AC7_def)
apply (rule allI)
```

```
apply (rule impI)
apply (case_tac "A=0", simp)
apply (rule AC7_AC6_lemma1)
apply (erule allE)
apply (blast del: notI
             intro!: AC7_AC6_lemma2 intro: eqpoll_sym eqpoll_trans
                    Sigma_fun_space_eqpoll)
done
```

```
lemma AC1_AC8_lemma1:
        "∀B ∈ A. ∃B1 B2. B=<B1,B2> & B1 ≈ B2
        ==> 0 ∉ { bij(fst(B),snd(B)). B ∈ A }"
apply (unfold eqpoll_def, auto)
done

lemma AC1_AC8_lemma2:
     "[| f ∈ (Π X ∈ RepFun(A,p). X); D ∈ A |] ==> (λx ∈ A. f`p(x))`D
∈ p(D)"
apply (simp, fast elim!: apply_type)
done

lemma AC1_AC8: "AC1 ==> AC8"
apply (unfold AC1_def AC8_def)
apply (fast dest: AC1_AC8_lemma1 AC1_AC8_lemma2)
done
```

```
lemma AC8_AC9_lemma:
     "∀B1 ∈ A. ∀B2 ∈ A. B1 ≈ B2
      ==> ∀B ∈ A*A. ∃B1 B2. B=<B1,B2> & B1 ≈ B2"
by fast

lemma AC8_AC9: "AC8 ==> AC9"
apply (unfold AC8_def AC9_def)
apply (intro allI impI)
apply (erule allE)
apply (erule impE, erule AC8_AC9_lemma, force)
done
```

```
lemma snd_lepoll_SigmaI: "b ∈ B ⟹ X ≲ B × X"
by (blast intro: lepoll_trans prod_lepoll_self eqpoll_imp_lepoll
                 prod_commute_eqpoll)

lemma nat_lepoll_lemma:
     "[|0 ∉ A; B ∈ A|] ==> nat ≲ ((nat → Union(A)) × B) × nat"
by (blast dest: Sigma_fun_space_not0 intro: snd_lepoll_SigmaI)

lemma AC9_AC1_lemma1:
     "[| 0∉A;  A≠0;
         C = {((nat->Union(A))*B)*nat. B ∈ A}  Un
             {cons(0,((nat->Union(A))*B)*nat). B ∈ A};
         B1 ∈ C;  B2 ∈ C |]
      ==> B1 ≈ B2"
by (blast intro!: nat_lepoll_lemma Sigma_fun_space_eqpoll
                    nat_cons_eqpoll [THEN eqpoll_trans]
                    prod_eqpoll_cong [OF _ eqpoll_refl]
             intro: eqpoll_trans eqpoll_sym )

lemma AC9_AC1_lemma2:
     "∀B1 ∈ {(F*B)*N. B ∈ A} Un {cons(0,(F*B)*N). B ∈ A}.
      ∀B2 ∈ {(F*B)*N. B ∈ A} Un {cons(0,(F*B)*N). B ∈ A}.
        f‘<B1,B2> ∈ bij(B1, B2)
      ==> (λB ∈ A. snd(fst((f‘<cons(0,(F*B)*N),(F*B)*N>)‘0))) ∈ (Π X
∈ A. X)"
apply (intro lam_type snd_type fst_type)
apply (rule apply_type [OF _ consI1])
apply (fast intro!: fun_weaken_type bij_is_fun)
done

lemma AC9_AC1: "AC9 ==> AC1"
apply (unfold AC1_def AC9_def)
apply (intro allI impI)
apply (erule allE)
apply (case_tac "A~=0")
apply (blast dest: AC9_AC1_lemma1 AC9_AC1_lemma2, force)
done

end
```

theory *WO1_AC*
imports *AC_Equiv*
begin

theorem *WO1_AC1: "WO1 ==> AC1"*
by *(unfold AC1_def WO1_def, fast elim!: ex_choice_fun)*

lemma *lemma1: "[| WO1; ∀B ∈ A. ∃C ∈ D(B). P(C,B) |] ==> ∃f. ∀B ∈ A. P(f‘B,B)"*
apply *(unfold WO1_def)*
apply *(erule_tac x = "Union ({{C ∈ D (B) . P (C,B) }. B ∈ A}) "* **in** *allE)*
apply *(erule exE, drule ex_choice_fun, fast)*
apply *(erule exE)*
apply *(rule_tac x = "λx ∈ A. f‘{C ∈ D (x) . P (C,x) }"* **in** *exI)*
apply *(simp, blast dest!: apply_type [OF _ RepFunI])*
done

lemma *lemma2_1: "[| ~Finite(B); WO1 |] ==> |B| + |B| ≈ B"*
apply *(unfold WO1_def)*
apply *(rule eqpoll_trans)*
prefer *2* **apply** *(fast elim!: well_ord_cardinal_eqpoll)*
apply *(rule eqpoll_sym [THEN eqpoll_trans])*
apply *(fast elim!: well_ord_cardinal_eqpoll)*
apply *(drule spec [of _ B])*
apply *(clarify dest!: eqpoll_imp_Finite_iff [OF well_ord_cardinal_eqpoll])*

apply *(simp add: cadd_def [symmetric]*
eqpoll_refl InfCard_cdouble_eq Card_cardinal Inf_Card_is_InfCard)

done

lemma *lemma2_2:*
"f ∈ bij(D+D, B) ==> {{f‘Inl(i), f‘Inr(i)}. i ∈ D} ∈ Pow(Pow(B))"
by *(fast elim!: bij_is_fun [THEN apply_type])*

lemma *lemma2_3:*
"f ∈ bij(D+D, B) ==> pairwise_disjoint({{f‘Inl(i), f‘Inr(i)}.

```
i ∈ D})"
apply (unfold pairwise_disjoint_def)
apply (blast dest: bij_is_inj [THEN inj_apply_equality])
done

lemma lemma2_4:
     "[| f ∈ bij(D+D, B); 1≤n |]
      ==> sets_of_size_between({{f`Inl(i), f`Inr(i)}. i ∈ D}, 2, succ(n))"
apply (simp (no_asm_simp) add: sets_of_size_between_def succ_def)
apply (blast intro!: cons_lepoll_cong
            intro: singleton_eqpoll_1 [THEN eqpoll_imp_lepoll]
                   le_imp_subset [THEN subset_imp_lepoll]  lepoll_trans

            dest: bij_is_inj [THEN inj_apply_equality] elim!: mem_irrefl)
done

lemma lemma2_5:
     "f ∈ bij(D+D, B) ==> Union({{f`Inl(i), f`Inr(i)}. i ∈ D})=B"
apply (unfold bij_def surj_def)
apply (fast elim!: inj_is_fun [THEN apply_type])
done

lemma lemma2:
     "[| WO1; ~Finite(B); 1≤n  |]
      ==> ∃C ∈ Pow(Pow(B)). pairwise_disjoint(C) &
                 sets_of_size_between(C, 2, succ(n)) &
                 Union(C)=B"
apply (drule lemma2_1 [THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]],

       assumption)
apply (blast intro!: lemma2_2 lemma2_3 lemma2_4 lemma2_5)
done

theorem WO1_AC10: "[| WO1; 1≤n |] ==> AC10(n)"
apply (unfold AC10_def)
apply (fast intro!: lemma1 elim!: lemma2)
done

end



theory Hartog
imports AC_Equiv
begin

definition
  Hartog :: "i => i"  where
```

```
  "Hartog(X) == LEAST i. ~ i ≲ X"

lemma Ords_in_set: "∀a. Ord(a) --> a ∈ X ==> P"
apply (rule_tac X1 = "{y ∈ X. Ord (y) }" in ON_class [THEN revcut_rl])
apply fast
done

lemma Ord_lepoll_imp_ex_well_ord:
     "[| Ord(a); a ≲ X |]
      ==> ∃Y. Y ⊆ X & (∃R. well_ord(Y,R) & ordertype(Y,R)=a)"
apply (unfold lepoll_def)
apply (erule exE)
apply (intro exI conjI)
  apply (erule inj_is_fun [THEN fun_is_rel, THEN image_subset])
 apply (rule well_ord_rvimage [OF bij_is_inj well_ord_Memrel])
  apply (erule restrict_bij [THEN bij_converse_bij])
apply (rule subset_refl, assumption)
apply (rule trans)
apply (rule bij_ordertype_vimage)
apply (erule restrict_bij [THEN bij_converse_bij])
apply (rule subset_refl)
apply (erule well_ord_Memrel)
apply (erule ordertype_Memrel)
done

lemma Ord_lepoll_imp_eq_ordertype:
     "[| Ord(a); a ≲ X |] ==> ∃Y. Y ⊆ X & (∃R. R ⊆ X*X & ordertype(Y,R)=a)"
apply (drule Ord_lepoll_imp_ex_well_ord, assumption, clarify)
apply (intro exI conjI)
apply (erule_tac [3] ordertype_Int, auto)
done

lemma Ords_lepoll_set_lemma:
     "(∀a. Ord(a) --> a ≲ X) ==>
       ∀a. Ord(a) -->
        a ∈ {b. Z ∈ Pow(X)*Pow(X*X), ∃Y R. Z=<Y,R> & ordertype(Y,R)=b}"
apply (intro allI impI)
apply (elim allE impE, assumption)
apply (blast dest!: Ord_lepoll_imp_eq_ordertype intro: sym)
done

lemma Ords_lepoll_set: "∀a. Ord(a) --> a ≲ X ==> P"
by (erule Ords_lepoll_set_lemma [THEN Ords_in_set])

lemma ex_Ord_not_lepoll: "∃a. Ord(a) & ~a ≲ X"
apply (rule ccontr)
apply (best intro: Ords_lepoll_set)
done
```

```
lemma not_Hartog_lepoll_self: "~ Hartog(A) ≲ A"
apply (unfold Hartog_def)
apply (rule ex_Ord_not_lepoll [THEN exE])
apply (rule LeastI, auto)
done

lemmas Hartog_lepoll_selfE = not_Hartog_lepoll_self [THEN notE, standard]

lemma Ord_Hartog: "Ord(Hartog(A))"
by (unfold Hartog_def, rule Ord_Least)

lemma less_HartogE1: "[| i < Hartog(A); ~ i ≲ A |] ==> P"
by (unfold Hartog_def, fast elim: less_LeastE)

lemma less_HartogE: "[| i < Hartog(A); i ≈ Hartog(A) |] ==> P"
by (blast intro: less_HartogE1 eqpoll_sym eqpoll_imp_lepoll
                 lepoll_trans [THEN Hartog_lepoll_selfE])

lemma Card_Hartog: "Card(Hartog(A))"
by (fast intro!: CardI Ord_Hartog elim: less_HartogE)

end



theory HH
imports AC_Equiv Hartog
begin

definition
  HH :: "[i, i, i] => i"  where
    "HH(f,x,a) == transrec(a, %b r. let z = x - (⋃c ∈ b. r`c)
                                    in  if f`z ∈ Pow(z)-{0} then f`z else
{x})"
```

0.1 Lemmas useful in each of the three proofs

```
lemma HH_def_satisfies_eq:
     "HH(f,x,a) = (let z = x - (⋃b ∈ a. HH(f,x,b))
                   in  if f`z ∈ Pow(z)-{0} then f`z else {x})"
by (rule HH_def [THEN def_transrec, THEN trans], simp)

lemma HH_values: "HH(f,x,a) ∈ Pow(x)-{0} | HH(f,x,a)={x}"
apply (rule HH_def_satisfies_eq [THEN ssubst])
apply (simp add: Let_def Diff_subset [THEN PowI], fast)
done

lemma subset_imp_Diff_eq:
     "B ⊆ A ==> X-(⋃a ∈ A. P(a)) = X-(⋃a ∈ A-B. P(a))-(⋃b ∈ B. P(b))"
```

```
by fast

lemma Ord_DiffE: "[| c ∈ a-b; b<a |] ==> c=b | b<c & c<a"
apply (erule ltE)
apply (drule Ord_linear [of _ c])
apply (fast elim: Ord_in_Ord)
apply (fast intro!: ltI intro: Ord_in_Ord)
done

lemma Diff_UN_eq_self: "(!!y. y∈A ==> P(y) = {x}) ==> x - (⋃y ∈ A.
P(y)) = x"
by (simp, fast elim!: mem_irrefl)

lemma HH_eq: "x - (⋃b ∈ a. HH(f,x,b)) = x - (⋃b ∈ a1. HH(f,x,b))
             ==> HH(f,x,a) = HH(f,x,a1)"
apply (subst HH_def_satisfies_eq [of _ _ a1])
apply (rule HH_def_satisfies_eq [THEN trans], simp)
done

lemma HH_is_x_gt_too: "[| HH(f,x,b)={x}; b<a |] ==> HH(f,x,a)={x}"
apply (rule_tac P = "b<a" in impE)
prefer 2 apply assumption+
apply (erule lt_Ord2 [THEN trans_induct])
apply (rule impI)
apply (rule HH_eq [THEN trans])
prefer 2 apply assumption+
apply (rule leI [THEN le_imp_subset, THEN subset_imp_Diff_eq, THEN ssubst],

       assumption)
apply (rule_tac t = "%z. z-?X" in subst_context)
apply (rule Diff_UN_eq_self)
apply (drule Ord_DiffE, assumption)
apply (fast elim: ltE, auto)
done

lemma HH_subset_x_lt_too:
     "[| HH(f,x,a) ∈ Pow(x)-{0}; b<a |] ==> HH(f,x,b) ∈ Pow(x)-{0}"
apply (rule HH_values [THEN disjE], assumption)
apply (drule HH_is_x_gt_too, assumption)
apply (drule subst, assumption)
apply (fast elim!: mem_irrefl)
done

lemma HH_subset_x_imp_subset_Diff_UN:
    "HH(f,x,a) ∈ Pow(x)-{0} ==> HH(f,x,a) ∈ Pow(x - (⋃b ∈ a. HH(f,x,b)))-{0}"
apply (drule HH_def_satisfies_eq [THEN subst])
apply (rule HH_def_satisfies_eq [THEN ssubst])
apply (simp add: Let_def Diff_subset [THEN PowI])
apply (drule split_if [THEN iffD1])
```

```
apply (fast elim!: mem_irrefl)
done

lemma HH_eq_arg_lt:
     "[| HH(f,x,v)=HH(f,x,w); HH(f,x,v) ∈ Pow(x)-{0}; v ∈ w |] ==> P"
apply (frule_tac P = "%y. y ∈ Pow (x) -{0}" in subst, assumption)
apply (drule_tac a = w in HH_subset_x_imp_subset_Diff_UN)
apply (drule subst_elem, assumption)
apply (fast intro!: singleton_iff [THEN iffD2] equals0I)
done

lemma HH_eq_imp_arg_eq:
  "[| HH(f,x,v)=HH(f,x,w); HH(f,x,w) ∈ Pow(x)-{0}; Ord(v); Ord(w) |] ==>
v=w"
apply (rule_tac j = w in Ord_linear_lt)
apply (simp_all (no_asm_simp))
 apply (drule subst_elem, assumption)
 apply (blast dest: ltD HH_eq_arg_lt)
apply (blast dest: HH_eq_arg_lt [OF sym] ltD)
done

lemma HH_subset_x_imp_lepoll:
     "[| HH(f, x, i) ∈ Pow(x)-{0}; Ord(i) |] ==> i lepoll Pow(x)-{0}"
apply (unfold lepoll_def inj_def)
apply (rule_tac x = "λj ∈ i. HH (f, x, j) " in exI)
apply (simp (no_asm_simp))
apply (fast del: DiffE
            elim!: HH_eq_imp_arg_eq Ord_in_Ord HH_subset_x_lt_too
            intro!: lam_type ballI ltI intro: bexI)
done

lemma HH_Hartog_is_x: "HH(f, x, Hartog(Pow(x)-{0})) = {x}"
apply (rule HH_values [THEN disjE])
prefer 2 apply assumption
apply (fast del: DiffE
            intro!: Ord_Hartog
            dest!: HH_subset_x_imp_lepoll
            elim!: Hartog_lepoll_selfE)
done

lemma HH_Least_eq_x: "HH(f, x, LEAST i. HH(f, x, i) = {x}) = {x}"
by (fast intro!: Ord_Hartog HH_Hartog_is_x LeastI)

lemma less_Least_subset_x:
     "a ∈ (LEAST i. HH(f,x,i)={x}) ==> HH(f,x,a) ∈ Pow(x)-{0}"
apply (rule HH_values [THEN disjE], assumption)
apply (rule less_LeastE)
apply (erule_tac [2] ltI [OF _ Ord_Least], assumption)
done
```

0.2 Lemmas used in the proofs of AC1 ==¿ WO2 and AC17 ==¿ AC1

