Machine Words in Isabelle/HOL

Jeremy Dawson, Paul Graunke, Brian Huffman, Gerwin Klein, and John Matthews

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Abstract

A formalisation of generic, fixed size machine words in Isabelle/HOL.
An earlier version of this formalisation is described in [1].

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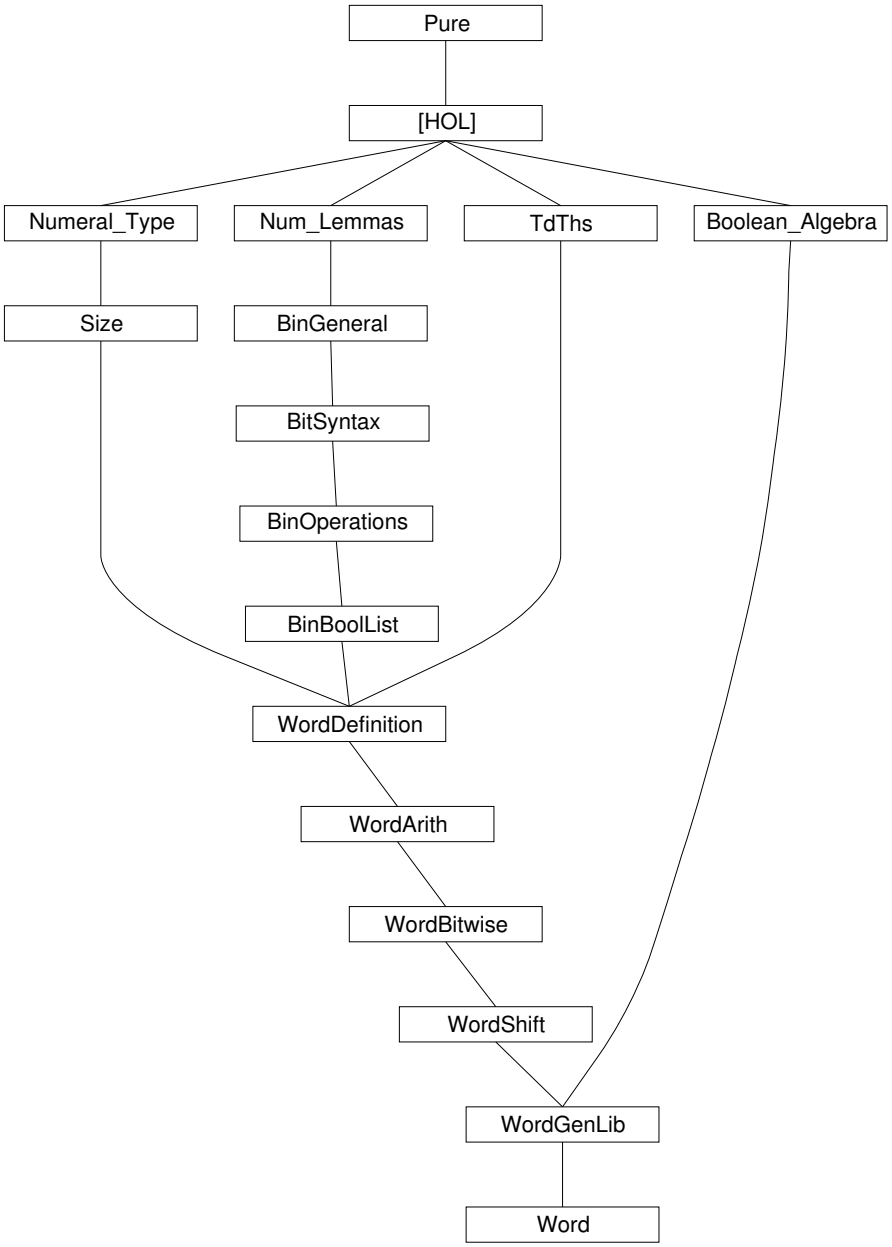
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1 Numeral-Type: Numeral Syntax for Types

theory *Numeral-Type*
imports *Main*
begin

1.1 Preliminary lemmas

lemma (**in** *type-definition*) *univ*:
UNIV = *Abs* ' *A*
⟨*proof*⟩

lemma (**in** *type-definition*) *card*: *card* (*UNIV* :: *'b set*) = *card A*
⟨*proof*⟩

1.2 Cardinalities of types

syntax -*type-card* :: *type* => *nat* ((*1CARD*/(*1'*(-'))))

translations *CARD*('*t*) => *CONST card* (*CONST UNIV* :: *'t set*)

⟨*ML*⟩

lemma *card-unit* [*simp*]: *CARD*(*unit*) = *1*
⟨*proof*⟩

lemma *card-bool* [*simp*]: *CARD*(*bool*) = *2*
⟨*proof*⟩

lemma *card-prod* [*simp*]: *CARD*('*a* × *'b*) = *CARD*('*a*::*finite*) * *CARD*('*b*::*finite*)
⟨*proof*⟩

lemma *card-sum* [*simp*]: *CARD*('*a* + *'b*) = *CARD*('*a*::*finite*) + *CARD*('*b*::*finite*)
⟨*proof*⟩

lemma *card-option* [*simp*]: *CARD*('*a option*) = *Suc CARD*('*a*::*finite*)
⟨*proof*⟩

lemma *card-set* [*simp*]: *CARD*('*a set*) = *2* ^ *CARD*('*a*::*finite*)
⟨*proof*⟩

lemma *card-nat* [*simp*]: *CARD*(*nat*) = *0*
⟨*proof*⟩

1.3 Classes with at least 1 and 2

Class finite already captures "at least 1"

lemma *zero-less-card-finite* [*simp*]: *0* < *CARD*('*a*::*finite*)
⟨*proof*⟩

lemma *one-le-card-finite* [*simp*]: *Suc 0* ≤ *CARD*('*a*::*finite*)
⟨*proof*⟩

Class for cardinality "at least 2"

class *card2* = *finite* +
assumes *two-le-card*: *2* ≤ *CARD*('*a*)

lemma *one-less-card*: *Suc 0* < *CARD*('*a*::*card2*)
⟨*proof*⟩

lemma *one-less-int-card*: *1* < *int CARD*('*a*::*card2*)
⟨*proof*⟩

1.4 Numeral Types

typedef (**open**) *num0* = *UNIV* :: *nat set* ⟨*proof*⟩
typedef (**open**) *num1* = *UNIV* :: *unit set* ⟨*proof*⟩

typedef (**open**) '*a bit0* = {*0* ..< *2* ∗ *int CARD*('*a*::*finite*)}
⟨*proof*⟩

typedef (**open**) '*a bit1* = {*0* ..< *1* + *2* ∗ *int CARD*('*a*::*finite*)}
⟨*proof*⟩

lemma *card-num0* [*simp*]: *CARD* (*num0*) = *0*
⟨*proof*⟩

lemma *card-num1* [*simp*]: *CARD*(*num1*) = *1*
⟨*proof*⟩

lemma *card-bit0* [*simp*]: *CARD*('*a bit0*) = *2* ∗ *CARD*('*a*::*finite*)
⟨*proof*⟩

lemma *card-bit1* [*simp*]: *CARD*('*a bit1*) = *Suc* (*2* ∗ *CARD*('*a*::*finite*))
⟨*proof*⟩

instance *num1* :: *finite*
⟨*proof*⟩

instance *bit0* :: (*finite*) *card2*
⟨*proof*⟩

instance *bit1* :: (*finite*) *card2*
⟨*proof*⟩

1.5 Locale for modular arithmetic subtypes

locale *mod-type* =

fixes $n :: int$
and $Rep :: 'a::\{zero,one,plus,times,uminus,minus\} \Rightarrow int$
and $Abs :: int \Rightarrow 'a::\{zero,one,plus,times,uminus,minus\}$
assumes *type*: *type-definition Rep Abs* $\{0..<n\}$
and *size1*: $1 < n$
and *zero-def*: $0 = Abs\ 0$
and *one-def*: $1 = Abs\ 1$
and *add-def*: $x + y = Abs\ ((Rep\ x + Rep\ y)\ mod\ n)$
and *mult-def*: $x * y = Abs\ ((Rep\ x * Rep\ y)\ mod\ n)$
and *diff-def*: $x - y = Abs\ ((Rep\ x - Rep\ y)\ mod\ n)$
and *minus-def*: $- x = Abs\ ((- Rep\ x)\ mod\ n)$
begin

lemma *size0*: $0 < n$
⟨*proof*⟩

lemmas *definitions* =
zero-def one-def add-def mult-def minus-def diff-def

lemma *Rep-less-n*: $Rep\ x < n$
⟨*proof*⟩

lemma *Rep-le-n*: $Rep\ x \leq n$
⟨*proof*⟩

lemma *Rep-inject-sym*: $x = y \longleftrightarrow Rep\ x = Rep\ y$
⟨*proof*⟩

lemma *Rep-inverse*: $Abs\ (Rep\ x) = x$
⟨*proof*⟩

lemma *Abs-inverse*: $m \in \{0..<n\} \Longrightarrow Rep\ (Abs\ m) = m$
⟨*proof*⟩

lemma *Rep-Abs-mod*: $Rep\ (Abs\ (m\ mod\ n)) = m\ mod\ n$
⟨*proof*⟩

lemma *Rep-Abs-0*: $Rep\ (Abs\ 0) = 0$
⟨*proof*⟩

lemma *Rep-0*: $Rep\ 0 = 0$
⟨*proof*⟩

lemma *Rep-Abs-1*: $Rep\ (Abs\ 1) = 1$
⟨*proof*⟩

lemma *Rep-1*: $Rep\ 1 = 1$
⟨*proof*⟩

lemma *Rep-mod*: *Rep x mod n = Rep x*
⟨*proof*⟩

lemmas *Rep-simps* =
Rep-inject-sym Rep-inverse Rep-Abs-mod Rep-mod Rep-Abs-0 Rep-Abs-1

lemma *comm-ring-1*: *OFCLASS('a, comm-ring-1-class)*
⟨*proof*⟩

end

locale *mod-ring = mod-type +*
constrains *n :: int*
and *Rep :: 'a::{number-ring} ⇒ int*
and *Abs :: int ⇒ 'a::{number-ring}*
begin

lemma *of-nat-eq*: *of-nat k = Abs (int k mod n)*
⟨*proof*⟩

lemma *of-int-eq*: *of-int z = Abs (z mod n)*
⟨*proof*⟩

lemma *Rep-number-of*:
Rep (number-of w) = number-of w mod n
⟨*proof*⟩

lemma *iszero-number-of*:
iszero (number-of w::'a) ⟷ number-of w mod n = 0
⟨*proof*⟩

lemma *cases*:
assumes *1*: *⋀z. ⟦(x::'a) = of-int z; 0 ≤ z; z < n⟧ ⟹ P*
shows *P*
⟨*proof*⟩

lemma *induct*:
(⋀z. ⟦0 ≤ z; z < n⟧ ⟹ P (of-int z)) ⟹ P (x::'a)
⟨*proof*⟩

end

1.6 Number ring instances

Unfortunately a number ring instance is not possible for *num1*, since 0 and 1 are not distinct.

instantiation *num1* :: {*comm-ring*,*comm-monoid-mult*,*number*}
begin

lemma *num1-eq-iff*: $(x::num1) = (y::num1) \longleftrightarrow True$
⟨*proof*⟩

instance ⟨*proof*⟩

end

instantiation
bit0 **and** *bit1* :: (*finite*) {*zero*,*one*,*plus*,*times*,*uminus*,*minus*}
begin

definition *Abs-bit0'* :: *int* ⇒ *'a bit0* **where**
Abs-bit0' x = *Abs-bit0* (*x mod int CARD*(*'a bit0*))

definition *Abs-bit1'* :: *int* ⇒ *'a bit1* **where**
Abs-bit1' x = *Abs-bit1* (*x mod int CARD*(*'a bit1*))

definition *0* = *Abs-bit0 0*
definition *1* = *Abs-bit0 1*
definition $x + y$ = *Abs-bit0'* (*Rep-bit0 x* + *Rep-bit0 y*)
definition $x * y$ = *Abs-bit0'* (*Rep-bit0 x* ∗ *Rep-bit0 y*)
definition $x - y$ = *Abs-bit0'* (*Rep-bit0 x* − *Rep-bit0 y*)
definition $- x$ = *Abs-bit0'* (− *Rep-bit0 x*)

definition *0* = *Abs-bit1 0*
definition *1* = *Abs-bit1 1*
definition $x + y$ = *Abs-bit1'* (*Rep-bit1 x* + *Rep-bit1 y*)
definition $x * y$ = *Abs-bit1'* (*Rep-bit1 x* ∗ *Rep-bit1 y*)
definition $x - y$ = *Abs-bit1'* (*Rep-bit1 x* − *Rep-bit1 y*)
definition $- x$ = *Abs-bit1'* (− *Rep-bit1 x*)

instance ⟨*proof*⟩

end

interpretation *bit0*:
mod-type int CARD(*'a*::*finite bit0*)
Rep-bit0 :: *'a*::*finite bit0* ⇒ *int*
Abs-bit0 :: *int* ⇒ *'a*::*finite bit0*
⟨*proof*⟩

interpretation *bit1*:
mod-type int CARD(*'a*::*finite bit1*)
Rep-bit1 :: *'a*::*finite bit1* ⇒ *int*
Abs-bit1 :: *int* ⇒ *'a*::*finite bit1*
⟨*proof*⟩

instance *bit0* :: (*finite*) *comm-ring-1*
⟨*proof*⟩

instance *bit1* :: (*finite*) *comm-ring-1*
⟨*proof*⟩

instantiation *bit0* **and** *bit1* :: (*finite*) *number-ring*
begin

definition (*number-of w* :: - *bit0*) = *of-int w*

definition (*number-of w* :: - *bit1*) = *of-int w*

instance ⟨*proof*⟩

end

interpretation *bit0*:
mod-ring int CARD('*a*::*finite bit0*)
Rep-bit0 :: '*a*::*finite bit0* ⇒ *int*
Abs-bit0 :: *int* ⇒ '*a*::*finite bit0*
⟨*proof*⟩

interpretation *bit1*:
mod-ring int CARD('*a*::*finite bit1*)
Rep-bit1 :: '*a*::*finite bit1* ⇒ *int*
Abs-bit1 :: *int* ⇒ '*a*::*finite bit1*
⟨*proof*⟩

Set up cases, induction, and arithmetic

lemmas *bit0-cases* [*case-names of-int*, *cases type*: *bit0*] = *bit0.cases*
lemmas *bit1-cases* [*case-names of-int*, *cases type*: *bit1*] = *bit1.cases*

lemmas *bit0-induct* [*case-names of-int*, *induct type*: *bit0*] = *bit0.induct*
lemmas *bit1-induct* [*case-names of-int*, *induct type*: *bit1*] = *bit1.induct*

lemmas *bit0-iszero-number-of* [*simp*] = *bit0.iszero-number-of*
lemmas *bit1-iszero-number-of* [*simp*] = *bit1.iszero-number-of*

1.7 Syntax

syntax
-NumeralType :: *num-const* => *type* (-)
-NumeralType0 :: *type* (*0*)
-NumeralType1 :: *type* (*1*)

translations
(*type*) *1* == (*type*) *num1*
(*type*) *0* == (*type*) *num0*

⟨*ML*⟩

1.8 Examples

lemma *CARD(0) = 0* ⟨*proof*⟩
lemma *CARD(17) = 17* ⟨*proof*⟩
lemma *8 * 11 ^ 3 - 6 = (2::5)* ⟨*proof*⟩

end

2 Size: The len classes

theory *Size*
imports *Numeral-Type*
begin

The aim of this is to allow any type as index type, but to provide a default instantiation for numeral types. This independence requires some duplication with the definitions in *Numeral-Type*.

class *len0* =
  fixes *len-of* :: *'a itself* ⇒ *nat*

Some theorems are only true on words with length greater 0.

class *len* = *len0* +
  assumes *len-gt-0* [*iff*]: *0 < len-of TYPE ('a)*

instantiation *num0* **and** *num1* :: *len0*
begin

definition
  len-num0: *len-of (x::num0 itself) = 0*

definition
  len-num1: *len-of (x::num1 itself) = 1*

instance ⟨*proof*⟩

end

instantiation *bit0* **and** *bit1* :: (*len0*) *len0*
begin

definition
  len-bit0: *len-of (x::'a::len0 bit0 itself) = 2 * len-of TYPE ('a)*

definition
  len-bit1: *len-of (x::'a::len0 bit1 itself) = 2 * len-of TYPE ('a) + 1*

instance ⟨*proof*⟩

end

lemmas *len-of-numeral-defs* [*simp*] = *len-num0 len-num1 len-bit0 len-bit1*

instance *num1* :: *len* ⟨*proof*⟩
instance *bit0* :: (*len*) *len* ⟨*proof*⟩
instance *bit1* :: (*len0*) *len* ⟨*proof*⟩
lemma *len-of TYPE*(*17*) = *17* ⟨*proof*⟩
lemma *len-of TYPE*(*0*) = *0* ⟨*proof*⟩
lemma *len-of TYPE*('*a*::*len0*) = *x*
⟨*proof*⟩

end

3 Num-Lemmas: Useful Numerical Lemmas

theory *Num-Lemmas*
imports *Main Parity*
begin

lemma *contentsI*: *y* = {*x*} ==> *contents y* = *x*
⟨*proof*⟩

lemmas *split-split* = *prod.split* [*unfolded prod-case-split*]
lemmas *split-split-asm* = *prod.split-asm* [*unfolded prod-case-split*]
lemmas *split-splits* = *split-split split-split-asm*

lemmas *funpow-0* = *funpow.simps*(*1*)
lemmas *funpow-Suc* = *funpow.simps*(*2*)

lemma *nonemptyE*: *S* ~= {} ==> (!!*x*. *x* : *S* ==> *R*) ==> *R* ⟨*proof*⟩

lemma *gt-or-eq-0*: *0* < *y* ∨ *0* = (*y*::*nat*) ⟨*proof*⟩

declare *iszero-0* [*iff*]

lemmas *xtr1* = *xtrans*(*1*)
lemmas *xtr2* = *xtrans*(*2*)
lemmas *xtr3* = *xtrans*(*3*)
lemmas *xtr4* = *xtrans*(*4*)
lemmas *xtr5* = *xtrans*(*5*)
lemmas *xtr6* = *xtrans*(*6*)
lemmas *xtr7* = *xtrans*(*7*)
lemmas *xtr8* = *xtrans*(*8*)

lemmas *nat-simps* = *diff-add-inverse2 diff-add-inverse*
lemmas *nat-iffs* = *le-add1 le-add2*

lemma *sum-imp-diff*: $j = k + i \Longrightarrow j - i = (k :: nat)$ ⟨*proof*⟩

lemma *nobm1*:
$0 < (number\text{-}of\ w :: nat) \Longrightarrow$
$number\text{-}of\ w - (1 :: nat) = number\text{-}of\ (Int.pred\ w)$
⟨*proof*⟩

lemma *zless2*: $0 < (2 :: int)$ ⟨*proof*⟩

lemmas *zless2p* [*simp*] = *zless2* [*THEN zero-less-power*]
lemmas *zle2p* [*simp*] = *zless2p* [*THEN order-less-imp-le*]

lemmas *pos-mod-sign2* = *zless2* [*THEN pos-mod-sign* [**where** *b* = *2*::*int*]]
lemmas *pos-mod-bound2* = *zless2* [*THEN pos-mod-bound* [**where** *b* = *2*::*int*]]