```
lemma lam_Least_HH_inj_Pow:
        "(λa ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,a))
         ∈ inj(LEAST i. HH(f,x,i)={x}, Pow(x)-{0})"
apply (unfold inj_def, simp)
apply (fast intro!: lam_type dest: less_Least_subset_x
            elim!: HH_eq_imp_arg_eq Ord_Least [THEN Ord_in_Ord])
done

lemma lam_Least_HH_inj:
     "∀a ∈ (LEAST i. HH(f,x,i)={x}). ∃z ∈ x. HH(f,x,a) = {z}
      ==> (λa ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,a))
           ∈ inj(LEAST i. HH(f,x,i)={x}, {{y}. y ∈ x})"
by (rule lam_Least_HH_inj_Pow [THEN inj_strengthen_type], simp)

lemma lam_surj_sing:
        "[| x - (⋃a ∈ A. F(a)) = 0;  ∀a ∈ A. ∃z ∈ x. F(a) = {z} |]

         ==> (λa ∈ A. F(a)) ∈ surj(A, {{y}. y ∈ x})"
apply (simp add: surj_def lam_type Diff_eq_0_iff)
apply (blast elim: equalityE)
done

lemma not_emptyI2: "y ∈ Pow(x)-{0} ==> x ≠ 0"
by auto

lemma f_subset_imp_HH_subset:
     "f‘(x - (⋃j ∈ i. HH(f,x,j))) ∈ Pow(x - (⋃j ∈ i. HH(f,x,j)))-{0}

      ==> HH(f, x, i) ∈ Pow(x) - {0}"
apply (rule HH_def_satisfies_eq [THEN ssubst])
apply (simp add: Let_def Diff_subset [THEN PowI] not_emptyI2 [THEN if_P],
fast)
done

lemma f_subsets_imp_UN_HH_eq_x:
     "∀z ∈ Pow(x)-{0}. f‘z ∈ Pow(z)-{0}
      ==> x - (⋃j ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,j)) = 0"
apply (case_tac "?P ∈ {0}", fast)
apply (drule Diff_subset [THEN PowI, THEN DiffI])
apply (drule bspec, assumption)
apply (drule f_subset_imp_HH_subset)
apply (blast dest!: subst_elem [OF _ HH_Least_eq_x [symmetric]]
             elim!: mem_irrefl)
done

lemma HH_values2: "HH(f,x,i) = f‘(x - (⋃j ∈ i. HH(f,x,j))) | HH(f,x,i)={x}"
apply (rule HH_def_satisfies_eq [THEN ssubst])
```

```
apply (simp add: Let_def Diff_subset [THEN PowI])
done

lemma HH_subset_imp_eq:
     "HH(f,x,i): Pow(x)-{0} ==> HH(f,x,i)=f`(x - (⋃j ∈ i. HH(f,x,j)))"
apply (rule HH_values2 [THEN disjE], assumption)
apply (fast elim!: equalityE mem_irrefl dest!: singleton_subsetD)
done

lemma f_sing_imp_HH_sing:
     "[| f ∈ (Pow(x)-{0}) -> {{z}. z ∈ x};
         a ∈ (LEAST i. HH(f,x,i)={x}) |] ==> ∃z ∈ x. HH(f,x,a) = {z}"
apply (drule less_Least_subset_x)
apply (frule HH_subset_imp_eq)
apply (drule apply_type)
apply (rule Diff_subset [THEN PowI, THEN DiffI])
apply (fast dest!: HH_subset_x_imp_subset_Diff_UN [THEN not_emptyI2],
force)
done

lemma f_sing_lam_bij:
     "[| x - (⋃j ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,j)) = 0;
         f ∈ (Pow(x)-{0}) -> {{z}. z ∈ x} |]
      ==> (λa ∈ (LEAST i. HH(f,x,i)={x}). HH(f,x,a))
          ∈ bij(LEAST i. HH(f,x,i)={x}, {{y}. y ∈ x})"
apply (unfold bij_def)
apply (fast intro!: lam_Least_HH_inj lam_surj_sing f_sing_imp_HH_sing)
done

lemma lam_singI:
     "f ∈ (Π X ∈ Pow(x)-{0}. F(X))
      ==> (λX ∈ Pow(x)-{0}. {f`X}) ∈ (Π X ∈ Pow(x)-{0}. {{z}. z ∈ F(X)})"
by (fast del: DiffI DiffE
           intro!: lam_type singleton_eq_iff [THEN iffD2] dest: apply_type)


lemmas bij_Least_HH_x =
    comp_bij [OF f_sing_lam_bij [OF _ lam_singI]
              lam_sing_bij [THEN bij_converse_bij], standard]
```

0.3 The proof of AC1 ==¿ WO2

```
lemma bijection:
     "f ∈ (Π X ∈ Pow(x) - {0}. X)
      ==> ∃g. g ∈ bij(x, LEAST i. HH(λX ∈ Pow(x)-{0}. {f`X}, x, i) =
{x})"
apply (rule exI)
apply (rule bij_Least_HH_x [THEN bij_converse_bij])
apply (rule f_subsets_imp_UN_HH_eq_x)
```

```
apply (intro ballI apply_type)
apply (fast intro: lam_type apply_type del: DiffE, assumption)
apply (fast intro: Pi_weaken_type)
done

lemma AC1_WO2: "AC1 ==> WO2"
apply (unfold AC1_def WO2_def eqpoll_def)
apply (intro allI)
apply (drule_tac x = "Pow(A) - {0}" in spec)
apply (blast dest: bijection)
done

end
```

```
theory AC15_WO6
imports HH Cardinal_aux
begin
```

```
lemma lepoll_Sigma: "A≠0 ==> B ≲ A*B"
apply (unfold lepoll_def)
apply (erule not_emptyE)
apply (rule_tac x = "λz ∈ B. <x,z>" in exI)
apply (fast intro!: snd_conv lam_injective)
done

lemma cons_times_nat_not_Finite:
     "0∉A ==> ∀B ∈ {cons(0,x*nat). x ∈ A}. ~Finite(B)"
apply clarify
apply (rule nat_not_Finite [THEN notE] )
apply (subgoal_tac "x ~= 0")
 apply (blast intro: lepoll_Sigma [THEN lepoll_Finite])+
done

lemma lemma1: "[| Union(C)=A; a ∈ A |] ==> ∃B ∈ C. a ∈ B & B ⊆ A"
by fast

lemma lemma2:
        "[| pairwise_disjoint(A); B ∈ A; C ∈ A; a ∈ B; a ∈ C |] ==>
```

```
B=C"
by (unfold pairwise_disjoint_def, blast)

lemma lemma3:
     "∀B ∈ {cons(0, x*nat). x ∈ A}. pairwise_disjoint(f`B) &
             sets_of_size_between(f`B, 2, n) & Union(f`B)=B
     ==> ∀B ∈ A. ∃! u. u ∈ f`cons(0, B*nat) & u ⊆ cons(0, B*nat) &

             0 ∈ u & 2 ≲ u & u ≲ n"
apply (unfold sets_of_size_between_def)
apply (rule ballI)
apply (erule_tac x="cons(0, B*nat)" in ballE)
 apply (blast dest: lemma1 intro!: lemma2, blast)
done

lemma lemma4: "[| A ≲ i; Ord(i) |] ==> {P(a). a ∈ A} ≲ i"
apply (unfold lepoll_def)
apply (erule exE)
apply (rule_tac x = "λx ∈ RepFun(A,P). LEAST j. ∃a∈A. x=P(a) & f`a=j"

       in exI)
apply (rule_tac d = "%y. P (converse (f) `y) " in lam_injective)
apply (erule RepFunE)
apply (frule inj_is_fun [THEN apply_type], assumption)
apply (fast intro: LeastI2 elim!: Ord_in_Ord inj_is_fun [THEN apply_type])
apply (erule RepFunE)
apply (rule LeastI2)
  apply fast
 apply (fast elim!: Ord_in_Ord inj_is_fun [THEN apply_type])
apply (fast elim: sym left_inverse [THEN ssubst])
done

lemma lemma5_1:
     "[| B ∈ A; 2 ≲ u(B) |] ==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})`B ≠
0"
apply simp
apply (fast dest: lepoll_Diff_sing
            elim: lepoll_trans [THEN succ_lepoll_natE] ssubst
            intro!: lepoll_refl)
done

lemma lemma5_2:
     "[|  B ∈ A; u(B) ⊆ cons(0, B*nat) |]
      ==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})`B ⊆ B"
apply auto
done

lemma lemma5_3:
     "[| n ∈ nat; B ∈ A; 0 ∈ u(B); u(B) ≲ succ(n) |]
```

```
      ==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})`B ≲ n"
apply simp
apply (fast elim!: Diff_lepoll [THEN lemma4 [OF _ nat_into_Ord]])
done

lemma ex_fun_AC13_AC15:
     "[| ∀B ∈ {cons(0, x*nat). x ∈ A}.
                pairwise_disjoint(f`B) &
                sets_of_size_between(f`B, 2, succ(n)) & Union(f`B)=B;

         n ∈ nat |]
      ==> ∃f. ∀B ∈ A. f`B ≠ 0 & f`B ⊆ B & f`B ≲ n"
by (fast del: subsetI notI
         dest!: lemma3 theI intro!: lemma5_1 lemma5_2 lemma5_3)
```

```
theorem AC10_AC11: "[| n ∈ nat; 1≤n; AC10(n) |] ==> AC11"
by (unfold AC10_def AC11_def, blast)
```

```
theorem AC11_AC12: "AC11 ==> AC12"
by (unfold AC10_def AC11_def AC11_def AC12_def, blast)
```

```
theorem AC12_AC15: "AC12 ==> AC15"
apply (unfold AC12_def AC15_def)
apply (blast del: ballI
            intro!: cons_times_nat_not_Finite ex_fun_AC13_AC15)
done
```

```
lemma OUN_eq_UN: "Ord(x) ==> (⋃a<x. F(a)) = (⋃a ∈ x. F(a))"
```

```
by (fast intro!: ltI dest!: ltD)

lemma AC15_WO6_aux1:
     "∀x ∈ Pow(A)-{0}. f`x≠0 & f`x ⊆ x & f`x ≲ m
      ==> (⋃i<LEAST x. HH(f,A,x)={A}. HH(f,A,i)) = A"
apply (simp add: Ord_Least [THEN OUN_eq_UN])
apply (rule equalityI)
apply (fast dest!: less_Least_subset_x)
apply (blast del: subsetI
           intro!: f_subsets_imp_UN_HH_eq_x [THEN Diff_eq_0_iff [THEN
iffD1]])
done

lemma AC15_WO6_aux2:
     "∀x ∈ Pow(A)-{0}. f`x≠0 & f`x ⊆ x & f`x ≲ m
      ==> ∀x < (LEAST x. HH(f,A,x)={A}). HH(f,A,x) ≲ m"
apply (rule oallI)
apply (drule ltD [THEN less_Least_subset_x])
apply (frule HH_subset_imp_eq)
apply (erule ssubst)
apply (blast dest!: HH_subset_x_imp_subset_Diff_UN [THEN not_emptyI2])

done

theorem AC15_WO6: "AC15 ==> WO6"
apply (unfold AC15_def WO6_def)
apply (rule allI)
apply (erule_tac x = "Pow (A) -{0}" in allE)
apply (erule impE, fast)
apply (elim bexE conjE exE)
apply (rule bexI)
 apply (rule conjI, assumption)
 apply (rule_tac x = "LEAST i. HH (f,A,i) ={A}" in exI)
 apply (rule_tac x = "λj ∈ (LEAST i. HH (f,A,i) ={A}) . HH (f,A,j) "
in exI)
 apply (simp_all add: ltD)
apply (fast intro!: Ord_Least lam_type [THEN domain_of_fun]
            elim!: less_Least_subset_x AC15_WO6_aux1 AC15_WO6_aux2)
done
```

theorem *AC10_AC13: "[| n ∈ nat; 1≤n; AC10(n) |] ==> AC13(n)"*
apply *(unfold AC10_def AC13_def, safe)*
apply *(erule allE)*
apply *(erule impE [OF _ cons_times_nat_not_Finite], assumption)*
apply *(fast elim!: impE [OF _ cons_times_nat_not_Finite]*
dest!: ex_fun_AC13_AC15)
done

lemma *AC1_AC13: "AC1 ==> AC13(1)"*
apply *(unfold AC1_def AC13_def)*
apply *(rule allI)*
apply *(erule allE)*
apply *(rule impI)*
apply *(drule mp, assumption)*
apply *(elim exE)*
apply *(rule_tac x = "λx ∈ A. {f`x}"* **in** *exI)*
apply *(simp add: singleton_eqpoll_1 [THEN eqpoll_imp_lepoll])*
done

lemma *AC13_mono: "[| m≤n; AC13(m) |] ==> AC13(n)"*
apply *(unfold AC13_def)*
apply *(drule le_imp_lepoll)*
apply *(fast elim!: lepoll_trans)*
done

```
theorem AC13_AC14: "[| n ∈ nat; 1≤n; AC13(n) |] ==> AC14"
by (unfold AC13_def AC14_def, auto)
```

```
theorem AC14_AC15: "AC14 ==> AC15"
by (unfold AC13_def AC14_def AC15_def, fast)
```

```
lemma lemma_aux: "[| A≠0; A ≲ 1 |] ==> ∃a. A={a}"
by (fast elim!: not_emptyE lepoll_1_is_sing)

lemma AC13_AC1_lemma:
     "∀B ∈ A. f(B)≠0 & f(B)<=B & f(B) ≲ 1
      ==> (λx ∈ A. THE y. f(x)={y}) ∈ (Π X ∈ A. X)"
apply (rule lam_type)
apply (drule bspec, assumption)
apply (elim conjE)
apply (erule lemma_aux [THEN exE], assumption)
apply (simp add: the_equality)
done

theorem AC13_AC1: "AC13(1) ==> AC1"
apply (unfold AC13_def AC1_def)
apply (fast elim!: AC13_AC1_lemma)
done
```

```
theorem AC11_AC14: "AC11 ==> AC14"
apply (unfold AC11_def AC14_def)
apply (fast intro!: AC10_AC13)
done

end
```

```
theory AC16_lemmas
imports AC_Equiv Hartog Cardinal_aux
begin

lemma cons_Diff_eq: "a∉A ==> cons(a,A)-{a}=A"
by fast

lemma nat_1_lepoll_iff: "1≲X <-> (∃x. x ∈ X)"
apply (unfold lepoll_def)
apply (rule iffI)
apply (fast intro: inj_is_fun [THEN apply_type])
apply (erule exE)
apply (rule_tac x = "λa ∈ 1. x" in exI)
apply (fast intro!: lam_injective)
done

lemma eqpoll_1_iff_singleton: "X≈1 <-> (∃x. X={x})"
apply (rule iffI)
apply (erule eqpollE)
apply (drule nat_1_lepoll_iff [THEN iffD1])
apply (fast intro!: lepoll_1_is_sing)
apply (fast intro!: singleton_eqpoll_1)
done

lemma cons_eqpoll_succ: "[| x≈n; y∉x |] ==> cons(y,x)≈succ(n)"
apply (unfold succ_def)
apply (fast elim!: cons_eqpoll_cong mem_irrefl)
done

lemma subsets_eqpoll_1_eq: "{Y ∈ Pow(X). Y≈1} = {{x}. x ∈ X}"
apply (rule equalityI)
apply (rule subsetI)
apply (erule CollectE)
apply (drule eqpoll_1_iff_singleton [THEN iffD1])
apply (fast intro!: RepFunI)
apply (rule subsetI)
apply (erule RepFunE)
apply (rule CollectI, fast)
apply (fast intro!: singleton_eqpoll_1)
done

lemma eqpoll_RepFun_sing: "X≈{{x}. x ∈ X}"
apply (unfold eqpoll_def bij_def)
apply (rule_tac x = "λx ∈ X. {x}" in exI)
apply (rule IntI)
apply (unfold inj_def surj_def, simp)
apply (fast intro!: lam_type RepFunI intro: singleton_eq_iff [THEN iffD1],
simp)
```

```
apply (fast intro!: lam_type)
done

lemma subsets_eqpoll_1_eqpoll: "{Y ∈ Pow(X). Y≈1}≈X"
apply (rule subsets_eqpoll_1_eq [THEN ssubst])
apply (rule eqpoll_RepFun_sing [THEN eqpoll_sym])
done

lemma InfCard_Least_in:
     "[| InfCard(x); y ⊆ x; y ≈ succ(z) |] ==> (LEAST i. i ∈ y) ∈ y"
apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll,
                         THEN succ_lepoll_imp_not_empty, THEN not_emptyE])
apply (fast intro: LeastI
            dest!: InfCard_is_Card [THEN Card_is_Ord]
            elim: Ord_in_Ord)
done

lemma subsets_lepoll_lemma1:
     "[| InfCard(x); n ∈ nat |]
      ==> {y ∈ Pow(x). y≈succ(succ(n))} ≲ x*{y ∈ Pow(x). y≈succ(n)}"
apply (unfold lepoll_def)
apply (rule_tac x = "λy ∈ {y ∈ Pow(x) . y≈succ (succ (n))}.
                    <LEAST i. i ∈ y, y-{LEAST i. i ∈ y}>" in exI)
apply (rule_tac d = "%z. cons (fst(z), snd(z))" in lam_injective)
 apply (blast intro!: Diff_sing_eqpoll intro: InfCard_Least_in)
apply (simp, blast intro: InfCard_Least_in)
done

lemma set_of_Ord_succ_Union: "(∀y ∈ z. Ord(y)) ==> z ⊆ succ(Union(z))"
apply (rule subsetI)
apply (case_tac "∀y ∈ z. y ⊆ x", blast )
apply (simp, erule bexE)
apply (rule_tac i=y and j=x in Ord_linear_le)
apply (blast dest: le_imp_subset elim: leE ltE)+
done

lemma subset_not_mem: "j ⊆ i ==> i ∉ j"
by (fast elim!: mem_irrefl)

lemma succ_Union_not_mem:
     "(!!y. y ∈ z ==> Ord(y)) ==> succ(Union(z)) ∉ z"
apply (rule set_of_Ord_succ_Union [THEN subset_not_mem], blast)
done

lemma Union_cons_eq_succ_Union:
     "Union(cons(succ(Union(z)),z)) = succ(Union(z))"
by fast

lemma Un_Ord_disj: "[| Ord(i); Ord(j) |] ==> i Un j = i | i Un j = j"
```

by *(fast dest!: le_imp_subset elim: Ord_linear_le)*

lemma *Union_eq_Un: "x ∈ X ==> Union(X) = x Un Union(X-{x})"*
by *fast*

lemma *Union_in_lemma [rule_format]:*
"n ∈ nat ==> ∀z. (∀y ∈ z. Ord(y)) & z≈n & z≠0 --> Union(z) ∈ z"
apply *(induct_tac "n")*
apply *(fast dest!: eqpoll_imp_lepoll [THEN lepoll_0_is_0])*
apply *(intro allI impI)*
apply *(erule natE)*
apply *(fast dest!: eqpoll_1_iff_singleton [THEN iffD1]*
intro!: Union_singleton, clarify)
apply *(elim not_emptyE)*
apply *(erule_tac x = "z-{xb}"* **in** *allE)*
apply *(erule impE)*
apply *(fast elim!: Diff_sing_eqpoll*
Diff_sing_eqpoll [THEN eqpoll_succ_imp_not_empty])
apply *(subgoal_tac "xb ∪ ⋃(z - {xb}) ∈ z")*
apply *(simp add: Union_eq_Un [symmetric])*
apply *(frule bspec, assumption)*
apply *(drule bspec)*
apply *(erule Diff_subset [THEN subsetD])*
apply *(drule Un_Ord_disj, assumption, auto)*
done

lemma *Union_in: "[| ∀x ∈ z. Ord(x); z≈n; z≠0; n ∈ nat |] ==> Union(z) ∈ z"*
by *(blast intro: Union_in_lemma)*

lemma *succ_Union_in_x:*
"[| InfCard(x); z ∈ Pow(x); z≈n; n ∈ nat |] ==> succ(Union(z)) ∈ x"
apply *(rule Limit_has_succ [THEN ltE])*
prefer *3* **apply** *assumption*
apply *(erule InfCard_is_Limit)*
apply *(case_tac "z=0")*
apply *(simp, fast intro!: InfCard_is_Limit [THEN Limit_has_0])*
apply *(rule ltI [OF PowD [THEN subsetD] InfCard_is_Card [THEN Card_is_Ord]], assumption)*
apply *(blast intro: Union_in*
InfCard_is_Card [THEN Card_is_Ord, THEN Ord_in_Ord])+
done

lemma *succ_lepoll_succ_succ:*
"[| InfCard(x); n ∈ nat |]
==> {y ∈ Pow(x). y≈succ(n)} ≲ {y ∈ Pow(x). y≈succ(succ(n))}"
apply *(unfold lepoll_def)*

```
apply (rule_tac x = "λz ∈ {y∈Pow(x). y≈succ(n)}. cons(succ(Union(z)),
z)"
       in exI)
apply (rule_tac d = "%z. z-{Union (z) }" in lam_injective)
apply (blast intro!: succ_Union_in_x succ_Union_not_mem
             intro: cons_eqpoll_succ Ord_in_Ord
             dest!: InfCard_is_Card [THEN Card_is_Ord])
apply (simp only: Union_cons_eq_succ_Union)
apply (rule cons_Diff_eq)
apply (fast dest!: InfCard_is_Card [THEN Card_is_Ord]
            elim: Ord_in_Ord
            intro!: succ_Union_not_mem)
done

lemma subsets_eqpoll_X:
     "[| InfCard(X); n ∈ nat |] ==> {Y ∈ Pow(X). Y≈succ(n)} ≈ X"
apply (induct_tac "n")
apply (rule subsets_eqpoll_1_eqpoll)
apply (rule eqpollI)
apply (rule subsets_lepoll_lemma1 [THEN lepoll_trans], assumption+)
apply (rule eqpoll_trans [THEN eqpoll_imp_lepoll])
 apply (erule eqpoll_refl [THEN prod_eqpoll_cong])
apply (erule InfCard_square_eqpoll)
apply (fast elim: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_trans]

            intro!: succ_lepoll_succ_succ)
done

lemma image_vimage_eq:
     "[| f ∈ surj(A,B); y ⊆ B |] ==> f``(converse(f)``y) = y"
apply (unfold surj_def)
apply (fast dest: apply_equality2 elim: apply_iff [THEN iffD2])
done

lemma vimage_image_eq: "[| f ∈ inj(A,B); y ⊆ A |] ==> converse(f)``(f``y)
= y"
by (fast elim!: inj_is_fun [THEN apply_Pair] dest: inj_equality)

lemma subsets_eqpoll:
     "A≈B ==> {Y ∈ Pow(A). Y≈n}≈{Y ∈ Pow(B). Y≈n}"
apply (unfold eqpoll_def)
apply (erule exE)
apply (rule_tac x = "λX ∈ {Y ∈ Pow (A) . ∃f. f ∈ bij (Y, n) }. f``X"
in exI)
apply (rule_tac d = "%Z. converse (f) ``Z" in lam_bijective)
apply (fast intro!: bij_is_inj [THEN restrict_bij, THEN bij_converse_bij,

                                THEN comp_bij]
            elim!: bij_is_fun [THEN fun_is_rel, THEN image_subset])
```

```
apply (blast intro!:  bij_is_inj [THEN restrict_bij]
                      comp_bij bij_converse_bij
                      bij_is_fun [THEN fun_is_rel, THEN image_subset])
apply (fast elim!: bij_is_inj [THEN vimage_image_eq])
apply (fast elim!: bij_is_surj [THEN image_vimage_eq])
done

lemma WO2_imp_ex_Card: "WO2 ==> ∃a. Card(a) & X≈a"
apply (unfold WO2_def)
apply (drule spec [of _ X])
apply (blast intro: Card_cardinal eqpoll_trans
         well_ord_Memrel [THEN well_ord_cardinal_eqpoll, THEN eqpoll_sym])
done

lemma lepoll_infinite: "[| X≲Y; ~Finite(X) |] ==> ~Finite(Y)"
by (blast intro: lepoll_Finite)

lemma infinite_Card_is_InfCard: "[| ~Finite(X); Card(X) |] ==> InfCard(X)"
apply (unfold InfCard_def)
apply (fast elim!: Card_is_Ord [THEN nat_le_infinite_Ord])
done

lemma WO2_infinite_subsets_eqpoll_X: "[| WO2; n ∈ nat; ~Finite(X) |]

        ==> {Y ∈ Pow(X). Y≈succ(n)}≈X"
apply (drule WO2_imp_ex_Card)
apply (elim allE exE conjE)
apply (frule eqpoll_imp_lepoll [THEN lepoll_infinite], assumption)
apply (drule infinite_Card_is_InfCard, assumption)
apply (blast intro: subsets_eqpoll subsets_eqpoll_X eqpoll_sym eqpoll_trans)

done

lemma well_ord_imp_ex_Card: "well_ord(X,R) ==> ∃a. Card(a) & X≈a"
by (fast elim!: well_ord_cardinal_eqpoll [THEN eqpoll_sym]
         intro!: Card_cardinal)

lemma well_ord_infinite_subsets_eqpoll_X:
     "[| well_ord(X,R); n ∈ nat; ~Finite(X) |] ==> {Y ∈ Pow(X). Y≈succ(n)}≈X"
apply (drule well_ord_imp_ex_Card)
apply (elim allE exE conjE)
apply (frule eqpoll_imp_lepoll [THEN lepoll_infinite], assumption)
apply (drule infinite_Card_is_InfCard, assumption)
apply (blast intro: subsets_eqpoll subsets_eqpoll_X eqpoll_sym eqpoll_trans)

done

end
```

```
theory WO2_AC16 imports AC_Equiv AC16_lemmas Cardinal_aux begin


definition
  recfunAC16 :: "[i,i,i,i] => i"  where
    "recfunAC16(f,h,i,a) ==
         transrec2(i, 0,
              %g r. if (∃y ∈ r. h‘g ⊆ y) then r
                    else r Un {f‘(LEAST i. h‘g ⊆ f‘i &
                         (∀b<a. (h‘b ⊆ f‘i --> (∀t ∈ r. ~ h‘b ⊆ t))))})"


lemma recfunAC16_0: "recfunAC16(f,h,0,a) = 0"
by (simp add: recfunAC16_def)

lemma recfunAC16_succ:
     "recfunAC16(f,h,succ(i),a) =
      (if (∃y ∈ recfunAC16(f,h,i,a). h ‘ i ⊆ y) then recfunAC16(f,h,i,a)

       else recfunAC16(f,h,i,a) Un
            {f ‘ (LEAST j. h ‘ i ⊆ f ‘ j &
             (∀b<a. (h‘b ⊆ f‘j
              --> (∀t ∈ recfunAC16(f,h,i,a). ~ h‘b ⊆ t))))})"
apply (simp add: recfunAC16_def)
done

lemma recfunAC16_Limit: "Limit(i)
        ==> recfunAC16(f,h,i,a) = (⋃j<i. recfunAC16(f,h,j,a))"
by (simp add: recfunAC16_def transrec2_Limit)


lemma transrec2_mono_lemma [rule_format]:
     "[| !!g r. r ⊆ B(g,r);  Ord(i) |]
      ==> j<i --> transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
apply (erule trans_induct)
apply (rule Ord_cases, assumption+, fast)
apply (simp (no_asm_simp))
apply (blast elim!: leE)
apply (simp add: transrec2_Limit)
apply (blast intro: OUN_I ltI Ord_in_Ord [THEN le_refl]
```

```
              elim!: Limit_has_succ [THEN ltE])
done

lemma transrec2_mono:
     "[| !!g r. r ⊆ B(g,r); j≤i |]
      ==> transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
apply (erule leE)
apply (rule transrec2_mono_lemma)
apply (auto intro: lt_Ord2 )
done

lemma recfunAC16_mono:
       "i≤j ==> recfunAC16(f, g, i, a) ⊆ recfunAC16(f, g, j, a)"
apply (unfold recfunAC16_def)
apply (rule transrec2_mono, auto)
done

lemma lemma3_1:
    "[| ∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) --> (∃! Y. Y ∈ F(y)
& f(z)<=Y);
        ∀i j. i≤j --> F(i) ⊆ F(j); j≤i; i<x; z<a;
        V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W |]
     ==> V = W"
apply (erule asm_rl allE impE)+
apply (drule subsetD, assumption, blast)
done

lemma lemma3:
    "[| ∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) --> (∃! Y. Y ∈ F(y)
& f(z)<=Y);
        ∀i j. i≤j --> F(i) ⊆ F(j); i<x; j<x; z<a;
        V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W |]
     ==> V = W"
apply (rule_tac j=j in Ord_linear_le [OF lt_Ord lt_Ord], assumption+)
apply (erule lemma3_1 [symmetric], assumption+)
apply (erule lemma3_1, assumption+)
done

lemma lemma4:
```

```
     "[| ∀y<x. F(y) ⊆ X &
                (∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) -->
                 (∃! Y. Y ∈ F(y) & h(x) ⊆ Y));
         x < a |]
      ==> ∀y<x. ∀z<a. z < y | (∃Y ∈ F(y). h(z) ⊆ Y) -->
                      (∃! Y. Y ∈ F(y) & h(z) ⊆ Y)"
apply (intro oallI impI)
apply (drule ospec, assumption, clarify)
apply (blast elim!: oallE )
done
```

```
lemma lemma5:
     "[| ∀y<x. F(y) ⊆ X &
               (∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) -->
                (∃! Y. Y ∈ F(y) & h(x) ⊆ Y));
         x < a; Limit(x); ∀i j. i≤j --> F(i) ⊆ F(j) |]
      ==> (⋃x<x. F(x)) ⊆ X &
          (∀xa<a. xa < x | (∃x ∈ ⋃x<x. F(x). h(xa) ⊆ x)
                --> (∃! Y. Y ∈ (⋃x<x. F(x)) & h(xa) ⊆ Y))"
apply (rule conjI)
apply (rule subsetI)
apply (erule OUN_E)
apply (drule ospec, assumption, fast)
apply (drule lemma4, assumption)
apply (rule oallI)
apply (rule impI)
apply (erule disjE)
apply (frule ospec, erule Limit_has_succ, assumption)
apply (drule_tac A = a and x = xa in ospec, assumption)
apply (erule impE, rule le_refl [THEN disjI1], erule lt_Ord)
apply (blast intro: lemma3 Limit_has_succ)
apply (blast intro: lemma3)
done
```

```
lemma dbl_Diff_eqpoll_Card:
      "[| A≈a; Card(a); ~Finite(a); B≺a; C≺a |] ==> A - B - C≈a"
by (blast intro: Diff_lesspoll_eqpoll_Card)
```

```
lemma Finite_lesspoll_infinite_Ord:
      "[| Finite(X); ~Finite(a); Ord(a) |] ==> X≺a"
apply (unfold lesspoll_def)
apply (rule conjI)
apply (drule nat_le_infinite_Ord [THEN le_imp_lepoll], assumption)
apply (unfold Finite_def)
apply (blast intro: leI [THEN le_imp_subset, THEN subset_imp_lepoll]
                    ltI eqpoll_imp_lepoll lepoll_trans)
apply (blast intro: eqpoll_sym [THEN eqpoll_trans])
done

lemma Union_lesspoll:
     "[| ∀x ∈ X. x lepoll n & x ⊆ T; well_ord(T, R); X lepoll b;
         b<a; ~Finite(a); Card(a); n ∈ nat |]
      ==> Union(X)≺a"
apply (case_tac "Finite (X)")
apply (blast intro: Card_is_Ord Finite_lesspoll_infinite_Ord
                    lepoll_nat_imp_Finite Finite_Union)
apply (drule lepoll_imp_ex_le_eqpoll)
apply (erule lt_Ord)
apply (elim exE conjE)
apply (frule eqpoll_imp_lepoll [THEN lepoll_infinite], assumption)
apply (erule eqpoll_sym [THEN eqpoll_def [THEN def_imp_iff, THEN iffD1],
                         THEN exE])
apply (frule bij_is_surj [THEN surj_image_eq])
apply (drule image_fun [OF bij_is_fun subset_refl])
apply (drule sym [THEN trans], assumption)
apply (blast intro: lt_Ord UN_lepoll lt_Card_imp_lesspoll
                    lt_trans1 lesspoll_trans1)
done
```