— the inverse(s) of *number-of*
lemma *nmod2*: $n\ mod\ (2::int) = 0 \mid n\ mod\ 2 = 1$ ⟨*proof*⟩

lemma *emep1*:
$even\ n \Longrightarrow even\ d \Longrightarrow 0 <= d \Longrightarrow (n + 1)\ mod\ (d :: int) = (n\ mod\ d) + 1$
⟨*proof*⟩

lemmas *eme1p* = *emep1* [*simplified add-commute*]

lemma *le-diff-eq'*: $(a \leq c - b) = (b + a \leq (c::int))$ ⟨*proof*⟩

lemma *less-diff-eq'*: $(a < c - b) = (b + a < (c::int))$ ⟨*proof*⟩

lemma *diff-le-eq'*: $(a - b \leq c) = (a \leq b + (c::int))$ ⟨*proof*⟩

lemma *diff-less-eq'*: $(a - b < c) = (a < b + (c::int))$ ⟨*proof*⟩

lemmas *m1mod2k* = *zless2p* [*THEN zmod-minus1*]
lemmas *m1mod22k* = *mult-pos-pos* [*OF zless2 zless2p*, *THEN zmod-minus1*]
lemmas *p1mod22k'* = *zless2p* [*THEN order-less-imp-le*, *THEN pos-zmod-mult-2*]
lemmas *z1pmod2'* = *zero-le-one* [*THEN pos-zmod-mult-2*, *simplified*]
lemmas *z1pdiv2'* = *zero-le-one* [*THEN pos-zdiv-mult-2*, *simplified*]

lemma *p1mod22k*:
$(2 * b + 1)\ mod\ (2 * 2\ \hat{}\ n) = 2 * (b\ mod\ 2\ \hat{}\ n) + (1::int)$
⟨*proof*⟩

lemma *z1pmod2*:
$(2 * b + 1)\ mod\ 2 = (1::int)$ ⟨*proof*⟩

lemma *z1pdiv2*:
$(2 * b + 1)\ div\ 2 = (b::int)$ ⟨*proof*⟩

lemmas *zdiv-le-dividend* = *xtr3* [*OF div-by-1* [*symmetric*] *zdiv-mono2*, *simplified int-one-le-iff-zero-less*, *simplified*, *standard*]

lemma *axxbyy*:
$a + m + m = b + n + n \Longrightarrow (a = 0 \mid a = 1) \Longrightarrow (b = 0 \mid b = 1) \Longrightarrow a = b \;\&\; m = (n :: int)$ ⟨*proof*⟩

lemma *axxmod2*:
$(1 + x + x) \bmod 2 = (1 :: int) \;\&\; (0 + x + x) \bmod 2 = (0 :: int)$ ⟨*proof*⟩

lemma *axxdiv2*:
$(1 + x + x) \operatorname{div} 2 = (x :: int) \;\&\; (0 + x + x) \operatorname{div} 2 = (x :: int)$ ⟨*proof*⟩

lemmas *iszero-minus* = *trans* [*THEN trans*, *OF iszero-def neg-equal-0-iff-equal iszero-def* [*symmetric*], *standard*]

lemmas *zadd-diff-inverse* = *trans* [*OF diff-add-cancel* [*symmetric*] *add-commute*, *standard*]

lemmas *add-diff-cancel2* = *add-commute* [*THEN diff-eq-eq* [*THEN iffD2*], *standard*]

lemma *zmod-uminus*: $-((a :: int) \bmod b) \bmod b = -a \bmod b$
⟨*proof*⟩

lemma *zmod-zsub-distrib*: $((a::int) - b) \bmod c = (a \bmod c - b \bmod c) \bmod c$
⟨*proof*⟩

lemma *zmod-zsub-right-eq*: $((a::int) - b) \bmod c = (a - b \bmod c) \bmod c$
⟨*proof*⟩

lemma *zmod-zsub-left-eq*: $((a::int) - b) \bmod c = (a \bmod c - b) \bmod c$
⟨*proof*⟩

lemma *zmod-zsub-self* [*simp*]:
$((b :: int) - a) \bmod a = b \bmod a$
⟨*proof*⟩

lemma *zmod-zmult1-eq-rev*:
$b * a \bmod c = b \bmod c * a \bmod (c::int)$
⟨*proof*⟩

lemmas *rdmods* [*symmetric*] = *zmod-uminus* [*symmetric*]
zmod-zsub-left-eq zmod-zsub-right-eq mod-add-left-eq
mod-add-right-eq zmod-zmult1-eq zmod-zmult1-eq-rev

lemma *mod-plus-right*:
$((a + x) \bmod m = (b + x) \bmod m) = (a \bmod m = b \bmod (m :: nat))$

⟨*proof*⟩

lemma *nat-minus-mod*: $(n - n \bmod m) \bmod m = (0 :: nat)$
⟨*proof*⟩

lemmas *nat-minus-mod-plus-right* = *trans* [*OF nat-minus-mod mod-0* [*symmetric*], *THEN mod-plus-right* [*THEN iffD2*], *standard*, *simplified*]

lemmas *push-mods'* = *mod-add-eq* [*standard*]
mod-mult-eq [*standard*] *zmod-zsub-distrib* [*standard*]
zmod-uminus [*symmetric*, *standard*]

lemmas *push-mods* = *push-mods'* [*THEN eq-reflection*, *standard*]
lemmas *pull-mods* = *push-mods* [*symmetric*] *rdmods* [*THEN eq-reflection*, *standard*]
lemmas *mod-simps* =
mod-mult-self2-is-0 [*THEN eq-reflection*]
mod-mult-self1-is-0 [*THEN eq-reflection*]
mod-mod-trivial [*THEN eq-reflection*]

lemma *nat-mod-eq*:
$!!b.\ b < n \Longrightarrow a \bmod n = b \bmod n \Longrightarrow a \bmod n = (b :: nat)$
⟨*proof*⟩

lemmas *nat-mod-eq'* = *refl* [*THEN* [*2*] *nat-mod-eq*]

lemma *nat-mod-lem*:
$(0 :: nat) < n \Longrightarrow b < n = (b \bmod n = b)$
⟨*proof*⟩

lemma *mod-nat-add*:
$(x :: nat) < z \Longrightarrow y < z \Longrightarrow$
$(x + y) \bmod z = (\mathit{if}\ x + y < z\ \mathit{then}\ x + y\ \mathit{else}\ x + y - z)$
⟨*proof*⟩

lemma *mod-nat-sub*:
$(x :: nat) < z \Longrightarrow (x - y) \bmod z = x - y$
⟨*proof*⟩

lemma *int-mod-lem*:
$(0 :: int) < n \Longrightarrow (0 <= b\ \&\ b < n) = (b \bmod n = b)$
⟨*proof*⟩

lemma *int-mod-eq*:
$(0 :: int) <= b \Longrightarrow b < n \Longrightarrow a \bmod n = b \bmod n \Longrightarrow a \bmod n = b$
⟨*proof*⟩

lemmas *int-mod-eq'* = *refl* [*THEN* [*3*] *int-mod-eq*]

lemma *int-mod-le*: $0 <= a ==> 0 < (n :: int) ==> a\ mod\ n <= a$
⟨*proof*⟩

lemma *int-mod-le'*: $0 <= b - n ==> 0 < (n :: int) ==> b\ mod\ n <= b - n$
⟨*proof*⟩

lemma *int-mod-ge*: $a < n ==> 0 < (n :: int) ==> a <= a\ mod\ n$
⟨*proof*⟩

lemma *int-mod-ge'*: $b < 0 ==> 0 < (n :: int) ==> b + n <= b\ mod\ n$
⟨*proof*⟩

lemma *mod-add-if-z*:
$(x :: int) < z ==> y < z ==> 0 <= y ==> 0 <= x ==> 0 <= z ==>$
$(x + y)\ mod\ z = (if\ x + y < z\ then\ x + y\ else\ x + y - z)$
⟨*proof*⟩

lemma *mod-sub-if-z*:
$(x :: int) < z ==> y < z ==> 0 <= y ==> 0 <= x ==> 0 <= z ==>$
$(x - y)\ mod\ z = (if\ y <= x\ then\ x - y\ else\ x - y + z)$
⟨*proof*⟩

lemmas *zmde* = *zmod-zdiv-equality* [*THEN diff-eq-eq* [*THEN iffD2*], *symmetric*]
lemmas *mcl* = *mult-cancel-left* [*THEN iffD1*, *THEN make-pos-rule*]

lemma *zdiv-mult-self*: $m \sim= (0 :: int) ==> (a + m * n)\ div\ m = a\ div\ m + n$
⟨*proof*⟩

lemma *eqne*: *equiv A r* ==> $X : A // r$ ==> $X \sim= \{\}$
⟨*proof*⟩

lemmas *Rep-Integ-ne* = *Integ.Rep-Integ*
[*THEN equiv-intrel* [*THEN eqne*, *simplified Integ-def* [*symmetric*]], *standard*]

lemmas *riq* = *Integ.Rep-Integ* [*simplified Integ-def*]
lemmas *intrel-refl* = *refl* [*THEN equiv-intrel-iff* [*THEN iffD1*], *standard*]
lemmas *Rep-Integ-equiv* = *quotient-eq-iff*
[*OF equiv-intrel riq riq*, *simplified Integ.Rep-Integ-inject*, *standard*]
lemmas *Rep-Integ-same* =
Rep-Integ-equiv [*THEN intrel-refl* [*THEN rev-iffD2*], *standard*]

lemma *RI-int*: $(a, 0) : Rep\text{-}Integ\ (int\ a)$
⟨*proof*⟩

lemmas *RI-intrel* [*simp*] = *UNIV-I* [*THEN quotientI*,
THEN Integ.Abs-Integ-inverse [*simplified Integ-def*], *standard*]

lemma *RI-minus*: $(a,\ b) : \mathit{Rep\text{-}Integ}\ x \Longrightarrow (b,\ a) : \mathit{Rep\text{-}Integ}\ (-\ x)$
⟨*proof*⟩

lemma *RI-add*:
$(a,\ b) : \mathit{Rep\text{-}Integ}\ x \Longrightarrow (c,\ d) : \mathit{Rep\text{-}Integ}\ y \Longrightarrow$
$(a + c,\ b + d) : \mathit{Rep\text{-}Integ}\ (x + y)$
⟨*proof*⟩

lemma *mem-same*: $a : S \Longrightarrow a = b \Longrightarrow b : S$
⟨*proof*⟩

lemma *RI-eq-diff'*: $(a,\ b) : \mathit{Rep\text{-}Integ}\ (\mathit{int}\ a - \mathit{int}\ b)$
⟨*proof*⟩

lemma *RI-eq-diff*: $((a,\ b) : \mathit{Rep\text{-}Integ}\ x) = (\mathit{int}\ a - \mathit{int}\ b = x)$
⟨*proof*⟩

lemma *mod-power-lem*:
$a > 1 \Longrightarrow a\ \hat{}\ n \bmod a\ \hat{}\ m = (\mathit{if}\ m <= n\ \mathit{then}\ 0\ \mathit{else}\ (a :: \mathit{int})\ \hat{}\ n)$
⟨*proof*⟩

lemma *min-pm* [*simp*]: $\mathit{min}\ a\ b + (a - b) = (a :: \mathit{nat})$ ⟨*proof*⟩

lemmas *min-pm1* [*simp*] = *trans* [*OF add-commute min-pm*]

lemma *rev-min-pm* [*simp*]: $\mathit{min}\ b\ a + (a - b) = (a{::}\mathit{nat})$ ⟨*proof*⟩

lemmas *rev-min-pm1* [*simp*] = *trans* [*OF add-commute rev-min-pm*]

lemma *pl-pl-rels*:
$a + b = c + d \Longrightarrow$
$a >= c\ \&\ b <= d \mid a <= c\ \&\ b >= (d :: \mathit{nat})$ ⟨*proof*⟩

lemmas *pl-pl-rels'* = *add-commute* [*THEN* [*2*] *trans*, *THEN pl-pl-rels*]

lemma *minus-eq*: $(m - k = m) = (k = 0 \mid m = (0 :: \mathit{nat}))$ ⟨*proof*⟩

lemma *pl-pl-mm*: $(a :: \mathit{nat}) + b = c + d \Longrightarrow a - c = d - b$ ⟨*proof*⟩

lemmas *pl-pl-mm'* = *add-commute* [*THEN* [*2*] *trans*, *THEN pl-pl-mm*]

lemma *min-minus* [*simp*] : $\mathit{min}\ m\ (m - k) = (m - k :: \mathit{nat})$ ⟨*proof*⟩

lemmas *min-minus'* [*simp*] = *trans* [*OF min-max.inf-commute min-minus*]

lemma *nat-no-eq-iff*:
$(\mathit{number\text{-}of}\ b :: \mathit{int}) >= 0 \Longrightarrow (\mathit{number\text{-}of}\ c :: \mathit{int}) >= 0 \Longrightarrow$
$(\mathit{number\text{-}of}\ b = (\mathit{number\text{-}of}\ c :: \mathit{nat})) = (b = c)$

⟨*proof*⟩

lemmas *dme* = *box-equals* [*OF div-mod-equality add-0-right add-0-right*]
lemmas *dtle* = *xtr3* [*OF dme* [*symmetric*] *le-add1*]
lemmas *th2* = *order-trans* [*OF order-refl* [*THEN* [*2*] *mult-le-mono*] *dtle*]

lemma *td-gal*:
*0 < c ==> (a >= b * c) = (a div c >= (b :: nat))*
⟨*proof*⟩

lemmas *td-gal-lt* = *td-gal* [*simplified not-less* [*symmetric*], *simplified*]

lemma *div-mult-le*: *(a :: nat) div b * b <= a*
⟨*proof*⟩

lemmas *sdl* = *split-div-lemma* [*THEN iffD1*, *symmetric*]

lemma *given-quot*: *f > (0 :: nat) ==> (f * l + (f − 1)) div f = l*
⟨*proof*⟩

lemma *given-quot-alt*: *f > (0 :: nat) ==> (l * f + f − Suc 0) div f = l*
⟨*proof*⟩

lemma *diff-mod-le*: *(a::nat) < d ==> b dvd d ==> a − a mod b <= d − b*
⟨*proof*⟩

lemma *less-le-mult'*:
*w * c < b * c ==> 0 ≤ c ==> (w + 1) * c ≤ b * (c::int)*
⟨*proof*⟩

lemmas *less-le-mult* = *less-le-mult'* [*simplified left-distrib*, *simplified*]

lemmas *less-le-mult-minus* = *iffD2* [*OF le-diff-eq less-le-mult*,
simplified left-diff-distrib, *standard*]

lemma *lrlem'*:
assumes *d*: *(i::nat) ≤ j ∨ m < j'*
assumes *R1*: *i * k ≤ j * k ⟹ R*
assumes *R2*: *Suc m * k' ≤ j' * k' ⟹ R*
shows *R* ⟨*proof*⟩

lemma *lrlem*: *(0::nat) < sc ==>*
*(sc − n + (n + lb * n) <= m * n) = (sc + lb * n <= m * n)*
⟨*proof*⟩

lemma *gen-minus*: *0 < n ==> f n = f (Suc (n − 1))*
⟨*proof*⟩

lemma *mpl-lem*: *j <= (i :: nat) ==> k < j ==> i − j + k < i* ⟨*proof*⟩

lemma *nonneg-mod-div*:
0 <= a ==> 0 <= b ==> 0 <= (a mod b :: int) & 0 <= a div b
⟨*proof*⟩

end

4 BinGeneral: Basic Definitions for Binary Integers

theory *BinGeneral*
imports *Num-Lemmas*
begin

4.1 Further properties of numerals

datatype *bit = B0 | B1*

definition
Bit :: int ⇒ bit ⇒ int (**infixl** *BIT 90*) **where**
k BIT b = (case b of B0 ⇒ 0 | B1 ⇒ 1) + k + k

lemma *BIT-B0-eq-Bit0* [*simp*]: *w BIT B0 = Int.Bit0 w*
⟨*proof*⟩

lemma *BIT-B1-eq-Bit1* [*simp*]: *w BIT B1 = Int.Bit1 w*
⟨*proof*⟩

lemmas *BIT-simps = BIT-B0-eq-Bit0 BIT-B1-eq-Bit1*

hide-const (**open**) *B0 B1*

lemma *Min-ne-Pls* [*iff*]:
Int.Min ~= Int.Pls
⟨*proof*⟩

lemmas *Pls-ne-Min* [*iff*] = *Min-ne-Pls* [*symmetric*]

lemmas *PlsMin-defs* [*intro*!] =
Pls-def Min-def Pls-def [*symmetric*] *Min-def* [*symmetric*]

lemmas *PlsMin-simps* [*simp*] = *PlsMin-defs* [*THEN Eq-TrueI*]

lemma *number-of-False-cong*:
False ⟹ number-of x = number-of y
⟨*proof*⟩

lemma *BIT-eq*: $u\ BIT\ b = v\ BIT\ c \Longrightarrow u = v\ \&\ b = c$
⟨*proof*⟩

lemmas *BIT-eqE* [*elim*!] = *BIT-eq* [*THEN conjE*, *standard*]

lemma *BIT-eq-iff* [*simp*]:
$(u\ BIT\ b = v\ BIT\ c) = (u = v \wedge b = c)$
⟨*proof*⟩

lemmas *BIT-eqI* [*intro*!] = *conjI* [*THEN BIT-eq-iff* [*THEN iffD2*]]

lemma *less-Bits*:
$(v\ BIT\ b < w\ BIT\ c) = (v < w \mid v <= w\ \&\ b = bit.B0\ \&\ c = bit.B1)$
⟨*proof*⟩

lemma *le-Bits*:
$(v\ BIT\ b <= w\ BIT\ c) = (v < w \mid v <= w\ \&\ (b \sim= bit.B1 \mid c \sim= bit.B0))$
⟨*proof*⟩

lemma *no-no* [*simp*]: *number-of* (*number-of i*) = *i*
⟨*proof*⟩

lemma *Bit-B0*:
$k\ BIT\ bit.B0 = k + k$
⟨*proof*⟩

lemma *Bit-B1*:
$k\ BIT\ bit.B1 = k + k + 1$
⟨*proof*⟩

lemma *Bit-B0-2t*: $k\ BIT\ bit.B0 = 2 * k$
⟨*proof*⟩

lemma *Bit-B1-2t*: $k\ BIT\ bit.B1 = 2 * k + 1$
⟨*proof*⟩

lemma *B-mod-2'*:
$X = 2 \Longrightarrow (w\ BIT\ bit.B1)\ mod\ X = 1\ \&\ (w\ BIT\ bit.B0)\ mod\ X = 0$
⟨*proof*⟩

lemma *B1-mod-2* [*simp*]: $(Int.Bit1\ w)\ mod\ 2 = 1$
⟨*proof*⟩

lemma *B0-mod-2* [*simp*]: $(Int.Bit0\ w)\ mod\ 2 = 0$
⟨*proof*⟩

lemma *neB1E* [*elim*!]:

assumes *ne*: $y \neq bit.B1$
assumes *y*: $y = bit.B0 \Longrightarrow P$
shows P
⟨*proof*⟩

lemma *bin-ex-rl*: $EX\ w\ b.\ w\ BIT\ b = bin$
⟨*proof*⟩

lemma *bin-exhaust*:
assumes *Q*: $\bigwedge x\ b.\ bin = x\ BIT\ b \Longrightarrow Q$
shows Q
⟨*proof*⟩

4.2 Destructors for binary integers

definition *bin-rl* :: $int \Rightarrow int \times bit$ **where**
[*code del*]: $bin\text{-}rl\ w = (THE\ (r,\ l).\ w = r\ BIT\ l)$

lemma *bin-rl-char*: $(bin\text{-}rl\ w = (r,\ l)) = (r\ BIT\ l = w)$
⟨*proof*⟩

definition
bin-rest-def [*code del*]: $bin\text{-}rest\ w = fst\ (bin\text{-}rl\ w)$

definition
bin-last-def [*code del*] : $bin\text{-}last\ w = snd\ (bin\text{-}rl\ w)$

primrec *bin-nth* **where**
Z: $bin\text{-}nth\ w\ 0 = (bin\text{-}last\ w = bit.B1)$
| *Suc*: $bin\text{-}nth\ w\ (Suc\ n) = bin\text{-}nth\ (bin\text{-}rest\ w)\ n$

lemma *bin-rl*: $bin\text{-}rl\ w = (bin\text{-}rest\ w,\ bin\text{-}last\ w)$
⟨*proof*⟩

lemma *bin-rl-simps* [*simp*]:
$bin\text{-}rl\ Int.Pls = (Int.Pls,\ bit.B0)$
$bin\text{-}rl\ Int.Min = (Int.Min,\ bit.B1)$
$bin\text{-}rl\ (Int.Bit0\ r) = (r,\ bit.B0)$
$bin\text{-}rl\ (Int.Bit1\ r) = (r,\ bit.B1)$
$bin\text{-}rl\ (r\ BIT\ b) = (r,\ b)$
⟨*proof*⟩

declare *bin-rl-simps*(1−4) [*code*]

lemmas *bin-rl-simp* [*simp*] = *iffD1* [*OF bin-rl-char bin-rl*]

lemma *bin-abs-lem*:
$bin = (w\ BIT\ b) \Longrightarrow \sim bin = Int.Min \longrightarrow \sim bin = Int.Pls \longrightarrow$
$nat\ (abs\ w) < nat\ (abs\ bin)$

⟨*proof*⟩

lemma *bin-induct*:
assumes *PPls*: *P Int.Pls*
and *PMin*: *P Int.Min*
and *PBit*: !!*bin bit*. *P bin* ==> *P* (*bin BIT bit*)
shows *P bin*
⟨*proof*⟩

lemma *numeral-induct*:
assumes *Pls*: *P Int.Pls*
assumes *Min*: *P Int.Min*
assumes *Bit0*: ⋀*w*. ⟦*P w*; *w* ≠ *Int.Pls*⟧ ⟹ *P* (*Int.Bit0 w*)
assumes *Bit1*: ⋀*w*. ⟦*P w*; *w* ≠ *Int.Min*⟧ ⟹ *P* (*Int.Bit1 w*)
shows *P x*
⟨*proof*⟩

lemma *bin-rest-simps* [*simp*]:
bin-rest Int.Pls = *Int.Pls*
bin-rest Int.Min = *Int.Min*
bin-rest (*Int.Bit0 w*) = *w*
bin-rest (*Int.Bit1 w*) = *w*
bin-rest (*w BIT b*) = *w*
⟨*proof*⟩

declare *bin-rest-simps*(*1*−*4*) [*code*]

lemma *bin-last-simps* [*simp*]:
bin-last Int.Pls = *bit.B0*
bin-last Int.Min = *bit.B1*
bin-last (*Int.Bit0 w*) = *bit.B0*
bin-last (*Int.Bit1 w*) = *bit.B1*
bin-last (*w BIT b*) = *b*
⟨*proof*⟩

declare *bin-last-simps*(*1*−*4*) [*code*]

lemma *bin-r-l-extras* [*simp*]:
bin-last 0 = *bit.B0*
bin-last (− *1*) = *bit.B1*
bin-last −*1* = *bit.B1*
bin-last 1 = *bit.B1*
bin-rest 1 = *0*
bin-rest 0 = *0*
bin-rest (− *1*) = − *1*
bin-rest −*1* = −*1*
⟨*proof*⟩

lemma *bin-last-mod*:

bin-last w = (if w mod 2 = 0 then bit.B0 else bit.B1)
⟨*proof*⟩

lemma *bin-rest-div*:
bin-rest w = w div 2
⟨*proof*⟩

lemma *Bit-div2* [*simp*]: *(w BIT b) div 2 = w*
⟨*proof*⟩

lemma *Bit0-div2* [*simp*]: *(Int.Bit0 w) div 2 = w*
⟨*proof*⟩

lemma *Bit1-div2* [*simp*]: *(Int.Bit1 w) div 2 = w*
⟨*proof*⟩

lemma *bin-nth-lem* [*rule-format*]:
ALL y. bin-nth x = bin-nth y −−> x = y
⟨*proof*⟩

lemma *bin-nth-eq-iff*: *(bin-nth x = bin-nth y) = (x = y)*
⟨*proof*⟩

lemmas *bin-eqI = ext* [*THEN bin-nth-eq-iff* [*THEN iffD1*], *standard*]

lemma *bin-nth-Pls* [*simp*]: *~ bin-nth Int.Pls n*
⟨*proof*⟩

lemma *bin-nth-Min* [*simp*]: *bin-nth Int.Min n*
⟨*proof*⟩

lemma *bin-nth-0-BIT*: *bin-nth (w BIT b) 0 = (b = bit.B1)*
⟨*proof*⟩

lemma *bin-nth-Suc-BIT*: *bin-nth (w BIT b) (Suc n) = bin-nth w n*
⟨*proof*⟩

lemma *bin-nth-minus* [*simp*]: *0 < n ==> bin-nth (w BIT b) n = bin-nth w (n − 1)*
⟨*proof*⟩

lemma *bin-nth-minus-Bit0* [*simp*]:
0 < n ==> bin-nth (Int.Bit0 w) n = bin-nth w (n − 1)
⟨*proof*⟩

lemma *bin-nth-minus-Bit1* [*simp*]:
0 < n ==> bin-nth (Int.Bit1 w) n = bin-nth w (n − 1)
⟨*proof*⟩

lemmas *bin-nth-0* = *bin-nth.simps*(*1*)
lemmas *bin-nth-Suc* = *bin-nth.simps*(*2*)

lemmas *bin-nth-simps* =
bin-nth-0 bin-nth-Suc bin-nth-Pls bin-nth-Min bin-nth-minus
bin-nth-minus-Bit0 bin-nth-minus-Bit1

4.3 Recursion combinator for binary integers

lemma *brlem*: (*bin* = *Int.Min*) = (− *bin* + *Int.pred 0* = *0*)
⟨*proof*⟩

function
bin-rec :: *'a* ⇒ *'a* ⇒ (*int* ⇒ *bit* ⇒ *'a* ⇒ *'a*) ⇒ *int* ⇒ *'a*
where
bin-rec f1 f2 f3 bin = (*if bin* = *Int.Pls then f1*
else if bin = *Int.Min then f2*
else case bin-rl bin of (*w*, *b*) => *f3 w b* (*bin-rec f1 f2 f3 w*))
⟨*proof*⟩

termination
⟨*proof*⟩

declare *bin-rec.simps* [*simp del*]

lemma *bin-rec-PM*:
f = *bin-rec f1 f2 f3* ==> *f Int.Pls* = *f1* & *f Int.Min* = *f2*
⟨*proof*⟩

lemma *bin-rec-Pls*: *bin-rec f1 f2 f3 Int.Pls* = *f1*
⟨*proof*⟩

lemma *bin-rec-Min*: *bin-rec f1 f2 f3 Int.Min* = *f2*
⟨*proof*⟩

lemma *bin-rec-Bit0*:
f3 Int.Pls bit.B0 f1 = *f1* ⟹
bin-rec f1 f2 f3 (*Int.Bit0 w*) = *f3 w bit.B0* (*bin-rec f1 f2 f3 w*)
⟨*proof*⟩

lemma *bin-rec-Bit1*:
f3 Int.Min bit.B1 f2 = *f2* ⟹
bin-rec f1 f2 f3 (*Int.Bit1 w*) = *f3 w bit.B1* (*bin-rec f1 f2 f3 w*)
⟨*proof*⟩

lemma *bin-rec-Bit*:
f = *bin-rec f1 f2 f3* ==> *f3 Int.Pls bit.B0 f1* = *f1* ==>
f3 Int.Min bit.B1 f2 = *f2* ==> *f* (*w BIT b*) = *f3 w b* (*f w*)
⟨*proof*⟩

lemmas *bin-rec-simps = refl [THEN bin-rec-Bit] bin-rec-Pls bin-rec-Min*
bin-rec-Bit0 bin-rec-Bit1

4.4 Truncating binary integers

definition
bin-sign-def [code del] : bin-sign = bin-rec Int.Pls Int.Min (%w b s. s)

lemma *bin-sign-simps [simp]:*
bin-sign Int.Pls = Int.Pls
bin-sign Int.Min = Int.Min
bin-sign (Int.Bit0 w) = bin-sign w
bin-sign (Int.Bit1 w) = bin-sign w
bin-sign (w BIT b) = bin-sign w
⟨*proof*⟩

declare *bin-sign-simps(1−4) [code]*

lemma *bin-sign-rest [simp]:*
bin-sign (bin-rest w) = (bin-sign w)
⟨*proof*⟩

consts
bintrunc :: nat => int => int
primrec
Z : bintrunc 0 bin = Int.Pls
Suc : bintrunc (Suc n) bin = bintrunc n (bin-rest bin) BIT (bin-last bin)

consts
sbintrunc :: nat => int => int
primrec
Z : sbintrunc 0 bin =
(case bin-last bin of bit.B1 => Int.Min | bit.B0 => Int.Pls)
Suc : sbintrunc (Suc n) bin = sbintrunc n (bin-rest bin) BIT (bin-last bin)

lemma *sign-bintr:*
!!w. bin-sign (bintrunc n w) = Int.Pls
⟨*proof*⟩

lemma *bintrunc-mod2p:*
!!w. bintrunc n w = (w mod 2 ^ n :: int)
⟨*proof*⟩

lemma *sbintrunc-mod2p:*
!!w. sbintrunc n w = ((w + 2 ^ n) mod 2 ^ (Suc n) − 2 ^ n :: int)
⟨*proof*⟩

4.5 Simplifications for (s)bintrunc

lemma *bit-bool*:
(*b* = (*b'* = *bit.B1*)) = (*b'* = (*if b then bit.B1 else bit.B0*))
⟨*proof*⟩

lemmas *bit-bool1* [*simp*] = *refl* [*THEN bit-bool* [*THEN iffD1*], *symmetric*]

lemma *bin-sign-lem*:
!!*bin*. (*bin-sign* (*sbintrunc n bin*) = *Int.Min*) = *bin-nth bin n*
⟨*proof*⟩

lemma *nth-bintr*:
!!*w m*. *bin-nth* (*bintrunc m w*) *n* = (*n* < *m* & *bin-nth w n*)
⟨*proof*⟩

lemma *nth-sbintr*:
!!*w m*. *bin-nth* (*sbintrunc m w*) *n* =
(*if n* < *m then bin-nth w n else bin-nth w m*)
⟨*proof*⟩

lemma *bin-nth-Bit*:
bin-nth (*w BIT b*) *n* = (*n* = *0* & *b* = *bit.B1* | (*EX m*. *n* = *Suc m* & *bin-nth w m*))
⟨*proof*⟩

lemma *bin-nth-Bit0*:
bin-nth (*Int.Bit0 w*) *n* = (*EX m*. *n* = *Suc m* & *bin-nth w m*)
⟨*proof*⟩

lemma *bin-nth-Bit1*:
bin-nth (*Int.Bit1 w*) *n* = (*n* = *0* | (*EX m*. *n* = *Suc m* & *bin-nth w m*))
⟨*proof*⟩

lemma *bintrunc-bintrunc-l*:
n <= *m* ==> (*bintrunc m* (*bintrunc n w*) = *bintrunc n w*)
⟨*proof*⟩

lemma *sbintrunc-sbintrunc-l*:
n <= *m* ==> (*sbintrunc m* (*sbintrunc n w*) = *sbintrunc n w*)
⟨*proof*⟩

lemma *bintrunc-bintrunc-ge*:
n <= *m* ==> (*bintrunc n* (*bintrunc m w*) = *bintrunc n w*)
⟨*proof*⟩

lemma *bintrunc-bintrunc-min* [*simp*]:
bintrunc m (*bintrunc n w*) = *bintrunc* (*min m n*) *w*
⟨*proof*⟩

lemma *sbintrunc-sbintrunc-min* [*simp*]:
sbintrunc m (*sbintrunc n w*) = *sbintrunc* (*min m n*) *w*
⟨*proof*⟩

lemmas *bintrunc-Pls* =
bintrunc.Suc [**where** *bin*=*Int.Pls*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *bintrunc-Min* [*simp*] =
bintrunc.Suc [**where** *bin*=*Int.Min*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *bintrunc-BIT* [*simp*] =
bintrunc.Suc [**where** *bin*=*w BIT b*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemma *bintrunc-Bit0* [*simp*]:
bintrunc (*Suc n*) (*Int.Bit0 w*) = *Int.Bit0* (*bintrunc n w*)
⟨*proof*⟩

lemma *bintrunc-Bit1* [*simp*]:
bintrunc (*Suc n*) (*Int.Bit1 w*) = *Int.Bit1* (*bintrunc n w*)
⟨*proof*⟩

lemmas *bintrunc-Sucs* = *bintrunc-Pls bintrunc-Min bintrunc-BIT*
bintrunc-Bit0 bintrunc-Bit1

lemmas *sbintrunc-Suc-Pls* =
sbintrunc.Suc [**where** *bin*=*Int.Pls*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *sbintrunc-Suc-Min* =
sbintrunc.Suc [**where** *bin*=*Int.Min*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *sbintrunc-Suc-BIT* [*simp*] =
sbintrunc.Suc [**where** *bin*=*w BIT b*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemma *sbintrunc-Suc-Bit0* [*simp*]:
sbintrunc (*Suc n*) (*Int.Bit0 w*) = *Int.Bit0* (*sbintrunc n w*)
⟨*proof*⟩

lemma *sbintrunc-Suc-Bit1* [*simp*]:
sbintrunc (*Suc n*) (*Int.Bit1 w*) = *Int.Bit1* (*sbintrunc n w*)
⟨*proof*⟩

lemmas *sbintrunc-Sucs* = *sbintrunc-Suc-Pls sbintrunc-Suc-Min sbintrunc-Suc-BIT*
sbintrunc-Suc-Bit0 sbintrunc-Suc-Bit1

lemmas *sbintrunc-Pls* =
sbintrunc.Z [**where** *bin=Int.Pls*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemmas *sbintrunc-Min* =
sbintrunc.Z [**where** *bin=Int.Min*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemmas *sbintrunc-0-BIT-B0* [*simp*] =
sbintrunc.Z [**where** *bin=w BIT bit.B0*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemmas *sbintrunc-0-BIT-B1* [*simp*] =
sbintrunc.Z [**where** *bin=w BIT bit.B1*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemma *sbintrunc-0-Bit0* [*simp*]: *sbintrunc 0* (*Int.Bit0 w*) = *Int.Pls*
⟨*proof*⟩

lemma *sbintrunc-0-Bit1* [*simp*]: *sbintrunc 0* (*Int.Bit1 w*) = *Int.Min*
⟨*proof*⟩

lemmas *sbintrunc-0-simps* =
sbintrunc-Pls sbintrunc-Min sbintrunc-0-BIT-B0 sbintrunc-0-BIT-B1
sbintrunc-0-Bit0 sbintrunc-0-Bit1

lemmas *bintrunc-simps* = *bintrunc.Z bintrunc-Sucs*
lemmas *sbintrunc-simps* = *sbintrunc-0-simps sbintrunc-Sucs*

lemma *bintrunc-minus*:
0 < n ==> bintrunc (*Suc* (*n − 1*)) *w* = *bintrunc n w*
⟨*proof*⟩

lemma *sbintrunc-minus*:
0 < n ==> sbintrunc (*Suc* (*n − 1*)) *w* = *sbintrunc n w*
⟨*proof*⟩

lemmas *bintrunc-minus-simps* =
bintrunc-Sucs [*THEN* [*2*] *bintrunc-minus* [*symmetric*, *THEN trans*], *standard*]
lemmas *sbintrunc-minus-simps* =
sbintrunc-Sucs [*THEN* [*2*] *sbintrunc-minus* [*symmetric*, *THEN trans*], *standard*]

lemma *bintrunc-n-Pls* [*simp*]:
bintrunc n Int.Pls = *Int.Pls*
⟨*proof*⟩

lemma *sbintrunc-n-PM* [*simp*]:
sbintrunc n Int.Pls = *Int.Pls*

sbintrunc n Int.Min = Int.Min
⟨*proof*⟩

lemmas *thobini1 = arg-cong* [**where** *f = %w. w BIT b*, *standard*]

lemmas *bintrunc-BIT-I = trans* [*OF bintrunc-BIT thobini1*]
lemmas *bintrunc-Min-I = trans* [*OF bintrunc-Min thobini1*]

lemmas *bmsts = bintrunc-minus-simps*(*1−3*) [*THEN thobini1* [*THEN* [*2*] *trans*], *standard*]
lemmas *bintrunc-Pls-minus-I = bmsts*(*1*)
lemmas *bintrunc-Min-minus-I = bmsts*(*2*)
lemmas *bintrunc-BIT-minus-I = bmsts*(*3*)