```
lemma Un_sing_eq_cons: "A Un {a} = cons(a, A)"
by fast

lemma Un_lepoll_succ: "A lepoll B ==> A Un {a} lepoll succ(B)"
apply (simp add: Un_sing_eq_cons succ_def)
apply (blast elim!: mem_irrefl intro: cons_lepoll_cong)
done

lemma Diff_UN_succ_empty: "Ord(a) ==> F(a) - (⋃b<succ(a). F(b)) = 0"
```

by *(fast intro!: le_refl)*

lemma *Diff_UN_succ_subset: "Ord(a) ==> F(a) Un X - (⋃b<succ(a). F(b)) ⊆ X"*
by *blast*

lemma *recfunAC16_Diff_lepoll_1:*
"Ord(x)
==> recfunAC16(f, g, x, a) - (⋃i<x. recfunAC16(f, g, i, a)) lepoll 1"
apply *(erule Ord_cases)*
apply *(simp add: recfunAC16_0 empty_subsetI [THEN subset_imp_lepoll])*

prefer *2* **apply** *(simp add: recfunAC16_Limit Diff_cancel*
empty_subsetI [THEN subset_imp_lepoll])

apply *(simp add: recfunAC16_succ*
Diff_UN_succ_empty [of _ "%j. recfunAC16(f,g,j,a)"]
empty_subsetI [THEN subset_imp_lepoll])
apply *(best intro: Diff_UN_succ_subset [THEN subset_imp_lepoll]*
singleton_eqpoll_1 [THEN eqpoll_imp_lepoll] lepoll_trans)
done

lemma *in_Least_Diff:*
"[| z ∈ F(x); Ord(x) |]
==> z ∈ F(LEAST i. z ∈ F(i)) - (⋃j<(LEAST i. z ∈ F(i)). F(j))"
by *(fast elim: less_LeastE elim!: LeastI)*

lemma *Least_eq_imp_ex:*
"[| (LEAST i. w ∈ F(i)) = (LEAST i. z ∈ F(i));
w ∈ (⋃i<a. F(i)); z ∈ (⋃i<a. F(i)) |]
==> ∃b<a. w ∈ (F(b) - (⋃c<b. F(c))) & z ∈ (F(b) - (⋃c<b. F(c)))"
apply *(elim OUN_E)*
apply *(drule in_Least_Diff, erule lt_Ord)*
apply *(frule in_Least_Diff, erule lt_Ord)*
apply *(rule oexI, force)*
apply *(blast intro: lt_Ord Least_le [THEN lt_trans1])*
done

lemma *two_in_lepoll_1: "[| A lepoll 1; a ∈ A; b ∈ A |] ==> a=b"*
by *(fast dest!: lepoll_1_is_sing)*

lemma *UN_lepoll_index:*
"[| ∀i<a. F(i)-(⋃j<i. F(j)) lepoll 1; Limit(a) |]
==> (⋃x<a. F(x)) lepoll a"
apply *(rule lepoll_def [THEN def_imp_iff [THEN iffD2]])*
apply *(rule_tac x = "λz ∈ (⋃x<a. F (x)). LEAST i. z ∈ F (i) "* **in** *exI)*

```
apply (unfold inj_def)
apply (rule CollectI)
apply (rule lam_type)
apply (erule OUN_E)
apply (erule Least_in_Ord)
apply (erule ltD)
apply (erule lt_Ord2)
apply (intro ballI)
apply (simp (no_asm_simp))
apply (rule impI)
apply (drule Least_eq_imp_ex, assumption+)
apply (fast elim!: two_in_lepoll_1)
done


lemma recfunAC16_lepoll_index: "Ord(y) ==> recfunAC16(f, h, y, a) lepoll
y"
apply (erule trans_induct3)

apply (simp (no_asm_simp) add: recfunAC16_0 lepoll_refl)

apply (simp (no_asm_simp) add: recfunAC16_succ)
apply (blast dest!: succI1 [THEN rev_bspec]
        intro: subset_succI [THEN subset_imp_lepoll] Un_lepoll_succ

               lepoll_trans)
apply (simp (no_asm_simp) add: recfunAC16_Limit)
apply (blast intro: lt_Ord [THEN recfunAC16_Diff_lepoll_1] UN_lepoll_index)
done


lemma Union_recfunAC16_lesspoll:
     "[| recfunAC16(f,g,y,a) ⊆ {X ∈ Pow(A). X≈n};
         A≈a;  y<a;  ~Finite(a);  Card(a);  n ∈ nat |]
      ==> Union(recfunAC16(f,g,y,a))≺a"
apply (erule eqpoll_def [THEN def_imp_iff, THEN iffD1, THEN exE])
apply (rule_tac T=A in Union_lesspoll, simp_all)
apply (blast intro!: eqpoll_imp_lepoll)
apply (blast intro: bij_is_inj Card_is_Ord [THEN well_ord_Memrel]
               well_ord_rvimage)
apply (erule lt_Ord [THEN recfunAC16_lepoll_index])
done

lemma dbl_Diff_eqpoll:
     "[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
         Card(a); ~ Finite(a); A≈a;
         k ∈ nat;  y<a;
         h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)}) |]
      ==> A - Union(recfunAC16(f, h, y, a)) - h`y≈a"
```

```
apply (rule dbl_Diff_eqpoll_Card, simp_all)
apply (simp add: Union_recfunAC16_lesspoll)
apply (rule Finite_lesspoll_infinite_Ord)
apply (rule Finite_def [THEN def_imp_iff, THEN iffD2])
apply (blast dest: ltD bij_is_fun [THEN apply_type], assumption)
apply (blast intro: Card_is_Ord)
done
```

```
lemmas disj_Un_eqpoll_nat_sum =
    eqpoll_trans [THEN eqpoll_trans,
                  OF disj_Un_eqpoll_sum sum_eqpoll_cong nat_sum_eqpoll_sum,
                  standard]
```

```
lemma Un_in_Collect: "[| x ∈ Pow(A - B - h`i); x≈m;
        h ∈ bij(a, {x ∈ Pow(A) . x≈k}); i<a; k ∈ nat; m ∈ nat |]
        ==> h ` i Un x ∈ {x ∈ Pow(A) . x≈k #+ m}"
by (blast intro: disj_Un_eqpoll_nat_sum
          dest:  ltD bij_is_fun [THEN apply_type])
```

```
lemma lemma6:
     "[| ∀y<succ(j). F(y)<=X & (∀x<a. x<y | P(x,y) --> Q(x,y)); succ(j)<a
|]
      ==> F(j)<=X & (∀x<a. x<j | P(x,j) --> Q(x,j))"
by (blast intro!: lt_Ord succI1 [THEN ltI, THEN lt_Ord, THEN le_refl])
```

```
lemma lemma7:
     "[| ∀x<a. x<j | P(x,j) --> Q(x,j);  succ(j)<a |]
      ==> P(j,j) --> (∀x<a. x≤j | P(x,j) --> Q(x,j))"
by (fast elim!: leE)
```

```
lemma ex_subset_eqpoll:
     "[| A≈a; ~ Finite(a); Ord(a); m ∈ nat |] ==> ∃X ∈ Pow(A). X≈m"
apply (rule lepoll_imp_eqpoll_subset [of m A, THEN exE])
 apply (rule lepoll_trans, rule leI [THEN le_imp_lepoll])
```

```
  apply (blast intro: lt_trans2 [OF ltI nat_le_infinite_Ord] Ord_nat)
apply (erule eqpoll_sym [THEN eqpoll_imp_lepoll])
apply (fast elim!: eqpoll_sym)
done

lemma subset_Un_disjoint: "[| A ⊆ B Un C; A Int C = 0 |] ==> A ⊆ B"
by blast


lemma Int_empty:
     "[| X ∈ Pow(A - Union(B) -C); T ∈ B; F ⊆ T |] ==> F Int X = 0"
by blast
```

```
lemma subset_imp_eq_lemma:
     "m ∈ nat ==> ∀A B. A ⊆ B & m lepoll A & B lepoll m --> A=B"
apply (induct_tac "m")
apply (fast dest!: lepoll_0_is_0)
apply (intro allI impI)
apply (elim conjE)
apply (rule succ_lepoll_imp_not_empty [THEN not_emptyE], assumption)
apply (frule subsetD [THEN Diff_sing_lepoll], assumption+)
apply (frule lepoll_Diff_sing)
apply (erule allE impE)+
apply (rule conjI)
prefer 2 apply fast
apply fast
apply (blast elim: equalityE)
done


lemma subset_imp_eq: "[| A ⊆ B; m lepoll A; B lepoll m; m ∈ nat |] ==>
A=B"
by (blast dest!: subset_imp_eq_lemma)


lemma bij_imp_arg_eq:
     "[| f ∈ bij(a, {Y ∈ X. Y≈succ(k)}); k ∈ nat; f`b ⊆ f`y; b<a; y<a
|]
      ==> b=y"
apply (drule subset_imp_eq)
apply (erule_tac [3] nat_succI)
apply (unfold bij_def inj_def)
apply (blast intro: eqpoll_sym eqpoll_imp_lepoll
```

```
              dest:  ltD apply_type)+
done
```

lemma *ex_next_set:*

```
     "[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
         Card(a); ~ Finite(a); A≈a;
         k ∈ nat; m ∈ nat; y<a;
         h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
         ~ (∃Y ∈ recfunAC16(f, h, y, a). h`y ⊆ Y) |]
      ==> ∃X ∈ {Y ∈ Pow(A). Y≈succ(k #+ m)}. h`y ⊆ X &
                (∀b<a. h`b ⊆ X -->
                (∀T ∈ recfunAC16(f, h, y, a). ~ h`b ⊆ T))"
apply (erule_tac m1=m in dbl_Diff_eqpoll [THEN ex_subset_eqpoll, THEN
bexE],
       assumption+)
apply (erule Card_is_Ord, assumption)
apply (frule Un_in_Collect, (erule asm_rl nat_succI)+)
apply (erule CollectE)
apply (rule rev_bexI, simp)
apply (rule conjI, blast)
apply (intro ballI impI oallI notI)
apply (drule subset_Un_disjoint, rule Int_empty, assumption+)
apply (blast dest: bij_imp_arg_eq)
done
```

lemma *ex_next_Ord:*

```
     "[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
         Card(a); ~ Finite(a); A≈a;
         k ∈ nat; m ∈ nat; y<a;
         h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
         f ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k #+ m)});
         ~ (∃Y ∈ recfunAC16(f, h, y, a). h`y ⊆ Y) |]
      ==> ∃c<a. h`y ⊆ f`c &
                (∀b<a. h`b ⊆ f`c -->
                (∀T ∈ recfunAC16(f, h, y, a). ~ h`b ⊆ T))"
apply (drule ex_next_set, assumption+)
apply (erule bexE)
apply (rule_tac x="converse(f)`X" in oexI)
apply (simp add: right_inverse_bij)
apply (blast intro: bij_converse_bij bij_is_fun [THEN apply_type] ltI
                    Card_is_Ord)
done
```

```
lemma lemma8:
     "[| ∀x<a. x<j | (∃xa ∈ F(j). P(x, xa))
         --> (∃! Y. Y ∈ F(j) & P(x, Y)); F(j) ⊆ X;
         L ∈ X; P(j, L) & (∀x<a. P(x, L) --> (∀xa ∈ F(j). ~P(x, xa)))
|]
      ==> F(j) Un {L} ⊆ X &
          (∀x<a. x≤j | (∃xa ∈ (F(j) Un {L}). P(x, xa)) -->
                 (∃! Y. Y ∈ (F(j) Un {L}) & P(x, Y)))"
apply (rule conjI)
apply (fast intro!: singleton_subsetI)
apply (rule oallI)
apply (blast elim!: leE oallE)
done
```

```
lemma main_induct:
     "[| b < a; f ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k #+ m)});
         h ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k)});
         ~Finite(a); Card(a); A≈a; k ∈ nat; m ∈ nat |]
      ==> recfunAC16(f, h, b, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)} &
          (∀x<a. x < b | (∃Y ∈ recfunAC16(f, h, b, a). h ` x ⊆ Y) -->

          (∃! Y. Y ∈ recfunAC16(f, h, b, a) & h ` x ⊆ Y))"
apply (erule lt_induct)
apply (frule lt_Ord)
apply (erule Ord_cases)

apply (simp add: recfunAC16_0)

prefer 2  apply (simp add: recfunAC16_Limit)
          apply (rule lemma5, assumption+)
          apply (blast dest!: recfunAC16_mono)

apply clarify
apply (erule lemma6 [THEN conjE], assumption)
apply (simp (no_asm_simp) split del: split_if add: recfunAC16_succ)
apply (rule conjI [THEN split_if [THEN iffD2]])
 apply (simp, erule lemma7, assumption)
apply (rule impI)
apply (rule ex_next_Ord [THEN oexE],
```

```
      assumption+, rule le_refl [THEN lt_trans], assumption+)
apply (erule lemma8, assumption)
 apply (rule bij_is_fun [THEN apply_type], assumption)
 apply (erule Least_le [THEN lt_trans2, THEN ltD])
  apply (erule lt_Ord)
 apply (erule succ_leI)
apply (erule LeastI)
apply (erule lt_Ord)
done
```

```
lemma lemma_simp_induct:
     "[| ∀b. b<a --> F(b) ⊆ S & (∀x<a. (x<b | (∃Y ∈ F(b). f`x ⊆ Y))

                                --> (∃! Y. Y ∈ F(b) & f`x ⊆ Y));
        f ∈ a->f``(a); Limit(a);
        ∀i j. i≤j --> F(i) ⊆ F(j) |]
       ==> (⋃j<a. F(j)) ⊆ S &
           (∀x ∈ f``a. ∃! Y. Y ∈ (⋃j<a. F(j)) & x ⊆ Y)"
apply (rule conjI)
apply (rule subsetI)
apply (erule OUN_E, blast)
apply (rule ballI)
apply (erule imageE)
apply (drule ltI, erule Limit_is_Ord)
apply (drule Limit_has_succ, assumption)
apply (frule_tac x1="succ(xa)" in spec [THEN mp], assumption)
apply (erule conjE)
apply (drule ospec)

apply (erule leI [THEN succ_leE])
apply (erule impE)
apply (fast elim!: leI [THEN succ_leE, THEN lt_Ord, THEN le_refl])
apply (drule apply_equality, assumption)
apply (elim conjE ex1E)

apply (rule ex1I, blast)
apply (elim conjE OUN_E)
apply (erule_tac i="succ(xa)" and j=aa
       in Ord_linear_le [OF lt_Ord lt_Ord], assumption)
prefer 2
apply (drule spec [THEN spec, THEN mp, THEN subsetD], assumption+, blast)


apply (drule_tac x1=aa in spec [THEN mp], assumption)
```

```
apply (frule succ_leE)
apply (drule spec [THEN spec, THEN mp, THEN subsetD], assumption+, blast)

done
```

```
theorem WO2_AC16: "[| WO2; 0<m; k ∈ nat; m ∈ nat |] ==> AC16(k #+ m,k)"
apply (unfold AC16_def)
apply (rule allI)
apply (rule impI)
apply (frule WO2_infinite_subsets_eqpoll_X, assumption+)
apply (frule_tac n="k #+ m" in WO2_infinite_subsets_eqpoll_X, simp, simp)

apply (frule WO2_imp_ex_Card)
apply (elim exE conjE)
apply (drule eqpoll_trans [THEN eqpoll_sym,
                          THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]],
       assumption)
apply (drule eqpoll_trans [THEN eqpoll_sym,
                          THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]],

       assumption+)
apply (elim exE)
apply (rename_tac h)
apply (rule_tac x = "⋃j<a. recfunAC16 (h,f,j,a) " in exI)
apply (rule_tac P="%z. ?Y & (∀x ∈ z. ?Z (x))"
       in bij_is_surj [THEN surj_image_eq, THEN subst], assumption)
apply (rule lemma_simp_induct)
apply (blast del: conjI notI
             intro!: main_induct eqpoll_imp_lepoll [THEN lepoll_infinite]
)
apply (blast intro: bij_is_fun [THEN surj_image, THEN surj_is_fun])
apply (erule eqpoll_imp_lepoll [THEN lepoll_infinite,
                                THEN infinite_Card_is_InfCard,
                                THEN InfCard_is_Limit],
       assumption+)
apply (blast dest!: recfunAC16_mono)
done

end
```

```
theory AC16_WO4
imports AC16_lemmas
begin
```

```
lemma lemma1:
     "[| Finite(A); 0<m; m ∈ nat |]
      ==> ∃a f. Ord(a) & domain(f) = a &
                (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m)"
apply (unfold Finite_def)
apply (erule bexE)
apply (drule eqpoll_sym [THEN eqpoll_def [THEN def_imp_iff, THEN iffD1]])
apply (erule exE)
apply (rule_tac x = n in exI)
apply (rule_tac x = "λi ∈ n. {f`i}" in exI)
apply (simp add: ltD bij_def surj_def)
apply (fast intro!: ltI nat_into_Ord lam_funtype [THEN domain_of_fun]

               singleton_eqpoll_1 [THEN eqpoll_imp_lepoll, THEN lepoll_trans]

                    nat_1_lepoll_iff [THEN iffD2]
          elim!: apply_type ltE)
done
```

```
lemmas well_ord_paired = paired_bij [THEN bij_is_inj, THEN well_ord_rvimage]

lemma lepoll_trans1: "[| A ≲ B; ~ A ≲ C |] ==> ~ B ≲ C"
by (blast intro: lepoll_trans)
```