lemma *bintrunc-0-Min*: *bintrunc 0 Int.Min = Int.Pls*
⟨*proof*⟩
lemma *bintrunc-0-BIT*: *bintrunc 0* (*w BIT b*) *= Int.Pls*
⟨*proof*⟩

lemma *bintrunc-Suc-lem*:
bintrunc (*Suc n*) *x = y ==> m = Suc n ==> bintrunc m x = y*
⟨*proof*⟩

lemmas *bintrunc-Suc-Ialts =*
bintrunc-Min-I [*THEN bintrunc-Suc-lem*, *standard*]
bintrunc-BIT-I [*THEN bintrunc-Suc-lem*, *standard*]

lemmas *sbintrunc-BIT-I = trans* [*OF sbintrunc-Suc-BIT thobini1*]

lemmas *sbintrunc-Suc-Is =*
sbintrunc-Sucs(*1−3*) [*THEN thobini1* [*THEN* [*2*] *trans*], *standard*]

lemmas *sbintrunc-Suc-minus-Is =*
sbintrunc-minus-simps(*1−3*) [*THEN thobini1* [*THEN* [*2*] *trans*], *standard*]

lemma *sbintrunc-Suc-lem*:
sbintrunc (*Suc n*) *x = y ==> m = Suc n ==> sbintrunc m x = y*
⟨*proof*⟩

lemmas *sbintrunc-Suc-Ialts =*
sbintrunc-Suc-Is [*THEN sbintrunc-Suc-lem*, *standard*]

lemma *sbintrunc-bintrunc-lt*:
m > n ==> sbintrunc n (*bintrunc m w*) *= sbintrunc n w*
⟨*proof*⟩

lemma *bintrunc-sbintrunc-le*:
m <= Suc n ==> bintrunc m (*sbintrunc n w*) *= bintrunc m w*
⟨*proof*⟩

lemmas *bintrunc-sbintrunc* [*simp*] = *order-refl* [*THEN bintrunc-sbintrunc-le*]
lemmas *sbintrunc-bintrunc* [*simp*] = *lessI* [*THEN sbintrunc-bintrunc-lt*]
lemmas *bintrunc-bintrunc* [*simp*] = *order-refl* [*THEN bintrunc-bintrunc-l*]
lemmas *sbintrunc-sbintrunc* [*simp*] = *order-refl* [*THEN sbintrunc-sbintrunc-l*]

lemma *bintrunc-sbintrunc'* [*simp*]:
0 < n ⟹ bintrunc n (sbintrunc (n − 1) w) = bintrunc n w
⟨*proof*⟩

lemma *sbintrunc-bintrunc'* [*simp*]:
0 < n ⟹ sbintrunc (n − 1) (bintrunc n w) = sbintrunc (n − 1) w
⟨*proof*⟩

lemma *bin-sbin-eq-iff*:
bintrunc (Suc n) x = bintrunc (Suc n) y <−>
sbintrunc n x = sbintrunc n y
⟨*proof*⟩

lemma *bin-sbin-eq-iff'*:
0 < n ⟹ bintrunc n x = bintrunc n y <−>
sbintrunc (n − 1) x = sbintrunc (n − 1) y
⟨*proof*⟩

lemmas *bintrunc-sbintruncS0* [*simp*] = *bintrunc-sbintrunc'* [*unfolded One-nat-def*]
lemmas *sbintrunc-bintruncS0* [*simp*] = *sbintrunc-bintrunc'* [*unfolded One-nat-def*]

lemmas *bintrunc-bintrunc-l'* = *le-add1* [*THEN bintrunc-bintrunc-l*]
lemmas *sbintrunc-sbintrunc-l'* = *le-add1* [*THEN sbintrunc-sbintrunc-l*]

lemmas *nat-non0-gr* =
trans [*OF iszero-def* [*THEN Not-eq-iff* [*THEN iffD2*]] *refl*, *standard*]

lemmas *bintrunc-pred-simps* [*simp*] =
bintrunc-minus-simps [*of number-of bin*, *simplified nobm1*, *standard*]

lemmas *sbintrunc-pred-simps* [*simp*] =
sbintrunc-minus-simps [*of number-of bin*, *simplified nobm1*, *standard*]

lemma *no-bintr-alt*:
number-of (bintrunc n w) = w mod 2 ^ n
⟨*proof*⟩

lemma *no-bintr-alt1*: *bintrunc n* = (%*w*. *w mod 2 ^ n* :: *int*)
⟨*proof*⟩

lemma *range-bintrunc*: *range (bintrunc n)* = {*i*. *0 <= i & i < 2 ^ n*}

⟨*proof*⟩

lemma *no-bintr*:
number-of (*bintrunc n w*) = (*number-of w mod 2 ^ n* :: *int*)
⟨*proof*⟩

lemma *no-sbintr-alt2*:
sbintrunc n = (%*w*. (*w* + *2 ^ n*) *mod 2 ^ Suc n* − *2 ^ n* :: *int*)
⟨*proof*⟩

lemma *no-sbintr*:
number-of (*sbintrunc n w*) =
((*number-of w* + *2 ^ n*) *mod 2 ^ Suc n* − *2 ^ n* :: *int*)
⟨*proof*⟩

lemma *range-sbintrunc*:
range (*sbintrunc n*) = {*i*. − (*2 ^ n*) <= *i* & *i* < *2 ^ n*}
⟨*proof*⟩

lemma *sb-inc-lem*:
(*a*::*int*) + *2^k* < *0* ⟹ *a* + *2^k* + *2^*(*Suc k*) <= (*a* + *2^k*) *mod 2^*(*Suc k*)
⟨*proof*⟩

lemma *sb-inc-lem'*:
(*a*::*int*) < − (*2^k*) ⟹ *a* + *2^k* + *2^*(*Suc k*) <= (*a* + *2^k*) *mod 2^*(*Suc k*)
⟨*proof*⟩

lemma *sbintrunc-inc*:
x < − (*2^n*) ==> *x* + *2^*(*Suc n*) <= *sbintrunc n x*
⟨*proof*⟩

lemma *sb-dec-lem*:
(*0*::*int*) <= − (*2^k*) + *a* ==> (*a* + *2^k*) *mod* (*2* ∗ *2 ^ k*) <= − (*2 ^ k*) + *a*
⟨*proof*⟩

lemma *sb-dec-lem'*:
(*2*::*int*) ^ *k* <= *a* ==> (*a* + *2 ^ k*) *mod* (*2* ∗ *2 ^ k*) <= − (*2 ^ k*) + *a*
⟨*proof*⟩

lemma *sbintrunc-dec*:
x >= (*2 ^ n*) ==> *x* − *2 ^* (*Suc n*) >= *sbintrunc n x*
⟨*proof*⟩

lemmas *zmod-uminus'* = *zmod-uminus* [**where** *b*=*c*, *standard*]
lemmas *zpower-zmod'* = *zpower-zmod* [**where** *m*=*c* **and** *y*=*k*, *standard*]

lemmas *brdmod1s'* [*symmetric*] =
mod-add-left-eq mod-add-right-eq
zmod-zsub-left-eq zmod-zsub-right-eq

zmod-zmult1-eq zmod-zmult1-eq-rev

lemmas *brdmods'* [*symmetric*] =
zpower-zmod' [*symmetric*]
trans [*OF mod-add-left-eq mod-add-right-eq*]
trans [*OF zmod-zsub-left-eq zmod-zsub-right-eq*]
trans [*OF zmod-zmult1-eq zmod-zmult1-eq-rev*]
zmod-uminus' [*symmetric*]
mod-add-left-eq [**where** *b* = *1*::*int*]
zmod-zsub-left-eq [**where** *b* = *1*]

lemmas *bintr-arith1s* =
brdmod1s' [**where** *c=2^n*::*int*, *folded pred-def succ-def bintrunc-mod2p*, *standard*]
lemmas *bintr-ariths* =
brdmods' [**where** *c=2^n*::*int*, *folded pred-def succ-def bintrunc-mod2p*, *standard*]

lemmas *m2pths* = *pos-mod-sign pos-mod-bound* [*OF zless2p*, *standard*]

lemma *bintr-ge0*: (*0* :: *int*) <= *number-of* (*bintrunc n w*)
⟨*proof*⟩

lemma *bintr-lt2p*: *number-of* (*bintrunc n w*) < (*2* ^ *n* :: *int*)
⟨*proof*⟩

lemma *bintr-Min*:
number-of (*bintrunc n Int.Min*) = (*2* ^ *n* :: *int*) − *1*
⟨*proof*⟩

lemma *sbintr-ge*: (− (*2* ^ *n*) :: *int*) <= *number-of* (*sbintrunc n w*)
⟨*proof*⟩

lemma *sbintr-lt*: *number-of* (*sbintrunc n w*) < (*2* ^ *n* :: *int*)
⟨*proof*⟩

lemma *bintrunc-Suc*:
bintrunc (*Suc n*) *bin* = *bintrunc n* (*bin-rest bin*) *BIT bin-last bin*
⟨*proof*⟩

lemma *sign-Pls-ge-0*:
(*bin-sign bin* = *Int.Pls*) = (*number-of bin* >= (*0* :: *int*))
⟨*proof*⟩

lemma *sign-Min-lt-0*:
(*bin-sign bin* = *Int.Min*) = (*number-of bin* < (*0* :: *int*))
⟨*proof*⟩

lemmas *sign-Min-neg* = *trans* [*OF sign-Min-lt-0 neg-def* [*symmetric*]]

lemma *bin-rest-trunc*:

!!*bin*. (*bin-rest* (*bintrunc n bin*)) = *bintrunc* (*n* − *1*) (*bin-rest bin*)
⟨*proof*⟩

lemma *bin-rest-power-trunc* [*rule-format*] :
(*bin-rest* ^^ *k*) (*bintrunc n bin*) =
bintrunc (*n* − *k*) ((*bin-rest* ^^ *k*) *bin*)
⟨*proof*⟩

lemma *bin-rest-trunc-i*:
bintrunc n (*bin-rest bin*) = *bin-rest* (*bintrunc* (*Suc n*) *bin*)
⟨*proof*⟩

lemma *bin-rest-strunc*:
!!*bin*. *bin-rest* (*sbintrunc* (*Suc n*) *bin*) = *sbintrunc n* (*bin-rest bin*)
⟨*proof*⟩

lemma *bintrunc-rest* [*simp*]:
!!*bin*. *bintrunc n* (*bin-rest* (*bintrunc n bin*)) = *bin-rest* (*bintrunc n bin*)
⟨*proof*⟩

lemma *sbintrunc-rest* [*simp*]:
!!*bin*. *sbintrunc n* (*bin-rest* (*sbintrunc n bin*)) = *bin-rest* (*sbintrunc n bin*)
⟨*proof*⟩

lemma *bintrunc-rest′*:
bintrunc n o bin-rest o bintrunc n = *bin-rest o bintrunc n*
⟨*proof*⟩

lemma *sbintrunc-rest′* :
sbintrunc n o bin-rest o sbintrunc n = *bin-rest o sbintrunc n*
⟨*proof*⟩

lemma *rco-lem*:
f o g o f = *g o f* ==> *f o* (*g o f*) ^^ *n* = *g* ^^ *n o f*
⟨*proof*⟩

lemma *rco-alt*: (*f o g*) ^^ *n o f* = *f o* (*g o f*) ^^ *n*
⟨*proof*⟩

lemmas *rco-bintr* = *bintrunc-rest′*
[*THEN rco-lem* [*THEN fun-cong*], *unfolded o-def*]
lemmas *rco-sbintr* = *sbintrunc-rest′*
[*THEN rco-lem* [*THEN fun-cong*], *unfolded o-def*]

4.6 Splitting and concatenation

primrec *bin-split* :: *nat* ⇒ *int* ⇒ *int* × *int* **where**
Z: *bin-split 0 w* = (*w*, *Int.Pls*)
| *Suc*: *bin-split* (*Suc n*) *w* = (*let* (*w1*, *w2*) = *bin-split n* (*bin-rest w*)

in (*w1*, *w2 BIT bin-last w*))

primrec *bin-cat* :: *int* ⇒ *nat* ⇒ *int* ⇒ *int* **where**
Z: *bin-cat w 0 v* = *w*
| *Suc*: *bin-cat w* (*Suc n*) *v* = *bin-cat w n* (*bin-rest v*) *BIT bin-last v*

4.7 Miscellaneous lemmas

lemma *funpow-minus-simp*:
0 < *n* ⟹ *f* ^^ *n* = *f* ∘ *f* ^^ (*n* − *1*)
⟨*proof*⟩

lemmas *funpow-pred-simp* [*simp*] =
funpow-minus-simp [*of number-of bin*, *simplified nobm1*, *standard*]

lemmas *replicate-minus-simp* =
trans [*OF gen-minus* [**where** *f* = %*n*. *replicate n x*] *replicate.replicate-Suc*,
standard]

lemmas *replicate-pred-simp* [*simp*] =
replicate-minus-simp [*of number-of bin*, *simplified nobm1*, *standard*]

lemmas *power-Suc-no* [*simp*] = *power-Suc* [*of number-of a*, *standard*]

lemmas *power-minus-simp* =
trans [*OF gen-minus* [**where** *f* = *power f*] *power-Suc*, *standard*]

lemmas *power-pred-simp* =
power-minus-simp [*of number-of bin*, *simplified nobm1*, *standard*]
lemmas *power-pred-simp-no* [*simp*] = *power-pred-simp* [**where** *f*= *number-of f*, *standard*]

lemma *list-exhaust-size-gt0*:
assumes *y*: ⋀*a list*. *y* = *a* # *list* ⟹ *P*
shows *0* < *length y* ⟹ *P*
⟨*proof*⟩

lemma *list-exhaust-size-eq0*:
assumes *y*: *y* = [] ⟹ *P*
shows *length y* = *0* ⟹ *P*
⟨*proof*⟩

lemma *size-Cons-lem-eq*:
y = *xa* # *list* ==> *size y* = *Suc k* ==> *size list* = *k*
⟨*proof*⟩

lemma *size-Cons-lem-eq-bin*:
y = *xa* # *list* ==> *size y* = *number-of* (*Int.succ k*) ==>
size list = *number-of k*

⟨*proof*⟩

lemmas *ls-splits* =
prod.split split-split prod.split-asm split-split-asm split-if-asm

lemma *not-B1-is-B0*: $y \neq bit.B1 \Longrightarrow y = bit.B0$
⟨*proof*⟩

lemma *B1-ass-B0*:
assumes *y*: $y = bit.B0 \Longrightarrow y = bit.B1$
shows $y = bit.B1$
⟨*proof*⟩
lemmas *n2s-ths* [*THEN eq-reflection*] = *add-2-eq-Suc add-2-eq-Suc'*

lemmas *s2n-ths* = *n2s-ths* [*symmetric*]

end

5 BitSyntax: Syntactic classes for bitwise operations

theory *BitSyntax*
imports *BinGeneral*
begin

class *bit* =
fixes *bitNOT* :: $'a \Rightarrow 'a$ (*NOT* - [*70*] *71*)
and *bitAND* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** *AND 64*)
and *bitOR* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** *OR 59*)
and *bitXOR* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** *XOR 59*)

We want the bitwise operations to bind slightly weaker than + and −, but ~~ to bind slightly stronger than ∗.

Testing and shifting operations.

class *bits* = *bit* +
fixes *test-bit* :: $'a \Rightarrow nat \Rightarrow bool$ (**infixl** !! *100*)
and *lsb* :: $'a \Rightarrow bool$
and *set-bit* :: $'a \Rightarrow nat \Rightarrow bool \Rightarrow 'a$
and *set-bits* :: $(nat \Rightarrow bool) \Rightarrow 'a$ (**binder** *BITS 10*)
and *shiftl* :: $'a \Rightarrow nat \Rightarrow 'a$ (**infixl** << *55*)
and *shiftr* :: $'a \Rightarrow nat \Rightarrow 'a$ (**infixl** >> *55*)

class *bitss* = *bits* +
fixes *msb* :: $'a \Rightarrow bool$

5.1 Bitwise operations on *bit*

instantiation *bit* :: *bit*
begin

primrec *bitNOT-bit* **where**
NOT bit.B0 = bit.B1
| *NOT bit.B1 = bit.B0*

primrec *bitAND-bit* **where**
bit.B0 AND y = bit.B0
| *bit.B1 AND y = y*

primrec *bitOR-bit* **where**
bit.B0 OR y = y
| *bit.B1 OR y = bit.B1*

primrec *bitXOR-bit* **where**
bit.B0 XOR y = y
| *bit.B1 XOR y = NOT y*

instance ⟨*proof*⟩

end

lemmas *bit-simps* =
bitNOT-bit.simps bitAND-bit.simps bitOR-bit.simps bitXOR-bit.simps

lemma *bit-extra-simps* [*simp*]:
x AND bit.B0 = bit.B0
x AND bit.B1 = x
x OR bit.B1 = bit.B1
x OR bit.B0 = x
x XOR bit.B1 = NOT x
x XOR bit.B0 = x
⟨*proof*⟩

lemma *bit-ops-comm*:
(*x*::*bit*) *AND y = y AND x*
(*x*::*bit*) *OR y = y OR x*
(*x*::*bit*) *XOR y = y XOR x*
⟨*proof*⟩

lemma *bit-ops-same* [*simp*]:
(*x*::*bit*) *AND x = x*
(*x*::*bit*) *OR x = x*
(*x*::*bit*) *XOR x = bit.B0*
⟨*proof*⟩

lemma *bit-not-not* [*simp*]: *NOT* (*NOT* (*x*::*bit*)) = *x*

⟨*proof*⟩

end

6 BinOperations: Bitwise Operations on Binary Integers

theory *BinOperations*
imports *BinGeneral BitSyntax*
begin

6.1 Logical operations

bit-wise logical operations on the int type

instantiation *int* :: *bit*
begin

definition
int-not-def [*code del*]: *bitNOT* = *bin-rec Int.Min Int.Pls*
(λ*w b s*. *s BIT* (*NOT b*))

definition
int-and-def [*code del*]: *bitAND* = *bin-rec* (λ*x*. *Int.Pls*) (λ*y*. *y*)
(λ*w b s y*. *s* (*bin-rest y*) *BIT* (*b AND bin-last y*))

definition
int-or-def [*code del*]: *bitOR* = *bin-rec* (λ*x*. *x*) (λ*y*. *Int.Min*)
(λ*w b s y*. *s* (*bin-rest y*) *BIT* (*b OR bin-last y*))

definition
int-xor-def [*code del*]: *bitXOR* = *bin-rec* (λ*x*. *x*) *bitNOT*
(λ*w b s y*. *s* (*bin-rest y*) *BIT* (*b XOR bin-last y*))

instance ⟨*proof*⟩

end

lemma *int-not-simps* [*simp*]:
NOT Int.Pls = *Int.Min*
NOT Int.Min = *Int.Pls*
NOT (*Int.Bit0 w*) = *Int.Bit1* (*NOT w*)
NOT (*Int.Bit1 w*) = *Int.Bit0* (*NOT w*)
NOT (*w BIT b*) = (*NOT w*) *BIT* (*NOT b*)
⟨*proof*⟩

declare *int-not-simps*(1−4) [*code*]

lemma *int-xor-Pls* [*simp*, *code*]:
Int.Pls XOR x = x
⟨*proof*⟩

lemma *int-xor-Min* [*simp*, *code*]:
Int.Min XOR x = NOT x
⟨*proof*⟩

lemma *int-xor-Bits* [*simp*]:
(x BIT b) XOR (y BIT c) = (x XOR y) BIT (b XOR c)
⟨*proof*⟩

lemma *int-xor-Bits2* [*simp*, *code*]:
(Int.Bit0 x) XOR (Int.Bit0 y) = Int.Bit0 (x XOR y)
(Int.Bit0 x) XOR (Int.Bit1 y) = Int.Bit1 (x XOR y)
(Int.Bit1 x) XOR (Int.Bit0 y) = Int.Bit1 (x XOR y)
(Int.Bit1 x) XOR (Int.Bit1 y) = Int.Bit0 (x XOR y)
⟨*proof*⟩

lemma *int-xor-x-simps'*:
w XOR (Int.Pls BIT bit.B0) = w
w XOR (Int.Min BIT bit.B1) = NOT w
⟨*proof*⟩

lemma *int-xor-extra-simps* [*simp*, *code*]:
w XOR Int.Pls = w
w XOR Int.Min = NOT w
⟨*proof*⟩

lemma *int-or-Pls* [*simp*, *code*]:
Int.Pls OR x = x
⟨*proof*⟩

lemma *int-or-Min* [*simp*, *code*]:
Int.Min OR x = Int.Min
⟨*proof*⟩

lemma *int-or-Bits* [*simp*]:
(x BIT b) OR (y BIT c) = (x OR y) BIT (b OR c)
⟨*proof*⟩

lemma *int-or-Bits2* [*simp*, *code*]:
(Int.Bit0 x) OR (Int.Bit0 y) = Int.Bit0 (x OR y)
(Int.Bit0 x) OR (Int.Bit1 y) = Int.Bit1 (x OR y)
(Int.Bit1 x) OR (Int.Bit0 y) = Int.Bit1 (x OR y)
(Int.Bit1 x) OR (Int.Bit1 y) = Int.Bit1 (x OR y)
⟨*proof*⟩

lemma *int-or-x-simps'*:

w OR (Int.Pls BIT bit.B0) = w
w OR (Int.Min BIT bit.B1) = Int.Min
⟨*proof*⟩

lemma *int-or-extra-simps* [*simp*, *code*]:
w OR Int.Pls = w
w OR Int.Min = Int.Min
⟨*proof*⟩

lemma *int-and-Pls* [*simp*, *code*]:
Int.Pls AND x = Int.Pls
⟨*proof*⟩

lemma *int-and-Min* [*simp*, *code*]:
Int.Min AND x = x
⟨*proof*⟩

lemma *int-and-Bits* [*simp*]:
(x BIT b) AND (y BIT c) = (x AND y) BIT (b AND c)
⟨*proof*⟩

lemma *int-and-Bits2* [*simp*, *code*]:
(Int.Bit0 x) AND (Int.Bit0 y) = Int.Bit0 (x AND y)
(Int.Bit0 x) AND (Int.Bit1 y) = Int.Bit0 (x AND y)
(Int.Bit1 x) AND (Int.Bit0 y) = Int.Bit0 (x AND y)
(Int.Bit1 x) AND (Int.Bit1 y) = Int.Bit1 (x AND y)
⟨*proof*⟩

lemma *int-and-x-simps'*:
w AND (Int.Pls BIT bit.B0) = Int.Pls
w AND (Int.Min BIT bit.B1) = w
⟨*proof*⟩

lemma *int-and-extra-simps* [*simp*, *code*]:
w AND Int.Pls = Int.Pls
w AND Int.Min = w
⟨*proof*⟩

lemma *bin-ops-comm*:
shows
int-and-comm: !!*y*::*int*. *x AND y = y AND x* **and**
int-or-comm: !!*y*::*int*. *x OR y = y OR x* **and**
int-xor-comm: !!*y*::*int*. *x XOR y = y XOR x*
⟨*proof*⟩

lemma *bin-ops-same* [*simp*]:
(x::*int) AND x = x*
(x::*int) OR x = x*

$(x::int)\ XOR\ x = Int.Pls$
⟨*proof*⟩

lemma *int-not-not* [*simp*]: $NOT\ (NOT\ (x::int)) = x$
⟨*proof*⟩

lemmas *bin-log-esimps* =
int-and-extra-simps int-or-extra-simps int-xor-extra-simps
int-and-Pls int-and-Min int-or-Pls int-or-Min int-xor-Pls int-xor-Min

lemma *bbw-ao-absorb*:
$!!y::int.\ x\ AND\ (y\ OR\ x) = x\ \&\ x\ OR\ (y\ AND\ x) = x$
⟨*proof*⟩

lemma *bbw-ao-absorbs-other*:
$x\ AND\ (x\ OR\ y) = x \wedge (y\ AND\ x)\ OR\ x = (x::int)$
$(y\ OR\ x)\ AND\ x = x \wedge x\ OR\ (x\ AND\ y) = (x::int)$
$(x\ OR\ y)\ AND\ x = x \wedge (x\ AND\ y)\ OR\ x = (x::int)$
⟨*proof*⟩

lemmas *bbw-ao-absorbs* [*simp*] = *bbw-ao-absorb bbw-ao-absorbs-other*

lemma *int-xor-not*:
$!!y::int.\ (NOT\ x)\ XOR\ y = NOT\ (x\ XOR\ y)\ \&$
$x\ XOR\ (NOT\ y) = NOT\ (x\ XOR\ y)$
⟨*proof*⟩

lemma *bbw-assocs'*:
$!!y\ z::int.\ (x\ AND\ y)\ AND\ z = x\ AND\ (y\ AND\ z)\ \&$
$(x\ OR\ y)\ OR\ z = x\ OR\ (y\ OR\ z)\ \&$
$(x\ XOR\ y)\ XOR\ z = x\ XOR\ (y\ XOR\ z)$
⟨*proof*⟩

lemma *int-and-assoc*:
$(x\ AND\ y)\ AND\ (z::int) = x\ AND\ (y\ AND\ z)$
⟨*proof*⟩

lemma *int-or-assoc*:
$(x\ OR\ y)\ OR\ (z::int) = x\ OR\ (y\ OR\ z)$
⟨*proof*⟩

lemma *int-xor-assoc*:
$(x\ XOR\ y)\ XOR\ (z::int) = x\ XOR\ (y\ XOR\ z)$
⟨*proof*⟩

lemmas *bbw-assocs* = *int-and-assoc int-or-assoc int-xor-assoc*

lemma *bbw-lcs* [*simp*]:
(*y*::*int*) *AND* (*x AND z*) = *x AND* (*y AND z*)
(*y*::*int*) *OR* (*x OR z*) = *x OR* (*y OR z*)
(*y*::*int*) *XOR* (*x XOR z*) = *x XOR* (*y XOR z*)
⟨*proof*⟩

lemma *bbw-not-dist*:
!!*y*::*int*. *NOT* (*x OR y*) = (*NOT x*) *AND* (*NOT y*)
!!*y*::*int*. *NOT* (*x AND y*) = (*NOT x*) *OR* (*NOT y*)
⟨*proof*⟩

lemma *bbw-oa-dist*:
!!*y z*::*int*. (*x AND y*) *OR z* =
(*x OR z*) *AND* (*y OR z*)
⟨*proof*⟩

lemma *bbw-ao-dist*:
!!*y z*::*int*. (*x OR y*) *AND z* =
(*x AND z*) *OR* (*y AND z*)
⟨*proof*⟩

lemma *plus-and-or* [*rule-format*]:
ALL y::*int*. (*x AND y*) + (*x OR y*) = *x* + *y*
⟨*proof*⟩

lemma *le-int-or*:
!!*x*. *bin-sign y* = *Int.Pls* ==> *x* <= *x OR y*
⟨*proof*⟩

lemmas *int-and-le* =
xtr3 [*OF bbw-ao-absorbs* (*2*) [*THEN conjunct2*, *symmetric*] *le-int-or*]

lemma *bin-nth-ops*:
!!*x y*. *bin-nth* (*x AND y*) *n* = (*bin-nth x n* & *bin-nth y n*)
!!*x y*. *bin-nth* (*x OR y*) *n* = (*bin-nth x n* | *bin-nth y n*)
!!*x y*. *bin-nth* (*x XOR y*) *n* = (*bin-nth x n* ~= *bin-nth y n*)
!!*x*. *bin-nth* (*NOT x*) *n* = (~ *bin-nth x n*)
⟨*proof*⟩

lemma *bin-add-not*: *x* + *NOT x* = *Int.Min*
⟨*proof*⟩

lemma *bin-trunc-ao*:
!!*x y*. (*bintrunc n x*) *AND* (*bintrunc n y*) = *bintrunc n* (*x AND y*)

!!*x y*. (*bintrunc n x*) *OR* (*bintrunc n y*) = *bintrunc n* (*x OR y*)
⟨*proof*⟩

lemma *bin-trunc-xor*:
!!*x y*. *bintrunc n* (*bintrunc n x XOR bintrunc n y*) =
bintrunc n (*x XOR y*)
⟨*proof*⟩

lemma *bin-trunc-not*:
!!*x*. *bintrunc n* (*NOT* (*bintrunc n x*)) = *bintrunc n* (*NOT x*)
⟨*proof*⟩

lemma *bintr-bintr-i*:
x = *bintrunc n y* ==> *bintrunc n x* = *bintrunc n y*
⟨*proof*⟩

lemmas *bin-trunc-and* = *bin-trunc-ao*(*1*) [*THEN bintr-bintr-i*]
lemmas *bin-trunc-or* = *bin-trunc-ao*(*2*) [*THEN bintr-bintr-i*]

6.2 Setting and clearing bits

primrec
bin-sc :: *nat* => *bit* => *int* => *int*
where
Z: *bin-sc 0 b w* = *bin-rest w BIT b*
| *Suc*: *bin-sc* (*Suc n*) *b w* = *bin-sc n b* (*bin-rest w*) *BIT bin-last w*

lemma *bin-nth-sc* [*simp*]:
!!*w*. *bin-nth* (*bin-sc n b w*) *n* = (*b* = *bit.B1*)
⟨*proof*⟩

lemma *bin-sc-sc-same* [*simp*]:
!!*w*. *bin-sc n c* (*bin-sc n b w*) = *bin-sc n c w*
⟨*proof*⟩

lemma *bin-sc-sc-diff*:
!!*w m*. *m* ~= *n* ==>
bin-sc m c (*bin-sc n b w*) = *bin-sc n b* (*bin-sc m c w*)
⟨*proof*⟩

lemma *bin-nth-sc-gen*:
!!*w m*. *bin-nth* (*bin-sc n b w*) *m* = (*if m* = *n then b* = *bit.B1 else bin-nth w m*)
⟨*proof*⟩

lemma *bin-sc-nth* [*simp*]:
!!*w*. (*bin-sc n* (*If* (*bin-nth w n*) *bit.B1 bit.B0*) *w*) = *w*

⟨*proof*⟩

lemma *bin-sign-sc* [*simp*]:
!!*w*. *bin-sign* (*bin-sc n b w*) = *bin-sign w*
⟨*proof*⟩

lemma *bin-sc-bintr* [*simp*]:
!!*w m*. *bintrunc m* (*bin-sc n x* (*bintrunc m* (*w*))) = *bintrunc m* (*bin-sc n x w*)
⟨*proof*⟩

lemma *bin-clr-le*:
!!*w*. *bin-sc n bit.B0 w* <= *w*
⟨*proof*⟩

lemma *bin-set-ge*:
!!*w*. *bin-sc n bit.B1 w* >= *w*
⟨*proof*⟩

lemma *bintr-bin-clr-le*:
!!*w m*. *bintrunc n* (*bin-sc m bit.B0 w*) <= *bintrunc n w*
⟨*proof*⟩

lemma *bintr-bin-set-ge*:
!!*w m*. *bintrunc n* (*bin-sc m bit.B1 w*) >= *bintrunc n w*
⟨*proof*⟩

lemma *bin-sc-FP* [*simp*]: *bin-sc n bit.B0 Int.Pls* = *Int.Pls*
⟨*proof*⟩

lemma *bin-sc-TM* [*simp*]: *bin-sc n bit.B1 Int.Min* = *Int.Min*
⟨*proof*⟩

lemmas *bin-sc-simps* = *bin-sc.Z bin-sc.Suc bin-sc-TM bin-sc-FP*

lemma *bin-sc-minus*:
0 < *n* ==> *bin-sc* (*Suc* (*n* − *1*)) *b w* = *bin-sc n b w*
⟨*proof*⟩

lemmas *bin-sc-Suc-minus* =
trans [*OF bin-sc-minus* [*symmetric*] *bin-sc.Suc*, *standard*]

lemmas *bin-sc-Suc-pred* [*simp*] =
bin-sc-Suc-minus [*of number-of bin*, *simplified nobm1*, *standard*]

6.3 Operations on lists of booleans

primrec *bl-to-bin-aux* :: *bool list* ⇒ *int* ⇒ *int* **where**
Nil: *bl-to-bin-aux* [] *w* = *w*
| *Cons*: *bl-to-bin-aux* (*b* # *bs*) *w* =

```
    bl-to-bin-aux bs (w BIT (if b then bit.B1 else bit.B0))

definition bl-to-bin :: bool list ⇒ int where
  bl-to-bin-def : bl-to-bin bs = bl-to-bin-aux bs Int.Pls

primrec bin-to-bl-aux :: nat ⇒ int ⇒ bool list ⇒ bool list where
  Z: bin-to-bl-aux 0 w bl = bl
  | Suc: bin-to-bl-aux (Suc n) w bl =
      bin-to-bl-aux n (bin-rest w) ((bin-last w = bit.B1) # bl)

definition bin-to-bl :: nat ⇒ int ⇒ bool list where
  bin-to-bl-def : bin-to-bl n w = bin-to-bl-aux n w []

primrec bl-of-nth :: nat ⇒ (nat ⇒ bool) ⇒ bool list where
  Suc: bl-of-nth (Suc n) f = f n # bl-of-nth n f
  | Z: bl-of-nth 0 f = []

primrec takefill :: 'a ⇒ nat ⇒ 'a list ⇒ 'a list where
  Z: takefill fill 0 xs = []
  | Suc: takefill fill (Suc n) xs = (
      case xs of [] => fill # takefill fill n xs
        | y # ys => y # takefill fill n ys)

definition map2 :: ('a ⇒ 'b ⇒ 'c) ⇒ 'a list ⇒ 'b list ⇒ 'c list where
  map2 f as bs = map (split f) (zip as bs)
```

6.4 Splitting and concatenation

```
definition bin-rcat :: nat ⇒ int list ⇒ int where
  bin-rcat n = foldl (%u v. bin-cat u n v) Int.Pls

fun bin-rsplit-aux :: nat ⇒ nat ⇒ int ⇒ int list ⇒ int list where
  bin-rsplit-aux n m c bs =
    (if m = 0 | n = 0 then bs else
      let (a, b) = bin-split n c
      in bin-rsplit-aux n (m - n) a (b # bs))

definition bin-rsplit :: nat ⇒ nat × int ⇒ int list where
  bin-rsplit n w = bin-rsplit-aux n (fst w) (snd w) []

fun bin-rsplitl-aux :: nat ⇒ nat ⇒ int ⇒ int list ⇒ int list where
  bin-rsplitl-aux n m c bs =
    (if m = 0 | n = 0 then bs else
      let (a, b) = bin-split (min m n) c
      in bin-rsplitl-aux n (m - n) a (b # bs))

definition bin-rsplitl :: nat ⇒ nat × int ⇒ int list where
  bin-rsplitl n w = bin-rsplitl-aux n (fst w) (snd w) []
```

declare *bin-rsplit-aux.simps* [*simp del*]
declare *bin-rsplitl-aux.simps* [*simp del*]

lemma *bin-sign-cat*:
!!*y*. *bin-sign* (*bin-cat x n y*) = *bin-sign x*
⟨*proof*⟩

lemma *bin-cat-Suc-Bit*:
bin-cat w (*Suc n*) (*v BIT b*) = *bin-cat w n v BIT b*
⟨*proof*⟩

lemma *bin-nth-cat*:
!!*n y*. *bin-nth* (*bin-cat x k y*) *n* =
(*if n* < *k then bin-nth y n else bin-nth x* (*n* − *k*))
⟨*proof*⟩

lemma *bin-nth-split*:
!!*b c*. *bin-split n c* = (*a*, *b*) ==>
(*ALL k*. *bin-nth a k* = *bin-nth c* (*n* + *k*)) &
(*ALL k*. *bin-nth b k* = (*k* < *n* & *bin-nth c k*))
⟨*proof*⟩

lemma *bin-cat-assoc*:
!!*z*. *bin-cat* (*bin-cat x m y*) *n z* = *bin-cat x* (*m* + *n*) (*bin-cat y n z*)
⟨*proof*⟩

lemma *bin-cat-assoc-sym*: !!*z m*.
bin-cat x m (*bin-cat y n z*) = *bin-cat* (*bin-cat x* (*m* − *n*) *y*) (*min m n*) *z*
⟨*proof*⟩

lemma *bin-cat-Pls* [*simp*]:
!!*w*. *bin-cat Int.Pls n w* = *bintrunc n w*
⟨*proof*⟩

lemma *bintr-cat1*:
!!*b*. *bintrunc* (*k* + *n*) (*bin-cat a n b*) = *bin-cat* (*bintrunc k a*) *n b*
⟨*proof*⟩

lemma *bintr-cat*: *bintrunc m* (*bin-cat a n b*) =
bin-cat (*bintrunc* (*m* − *n*) *a*) *n* (*bintrunc* (*min m n*) *b*)
⟨*proof*⟩

lemma *bintr-cat-same* [*simp*]:
bintrunc n (*bin-cat a n b*) = *bintrunc n b*
⟨*proof*⟩

lemma *cat-bintr* [*simp*]:
!!*b*. *bin-cat a n* (*bintrunc n b*) = *bin-cat a n b*
⟨*proof*⟩

lemma *split-bintrunc*:
!!*b c*. *bin-split n c* = (*a*, *b*) ==> *b* = *bintrunc n c*
⟨*proof*⟩

lemma *bin-cat-split*:
!!*v w*. *bin-split n w* = (*u*, *v*) ==> *w* = *bin-cat u n v*
⟨*proof*⟩

lemma *bin-split-cat*:
!!*w*. *bin-split n* (*bin-cat v n w*) = (*v*, *bintrunc n w*)
⟨*proof*⟩

lemma *bin-split-Pls* [*simp*]:
bin-split n Int.Pls = (*Int.Pls*, *Int.Pls*)
⟨*proof*⟩

lemma *bin-split-Min* [*simp*]:
bin-split n Int.Min = (*Int.Min*, *bintrunc n Int.Min*)
⟨*proof*⟩

lemma *bin-split-trunc*:
!!*m b c*. *bin-split* (*min m n*) *c* = (*a*, *b*) ==>
bin-split n (*bintrunc m c*) = (*bintrunc* (*m* − *n*) *a*, *b*)
⟨*proof*⟩

lemma *bin-split-trunc1*:
!!*m b c*. *bin-split n c* = (*a*, *b*) ==>
bin-split n (*bintrunc m c*) = (*bintrunc* (*m* − *n*) *a*, *bintrunc m b*)
⟨*proof*⟩

lemma *bin-cat-num*:
!!*b*. *bin-cat a n b* = *a* ∗ *2* ^ *n* + *bintrunc n b*
⟨*proof*⟩

lemma *bin-split-num*:
!!*b*. *bin-split n b* = (*b div 2* ^ *n*, *b mod 2* ^ *n*)
⟨*proof*⟩