```
lemmas lepoll_paired = paired_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll]

lemma lemma2: "∃y R. well_ord(y,R) & x Int y = 0 & ~y ≲ z & ~Finite(y)"
apply (rule_tac x = "{{a,x}. a ∈ nat Un Hartog (z) }" in exI)
apply (rule well_ord_Un [OF Ord_nat [THEN well_ord_Memrel]
                        Ord_Hartog [THEN well_ord_Memrel], THEN exE])
apply (blast intro!: Ord_Hartog well_ord_Memrel well_ord_paired
                    lepoll_trans1 [OF _ not_Hartog_lepoll_self]
                    lepoll_trans [OF subset_imp_lepoll lepoll_paired]
       elim!: nat_not_Finite [THEN notE]
       elim: mem_asym
```

```
       dest!: Un_upper1 [THEN subset_imp_lepoll, THEN lepoll_Finite]
              lepoll_paired [THEN lepoll_Finite])
done

lemma infinite_Un: "~Finite(B) ==> ~Finite(A Un B)"
by (blast intro: subset_Finite)
```

```
lemma succ_not_lepoll_lemma:
     "[| ~(∃x ∈ A. f`x=y); f ∈ inj(A, B); y ∈ B |]
      ==> (λa ∈ succ(A). if(a=A, y, f`a)) ∈ inj(succ(A), B)"
apply (rule_tac d = "%z. if (z=y, A, converse (f) `z) " in lam_injective)
apply (force simp add: inj_is_fun [THEN apply_type])

apply (simp (no_asm_simp))
apply force
done

lemma succ_not_lepoll_imp_eqpoll: "[| ~A ≈ B; A ≲ B |] ==> succ(A)
≲ B"
apply (unfold lepoll_def eqpoll_def bij_def surj_def)
apply (fast elim!: succ_not_lepoll_lemma inj_is_fun)
done
```

```
lemmas ordertype_eqpoll =
       ordermap_bij [THEN exI [THEN eqpoll_def [THEN def_imp_iff, THEN
iffD2]]]

lemma cons_cons_subset:
     "[| a ⊆ y; b ∈ y-a; u ∈ x |] ==> cons(b, cons(u, a)) ∈ Pow(x Un
y)"
by fast
```

```
lemma cons_cons_eqpoll:
     "[| a ≈ k; a ⊆ y; b ∈ y-a; u ∈ x; x Int y = 0 |]
      ==> cons(b, cons(u, a)) ≈ succ(succ(k))"
by (fast intro!: cons_eqpoll_succ)

lemma set_eq_cons:
     "[| succ(k) ≈ A; k ≈ B; B ⊆ A; a ∈ A-B; k ∈ nat |] ==> A = cons(a,
B)"
apply (rule equalityI)
prefer 2 apply fast
apply (rule Diff_eq_0_iff [THEN iffD1])
apply (rule equals0I)
apply (drule eqpoll_sym [THEN eqpoll_imp_lepoll])
apply (drule eqpoll_sym [THEN cons_eqpoll_succ], fast)
apply (drule cons_eqpoll_succ, fast)
apply (fast elim!: lepoll_trans [THEN lepoll_trans, THEN succ_lepoll_natE,
         OF eqpoll_sym [THEN eqpoll_imp_lepoll] subset_imp_lepoll])
done

lemma cons_eqE: "[| cons(x,a) = cons(y,a); x ∉ a |] ==> x = y "
by (fast elim!: equalityE)

lemma eq_imp_Int_eq: "A = B ==> A Int C = B Int C"
by blast




lemma eqpoll_sum_imp_Diff_lepoll_lemma [rule_format]:
     "[| k ∈ nat; m ∈ nat |]
      ==> ∀A B. A ≈ k #+ m & k ≲ B & B ⊆ A --> A-B ≲ m"
apply (induct_tac "k")
apply (simp add: add_0)
apply (blast intro: eqpoll_imp_lepoll lepoll_trans
                    Diff_subset [THEN subset_imp_lepoll])
apply (intro allI impI)
apply (rule succ_lepoll_imp_not_empty [THEN not_emptyE], fast)
apply (erule_tac x = "A - {xa}" in allE)
apply (erule_tac x = "B - {xa}" in allE)
apply (erule impE)
apply (simp add: add_succ)
apply (fast intro!: Diff_sing_eqpoll lepoll_Diff_sing)
apply (subgoal_tac "A - {xa} - (B - {xa}) = A - B", simp)
apply blast
done

lemma eqpoll_sum_imp_Diff_lepoll:
```

```
     "[| A ≈ succ(k #+ m); B ⊆ A; succ(k) ≲ B;  k ∈ nat; m ∈ nat |]

      ==> A-B ≲ m"
apply (simp only: add_succ [symmetric])
apply (blast intro: eqpoll_sum_imp_Diff_lepoll_lemma)
done
```

```
lemma eqpoll_sum_imp_Diff_eqpoll_lemma [rule_format]:
     "[| k ∈ nat; m ∈ nat |]
      ==> ∀A B. A ≈ k #+ m & k ≈ B & B ⊆ A --> A-B ≈ m"
apply (induct_tac "k")
apply (force dest!: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_0_is_0])
apply (intro allI impI)
apply (rule succ_lepoll_imp_not_empty [THEN not_emptyE])
apply (fast elim!: eqpoll_imp_lepoll)
apply (erule_tac x = "A - {xa}" in allE)
apply (erule_tac x = "B - {xa}" in allE)
apply (erule impE)
apply (force intro: eqpoll_sym intro!: Diff_sing_eqpoll)
apply (subgoal_tac "A - {xa} - (B - {xa}) = A - B", simp)
apply blast
done

lemma eqpoll_sum_imp_Diff_eqpoll:
     "[| A ≈ succ(k #+ m); B ⊆ A; succ(k) ≈ B; k ∈ nat; m ∈ nat |]

      ==> A-B ≈ m"
apply (simp only: add_succ [symmetric])
apply (blast intro: eqpoll_sum_imp_Diff_eqpoll_lemma)
done
```

```
lemma subsets_lepoll_0_eq_unit: "{x ∈ Pow(X). x ≲ 0} = {0}"
by (fast dest!: lepoll_0_is_0 intro!: lepoll_refl)

lemma subsets_lepoll_succ:
     "n ∈ nat ==> {z ∈ Pow(y). z ≲ succ(n)} =
                  {z ∈ Pow(y). z ≲ n} Un {z ∈ Pow(y). z ≈ succ(n)}"
by (blast intro: leI le_imp_lepoll nat_into_Ord
                    lepoll_trans eqpoll_imp_lepoll
          dest!: lepoll_succ_disj)
```

```
lemma Int_empty:
     "n ∈ nat ==> {z ∈ Pow(y). z ≲ n} Int {z ∈ Pow(y). z ≈ succ(n)}
= 0"
by (blast intro: eqpoll_sym [THEN eqpoll_imp_lepoll, THEN lepoll_trans]

                 succ_lepoll_natE)

locale AC16 =
  fixes x and y and k and l and m and t_n and R and MM and LL and
GG and s
  defines k_def:      "k   == succ(l)"
      and MM_def:     "MM  == {v ∈ t_n. succ(k) ≲ v Int y}"
      and LL_def:     "LL  == {v Int y. v ∈ MM}"
      and GG_def:     "GG  == λv ∈ LL. (THE w. w ∈ MM & v ⊆ w) - v"
      and s_def:      "s(u) == {v ∈ t_n. u ∈ v & k ≲ v Int y}"
  assumes all_ex:     "∀z ∈ {z ∈ Pow(x Un y) . z ≈ succ(k)}.
                       ∃! w. w ∈ t_n & z ⊆ w "
    and disjoint[iff]:  "x Int y = 0"
    and "includes":  "t_n ⊆ {v ∈ Pow(x Un y). v ≈ succ(k #+ m)}"
    and WO_R[iff]:      "well_ord(y,R)"
    and lnat[iff]:      "l ∈ nat"
    and mnat[iff]:      "m ∈ nat"
    and mpos[iff]:      "0<m"
    and Infinite[iff]:  "~ Finite(y)"
    and noLepoll:  "~ y ≲ {v ∈ Pow(x). v ≈ m}"

lemma (in AC16) knat [iff]: "k ∈ nat"
by (simp add: k_def)

lemma (in AC16) Diff_Finite_eqpoll: "[| l ≈ a; a ⊆ y |] ==> y - a ≈
y"
apply (insert WO_R Infinite lnat)
apply (rule eqpoll_trans)
apply (rule Diff_lesspoll_eqpoll_Card)
apply (erule well_ord_cardinal_eqpoll [THEN eqpoll_sym])
apply (blast intro: lesspoll_trans1
            intro!: Card_cardinal
                    Card_cardinal [THEN Card_is_Ord, THEN nat_le_infinite_Ord,
                                   THEN le_imp_lepoll]
            dest: well_ord_cardinal_eqpoll
                  eqpoll_sym  eqpoll_imp_lepoll
                  n_lesspoll_nat [THEN lesspoll_trans2]
```

```
                    well_ord_cardinal_eqpoll [THEN eqpoll_sym,
                          THEN eqpoll_imp_lepoll, THEN lepoll_infinite])+
done


lemma (in AC16) s_subset: "s(u) ⊆ t_n"
by (unfold s_def, blast)

lemma (in AC16) sI:
      "[| w ∈ t_n; cons(b,cons(u,a)) ⊆ w; a ⊆ y; b ∈ y-a; l ≈ a |]

       ==> w ∈ s(u)"
apply (unfold s_def succ_def k_def)
apply (blast intro!: eqpoll_imp_lepoll [THEN cons_lepoll_cong]
             intro: subset_imp_lepoll lepoll_trans)
done

lemma (in AC16) in_s_imp_u_in: "v ∈ s(u) ==> u ∈ v"
by (unfold s_def, blast)


lemma (in AC16) ex1_superset_a:
     "[| l ≈ a;  a ⊆ y;  b ∈ y - a;  u ∈ x |]
      ==> ∃! c. c ∈ s(u) & a ⊆ c & b ∈ c"
apply (rule all_ex [simplified k_def, THEN ballE])
 apply (erule ex1E)
 apply (rule_tac a = w in ex1I, blast intro: sI)
 apply (blast dest: s_subset [THEN subsetD] in_s_imp_u_in)
apply (blast del: PowI
             intro!: cons_cons_subset eqpoll_sym [THEN cons_cons_eqpoll])
done

lemma (in AC16) the_eq_cons:
     "[| ∀v ∈ s(u). succ(l) ≈ v Int y;
         l ≈ a;  a ⊆ y;  b ∈ y - a;  u ∈ x |]
      ==> (THE c. c ∈ s(u) & a ⊆ c & b ∈ c) Int y = cons(b, a)"
apply (frule ex1_superset_a [THEN theI], assumption+)
apply (rule set_eq_cons)
apply (fast+)
done

lemma (in AC16) y_lepoll_subset_s:
     "[| ∀v ∈ s(u). succ(l) ≈ v Int y;
         l ≈ a;  a ⊆ y;  u ∈ x |]
      ==> y ≲ {v ∈ s(u). a ⊆ v}"
apply (rule Diff_Finite_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll,

                                  THEN lepoll_trans],  fast+)
apply (rule_tac  f3 = "λb ∈ y-a. THE c. c ∈ s (u) & a ⊆ c & b ∈ c"
```

```
        in exI [THEN lepoll_def [THEN def_imp_iff, THEN iffD2]])
apply (simp add: inj_def)
apply (rule conjI)
apply (rule lam_type)
apply (frule ex1_superset_a [THEN theI], fast+, clarify)
apply (rule cons_eqE [of _ a])
apply (drule_tac A = "THE c. ?P (c) " and C = y in eq_imp_Int_eq)
apply (simp_all add: the_eq_cons)
done
```

```
lemma (in AC16) x_imp_not_y [dest]: "a ∈ x ==> a ∉ y"
by (blast dest:  disjoint [THEN equalityD1, THEN subsetD, OF IntI])

lemma (in AC16) w_Int_eq_w_Diff:
     "w ⊆ x Un y ==> w Int (x - {u}) = w - cons(u, w Int y)"
by blast

lemma (in AC16) w_Int_eqpoll_m:
     "[| w ∈ {v ∈ s(u). a ⊆ v};
         l ≈ a;  u ∈ x;
         ∀v ∈ s(u). succ(l) ≈ v Int y |]
      ==> w Int (x - {u}) ≈ m"
apply (erule CollectE)
apply (subst w_Int_eq_w_Diff)
apply (fast dest!: s_subset [THEN subsetD]
                   "includes" [simplified k_def, THEN subsetD])
apply (blast dest: s_subset [THEN subsetD]
                   "includes" [simplified k_def, THEN subsetD]
            dest: eqpoll_sym [THEN cons_eqpoll_succ, THEN eqpoll_sym]

                  in_s_imp_u_in
           intro!: eqpoll_sum_imp_Diff_eqpoll)
done
```

```
lemma (in AC16) eqpoll_m_not_empty: "a ≈ m ==> a ≠ 0"
```

```
apply (insert mpos)
apply (fast elim!: zero_lt_natE dest!: eqpoll_succ_imp_not_empty)
done

lemma (in AC16) cons_cons_in:
     "[| z ∈ xa Int (x - {u}); l ≈ a; a ⊆ y; u ∈ x |]
      ==> ∃! w. w ∈ t_n & cons(z, cons(u, a)) ⊆ w"
apply (rule all_ex [THEN bspec])
apply (unfold k_def)
apply (fast intro!: cons_eqpoll_succ elim: eqpoll_sym)
done

lemma (in AC16) subset_s_lepoll_w:
     "[| ∀v ∈ s(u). succ(l) ≈ v Int y; a ⊆ y; l ≈ a; u ∈ x |]
      ==> {v ∈ s(u). a ⊆ v} ≲ {v ∈ Pow(x). v ≈ m}"
apply (rule_tac f3 = "λw ∈ {v ∈ s (u) . a ⊆ v}. w Int (x - {u})"
       in exI [THEN lepoll_def [THEN def_imp_iff, THEN iffD2]])
apply (simp add: inj_def)
apply (intro conjI lam_type CollectI)
  apply fast
 apply (blast intro: w_Int_eqpoll_m)
apply (intro ballI impI)