6.5 Miscellaneous lemmas

lemma *nth-2p-bin*:
!!*m*. *bin-nth* (*2* ^ *n*) *m* = (*m* = *n*)
⟨*proof*⟩

lemma *ex-eq-or*:
(*EX m*. *n* = *Suc m* & (*m* = *k* | *P m*)) = (*n* = *Suc k* | (*EX m*. *n* = *Suc m* & *P*

m))
⟨*proof*⟩

end

7 BinBoolList: Bool lists and integers

theory *BinBoolList*
imports *BinOperations*
begin

7.1 Arithmetic in terms of bool lists

primrec *rbl-succ* :: *bool list* => *bool list* **where**
Nil: *rbl-succ Nil* = *Nil*
| *Cons*: *rbl-succ* (*x* # *xs*) = (*if x then False* # *rbl-succ xs else True* # *xs*)

primrec *rbl-pred* :: *bool list* => *bool list* **where**
Nil: *rbl-pred Nil* = *Nil*
| *Cons*: *rbl-pred* (*x* # *xs*) = (*if x then False* # *xs else True* # *rbl-pred xs*)

primrec *rbl-add* :: *bool list* => *bool list* => *bool list* **where**

Nil: *rbl-add Nil x* = *Nil*
| *Cons*: *rbl-add* (*y* # *ys*) *x* = (*let ws* = *rbl-add ys* (*tl x*) *in*
(*y* ~= *hd x*) # (*if hd x* & *y then rbl-succ ws else ws*))

primrec *rbl-mult* :: *bool list* => *bool list* => *bool list* **where**

Nil: *rbl-mult Nil x* = *Nil*
| *Cons*: *rbl-mult* (*y* # *ys*) *x* = (*let ws* = *False* # *rbl-mult ys x in*
if y then rbl-add ws x else ws)

lemma *butlast-power*:
(*butlast* ^^ *n*) *bl* = *take* (*length bl* − *n*) *bl*
⟨*proof*⟩

lemma *bin-to-bl-aux-Pls-minus-simp* [*simp*]:
0 < *n* ==> *bin-to-bl-aux n Int.Pls bl* =
bin-to-bl-aux (*n* − *1*) *Int.Pls* (*False* # *bl*)
⟨*proof*⟩

lemma *bin-to-bl-aux-Min-minus-simp* [*simp*]:
0 < *n* ==> *bin-to-bl-aux n Int.Min bl* =
bin-to-bl-aux (*n* − *1*) *Int.Min* (*True* # *bl*)
⟨*proof*⟩

lemma *bin-to-bl-aux-Bit-minus-simp* [*simp*]:
0 < n ==> bin-to-bl-aux n (w BIT b) bl =
bin-to-bl-aux (n − 1) w ((b = bit.B1) # bl)
⟨*proof*⟩

lemma *bin-to-bl-aux-Bit0-minus-simp* [*simp*]:
0 < n ==> bin-to-bl-aux n (Int.Bit0 w) bl =
bin-to-bl-aux (n − 1) w (False # bl)
⟨*proof*⟩

lemma *bin-to-bl-aux-Bit1-minus-simp* [*simp*]:
0 < n ==> bin-to-bl-aux n (Int.Bit1 w) bl =
bin-to-bl-aux (n − 1) w (True # bl)
⟨*proof*⟩

lemma *bl-to-bin-aux-append*:
bl-to-bin-aux (bs @ cs) w = bl-to-bin-aux cs (bl-to-bin-aux bs w)
⟨*proof*⟩

lemma *bin-to-bl-aux-append*:
bin-to-bl-aux n w bs @ cs = bin-to-bl-aux n w (bs @ cs)
⟨*proof*⟩

lemma *bl-to-bin-append*:
bl-to-bin (bs @ cs) = bl-to-bin-aux cs (bl-to-bin bs)
⟨*proof*⟩

lemma *bin-to-bl-aux-alt*:
bin-to-bl-aux n w bs = bin-to-bl n w @ bs
⟨*proof*⟩

lemma *bin-to-bl-0*: *bin-to-bl 0 bs = []*
⟨*proof*⟩

lemma *size-bin-to-bl-aux*:
size (bin-to-bl-aux n w bs) = n + length bs
⟨*proof*⟩

lemma *size-bin-to-bl*: *size (bin-to-bl n w) = n*
⟨*proof*⟩

lemma *bin-bl-bin'*:
bl-to-bin (bin-to-bl-aux n w bs) =
bl-to-bin-aux bs (bintrunc n w)
⟨*proof*⟩

lemma *bin-bl-bin*: *bl-to-bin (bin-to-bl n w) = bintrunc n w*

⟨*proof*⟩

lemma *bl-bin-bl'*:
bin-to-bl (*n* + *length bs*) (*bl-to-bin-aux bs w*) =
bin-to-bl-aux n w bs
⟨*proof*⟩

lemma *bl-bin-bl*: *bin-to-bl* (*length bs*) (*bl-to-bin bs*) = *bs*
⟨*proof*⟩

declare
bin-to-bl-0 [*simp*]
size-bin-to-bl [*simp*]
bin-bl-bin [*simp*]
bl-bin-bl [*simp*]

lemma *bl-to-bin-inj*:
bl-to-bin bs = *bl-to-bin cs* ==> *length bs* = *length cs* ==> *bs* = *cs*
⟨*proof*⟩

lemma *bl-to-bin-False*: *bl-to-bin* (*False* # *bl*) = *bl-to-bin bl*
⟨*proof*⟩

lemma *bl-to-bin-Nil*: *bl-to-bin* [] = *Int.Pls*
⟨*proof*⟩

lemma *bin-to-bl-Pls-aux*:
bin-to-bl-aux n Int.Pls bl = *replicate n False* @ *bl*
⟨*proof*⟩

lemma *bin-to-bl-Pls*: *bin-to-bl n Int.Pls* = *replicate n False*
⟨*proof*⟩

lemma *bin-to-bl-Min-aux* [*rule-format*] :
ALL bl. bin-to-bl-aux n Int.Min bl = *replicate n True* @ *bl*
⟨*proof*⟩

lemma *bin-to-bl-Min*: *bin-to-bl n Int.Min* = *replicate n True*
⟨*proof*⟩

lemma *bl-to-bin-rep-F*:
bl-to-bin (*replicate n False* @ *bl*) = *bl-to-bin bl*
⟨*proof*⟩

lemma *bin-to-bl-trunc*:
n <= *m* ==> *bin-to-bl n* (*bintrunc m w*) = *bin-to-bl n w*
⟨*proof*⟩

declare

bin-to-bl-trunc [*simp*]
bl-to-bin-False [*simp*]
bl-to-bin-Nil [*simp*]

lemma *bin-to-bl-aux-bintr* [*rule-format*] :
ALL m bin bl. bin-to-bl-aux n (bintrunc m bin) bl =
replicate (n − m) False @ bin-to-bl-aux (min n m) bin bl
⟨*proof*⟩

lemmas *bin-to-bl-bintr* =
bin-to-bl-aux-bintr [**where** *bl* = [], *folded bin-to-bl-def*]

lemma *bl-to-bin-rep-False*: *bl-to-bin (replicate n False)* = *Int.Pls*
⟨*proof*⟩

lemma *len-bin-to-bl-aux*:
length (bin-to-bl-aux n w bs) = *n* + *length bs*
⟨*proof*⟩

lemma *len-bin-to-bl* [*simp*]: *length (bin-to-bl n w)* = *n*
⟨*proof*⟩

lemma *sign-bl-bin'*:
bin-sign (bl-to-bin-aux bs w) = *bin-sign w*
⟨*proof*⟩

lemma *sign-bl-bin*: *bin-sign (bl-to-bin bs)* = *Int.Pls*
⟨*proof*⟩

lemma *bl-sbin-sign-aux*:
hd (bin-to-bl-aux (Suc n) w bs) =
(bin-sign (sbintrunc n w) = *Int.Min)*
⟨*proof*⟩

lemma *bl-sbin-sign*:
hd (bin-to-bl (Suc n) w) = *(bin-sign (sbintrunc n w)* = *Int.Min)*
⟨*proof*⟩

lemma *bin-nth-of-bl-aux* [*rule-format*]:
∀ *w. bin-nth (bl-to-bin-aux bl w) n =*
(n < size bl & rev bl ! n | n >= length bl & bin-nth w (n − size bl))
⟨*proof*⟩

lemma *bin-nth-of-bl*: *bin-nth (bl-to-bin bl) n* = *(n < length bl & rev bl ! n)*
⟨*proof*⟩

lemma *bin-nth-bl* [*rule-format*] : *ALL m w. n < m −−>*
bin-nth w n = *nth (rev (bin-to-bl m w)) n*
⟨*proof*⟩

lemma *nth-rev* [*rule-format*] :
n < *length xs* −−> *rev xs* ! *n* = *xs* ! (*length xs* − *1* − *n*)
⟨*proof*⟩

lemmas *nth-rev-alt* = *nth-rev* [**where** *xs* = *rev ys*, *simplified*, *standard*]

lemma *nth-bin-to-bl-aux* [*rule-format*] :
ALL w n bl. *n* < *m* + *length bl* −−> (*bin-to-bl-aux m w bl*) ! *n* =
(*if n* < *m then bin-nth w* (*m* − *1* − *n*) *else bl* ! (*n* − *m*))
⟨*proof*⟩

lemma *nth-bin-to-bl*: *n* < *m* ==> (*bin-to-bl m w*) ! *n* = *bin-nth w* (*m* − *Suc n*)
⟨*proof*⟩

lemma *bl-to-bin-lt2p-aux* [*rule-format*]:
∀ *w*. *bl-to-bin-aux bs w* < (*w* + *1*) ∗ (*2* ^ *length bs*)
⟨*proof*⟩

lemma *bl-to-bin-lt2p*: *bl-to-bin bs* < (*2* ^ *length bs*)
⟨*proof*⟩

lemma *bl-to-bin-ge2p-aux* [*rule-format*] :
∀ *w*. *bl-to-bin-aux bs w* >= *w* ∗ (*2* ^ *length bs*)
⟨*proof*⟩

lemma *bl-to-bin-ge0*: *bl-to-bin bs* >= *0*
⟨*proof*⟩

lemma *butlast-rest-bin*:
butlast (*bin-to-bl n w*) = *bin-to-bl* (*n* − *1*) (*bin-rest w*)
⟨*proof*⟩

lemmas *butlast-bin-rest* = *butlast-rest-bin*
[**where** *w*=*bl-to-bin bl* **and** *n*=*length bl*, *simplified*, *standard*]

lemma *butlast-rest-bl2bin-aux*:
bl ~= [] ⟹
bl-to-bin-aux (*butlast bl*) *w* = *bin-rest* (*bl-to-bin-aux bl w*)
⟨*proof*⟩

lemma *butlast-rest-bl2bin*:
bl-to-bin (*butlast bl*) = *bin-rest* (*bl-to-bin bl*)
⟨*proof*⟩

lemma *trunc-bl2bin-aux* [*rule-format*]:
ALL w. *bintrunc m* (*bl-to-bin-aux bl w*) =
bl-to-bin-aux (*drop* (*length bl* − *m*) *bl*) (*bintrunc* (*m* − *length bl*) *w*)
⟨*proof*⟩

lemma *trunc-bl2bin*:
bintrunc m (bl-to-bin bl) = bl-to-bin (drop (length bl − m) bl)
⟨*proof*⟩

lemmas *trunc-bl2bin-len* [*simp*] =
trunc-bl2bin [*of length bl bl*, *simplified*, *standard*]

lemma *bl2bin-drop*:
bl-to-bin (drop k bl) = bintrunc (length bl − k) (bl-to-bin bl)
⟨*proof*⟩

lemma *nth-rest-power-bin* [*rule-format*] :
ALL n. bin-nth ((bin-rest ^^ k) w) n = bin-nth w (n + k)
⟨*proof*⟩

lemma *take-rest-power-bin*:
m <= n ==> take m (bin-to-bl n w) = bin-to-bl m ((bin-rest ^^ (n − m)) w)
⟨*proof*⟩

lemma *hd-butlast*: *size xs > 1 ==> hd (butlast xs) = hd xs*
⟨*proof*⟩

lemma *last-bin-last'*:
size xs > 0 ⟹ last xs = (bin-last (bl-to-bin-aux xs w) = bit.B1)
⟨*proof*⟩

lemma *last-bin-last*:
size xs > 0 ==> last xs = (bin-last (bl-to-bin xs) = bit.B1)
⟨*proof*⟩

lemma *bin-last-last*:
bin-last w = (if last (bin-to-bl (Suc n) w) then bit.B1 else bit.B0)
⟨*proof*⟩

lemma *map2-Nil* [*simp*]: *map2 f [] ys = []*
⟨*proof*⟩

lemma *map2-Cons* [*simp*]:
map2 f (x # xs) (y # ys) = f x y # map2 f xs ys
⟨*proof*⟩

lemma *bl-xor-aux-bin* [*rule-format*] : *ALL v w bs cs.*
map2 (%x y. x ~= y) (bin-to-bl-aux n v bs) (bin-to-bl-aux n w cs) =
bin-to-bl-aux n (v XOR w) (map2 (%x y. x ~= y) bs cs)
⟨*proof*⟩

lemma *bl-or-aux-bin* [*rule-format*] : *ALL v w bs cs.*
map2 (op |) (bin-to-bl-aux n v bs) (bin-to-bl-aux n w cs) =
bin-to-bl-aux n (v OR w) (map2 (op |) bs cs)
⟨*proof*⟩

lemma *bl-and-aux-bin* [*rule-format*] : *ALL v w bs cs.*
map2 (op &) (bin-to-bl-aux n v bs) (bin-to-bl-aux n w cs) =
bin-to-bl-aux n (v AND w) (map2 (op &) bs cs)
⟨*proof*⟩

lemma *bl-not-aux-bin* [*rule-format*] :
ALL w cs. map Not (bin-to-bl-aux n w cs) =
bin-to-bl-aux n (NOT w) (map Not cs)
⟨*proof*⟩

lemmas *bl-not-bin = bl-not-aux-bin*
[**where** *cs* = [], *unfolded bin-to-bl-def* [*symmetric*] *map.simps*]

lemmas *bl-and-bin = bl-and-aux-bin* [**where** *bs*=[] **and** *cs*=[],
unfolded map2-Nil, *folded bin-to-bl-def*]

lemmas *bl-or-bin = bl-or-aux-bin* [**where** *bs*=[] **and** *cs*=[],
unfolded map2-Nil, *folded bin-to-bl-def*]

lemmas *bl-xor-bin = bl-xor-aux-bin* [**where** *bs*=[] **and** *cs*=[],
unfolded map2-Nil, *folded bin-to-bl-def*]

lemma *drop-bin2bl-aux* [*rule-format*] :
ALL m bin bs. drop m (bin-to-bl-aux n bin bs) =
bin-to-bl-aux (n − m) bin (drop (m − n) bs)
⟨*proof*⟩

lemma *drop-bin2bl*: *drop m (bin-to-bl n bin) = bin-to-bl (n − m) bin*
⟨*proof*⟩

lemma *take-bin2bl-lem1* [*rule-format*] :
ALL w bs. take m (bin-to-bl-aux m w bs) = bin-to-bl m w
⟨*proof*⟩

lemma *take-bin2bl-lem* [*rule-format*] :
ALL w bs. take m (bin-to-bl-aux (m + n) w bs) =
take m (bin-to-bl (m + n) w)
⟨*proof*⟩

lemma *bin-split-take* [*rule-format*] :
ALL b c. bin-split n c = (a, b) −−>
bin-to-bl m a = take m (bin-to-bl (m + n) c)
⟨*proof*⟩

lemma *bin-split-take1*:
$k = m + n \Longrightarrow$ *bin-split* $n\ c = (a,\ b) \Longrightarrow$
bin-to-bl $m\ a =$ *take* m (*bin-to-bl* $k\ c$)
⟨*proof*⟩

lemma *nth-takefill* [*rule-format*] : *ALL* $m\ l.\ m < n \longrightarrow$
takefill fill $n\ l\ !\ m =$ (*if* $m <$ *length* l *then* $l\ !\ m$ *else fill*)
⟨*proof*⟩

lemma *takefill-alt* [*rule-format*] :
ALL $l.$ *takefill fill* $n\ l =$ *take* $n\ l$ @ *replicate* $(n -$ *length* $l)$ *fill*
⟨*proof*⟩

lemma *takefill-replicate* [*simp*]:
takefill fill n (*replicate* m *fill*) = *replicate* n *fill*
⟨*proof*⟩

lemma *takefill-le'* [*rule-format*] :
ALL $l\ n.\ n = m + k \longrightarrow$ *takefill* $x\ m$ (*takefill* $x\ n\ l$) = *takefill* $x\ m\ l$
⟨*proof*⟩

lemma *length-takefill* [*simp*]: *length* (*takefill fill* $n\ l$) = n
⟨*proof*⟩

lemma *take-takefill'*:
$!!w\ n.\ \ n = k + m \Longrightarrow$ *take* k (*takefill fill* $n\ w$) = *takefill fill* $k\ w$
⟨*proof*⟩

lemma *drop-takefill*:
$!!w.$ *drop* k (*takefill fill* $(m + k)\ w$) = *takefill fill* m (*drop* $k\ w$)
⟨*proof*⟩

lemma *takefill-le* [*simp*]:
$m \leq n \Longrightarrow$ *takefill* $x\ m$ (*takefill* $x\ n\ l$) = *takefill* $x\ m\ l$
⟨*proof*⟩

lemma *take-takefill* [*simp*]:
$m \leq n \Longrightarrow$ *take* m (*takefill fill* $n\ w$) = *takefill fill* $m\ w$
⟨*proof*⟩

lemma *takefill-append*:
takefill fill $(m +$ *length* $xs)$ $(xs$ @ $w) = xs$ @ (*takefill fill* $m\ w$)
⟨*proof*⟩

lemma *takefill-same'*:
$l =$ *length* $xs \Longrightarrow$ *takefill fill* $l\ xs = xs$
⟨*proof*⟩

lemmas *takefill-same* [*simp*] = *takefill-same'* [*OF refl*]

lemma *takefill-bintrunc*:
takefill False n bl = rev (bin-to-bl n (bl-to-bin (rev bl)))
⟨*proof*⟩

lemma *bl-bin-bl-rtf*:
bin-to-bl n (bl-to-bin bl) = rev (takefill False n (rev bl))
⟨*proof*⟩

lemmas *bl-bin-bl-rep-drop =*
bl-bin-bl-rtf [simplified takefill-alt,
simplified, simplified rev-take, simplified]

lemma *tf-rev*:
n + k = m + length bl ==> takefill x m (rev (takefill y n bl)) =
rev (takefill y m (rev (takefill x k (rev bl))))
⟨*proof*⟩

lemma *takefill-minus*:
0 < n ==> takefill fill (Suc (n − 1)) w = takefill fill n w
⟨*proof*⟩

lemmas *takefill-Suc-cases =*
list.cases [THEN takefill.Suc [THEN trans], standard]

lemmas *takefill-Suc-Nil = takefill-Suc-cases (1)*
lemmas *takefill-Suc-Cons = takefill-Suc-cases (2)*

lemmas *takefill-minus-simps = takefill-Suc-cases [THEN [2]*
takefill-minus [symmetric, THEN trans], standard]

lemmas *takefill-pred-simps [simp] =*
*takefill-minus-simps [***where** *n=number-of bin, simplified nobm1, standard]*

lemma *bl-to-bin-aux-cat*:
!!nv v. bl-to-bin-aux bs (bin-cat w nv v) =
bin-cat w (nv + length bs) (bl-to-bin-aux bs v)
⟨*proof*⟩

lemma *bin-to-bl-aux-cat*:
!!w bs. bin-to-bl-aux (nv + nw) (bin-cat v nw w) bs =
bin-to-bl-aux nv v (bin-to-bl-aux nw w bs)
⟨*proof*⟩

lemmas *bl-to-bin-aux-alt =*
*bl-to-bin-aux-cat [***where** *nv = 0* **and** *v = Int.Pls,*
simplified bl-to-bin-def [symmetric], simplified]

lemmas *bin-to-bl-cat* =
bin-to-bl-aux-cat [**where** *bs* = [], *folded bin-to-bl-def*]

lemmas *bl-to-bin-aux-app-cat* =
trans [*OF bl-to-bin-aux-append bl-to-bin-aux-alt*]

lemmas *bin-to-bl-aux-cat-app* =
trans [*OF bin-to-bl-aux-cat bin-to-bl-aux-alt*]

lemmas *bl-to-bin-app-cat* = *bl-to-bin-aux-app-cat*
[**where** *w* = *Int.Pls*, *folded bl-to-bin-def*]

lemmas *bin-to-bl-cat-app* = *bin-to-bl-aux-cat-app*
[**where** *bs* = [], *folded bin-to-bl-def*]

lemma *bl-to-bin-app-cat-alt*:
bin-cat (*bl-to-bin cs*) *n w* = *bl-to-bin* (*cs* @ *bin-to-bl n w*)
⟨*proof*⟩

lemma *mask-lem*: (*bl-to-bin* (*True* # *replicate n False*)) =
Int.succ (*bl-to-bin* (*replicate n True*))
⟨*proof*⟩

lemma *length-bl-of-nth* [*simp*]: *length* (*bl-of-nth n f*) = *n*
⟨*proof*⟩

lemma *nth-bl-of-nth* [*simp*]:
m < *n* ⟹ *rev* (*bl-of-nth n f*) ! *m* = *f m*
⟨*proof*⟩

lemma *bl-of-nth-inj*:
(!!*k*. *k* < *n* ==> *f k* = *g k*) ==> *bl-of-nth n f* = *bl-of-nth n g*
⟨*proof*⟩

lemma *bl-of-nth-nth-le* [*rule-format*] : *ALL xs*.
length xs >= *n* −−> *bl-of-nth n* (*nth* (*rev xs*)) = *drop* (*length xs* − *n*) *xs*
⟨*proof*⟩

lemmas *bl-of-nth-nth* [*simp*] = *order-refl* [*THEN bl-of-nth-nth-le*, *simplified*]

lemma *size-rbl-pred*: *length* (*rbl-pred bl*) = *length bl*
⟨*proof*⟩

lemma *size-rbl-succ*: *length* (*rbl-succ bl*) = *length bl*
⟨*proof*⟩

lemma *size-rbl-add*:
!!*cl*. *length* (*rbl-add bl cl*) = *length bl*
⟨*proof*⟩

lemma *size-rbl-mult*:
!!*cl*. *length* (*rbl-mult bl cl*) = *length bl*
⟨*proof*⟩

lemmas *rbl-sizes* [*simp*] =
size-rbl-pred size-rbl-succ size-rbl-add size-rbl-mult

lemmas *rbl-Nils* =
rbl-pred.Nil rbl-succ.Nil rbl-add.Nil rbl-mult.Nil

lemma *rbl-pred*:
!!*bin*. *rbl-pred* (*rev* (*bin-to-bl n bin*)) = *rev* (*bin-to-bl n* (*Int.pred bin*))
⟨*proof*⟩

lemma *rbl-succ*:
!!*bin*. *rbl-succ* (*rev* (*bin-to-bl n bin*)) = *rev* (*bin-to-bl n* (*Int.succ bin*))
⟨*proof*⟩

lemma *rbl-add*:
!!*bina binb*. *rbl-add* (*rev* (*bin-to-bl n bina*)) (*rev* (*bin-to-bl n binb*)) =
rev (*bin-to-bl n* (*bina* + *binb*))
⟨*proof*⟩

lemma *rbl-add-app2*:
!!*blb*. *length blb* >= *length bla* ==>
rbl-add bla (*blb* @ *blc*) = *rbl-add bla blb*
⟨*proof*⟩

lemma *rbl-add-take2*:
!!*blb*. *length blb* >= *length bla* ==>
rbl-add bla (*take* (*length bla*) *blb*) = *rbl-add bla blb*
⟨*proof*⟩

lemma *rbl-add-long*:
m >= *n* ==> *rbl-add* (*rev* (*bin-to-bl n bina*)) (*rev* (*bin-to-bl m binb*)) =
rev (*bin-to-bl n* (*bina* + *binb*))
⟨*proof*⟩

lemma *rbl-mult-app2*:
!!*blb*. *length blb* >= *length bla* ==>
rbl-mult bla (*blb* @ *blc*) = *rbl-mult bla blb*
⟨*proof*⟩

lemma *rbl-mult-take2*:
length blb >= *length bla* ==>

rbl-mult bla (take (length bla) blb) = rbl-mult bla blb
⟨*proof*⟩

lemma *rbl-mult-gt1*:
m >= length bl ==> rbl-mult bl (rev (bin-to-bl m binb)) =
rbl-mult bl (rev (bin-to-bl (length bl) binb))
⟨*proof*⟩

lemma *rbl-mult-gt*:
m > n ==> rbl-mult (rev (bin-to-bl n bina)) (rev (bin-to-bl m binb)) =
rbl-mult (rev (bin-to-bl n bina)) (rev (bin-to-bl n binb))
⟨*proof*⟩

lemmas *rbl-mult-Suc = lessI [THEN rbl-mult-gt]*

lemma *rbbl-Cons*:
b # rev (bin-to-bl n x) = rev (bin-to-bl (Suc n) (x BIT If b bit.B1 bit.B0))
⟨*proof*⟩

lemma *rbl-mult*: !!*bina binb*.
rbl-mult (rev (bin-to-bl n bina)) (rev (bin-to-bl n binb)) =
*rev (bin-to-bl n (bina * binb))*
⟨*proof*⟩

lemma *rbl-add-split*:
P (rbl-add (y # ys) (x # xs)) =
(ALL ws. length ws = length ys --> ws = rbl-add ys xs -->
(y --> ((x --> P (False # rbl-succ ws)) & (~ x --> P (True # ws)))) &
(~ y --> P (x # ws)))
⟨*proof*⟩

lemma *rbl-mult-split*:
P (rbl-mult (y # ys) xs) =
(ALL ws. length ws = Suc (length ys) --> ws = False # rbl-mult ys xs -->

(y --> P (rbl-add ws xs)) & (~ y --> P ws))
⟨*proof*⟩

lemma *and-len*: *xs = ys ==> xs = ys & length xs = length ys*
⟨*proof*⟩

lemma *size-if*: *size (if p then xs else ys) = (if p then size xs else size ys)*
⟨*proof*⟩

lemma *tl-if*: *tl (if p then xs else ys) = (if p then tl xs else tl ys)*
⟨*proof*⟩

lemma *hd-if*: *hd (if p then xs else ys) = (if p then hd xs else hd ys)*
⟨*proof*⟩

lemma *if-Not-x*: (*if p then ~ x else x*) = (*p* = (*~ x*))
⟨*proof*⟩

lemma *if-x-Not*: (*if p then x else ~ x*) = (*p* = *x*)
⟨*proof*⟩

lemma *if-same-and*: (*If p x y* & *If p u v*) = (*if p then x* & *u else y* & *v*)
⟨*proof*⟩

lemma *if-same-eq*: (*If p x y* = (*If p u v*)) = (*if p then x* = (*u*) *else y* = (*v*))
⟨*proof*⟩

lemma *if-same-eq-not*:
(*If p x y* = (*~ If p u v*)) = (*if p then x* = (*~u*) *else y* = (*~v*))
⟨*proof*⟩

lemma *if-Cons*: (*if p then x # xs else y # ys*) = *If p x y # If p xs ys*
⟨*proof*⟩

lemma *if-single*:
(*if xc then* [*xab*] *else* [*an*]) = [*if xc then xab else an*]
⟨*proof*⟩

lemma *if-bool-simps*:
If p True y = (*p* | *y*) & *If p False y* = (*~p* & *y*) &
If p y True = (*p* −−> *y*) & *If p y False* = (*p* & *y*)
⟨*proof*⟩

lemmas *if-simps* = *if-x-Not if-Not-x if-cancel if-True if-False if-bool-simps*

lemmas *seqr* = *eq-reflection* [**where** *x* = *size w*, *standard*]

lemmas *tl-Nil* = *tl.simps* (*1*)
lemmas *tl-Cons* = *tl.simps* (*2*)

7.2 Repeated splitting or concatenation

lemma *sclem*:
size (*concat* (*map* (*bin-to-bl n*) *xs*)) = *length xs* ∗ *n*
⟨*proof*⟩

lemma *bin-cat-foldl-lem* [*rule-format*] :
ALL x. *foldl* (%*u*. *bin-cat u n*) *x xs* =
bin-cat x (*size xs* ∗ *n*) (*foldl* (%*u*. *bin-cat u n*) *y xs*)
⟨*proof*⟩

lemma *bin-rcat-bl*:

(*bin-rcat n wl*) = *bl-to-bin* (*concat* (*map* (*bin-to-bl n*) *wl*))
⟨*proof*⟩

lemmas *bin-rsplit-aux-simps* = *bin-rsplit-aux.simps bin-rsplitl-aux.simps*
lemmas *rsplit-aux-simps* = *bin-rsplit-aux-simps*

lemmas *th-if-simp1* = *split-if* [**where** *P* = *op* = *l*,
THEN iffD1, *THEN conjunct1*, *THEN mp*, *standard*]
lemmas *th-if-simp2* = *split-if* [**where** *P* = *op* = *l*,
THEN iffD1, *THEN conjunct2*, *THEN mp*, *standard*]

lemmas *rsplit-aux-simp1s* = *rsplit-aux-simps* [*THEN th-if-simp1*]

lemmas *rsplit-aux-simp2ls* = *rsplit-aux-simps* [*THEN th-if-simp2*]

lemmas *bin-rsplit-aux-simp2s* [*simp*] = *rsplit-aux-simp2ls* [*unfolded Let-def*]
lemmas *rbscl* = *bin-rsplit-aux-simp2s* (*2*)

lemmas *rsplit-aux-0-simps* [*simp*] =
rsplit-aux-simp1s [*OF disjI1*] *rsplit-aux-simp1s* [*OF disjI2*]

lemma *bin-rsplit-aux-append*:
bin-rsplit-aux n m c (*bs* @ *cs*) = *bin-rsplit-aux n m c bs* @ *cs*
⟨*proof*⟩

lemma *bin-rsplitl-aux-append*:
bin-rsplitl-aux n m c (*bs* @ *cs*) = *bin-rsplitl-aux n m c bs* @ *cs*
⟨*proof*⟩

lemmas *rsplit-aux-apps* [**where** *bs* = []] =
bin-rsplit-aux-append bin-rsplitl-aux-append

lemmas *rsplit-def-auxs* = *bin-rsplit-def bin-rsplitl-def*

lemmas *rsplit-aux-alts* = *rsplit-aux-apps*
[*unfolded append-Nil rsplit-def-auxs* [*symmetric*]]

lemma *bin-split-minus*: *0* < *n* ==> *bin-split* (*Suc* (*n* − *1*)) *w* = *bin-split n w*
⟨*proof*⟩

lemmas *bin-split-minus-simp* =
bin-split.Suc [*THEN* [*2*] *bin-split-minus* [*symmetric*, *THEN trans*], *standard*]

lemma *bin-split-pred-simp* [*simp*]:
(*0*::*nat*) < *number-of bin* ⟹
bin-split (*number-of bin*) *w* =
(*let* (*w1*, *w2*) = *bin-split* (*number-of* (*Int.pred bin*)) (*bin-rest w*)
in (*w1*, *w2 BIT bin-last w*))
⟨*proof*⟩

declare *bin-split-pred-simp* [*simp*]

lemma *bin-rsplit-aux-simp-alt*:
bin-rsplit-aux n m c bs =
(*if m* = *0* ∨ *n* = *0*
then bs
else let (*a*, *b*) = *bin-split n c in bin-rsplit n* (*m* − *n*, *a*) @ *b* # *bs*)
⟨*proof*⟩

lemmas *bin-rsplit-simp-alt* =
trans [*OF bin-rsplit-def*
bin-rsplit-aux-simp-alt, *standard*]

lemmas *bthrs* = *bin-rsplit-simp-alt* [*THEN* [*2*] *trans*]

lemma *bin-rsplit-size-sign′* [*rule-format*] :
n > *0* ==> (*ALL nw w*. *rev sw* = *bin-rsplit n* (*nw*, *w*) −−>
(*ALL v*: *set sw*. *bintrunc n v* = *v*))
⟨*proof*⟩

lemmas *bin-rsplit-size-sign* = *bin-rsplit-size-sign′* [*OF asm-rl*
rev-rev-ident [*THEN trans*] *set-rev* [*THEN equalityD2* [*THEN subsetD*]],
standard]

lemma *bin-nth-rsplit* [*rule-format*] :
n > *0* ==> *m* < *n* ==> (*ALL w k nw*. *rev sw* = *bin-rsplit n* (*nw*, *w*) −−>
k < *size sw* −−> *bin-nth* (*sw* ! *k*) *m* = *bin-nth w* (*k* ∗ *n* + *m*))
⟨*proof*⟩

lemma *bin-rsplit-all*:
0 < *nw* ==> *nw* <= *n* ==> *bin-rsplit n* (*nw*, *w*) = [*bintrunc n w*]
⟨*proof*⟩

lemma *bin-rsplit-l* [*rule-format*] :
ALL bin. *bin-rsplitl n* (*m*, *bin*) = *bin-rsplit n* (*m*, *bintrunc m bin*)
⟨*proof*⟩

lemma *bin-rsplit-rcat* [*rule-format*] :
n > *0* −−> *bin-rsplit n* (*n* ∗ *size ws*, *bin-rcat n ws*) = *map* (*bintrunc n*) *ws*
⟨*proof*⟩

lemma *bin-rsplit-aux-len-le* [*rule-format*] :
∀ *ws m*. *n* ≠ *0* ⟶ *ws* = *bin-rsplit-aux n nw w bs* ⟶
length ws ≤ *m* ⟷ *nw* + *length bs* ∗ *n* ≤ *m* ∗ *n*
⟨*proof*⟩

lemma *bin-rsplit-len-le*:
n ≠ *0* −−> *ws* = *bin-rsplit n* (*nw*, *w*) −−> (*length ws* <= *m*) = (*nw* <= *m* ∗

n)
⟨*proof*⟩

lemma *bin-rsplit-aux-len* [*rule-format*] :
n≠0 −−> length (*bin-rsplit-aux n nw w cs*) =
(*nw* + *n* − *1*) *div n* + *length cs*
⟨*proof*⟩

lemma *bin-rsplit-len*:
n≠0 ==> length (*bin-rsplit n* (*nw*, *w*)) = (*nw* + *n* − *1*) *div n*
⟨*proof*⟩

lemma *bin-rsplit-aux-len-indep*:
n ≠ *0* ⟹ *length bs* = *length cs* ⟹
length (*bin-rsplit-aux n nw v bs*) =
length (*bin-rsplit-aux n nw w cs*)
⟨*proof*⟩

lemma *bin-rsplit-len-indep*:
n≠0 ==> length (*bin-rsplit n* (*nw*, *v*)) = *length* (*bin-rsplit n* (*nw*, *w*))
⟨*proof*⟩

end

8 TdThs: Type Definition Theorems

theory *TdThs*
imports *Main*
begin

9 More lemmas about normal type definitions

lemma
tdD1: *type-definition Rep Abs A* ⟹ ∀ *x*. *Rep x* ∈ *A* **and**
tdD2: *type-definition Rep Abs A* ⟹ ∀ *x*. *Abs* (*Rep x*) = *x* **and**
tdD3: *type-definition Rep Abs A* ⟹ ∀ *y*. *y* ∈ *A* ⟶ *Rep* (*Abs y*) = *y*
⟨*proof*⟩

lemma *td-nat-int*:
type-definition int nat (*Collect* (*op <= 0*))
⟨*proof*⟩

context *type-definition*
begin

lemmas *Rep′* [*iff*] = *Rep* [*simplified*]

declare *Rep-inverse* [*simp*] *Rep-inject* [*simp*]

lemma *Abs-eqD*: *Abs x = Abs y ==> x ∈ A ==> y ∈ A ==> x = y*
⟨*proof*⟩

lemma *Abs-inverse'*:
r : A ==> Abs r = a ==> Rep a = r
⟨*proof*⟩

lemma *Rep-comp-inverse*:
Rep o f = g ==> Abs o g = f
⟨*proof*⟩

lemma *Rep-eqD* [*elim*!]: *Rep x = Rep y ==> x = y*
⟨*proof*⟩

lemma *Rep-inverse'*: *Rep a = r ==> Abs r = a*
⟨*proof*⟩

lemma *comp-Abs-inverse*:
f o Abs = g ==> g o Rep = f
⟨*proof*⟩

lemma *set-Rep*:
A = range Rep
⟨*proof*⟩

lemma *set-Rep-Abs*: *A = range (Rep o Abs)*
⟨*proof*⟩

lemma *Abs-inj-on*: *inj-on Abs A*
⟨*proof*⟩

lemma *image*: *Abs ' A = UNIV*
⟨*proof*⟩

lemmas *td-thm = type-definition-axioms*

lemma *fns1*:
Rep o fa = fr o Rep | fa o Abs = Abs o fr ==> Abs o fr o Rep = fa
⟨*proof*⟩

lemmas *fns1a = disjI1* [*THEN fns1*]
lemmas *fns1b = disjI2* [*THEN fns1*]

lemma *fns4*:
Rep o fa o Abs = fr ==>
Rep o fa = fr o Rep & fa o Abs = Abs o fr
⟨*proof*⟩

end

interpretation *nat-int*: *type-definition int nat Collect* (*op <= 0*)
⟨*proof*⟩

declare
nat-int.Rep-cases [*cases del*]
nat-int.Abs-cases [*cases del*]
nat-int.Rep-induct [*induct del*]
nat-int.Abs-induct [*induct del*]