apply (rule w_Int_eqpoll_m [THEN eqpoll_m_not_empty, THEN not_emptyE])
apply (blast, assumption+)
apply (drule equalityD1 [THEN subsetD], assumption)
apply (frule cons_cons_in, assumption+)
apply (blast dest: ex1_two_eq intro: s_subset [THEN subsetD] in_s_imp_u_in)+
done




lemma (in AC16) well_ord_subsets_eqpoll_n:
     "n ∈ nat ==> ∃S. well_ord({z ∈ Pow(y) . z ≈ succ(n)}, S)"
apply (rule WO_R [THEN well_ord_infinite_subsets_eqpoll_X,
                  THEN eqpoll_def [THEN def_imp_iff, THEN iffD1], THEN
exE])
apply (fast intro: bij_is_inj [THEN well_ord_rvimage])+
done

lemma (in AC16) well_ord_subsets_lepoll_n:
     "n ∈ nat ==> ∃R. well_ord({z ∈ Pow(y). z ≲ n}, R)"
apply (induct_tac "n")
apply (force intro!: well_ord_unit simp add: subsets_lepoll_0_eq_unit)
apply (erule exE)
apply (rule well_ord_subsets_eqpoll_n [THEN exE], assumption)
```

apply *(simp add: subsets_lepoll_succ)*
apply *(drule well_ord_radd, assumption)*
apply *(erule Int_empty [THEN disj_Un_eqpoll_sum,*
THEN eqpoll_def [THEN def_imp_iff, THEN iffD1], THEN exE])
apply *(fast elim!: bij_is_inj [THEN well_ord_rvimage])*
done

lemma *(**in** AC16) LL_subset: "LL ⊆ {z ∈ Pow(y). z ≲ succ(k #+ m)}"*
apply *(unfold LL_def MM_def)*
apply *(insert "includes")*
apply *(blast intro: subset_imp_lepoll eqpoll_imp_lepoll lepoll_trans)*
done

lemma *(**in** AC16) well_ord_LL: "∃S. well_ord(LL,S)"*
apply *(rule well_ord_subsets_lepoll_n [THEN exE, of "succ(k#+m)"])*
apply *simp*
apply *(blast intro: well_ord_subset [OF _ LL_subset])*
done

lemma *(**in** AC16) unique_superset_in_MM:*
"v ∈ LL ==> ∃! w. w ∈ MM & v ⊆ w"
apply *(unfold MM_def LL_def, safe, fast)*
apply *(rule lepoll_imp_eqpoll_subset [THEN exE], assumption)*
apply *(rule_tac x = x **in** all_ex [THEN ballE])*
apply *(blast intro: eqpoll_sym)+*
done

lemma *(**in** AC16) Int_in_LL: "w ∈ MM ==> w Int y ∈ LL"*
by *(unfold LL_def, fast)*

lemma *(**in** AC16) in_LL_eq_Int:*
"v ∈ LL ==> v = (THE x. x ∈ MM & v ⊆ x) Int y"
apply *(unfold LL_def, clarify)*
apply *(subst unique_superset_in_MM [THEN the_equality2])*

apply *(auto simp add: Int_in_LL)*
done

lemma *(***in** *AC16) unique_superset1: "a ∈ LL ⟹ (THE x. x ∈ MM ∧ a ⊆ x) ∈ MM"*
by *(erule unique_superset_in_MM [THEN theI, THEN conjunct1])*

lemma *(***in** *AC16) the_in_MM_subset:*
"v ∈ LL ==> (THE x. x ∈ MM & v ⊆ x) ⊆ x Un y"
apply *(drule unique_superset1)*
apply *(unfold MM_def)*
apply *(fast dest!: unique_superset1 "includes" [THEN subsetD])*
done

lemma *(***in** *AC16) GG_subset: "v ∈ LL ==> GG ‘ v ⊆ x"*
apply *(unfold GG_def)*
apply *(frule the_in_MM_subset)*
apply *(frule in_LL_eq_Int)*
apply *(force elim: equalityE)*
done

lemma *(***in** *AC16) nat_lepoll_ordertype: "nat ≲ ordertype(y, R)"*
apply *(rule nat_le_infinite_Ord [THEN le_imp_lepoll])*
apply *(rule Ord_ordertype [OF WO_R])*
apply *(rule ordertype_eqpoll [THEN eqpoll_imp_lepoll, THEN lepoll_infinite])*

apply *(rule WO_R)*
apply *(rule Infinite)*
done

lemma *(***in** *AC16) ex_subset_eqpoll_n: "n ∈ nat ==> ∃z. z ⊆ y & n ≈ z"*
apply *(erule nat_lepoll_imp_ex_eqpoll_n)*
apply *(rule lepoll_trans [OF nat_lepoll_ordertype])*
apply *(rule ordertype_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll])*

apply *(rule WO_R)*
done

lemma *(***in** *AC16) exists_proper_in_s: "u ∈ x ==> ∃v ∈ s(u). succ(k) ≲ v Int y"*
apply *(rule ccontr)*
apply *(subgoal_tac "∀v ∈ s (u) . k ≈ v Int y")*
prefer *2* **apply** *(simp add: s_def, blast intro: succ_not_lepoll_imp_eqpoll)*
apply *(unfold k_def)*
apply *(insert all_ex "includes" lnat)*
apply *(rule ex_subset_eqpoll_n [THEN exE], assumption)*
apply *(rule noLepoll [THEN notE])*
apply *(blast intro: lepoll_trans [OF y_lepoll_subset_s subset_s_lepoll_w])*

```
done

lemma (in AC16) exists_in_MM: "u ∈ x ==> ∃w ∈ MM. u ∈ w"
apply (erule exists_proper_in_s [THEN bexE])
apply (unfold MM_def s_def, fast)
done

lemma (in AC16) exists_in_LL: "u ∈ x ==> ∃w ∈ LL. u ∈ GG`w"
apply (rule exists_in_MM [THEN bexE], assumption)
apply (rule bexI)
apply (erule_tac [2] Int_in_LL)
apply (unfold GG_def)
apply (simp add: Int_in_LL)
apply (subst unique_superset_in_MM [THEN the_equality2])
apply (fast elim!: Int_in_LL)+
done

lemma (in AC16) OUN_eq_x: "well_ord(LL,S) ==>
      (⋃b<ordertype(LL,S). GG ` (converse(ordermap(LL,S)) ` b)) = x"
apply (rule equalityI)
apply (rule subsetI)
apply (erule OUN_E)
apply (rule GG_subset [THEN subsetD])
prefer 2 apply assumption
apply (blast intro: ltD  ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun,
                                       THEN apply_type])
apply (rule subsetI)
apply (erule exists_in_LL [THEN bexE])
apply (force intro: ltI [OF _ Ord_ordertype]
                    ordermap_type [THEN apply_type]
             simp add: ordermap_bij [THEN bij_is_inj, THEN left_inverse])
done



lemma (in AC16) in_MM_eqpoll_n: "w ∈ MM ==> w ≈ succ(k #+ m)"
apply (unfold MM_def)
apply (fast dest: "includes" [THEN subsetD])
done

lemma (in AC16) in_LL_eqpoll_n: "w ∈ LL ==> succ(k) ≲ w"
by (unfold LL_def MM_def, fast)

lemma (in AC16) in_LL: "w ∈ LL ==> w ⊆ (THE x. x ∈ MM ∧ w ⊆ x)"
by (erule subset_trans [OF in_LL_eq_Int [THEN equalityD1] Int_lower1])

lemma (in AC16) all_in_lepoll_m:
```

```
    "well_ord(LL,S) ==>
     ∀b<ordertype(LL,S). GG ‘ (converse(ordermap(LL,S)) ‘ b) ≲ m"
apply (unfold GG_def)
apply (rule oallI)
apply (simp add: ltD ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun,
THEN apply_type])
apply (insert "includes")
apply (rule eqpoll_sum_imp_Diff_lepoll)
apply (blast del: subsetI
             dest!: ltD
             intro!: eqpoll_sum_imp_Diff_lepoll in_LL_eqpoll_n
             intro: in_LL   unique_superset1 [THEN in_MM_eqpoll_n]
                    ordermap_bij [THEN bij_converse_bij, THEN bij_is_fun,

                              THEN apply_type])+
done

lemma (in AC16) conclusion:
    "∃a f. Ord(a) & domain(f) = a & (⋃b<a. f ‘ b) = x & (∀b<a. f ‘
b ≲ m)"
apply (rule well_ord_LL [THEN exE])
apply (rename_tac S)
apply (rule_tac x = "ordertype (LL,S)" in exI)
apply (rule_tac x = "λb ∈ ordertype(LL,S).
                      GG ‘ (converse (ordermap (LL,S)) ‘ b)"  in exI)
apply (simp add: ltD)
apply (blast intro: lam_funtype [THEN domain_of_fun]
                    Ord_ordertype  OUN_eq_x  all_in_lepoll_m [THEN ospec])
done




term AC16

theorem AC16_WO4:
    "[| AC_Equiv.AC16(k #+ m, k); 0 < k; 0 < m; k ∈ nat; m ∈ nat |]
==> WO4(m)"
apply (unfold AC_Equiv.AC16_def WO4_def)
apply (rule allI)
apply (case_tac "Finite (A)")
apply (rule lemma1, assumption+)
apply (cut_tac lemma2)
apply (elim exE conjE)
apply (erule_tac x = "A Un y" in allE)
apply (frule infinite_Un, drule mp, assumption)
apply (erule zero_lt_natE, assumption, clarify)
```

```
apply (blast intro: AC16.conclusion [OF AC16.intro])
done

end
```

```
theory AC17_AC1
imports HH
begin
```

```
lemma AC0_AC1_lemma: "[| f:(Π X ∈ A. X); D ⊆ A |] ==> ∃g. g:(Π X ∈
D. X)"
by (fast intro!: lam_type apply_type)

lemma AC0_AC1: "AC0 ==> AC1"
apply (unfold AC0_def AC1_def)
apply (blast intro: AC0_AC1_lemma)
done

lemma AC1_AC0: "AC1 ==> AC0"
by (unfold AC0_def AC1_def, blast)
```

```
lemma AC1_AC17_lemma: "f ∈ (Π X ∈ Pow(A) - {0}. X) ==> f ∈ (Pow(A)
- {0} -> A)"
apply (rule Pi_type, assumption)
apply (drule apply_type, assumption, fast)
done

lemma AC1_AC17: "AC1 ==> AC17"
apply (unfold AC1_def AC17_def)
apply (rule allI)
apply (rule ballI)
apply (erule_tac x = "Pow (A) -{0}" in allE)
apply (erule impE, fast)
apply (erule exE)
apply (rule bexI)
apply (erule_tac [2] AC1_AC17_lemma)
apply (rule apply_type, assumption)
apply (fast dest!: AC1_AC17_lemma elim!: apply_type)
done
```

```
lemma UN_eq_imp_well_ord:
     "[| x - (⋃j ∈ LEAST i. HH(λX ∈ Pow(x)-{0}. {f`X}, x, i) = {x}.

        HH(λX ∈ Pow(x)-{0}. {f`X}, x, j)) = 0;
        f ∈ Pow(x)-{0} -> x |]
        ==> ∃r. well_ord(x,r)"
apply (rule exI)
apply (erule well_ord_rvimage
        [OF bij_Least_HH_x [THEN bij_converse_bij, THEN bij_is_inj]
            Ord_Least [THEN well_ord_Memrel]], assumption)
done
```

```
lemma not_AC1_imp_ex:
     "~AC1 ==> ∃A. ∀f ∈ Pow(A)-{0} -> A. ∃u ∈ Pow(A)-{0}. f`u ∉ u"
apply (unfold AC1_def)
apply (erule swap)
apply (rule allI)
apply (erule swap)
apply (rule_tac x = "Union (A)" in exI)
apply (blast intro: lam_type)
done
```

```
lemma AC17_AC1_aux1:
     "[| ∀f ∈ Pow(x) - {0} -> x. ∃u ∈ Pow(x) - {0}. f`u∉u;
         ∃f ∈ Pow(x)-{0}->x.
            x - (⋃a ∈ (LEAST i. HH(λX ∈ Pow(x)-{0}. {f`X},x,i)={x}).

            HH(λX ∈ Pow(x)-{0}. {f`X},x,a)) = 0 |]
        ==> P"
apply (erule bexE)
apply (erule UN_eq_imp_well_ord [THEN exE], assumption)
apply (erule ex_choice_fun_Pow [THEN exE])
apply (erule ballE)
apply (fast intro: apply_type del: DiffE)
apply (erule notE)
apply (rule Pi_type, assumption)
apply (blast dest: apply_type)
done
```

```
lemma AC17_AC1_aux2:
      "~ (∃f ∈ Pow(x)-{0}->x. x - F(f) = 0)
       ==> (λf ∈ Pow(x)-{0}->x . x - F(f))
            ∈ (Pow(x) -{0} -> x) -> Pow(x) - {0}"
by (fast intro!: lam_type dest!: Diff_eq_0_iff [THEN iffD1])

lemma AC17_AC1_aux3:
     "[| f`Z ∈ Z; Z ∈ Pow(x)-{0} |]
      ==> (λX ∈ Pow(x)-{0}. {f`X})`Z ∈ Pow(Z)-{0}"
by auto

lemma AC17_AC1_aux4:
     "∃f ∈ F. f`((λf ∈ F. Q(f))`f) ∈ (λf ∈ F. Q(f))`f
      ==> ∃f ∈ F. f`Q(f) ∈ Q(f)"
by simp

lemma AC17_AC1: "AC17 ==> AC1"
apply (unfold AC17_def)
apply (rule classical)
apply (erule not_AC1_imp_ex [THEN exE])
apply (case_tac
       "∃f ∈ Pow(x)-{0} -> x.
        x - (⋃a ∈ (LEAST i. HH (λX ∈ Pow (x) -{0}. {f`X},x,i) ={x})
. HH (λX ∈ Pow (x) -{0}. {f`X},x,a)) = 0")
apply (erule AC17_AC1_aux1, assumption)
apply (drule AC17_AC1_aux2)
apply (erule allE)
apply (drule bspec, assumption)
apply (drule AC17_AC1_aux4)
apply (erule bexE)
apply (drule apply_type, assumption)
apply (simp add: HH_Least_eq_x del: Diff_iff )
apply (drule AC17_AC1_aux3, assumption)
apply (fast dest!: subst_elem [OF _ HH_Least_eq_x [symmetric]]
                f_subset_imp_HH_subset elim!: mem_irrefl)
done
```

```
lemma AC1_AC2_aux1:
     "[| f:(Π X ∈ A. X);  B ∈ A;  0∉A |] ==> {f`B} ⊆ B Int {f`C. C
∈ A}"
by (fast elim!: apply_type)
```

```
lemma AC1_AC2_aux2:
        "[| pairwise_disjoint(A); B ∈ A; C ∈ A; D ∈ B; D ∈ C |] ==>
f`B = f`C"
by (unfold pairwise_disjoint_def, fast)

lemma AC1_AC2: "AC1 ==> AC2"
apply (unfold AC1_def AC2_def)
apply (rule allI)
apply (rule impI)
apply (elim asm_rl conjE allE exE impE, assumption)
apply (intro exI ballI equalityI)
prefer 2 apply (rule AC1_AC2_aux1, assumption+)
apply (fast elim!: AC1_AC2_aux2 elim: apply_type)
done
```

```
lemma AC2_AC1_aux1: "0∉A ==> 0 ∉ {B*{B}. B ∈ A}"
by (fast dest!: sym [THEN Sigma_empty_iff [THEN iffD1]])

lemma AC2_AC1_aux2: "[| X*{X} Int C = {y}; X ∈ A |]
               ==> (THE y. X*{X} Int C = {y}): X*A"
apply (rule subst_elem [of y])
apply (blast elim!: equalityE)
apply (auto simp add: singleton_eq_iff)
done

lemma AC2_AC1_aux3:
     "∀D ∈ {E*{E}. E ∈ A}. ∃y. D Int C = {y}
      ==> (λx ∈ A. fst(THE z. (x*{x} Int C = {z}))) ∈ (Π X ∈ A. X)"
apply (rule lam_type)
apply (drule bspec, blast)
apply (blast intro: AC2_AC1_aux2 fst_type)
done

lemma AC2_AC1: "AC2 ==> AC1"
apply (unfold AC1_def AC2_def pairwise_disjoint_def)
apply (intro allI impI)
apply (elim allE impE)
prefer 2 apply (fast elim!: AC2_AC1_aux3)
apply (blast intro!: AC2_AC1_aux1)
done
```

```
lemma empty_notin_images: "0 ∉ {R``{x}. x ∈ domain(R)}"
by blast

lemma AC1_AC4: "AC1 ==> AC4"
apply (unfold AC1_def AC4_def)
apply (intro allI impI)
apply (drule spec, drule mp [OF _ empty_notin_images])
apply (best intro!: lam_type elim!: apply_type)
done
```

```
lemma AC4_AC3_aux1: "f ∈ A->B ==> (⋃z ∈ A. {z}*f`z) ⊆ A*Union(B)"
by (fast dest!: apply_type)

lemma AC4_AC3_aux2: "domain(⋃z ∈ A. {z}*f(z)) = {a ∈ A. f(a)≠0}"
by blast

lemma AC4_AC3_aux3: "x ∈ A ==> (⋃z ∈ A. {z}*f(z))``{x} = f(x)"
by fast

lemma AC4_AC3: "AC4 ==> AC3"
apply (unfold AC3_def AC4_def)
apply (intro allI ballI)
apply (elim allE impE)
apply (erule AC4_AC3_aux1)
apply (simp add: AC4_AC3_aux2 AC4_AC3_aux3 cong add: Pi_cong)
done
```

```
lemma AC3_AC1_lemma:
     "b∉A ==> (Π x ∈ {a ∈ A. id(A)`a≠b}. id(A)`x) = (Π x ∈ A. x)"
apply (simp add: id_def cong add: Pi_cong)
apply (rule_tac b = A in subst_context, fast)
done

lemma AC3_AC1: "AC3 ==> AC1"
apply (unfold AC1_def AC3_def)
apply (fast intro!: id_type elim: AC3_AC1_lemma [THEN subst])
done
```

```
lemma AC4_AC5: "AC4 ==> AC5"
apply (unfold range_def AC4_def AC5_def)
apply (intro allI ballI)
apply (elim allE impE)
apply (erule fun_is_rel [THEN converse_type])
apply (erule exE)
apply (rename_tac g)
apply (rule_tac x=g in bexI)
apply (blast dest: apply_equality range_type)
apply (blast intro: Pi_type dest: apply_type fun_is_rel)
done
```

```
lemma AC5_AC4_aux1: "R ⊆ A*B ==> (λx ∈ R. fst(x)) ∈ R -> A"
by (fast intro!: lam_type fst_type)

lemma AC5_AC4_aux2: "R ⊆ A*B ==> range(λx ∈ R. fst(x)) = domain(R)"
by (unfold lam_def, force)

lemma AC5_AC4_aux3: "[| ∃f ∈ A->C. P(f,domain(f)); A=B |] ==> ∃f ∈
B->C. P(f,B)"
apply (erule bexE)
apply (frule domain_of_fun, fast)
done

lemma AC5_AC4_aux4: "[| R ⊆ A*B; g ∈ C->R; ∀x ∈ C. (λz ∈ R. fst(z))‘
(g‘x) = x |]
                ==> (λx ∈ C. snd(g‘x)): (Π x ∈ C. R‘‘{x})"
apply (rule lam_type)
apply (force dest: apply_type)
done

lemma AC5_AC4: "AC5 ==> AC4"
apply (unfold AC4_def AC5_def, clarify)
apply (elim allE ballE)
apply (drule AC5_AC4_aux3 [OF _ AC5_AC4_aux2], assumption)
apply (fast elim!: AC5_AC4_aux4)
apply (blast intro: AC5_AC4_aux1)
done
```

```
lemma AC1_iff_AC6: "AC1 <-> AC6"
by (unfold AC1_def AC6_def, blast)

end
```

```
theory AC18_AC19
imports AC_Equiv
begin

definition
  uu    :: "i => i" where
    "uu(a) == {c Un {0}. c ∈ a}"
```

```
lemma PROD_subsets:
     "[| f ∈ (Π b ∈ {P(a). a ∈ A}. b);  ∀a ∈ A. P(a)<=Q(a) |]
      ==> (λa ∈ A. f`P(a)) ∈ (Π a ∈ A. Q(a))"
by (rule lam_type, drule apply_type, auto)

lemma lemma_AC18:
     "[| ∀A. 0 ∉ A --> (∃f. f ∈ (Π X ∈ A. X)); A ≠ 0 |]
      ==> (⋂a ∈ A. ⋃b ∈ B(a). X(a, b)) ⊆
          (⋃f ∈ Π a ∈ A. B(a). ⋂a ∈ A. X(a, f`a))"
apply (rule subsetI)
apply (erule_tac x = "{{b ∈ B (a) . x ∈ X (a,b) }. a ∈ A}" in allE)
apply (erule impE, fast)
apply (erule exE)
apply (rule UN_I)
 apply (fast elim!: PROD_subsets)
apply (simp, fast elim!: not_emptyE dest: apply_type [OF _ RepFunI])
done

lemma AC1_AC18: "AC1 ==> PROP AC18"
apply (unfold AC1_def)
apply (rule AC18.intro)
apply (fast elim!: lemma_AC18 apply_type intro!: equalityI INT_I UN_I)
done
```

```
theorem (in AC18) AC19
apply (unfold AC19_def)
apply (intro allI impI)
apply (rule AC18 [of _ "%x. x", THEN mp], blast)
done
```

```
lemma RepRep_conj:
        "[| A ≠ 0; 0 ∉ A |] ==> {uu(a). a ∈ A} ≠ 0 & 0 ∉ {uu(a). a
∈ A}"
apply (unfold uu_def, auto)
apply (blast dest!: sym [THEN RepFun_eq_0_iff [THEN iffD1]])
done

lemma lemma1_1: "[|c ∈ a; x = c Un {0}; x ∉ a |] ==> x - {0} ∈ a"
apply clarify
apply (rule subst_elem, assumption)
apply (fast elim: notE subst_elem)
done

lemma lemma1_2:
        "[| f`(uu(a)) ∉ a; f ∈ (Π B ∈ {uu(a). a ∈ A}. B); a ∈ A |]

                ==> f`(uu(a))-{0} ∈ a"
apply (unfold uu_def, fast elim!: lemma1_1 dest!: apply_type)
done

lemma lemma1: "∃f. f ∈ (Π B ∈ {uu(a). a ∈ A}. B) ==> ∃f. f ∈ (Π
B ∈ A. B)"
apply (erule exE)
apply (rule_tac x = "λa∈A. if (f` (uu(a)) ∈ a, f` (uu(a)), f` (uu(a))-{0})"

       in exI)
apply (rule lam_type)
apply (simp add: lemma1_2)
done

lemma lemma2_1: "a≠0 ==> 0 ∈ (⋃b ∈ uu(a). b)"
by (unfold uu_def, auto)

lemma lemma2: "[| A≠0; 0∉A |] ==> (⋂x ∈ {uu(a). a ∈ A}. ⋃b ∈ x.
b) ≠ 0"
apply (erule not_emptyE)
```

```
apply (rule_tac a = 0 in not_emptyI)
apply (fast intro!: lemma2_1)
done

lemma AC19_AC1: "AC19 ==> AC1"
apply (unfold AC19_def AC1_def, clarify)
apply (case_tac "A=0", force)
apply (erule_tac x = "{uu (a) . a ∈ A}" in allE)
apply (erule impE)
apply (erule RepRep_conj, assumption)
apply (rule lemma1)
apply (drule lemma2, assumption, auto)
done

end
```

```
theory DC
imports AC_Equiv Hartog Cardinal_aux
begin

lemma RepFun_lepoll: "Ord(a) ==> {P(b). b ∈ a} ≲ a"
apply (unfold lepoll_def)
apply (rule_tac x = "λz ∈ RepFun (a,P) . LEAST i. z=P (i) " in exI)
apply (rule_tac d="%z. P (z)" in lam_injective)
 apply (fast intro!: Least_in_Ord)
apply (rule sym)
apply (fast intro: LeastI Ord_in_Ord)
done
```

Trivial in the presence of AC, but here we need a wellordering of X

```
lemma image_Ord_lepoll: "[| f ∈ X->Y; Ord(X) |] ==> f``X ≲ X"
apply (unfold lepoll_def)
apply (rule_tac x = "λx ∈ f``X. LEAST y. f`y = x" in exI)
apply (rule_tac d = "%z. f`z" in lam_injective)
apply (fast intro!: Least_in_Ord apply_equality, clarify)
apply (rule LeastI)
 apply (erule apply_equality, assumption+)
apply (blast intro: Ord_in_Ord)
done

lemma range_subset_domain:
      "[| R ⊆ X*X;   !!g. g ∈ X ==> ∃u. <g,u> ∈ R |]
       ==> range(R) ⊆ domain(R)"
by blast

lemma cons_fun_type: "g ∈ n->X ==> cons(<n,x>, g) ∈ succ(n) -> cons(x,
X)"
```

```
apply (unfold succ_def)
apply (fast intro!: fun_extend elim!: mem_irrefl)
done

lemma cons_fun_type2:
     "[| g ∈ n->X; x ∈ X |] ==> cons(<n,x>, g) ∈ succ(n) -> X"
by (erule cons_absorb [THEN subst], erule cons_fun_type)

lemma cons_image_n: "n ∈ nat ==> cons(<n,x>, g)``n = g``n"
by (fast elim!: mem_irrefl)

lemma cons_val_n: "g ∈ n->X ==> cons(<n,x>, g)`n = x"
by (fast intro!: apply_equality elim!: cons_fun_type)

lemma cons_image_k: "k ∈ n ==> cons(<n,x>, g)``k = g``k"
by (fast elim: mem_asym)

lemma cons_val_k: "[| k ∈ n; g ∈ n->X |] ==> cons(<n,x>, g)`k = g`k"
by (fast intro!: apply_equality consI2 elim!: cons_fun_type apply_Pair)

lemma domain_cons_eq_succ: "domain(f)=x ==> domain(cons(<x,y>, f)) =
succ(x)"
by (simp add: domain_cons succ_def)

lemma restrict_cons_eq: "g ∈ n->X ==> restrict(cons(<n,x>, g), n) =
g"
apply (simp add: restrict_def Pi_iff)
apply (blast intro: elim: mem_irrefl)
done

lemma succ_in_succ: "[| Ord(k); i ∈ k |] ==> succ(i) ∈ succ(k)"
apply (rule Ord_linear [of "succ(i)" "succ(k)", THEN disjE])
apply (fast elim: Ord_in_Ord mem_irrefl mem_asym)+
done

lemma restrict_eq_imp_val_eq:
      "[|restrict(f, domain(g)) = g; x ∈ domain(g)|]
       ==> f`x = g`x"
by (erule subst, simp add: restrict)

lemma domain_eq_imp_fun_type: "[| domain(f)=A; f ∈ B->C |] ==> f ∈ A->C"
by (frule domain_of_fun, fast)

lemma ex_in_domain: "[| R ⊆ A * B; R ≠ 0 |] ==> ∃x. x ∈ domain(R)"
by (fast elim!: not_emptyE)


definition
  DC  :: "i => o"  where
```

```
    "DC(a) == ∀X R. R ⊆ Pow(X)*X  &
                  (∀Y ∈ Pow(X). Y ≺ a --> (∃x ∈ X. <Y,x> ∈ R))
                  --> (∃f ∈ a->X. ∀b<a. <f``b,f`b> ∈ R)"

definition
  DC0 :: o  where
    "DC0 == ∀A B R. R ⊆ A*B & R≠0 & range(R) ⊆ domain(R)
                  --> (∃f ∈ nat->domain(R). ∀n ∈ nat. <f`n,f`succ(n)>:R)"

definition
  ff  :: "[i, i, i, i] => i"  where
    "ff(b, X, Q, R) ==
           transrec(b, %c r. THE x. first(x, {x ∈ X. <r``c, x> ∈ R},
Q))"


locale DC0_imp =
  fixes XX and RR and X and R

  assumes all_ex: "∀Y ∈ Pow(X). Y ≺ nat --> (∃x ∈ X. <Y, x> ∈ R)"

  defines XX_def: "XX == (⋃n ∈ nat. {f ∈ n->X. ∀k ∈ n. <f``k, f`k>
∈ R})"
     and RR_def:  "RR == {<z1,z2>:XX*XX. domain(z2)=succ(domain(z1))
                                  & restrict(z2, domain(z1)) = z1}"
```

```
lemma (in DC0_imp) lemma1_1: "RR ⊆ XX*XX"
by (unfold RR_def, fast)

lemma (in DC0_imp) lemma1_2: "RR ≠ 0"
apply (unfold RR_def XX_def)
apply (rule all_ex [THEN ballE])
apply (erule_tac [2] notE [OF _ empty_subsetI [THEN PowI]])
apply (erule_tac impE [OF _ nat_0I [THEN n_lesspoll_nat]])
apply (erule bexE)
apply (rule_tac a = "<0, {<0, x>}>" in not_emptyI)
apply (rule CollectI)
apply (rule SigmaI)
apply (rule nat_0I [THEN UN_I])
apply (simp (no_asm_simp) add: nat_0I [THEN UN_I])
apply (rule nat_1I [THEN UN_I])
apply (force intro!: singleton_fun [THEN Pi_type]
             simp add: singleton_0 [symmetric])
apply (simp add: singleton_0)
done

lemma (in DC0_imp) lemma1_3: "range(RR) ⊆ domain(RR)"
apply (unfold RR_def XX_def)
apply (rule range_subset_domain, blast, clarify)
apply (frule fun_is_rel [THEN image_subset, THEN PowI,
                         THEN all_ex [THEN bspec]])
apply (erule impE[OF _ lesspoll_trans1[OF image_Ord_lepoll
                                          [OF _ nat_into_Ord] n_lesspoll_nat]],
       assumption+)
apply (erule bexE)
apply (rule_tac x = "cons (<n,x>, g) " in exI)
apply (rule CollectI)
apply (force elim!: cons_fun_type2
             simp add: cons_image_n cons_val_n cons_image_k cons_val_k)
apply (simp add: domain_of_fun succ_def restrict_cons_eq)
done

lemma (in DC0_imp) lemma2:
     "[| ∀n ∈ nat. <f`n, f`succ(n)> ∈ RR;  f ∈ nat -> XX;  n ∈ nat |]

      ==> ∃k ∈ nat. f`succ(n) ∈ k -> X & n ∈ k
                 & <f`succ(n)``n, f`succ(n)`n> ∈ R"
apply (induct_tac "n")
apply (drule apply_type [OF _ nat_1I])
apply (drule bspec [OF _ nat_0I])
apply (simp add: XX_def, safe)
```

```
apply (rule rev_bexI, assumption)
apply (subgoal_tac "0 ∈ y", force)
apply (force simp add: RR_def
             intro: ltD elim!: nat_0_le [THEN leE])

apply (drule bspec [OF _ nat_succI], assumption)
apply (subgoal_tac "f ‘ succ (succ (x)) ∈ succ (k) ->X")
apply (drule apply_type [OF _ nat_succI [THEN nat_succI]], assumption)
apply (simp (no_asm_use) add: XX_def RR_def)
apply safe
apply (frule_tac a="succ(k)" in domain_of_fun [symmetric, THEN trans],

       assumption)
apply (frule_tac a=y in domain_of_fun [symmetric, THEN trans],
       assumption)
apply (fast elim!: nat_into_Ord [THEN succ_in_succ]
            dest!: bspec [OF _ nat_into_Ord [THEN succ_in_succ]])
apply (drule domain_of_fun)
apply (simp add: XX_def RR_def, clarify)
apply (blast dest: domain_of_fun [symmetric, THEN trans] )
done

lemma (in DC0_imp) lemma3_1:
     "[| ∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  m ∈ nat |]

      ==>  {f‘succ(x)‘x. x ∈ m} = {f‘succ(m)‘x. x ∈ m}"
apply (subgoal_tac "∀x ∈ m. f‘succ (m) ‘x = f‘succ (x) ‘x")
apply simp
apply (induct_tac "m", blast)
apply (rule ballI)
apply (erule succE)
 apply (rule restrict_eq_imp_val_eq)
  apply (drule bspec [OF _ nat_succI], assumption)
  apply (simp add: RR_def)
 apply (drule lemma2, assumption+)
 apply (fast dest!: domain_of_fun)
apply (drule_tac x = xa in bspec, assumption)
apply (erule sym [THEN trans, symmetric])
apply (rule restrict_eq_imp_val_eq [symmetric])
 apply (drule bspec [OF _ nat_succI], assumption)
 apply (simp add: RR_def)
apply (drule lemma2, assumption+)
apply (blast dest!: domain_of_fun
             intro: nat_into_Ord OrdmemD [THEN subsetD])
done

lemma (in DC0_imp) lemma3:
     "[| ∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  m ∈ nat |]
```

```
      ==> (λx ∈ nat. f`succ(x)`x) `` m = f`succ(m)``m"
apply (erule natE, simp)
apply (subst image_lam)
 apply (fast elim!: OrdmemD [OF nat_succI Ord_nat])
apply (subst lemma3_1, assumption+)
 apply fast
apply (fast dest!: lemma2
            elim!: image_fun [symmetric, OF _ OrdmemD [OF _ nat_into_Ord]])
done


theorem DC0_imp_DC_nat: "DC0 ==> DC(nat)"
apply (unfold DC_def DC0_def, clarify)
apply (elim allE)
apply (erule impE)

apply (blast intro!: DC0_imp.lemma1_1 [OF DC0_imp.intro] DC0_imp.lemma1_2
[OF DC0_imp.intro] DC0_imp.lemma1_3 [OF DC0_imp.intro])
apply (erule bexE)
apply (rule_tac x = "λn ∈ nat. f`succ (n) `n" in rev_bexI)
 apply (rule lam_type, blast dest!: DC0_imp.lemma2 [OF DC0_imp.intro]
intro: fun_weaken_type)
apply (rule oallI)
apply (frule DC0_imp.lemma2 [OF DC0_imp.intro], assumption)
  apply (blast intro: fun_weaken_type)
 apply (erule ltD)

apply (subst DC0_imp.lemma3 [OF DC0_imp.intro], assumption+)
  apply (fast elim!: fun_weaken_type)
 apply (erule ltD)
apply (force simp add: lt_def)
done




lemma singleton_in_funs:
 "x ∈ X ==> {<0,x>} ∈
            (⋃n ∈ nat. {f ∈ succ(n)->X. ∀k ∈ n. <f`k, f`succ(k)> ∈
R})"
apply (rule nat_0I [THEN UN_I])
apply (force simp add: singleton_0 [symmetric]
             intro!: singleton_fun [THEN Pi_type])
done


locale imp_DC0 =
  fixes XX and RR and x and R and f and allRR
  defines XX_def: "XX == (⋃n ∈ nat.
```

```
                    {f ∈ succ(n)->domain(R). ∀k ∈ n. <f`k, f`succ(k)>
∈ R})"
      and RR_def:
         "RR == {<z1,z2>:Fin(XX)*XX.
                  (domain(z2)=succ(⋃f ∈ z1. domain(f))
                    & (∀f ∈ z1. restrict(z2, domain(f)) = f))
                  | (~ (∃g ∈ XX. domain(g)=succ(⋃f ∈ z1. domain(f))

                    & (∀f ∈ z1. restrict(g, domain(f)) = f)) & z2={<0,x>})}"
      and allRR_def:
        "allRR == ∀b<nat.
                   <f``b, f`b> ∈
                    {<z1,z2>∈Fin(XX)*XX. (domain(z2)=succ(⋃f ∈ z1. domain(f))
                                  & (⋃f ∈ z1. domain(f)) = b
                                  & (∀f ∈ z1. restrict(z2,domain(f))
= f))}"

lemma (in imp_DC0) lemma4:
     "[| range(R) ⊆ domain(R);  x ∈ domain(R) |]
      ==> RR ⊆ Pow(XX)*XX &
             (∀Y ∈ Pow(XX). Y ≺ nat --> (∃x ∈ XX. <Y,x>:RR))"
apply (rule conjI)
apply (force dest!: FinD [THEN PowI] simp add: RR_def)
apply (rule impI [THEN ballI])
apply (drule Finite_Fin [OF lesspoll_nat_is_Finite PowD], assumption)
apply (case_tac
       "∃g ∈ XX. domain (g) =
             succ(⋃f ∈ Y. domain(f)) & (∀f∈Y. restrict(g, domain(f))
= f)")
apply (simp add: RR_def, blast)
apply (safe del: domainE)
apply (unfold XX_def RR_def)
apply (rule rev_bexI, erule singleton_in_funs)
apply (simp add: nat_0I [THEN rev_bexI] cons_fun_type2)
done

lemma (in imp_DC0) UN_image_succ_eq:
     "[| f ∈ nat->X; n ∈ nat |]
      ==> (⋃x ∈ f``succ(n). P(x)) =  P(f`n) Un (⋃x ∈ f``n. P(x))"
by (simp add: image_fun OrdmemD)

lemma (in imp_DC0) UN_image_succ_eq_succ:
     "[| (⋃x ∈ f``n. P(x)) = y; P(f`n) = succ(y);
         f ∈ nat -> X; n ∈ nat |] ==> (⋃x ∈ f``succ(n). P(x)) = succ(y)"
by (simp add: UN_image_succ_eq, blast)

lemma (in imp_DC0) apply_domain_type:
     "[| h ∈ succ(n) -> D;  n ∈ nat; domain(h)=succ(y) |] ==> h`y ∈ D"
by (fast elim: apply_type dest!: trans [OF sym domain_of_fun])
```

```
lemma (in imp_DC0) image_fun_succ:
     "[| h ∈ nat -> X; n ∈ nat |] ==> h``succ(n) = cons(h`n, h``n)"
by (simp add: image_fun OrdmemD)

lemma (in imp_DC0) f_n_type:
     "[| domain(f`n) = succ(k); f ∈ nat -> XX;  n ∈ nat |]
      ==> f`n ∈ succ(k) -> domain(R)"
apply (unfold XX_def)
apply (drule apply_type, assumption)
apply (fast elim: domain_eq_imp_fun_type)
done

lemma (in imp_DC0) f_n_pairs_in_R [rule_format]:
     "[| h ∈ nat -> XX;  domain(h`n) = succ(k);  n ∈ nat |]
      ==> ∀ i ∈ k. <h`n`i, h`n`succ(i)> ∈ R"
apply (unfold XX_def)
apply (drule apply_type, assumption)
apply (elim UN_E CollectE)
apply (drule domain_of_fun [symmetric, THEN trans], assumption, simp)
done

lemma (in imp_DC0) restrict_cons_eq_restrict:
     "[| restrict(h, domain(u))=u;  h ∈ n->X;  domain(u) ⊆ n |]
      ==> restrict(cons(<n, y>, h), domain(u)) = u"
apply (unfold restrict_def)
apply (simp add: restrict_def Pi_iff)
apply (erule sym [THEN trans, symmetric])
apply (blast elim: mem_irrefl)
done

lemma (in imp_DC0) all_in_image_restrict_eq:
     "[| ∀x ∈ f``n. restrict(f`n, domain(x))=x;
         f ∈ nat -> XX;
         n ∈ nat;  domain(f`n) = succ(n);
         (⋃x ∈ f``n. domain(x)) ⊆ n |]
      ==> ∀x ∈ f``succ(n). restrict(cons(<succ(n),y>, f`n), domain(x))
= x"
apply (rule ballI)
apply (simp add: image_fun_succ)
apply (drule f_n_type, assumption+)
apply (erule disjE)
 apply (simp add: domain_of_fun restrict_cons_eq)
apply (blast intro!: restrict_cons_eq_restrict)
done

lemma (in imp_DC0) simplify_recursion:
     "[| ∀b<nat. <f``b, f`b> ∈ RR;
         f ∈ nat -> XX; range(R) ⊆ domain(R); x ∈ domain(R)|]
```

```
      ==> allRR"
apply (unfold RR_def allRR_def)
apply (rule oallI, drule ltD)
apply (erule nat_induct)
apply (drule_tac x=0 in ospec, blast intro: Limit_has_0)
apply (force simp add: singleton_fun [THEN domain_of_fun] singleton_in_funs)



apply (simp only: separation split)
apply (drule_tac x="succ(xa)" in ospec, blast intro: ltI)
apply (elim conjE disjE)
apply (force elim!: trans subst_context
             intro!: UN_image_succ_eq_succ)
apply (erule notE)
apply (simp add: XX_def UN_image_succ_eq_succ)
apply (elim conjE bexE)
apply (drule apply_domain_type, assumption+)
apply (erule domainE)+
apply (frule f_n_type)
apply (simp add: XX_def, assumption+)
apply (rule rev_bexI, erule nat_succI)
apply (rename_tac m i j y z)
apply (rule_tac x = "cons(<succ(m), z>, f`m)" in bexI)
prefer 2 apply (blast intro: cons_fun_type2)
apply (rule conjI)
prefer 2 apply (fast del: ballI subsetI
                 elim: trans [OF _ subst_context, THEN domain_cons_eq_succ]
                       subst_context
                       all_in_image_restrict_eq [simplified XX_def]
                       trans equalityD1)

apply (rule ballI)
apply (erule succE)

 apply (simp add: cons_val_n cons_val_k)

apply (drule f_n_pairs_in_R [simplified XX_def, OF _ domain_of_fun],
       assumption, assumption, assumption)
apply (simp add: nat_into_Ord [THEN succ_in_succ] succI2 cons_val_k)
done


lemma (in imp_DC0) lemma2:
     "[| allRR; f ∈ nat->XX; range(R) ⊆ domain(R); x ∈ domain(R); n
∈ nat |]
      ==> f`n ∈ succ(n) -> domain(R) & (∀i ∈ n. <f`n`i, f`n`succ(i)>:R)"
apply (unfold allRR_def)
apply (drule ospec)
```

```
apply (erule ltI [OF _ Ord_nat])
apply (erule CollectE, simp)
apply (rule conjI)
prefer 2 apply (fast elim!: f_n_pairs_in_R trans subst_context)
apply (unfold XX_def)
apply (fast elim!: trans [THEN domain_eq_imp_fun_type] subst_context)
done

lemma (in imp_DC0) lemma3:
     "[| allRR; f ∈ nat->XX; n∈nat; range(R) ⊆ domain(R);  x ∈ domain(R)
|]
      ==> f`n`n = f`succ(n)`n"
apply (frule lemma2 [THEN conjunct1, THEN domain_of_fun], assumption+)
apply (unfold allRR_def)
apply (drule ospec)
apply (drule ltI [OF nat_succI Ord_nat], assumption, simp)
apply (elim conjE ballE)
apply (erule restrict_eq_imp_val_eq [symmetric], force)
apply (simp add: image_fun OrdmemD)
done


theorem DC_nat_imp_DC0: "DC(nat) ==> DC0"
apply (unfold DC_def DC0_def)
apply (intro allI impI)
apply (erule asm_rl conjE ex_in_domain [THEN exE] allE)+
apply (erule impE [OF _ imp_DC0.lemma4], assumption+)
apply (erule bexE)
apply (drule imp_DC0.simplify_recursion, assumption+)
apply (rule_tac x = "λn ∈ nat. f`n`n" in bexI)
apply (rule_tac [2] lam_type)
apply (erule_tac [2] apply_type [OF imp_DC0.lemma2 [THEN conjunct1] succI1])
apply (rule ballI)
apply (frule_tac n="succ(n)" in imp_DC0.lemma2,
       (assumption|erule nat_succI)+)
apply (drule imp_DC0.lemma3, auto)
done




lemma fun_Ord_inj:
      "[| f ∈ a->X;  Ord(a);
          !!b c. [| b<c; c ∈ a |] ==> f`b≠f`c |]
       ==> f ∈ inj(a, X)"
apply (unfold inj_def, simp)
apply (intro ballI impI)
apply (rule_tac j=x in Ord_in_Ord [THEN Ord_linear_lt], assumption+)
```

```
apply (blast intro: Ord_in_Ord, auto)
apply (atomize, blast dest: not_sym)
done

lemma value_in_image: "[| f ∈ X->Y; A ⊆ X; a ∈ A |] ==> f`a ∈ f``A"
by (fast elim!: image_fun [THEN ssubst])

theorem DC_WO3: "(∀K. Card(K) --> DC(K)) ==> WO3"
apply (unfold DC_def WO3_def)
apply (rule allI)
apply (case_tac "A ≺ Hartog (A)")
apply (fast dest!: lesspoll_imp_ex_lt_eqpoll
            intro!: Ord_Hartog leI [THEN le_imp_subset])
apply (erule allE impE)+
apply (rule Card_Hartog)
apply (erule_tac x = A in allE)
apply (erule_tac x = "{<z1,z2> ∈ Pow (A) *A . z1 ≺ Hartog (A) & z2 ∉
z1}"
                 in allE)
apply simp
apply (erule impE, fast elim: lesspoll_lemma [THEN not_emptyE])
apply (erule bexE)
apply (rule Hartog_lepoll_selfE)
apply (rule lepoll_def [THEN def_imp_iff, THEN iffD2])
apply (rule exI, rule fun_Ord_inj, assumption, rule Ord_Hartog)
apply (drule value_in_image)
apply (drule OrdmemD, rule Ord_Hartog, assumption+, erule ltD)
apply (drule ospec)
apply (blast intro: ltI Ord_Hartog, force)
done




lemma images_eq:
     "[| ∀x ∈ A. f`x=g`x; f ∈ Df->Cf; g ∈ Dg->Cg; A ⊆ Df; A ⊆ Dg |]

      ==> f``A = g``A"
apply (simp (no_asm_simp) add: image_fun)
done

lemma lam_images_eq:
     "[| Ord(a); b ∈ a |] ==> (λx ∈ a. h(x))``b = (λx ∈ b. h(x))``b"
apply (rule images_eq)
    apply (rule ballI)
    apply (drule OrdmemD [THEN subsetD], assumption+)
    apply simp
   apply (fast elim!: RepFunI OrdmemD intro!: lam_type)+
```

```
done

lemma lam_type_RepFun: "(λb ∈ a. h(b)) ∈ a -> {h(b). b ∈ a}"
by (fast intro!: lam_type RepFunI)

lemma lemmaX:
     "[| ∀Y ∈ Pow(X). Y ≺ K --> (∃x ∈ X. <Y, x> ∈ R);
         b ∈ K; Z ∈ Pow(X); Z ≺ K |]
      ==> {x ∈ X. <Z,x> ∈ R} ≠ 0"
by blast


lemma WO1_DC_lemma:
     "[| Card(K); well_ord(X,Q);
         ∀Y ∈ Pow(X). Y ≺ K --> (∃x ∈ X. <Y, x> ∈ R); b ∈ K |]
      ==> ff(b, X, Q, R) ∈ {x ∈ X. <(λc ∈ b. ff(c, X, Q, R))``b, x>
∈ R}"
apply (rule_tac P = "b ∈ K" in impE, (erule_tac [2] asm_rl)+)
apply (rule_tac i=b in Card_is_Ord [THEN Ord_in_Ord, THEN trans_induct],

       assumption+)
apply (rule impI)
apply (rule ff_def [THEN def_transrec, THEN ssubst])
apply (erule the_first_in, fast)
apply (simp add: image_fun [OF lam_type_RepFun subset_refl])
apply (erule lemmaX, assumption)
 apply (blast intro: Card_is_Ord OrdmemD [THEN subsetD])
apply (blast intro: lesspoll_trans1 in_Card_imp_lesspoll RepFun_lepoll)
done

theorem WO1_DC_Card: "WO1 ==> ∀K. Card(K) --> DC(K)"
apply (unfold DC_def WO1_def)
apply (rule allI impI)+
apply (erule allE exE conjE)+
apply (rule_tac x = "λb ∈ K. ff (b, X, Ra, R) " in bexI)
 apply (simp add: lam_images_eq [OF Card_is_Ord ltD])
 apply (fast elim!: ltE WO1_DC_lemma [THEN CollectD2])
apply (rule_tac lam_type)
apply (rule WO1_DC_lemma [THEN CollectD1], assumption+)
done

end
```

References

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[2] Herman Rubin and Jean E. Rubin. *Equivalents of the Axiom of Choice, II.* North-Holland, 1985.