9.1 Extended form of type definition predicate

lemma *td-conds*:
norm o norm = norm ==> (fr o norm = norm o fr) =
(norm o fr o norm = fr o norm & norm o fr o norm = norm o fr)
⟨*proof*⟩

lemma *fn-comm-power*:
fa o tr = tr o fr ==> fa ^^ n o tr = tr o fr ^^ n
⟨*proof*⟩

lemmas *fn-comm-power′ =*
ext [*THEN fn-comm-power*, *THEN fun-cong*, *unfolded o-def*, *standard*]

locale *td-ext = type-definition +*
fixes *norm*
assumes *eq-norm*: ⋀*x*. *Rep* (*Abs x*) = *norm x*
begin

lemma *Abs-norm* [*simp*]:
Abs (*norm x*) = *Abs x*
⟨*proof*⟩

lemma *td-th*:
g o Abs = f ==> f (*Rep x*) = *g x*
⟨*proof*⟩

lemma *eq-norm′*: *Rep o Abs = norm*
⟨*proof*⟩

lemma *norm-Rep* [*simp*]: *norm* (*Rep x*) = *Rep x*
⟨*proof*⟩

lemmas *td = td-thm*

lemma *set-iff-norm*: *w* : *A* <−> *w = norm w*

⟨*proof*⟩

lemma *inverse-norm*:
(*Abs n* = *w*) = (*Rep w* = *norm n*)
⟨*proof*⟩

lemma *norm-eq-iff*:
(*norm x* = *norm y*) = (*Abs x* = *Abs y*)
⟨*proof*⟩

lemma *norm-comps*:
Abs o norm = *Abs*
norm o Rep = *Rep*
norm o norm = *norm*
⟨*proof*⟩

lemmas *norm-norm* [*simp*] = *norm-comps*

lemma *fns5*:
Rep o fa o Abs = *fr* ==>
fr o norm = *fr* & *norm o fr* = *fr*
⟨*proof*⟩

lemma *fns2*:
Abs o fr o Rep = *fa* ==>
(*norm o fr o norm* = *fr o norm*) = (*Rep o fa* = *fr o Rep*)
⟨*proof*⟩

lemma *fns3*:
Abs o fr o Rep = *fa* ==>
(*norm o fr o norm* = *norm o fr*) = (*fa o Abs* = *Abs o fr*)
⟨*proof*⟩

lemma *fns*:
fr o norm = *norm o fr* ==>
(*fa o Abs* = *Abs o fr*) = (*Rep o fa* = *fr o Rep*)
⟨*proof*⟩

lemma *range-norm*:
range (*Rep o Abs*) = *A*
⟨*proof*⟩

end

lemmas *td-ext-def'* =
td-ext-def [*unfolded type-definition-def td-ext-axioms-def*]

end

10 WordDefinition: Definition of Word Type

theory *WordDefinition*
imports *Size BinBoolList TdThs*
begin

10.1 Type definition

typedef (**open** *word*) *'a word* = {(*0*::*int*) *..< 2^len-of TYPE('a::len0)*}
  morphisms *uint Abs-word* ⟨*proof*⟩

definition *word-of-int* :: *int* ⇒ *'a*::*len0 word* **where**
  — representation of words using unsigned or signed bins, only difference in these is the type class
  word-of-int w = *Abs-word* (*bintrunc* (*len-of TYPE* (*'a*)) *w*)

lemma *uint-word-of-int* [*code*]: *uint* (*word-of-int w* :: *'a*::*len0 word*) = *w mod 2* ^ *len-of TYPE('a)*
  ⟨*proof*⟩

code-datatype *word-of-int*

notation *fcomp* (**infixl** *o>* *60*)
notation *scomp* (**infixl** *o→* *60*)

instantiation *word* :: ({*len0*, *typerep*}) *random*
begin

definition
  random-word i = *Random.range* (*max i* (*2* ^ *len-of TYPE('a)*)) *o→* (λ*k*. *Pair* (
    let j = *word-of-int* (*Code-Numeral.int-of k*) :: *'a word*
    in (*j*, λ-::*unit*. *Code-Evaluation.term-of j*)))

instance ⟨*proof*⟩

end

no-notation *fcomp* (**infixl** *o>* *60*)
no-notation *scomp* (**infixl** *o→* *60*)

10.2 Type conversions and casting

definition *sint* :: *'a* :: *len word* => *int* **where**
  — treats the most-significant-bit as a sign bit
  sint-uint: *sint w* = *sbintrunc* (*len-of TYPE* (*'a*) − *1*) (*uint w*)

definition *unat* :: *'a* :: *len0 word* => *nat* **where**
unat w = *nat* (*uint w*)

definition *uints* :: *nat* => *int set* **where**
— the sets of integers representing the words
uints n = *range* (*bintrunc n*)

definition *sints* :: *nat* => *int set* **where**
sints n = *range* (*sbintrunc* (*n* − *1*))

definition *unats* :: *nat* => *nat set* **where**
unats n = {*i*. *i* < *2* ^ *n*}

definition *norm-sint* :: *nat* => *int* => *int* **where**
norm-sint n w = (*w* + *2* ^ (*n* − *1*)) *mod 2* ^ *n* − *2* ^ (*n* − *1*)

definition *scast* :: *'a* :: *len word* => *'b* :: *len word* **where**
— cast a word to a different length
scast w = *word-of-int* (*sint w*)

definition *ucast* :: *'a* :: *len0 word* => *'b* :: *len0 word* **where**
ucast w = *word-of-int* (*uint w*)

instantiation *word* :: (*len0*) *size*
begin

definition
word-size: *size* (*w* :: *'a word*) = *len-of TYPE*(*'a*)

instance ⟨*proof*⟩

end

definition *source-size* :: (*'a* :: *len0 word* => *'b*) => *nat* **where**
— whether a cast (or other) function is to a longer or shorter length
source-size c = (*let arb* = *undefined* ; *x* = *c arb in size arb*)

definition *target-size* :: (*'a* => *'b* :: *len0 word*) => *nat* **where**
target-size c = *size* (*c undefined*)

definition *is-up* :: (*'a* :: *len0 word* => *'b* :: *len0 word*) => *bool* **where**
is-up c ⟷ *source-size c* <= *target-size c*

definition *is-down* :: (*'a* :: *len0 word* => *'b* :: *len0 word*) => *bool* **where**
is-down c ⟷ *target-size c* <= *source-size c*

definition *of-bl* :: *bool list* => *'a* :: *len0 word* **where**
of-bl bl = *word-of-int* (*bl-to-bin bl*)

definition *to-bl* :: *'a* :: *len0 word* => *bool list* **where**
to-bl w = *bin-to-bl* (*len-of TYPE* (*'a*)) (*uint w*)

definition *word-reverse* :: *'a* :: *len0 word* => *'a word* **where**
word-reverse w = *of-bl* (*rev* (*to-bl w*))

definition *word-int-case* :: (*int* => *'b*) => (*'a* :: *len0 word*) => *'b* **where**
word-int-case f w = *f* (*uint w*)

syntax
of-int :: *int* => *'a*
translations
case x of CONST of-int y => *b* == *CONST word-int-case* (%*y*. *b*) *x*

10.3 Arithmetic operations

instantiation *word* :: (*len0*) {*number*, *uminus*, *minus*, *plus*, *one*, *zero*, *times*, *Divides.div*, *ord*, *bit*}
begin

definition
word-0-wi: *0* = *word-of-int 0*

definition
word-1-wi: *1* = *word-of-int 1*

definition
word-add-def: *a* + *b* = *word-of-int* (*uint a* + *uint b*)

definition
word-sub-wi: *a* − *b* = *word-of-int* (*uint a* − *uint b*)

definition
word-minus-def: − *a* = *word-of-int* (− *uint a*)

definition
word-mult-def: *a* ∗ *b* = *word-of-int* (*uint a* ∗ *uint b*)

definition
word-div-def: *a div b* = *word-of-int* (*uint a div uint b*)

definition
word-mod-def: *a mod b* = *word-of-int* (*uint a mod uint b*)

definition
word-number-of-def: *number-of w* = *word-of-int w*

definition
word-le-def: *a* ≤ *b* ⟷ *uint a* ≤ *uint b*

definition
word-less-def: $x < y \longleftrightarrow x \leq y \wedge x \neq (y :: 'a\ word)$

definition
word-and-def:
$(a::'a\ word)\ AND\ b = word\text{-}of\text{-}int\ (uint\ a\ AND\ uint\ b)$

definition
word-or-def:
$(a::'a\ word)\ OR\ b = word\text{-}of\text{-}int\ (uint\ a\ OR\ uint\ b)$

definition
word-xor-def:
$(a::'a\ word)\ XOR\ b = word\text{-}of\text{-}int\ (uint\ a\ XOR\ uint\ b)$

definition
word-not-def:
$NOT\ (a::'a\ word) = word\text{-}of\text{-}int\ (NOT\ (uint\ a))$

instance ⟨*proof*⟩

end

definition
$word\text{-}succ :: 'a :: len0\ word => 'a\ word$
where
$word\text{-}succ\ a = word\text{-}of\text{-}int\ (Int.succ\ (uint\ a))$

definition
$word\text{-}pred :: 'a :: len0\ word => 'a\ word$
where
$word\text{-}pred\ a = word\text{-}of\text{-}int\ (Int.pred\ (uint\ a))$

definition $udvd :: 'a::len\ word => 'a::len\ word => bool$ (**infixl** $udvd\ 50$) **where**
$a\ udvd\ b == EX\ n>=0.\ uint\ b = n * uint\ a$

definition $word\text{-}sle :: 'a :: len\ word => 'a\ word => bool\ ((-/ <=s -)\ [50, 51]\ 50)$
where
$a <=s\ b == sint\ a <= sint\ b$

definition $word\text{-}sless :: 'a :: len\ word => 'a\ word => bool\ ((-/ <s -)\ [50, 51]\ 50)$
where
$(x <s\ y) == (x <=s\ y\ \&\ x \sim= y)$

10.4 Bit-wise operations

instantiation $word :: (len0)\ bits$
begin

definition
word-test-bit-def: *test-bit a* = *bin-nth* (*uint a*)

definition
word-set-bit-def: *set-bit a n x* =
word-of-int (*bin-sc n* (*If x bit.B1 bit.B0*) (*uint a*))

definition
word-set-bits-def: (*BITS n. f n*) = *of-bl* (*bl-of-nth* (*len-of TYPE* ('*a*)) *f*)

definition
word-lsb-def: *lsb a* ⟷ *bin-last* (*uint a*) = *bit.B1*

definition *shiftl1* :: '*a word* ⇒ '*a word* **where**
shiftl1 w = *word-of-int* (*uint w BIT bit.B0*)

definition *shiftr1* :: '*a word* ⇒ '*a word* **where**
— shift right as unsigned or as signed, ie logical or arithmetic
shiftr1 w = *word-of-int* (*bin-rest* (*uint w*))

definition
shiftl-def: *w* << *n* = (*shiftl1* ^^ *n*) *w*

definition
shiftr-def: *w* >> *n* = (*shiftr1* ^^ *n*) *w*

instance ⟨*proof*⟩

end

instantiation *word* :: (*len*) *bitss*
begin

definition
word-msb-def:
msb a ⟷ *bin-sign* (*sint a*) = *Int.Min*

instance ⟨*proof*⟩

end

definition *setBit* :: '*a* :: *len0 word* => *nat* => '*a word* **where**
setBit w n == *set-bit w n True*

definition *clearBit* :: '*a* :: *len0 word* => *nat* => '*a word* **where**
clearBit w n == *set-bit w n False*

10.5 Shift operations

definition *sshiftr1* :: *'a* :: *len word* => *'a word* **where**
sshiftr1 w == *word-of-int* (*bin-rest* (*sint w*))

definition *bshiftr1* :: *bool* => *'a* :: *len word* => *'a word* **where**
bshiftr1 b w == *of-bl* (*b* # *butlast* (*to-bl w*))

definition *sshiftr* :: *'a* :: *len word* => *nat* => *'a word* (**infixl** >>> *55*) **where**
w >>> *n* == (*sshiftr1* ^^ *n*) *w*

definition *mask* :: *nat* => *'a*::*len word* **where**
mask n == (*1* << *n*) − *1*

definition *revcast* :: *'a* :: *len0 word* => *'b* :: *len0 word* **where**
revcast w == *of-bl* (*takefill False* (*len-of TYPE*('*b*)) (*to-bl w*))

definition *slice1* :: *nat* => *'a* :: *len0 word* => *'b* :: *len0 word* **where**
slice1 n w == *of-bl* (*takefill False n* (*to-bl w*))

definition *slice* :: *nat* => *'a* :: *len0 word* => *'b* :: *len0 word* **where**
slice n w == *slice1* (*size w* − *n*) *w*

10.6 Rotation

definition *rotater1* :: *'a list* => *'a list* **where**
rotater1 ys ==
case ys of [] => [] | *x* # *xs* => *last ys* # *butlast ys*

definition *rotater* :: *nat* => *'a list* => *'a list* **where**
rotater n == *rotater1* ^^ *n*

definition *word-rotr* :: *nat* => *'a* :: *len0 word* => *'a* :: *len0 word* **where**
word-rotr n w == *of-bl* (*rotater n* (*to-bl w*))

definition *word-rotl* :: *nat* => *'a* :: *len0 word* => *'a* :: *len0 word* **where**
word-rotl n w == *of-bl* (*rotate n* (*to-bl w*))

definition *word-roti* :: *int* => *'a* :: *len0 word* => *'a* :: *len0 word* **where**
word-roti i w == *if i* >= *0 then word-rotr* (*nat i*) *w*
else word-rotl (*nat* (− *i*)) *w*

10.7 Split and cat operations

definition *word-cat* :: *'a* :: *len0 word* => *'b* :: *len0 word* => *'c* :: *len0 word* **where**
word-cat a b == *word-of-int* (*bin-cat* (*uint a*) (*len-of TYPE* ('*b*)) (*uint b*))

definition *word-split* :: *'a* :: *len0 word* => (*'b* :: *len0 word*) ∗ (*'c* :: *len0 word*)
where
word-split a ==

case bin-split (len-of TYPE ('c)) (uint a) of
(u, v) => (word-of-int u, word-of-int v)

definition *word-rcat :: 'a :: len0 word list => 'b :: len0 word* **where**
word-rcat ws ==
word-of-int (bin-rcat (len-of TYPE ('a)) (map uint ws))

definition *word-rsplit :: 'a :: len0 word => 'b :: len word list* **where**
word-rsplit w ==
map word-of-int (bin-rsplit (len-of TYPE ('b)) (len-of TYPE ('a), uint w))

definition *max-word :: 'a::len word* — Largest representable machine integer. **where**
max-word ≡ word-of-int (2 ^ len-of TYPE('a) − 1)

primrec *of-bool :: bool ⇒ 'a::len word* **where**
of-bool False = 0
| *of-bool True = 1*

lemmas *of-nth-def = word-set-bits-def*

lemmas *word-size-gt-0 [iff] =*
xtr1 [OF word-size len-gt-0, standard]
lemmas *lens-gt-0 = word-size-gt-0 len-gt-0*
lemmas *lens-not-0 [iff] = lens-gt-0 [THEN gr-implies-not0, standard]*

lemma *uints-num: uints n = {i. 0 ≤ i ∧ i < 2 ^ n}*
⟨*proof*⟩

lemma *sints-num: sints n = {i. − (2 ^ (n − 1)) ≤ i ∧ i < 2 ^ (n − 1)}*
⟨*proof*⟩

lemmas *atLeastLessThan-alt = atLeastLessThan-def [unfolded*
atLeast-def lessThan-def Collect-conj-eq [symmetric]]

lemma *mod-in-reps: m > 0 ==> y mod m : {0::int ..< m}*
⟨*proof*⟩

lemma
uint-0:0 <= uint x **and**
uint-lt: uint (x::'a::len0 word) < 2 ^ len-of TYPE('a)
⟨*proof*⟩

lemma *uint-mod-same:*
uint x mod 2 ^ len-of TYPE('a) = uint (x::'a::len0 word)
⟨*proof*⟩

lemma *td-ext-uint:*

td-ext (*uint* :: *'a word* => *int*) *word-of-int* (*uints* (*len-of TYPE*(*'a*::*len0*)))
(%*w*::*int*. *w mod 2* ^ *len-of TYPE*(*'a*))
⟨*proof*⟩

lemmas *int-word-uint* = *td-ext-uint* [*THEN td-ext.eq-norm*, *standard*]

interpretation *word-uint*:
td-ext uint::*'a*::*len0 word* ⇒ *int*
word-of-int
uints (*len-of TYPE*(*'a*::*len0*))
λ*w*. *w mod 2* ^ *len-of TYPE*(*'a*::*len0*)
⟨*proof*⟩

lemmas *td-uint* = *word-uint.td-thm*

lemmas *td-ext-ubin* = *td-ext-uint*
[*simplified len-gt-0 no-bintr-alt1* [*symmetric*]]

interpretation *word-ubin*:
td-ext uint::*'a*::*len0 word* ⇒ *int*
word-of-int
uints (*len-of TYPE*(*'a*::*len0*))
bintrunc (*len-of TYPE*(*'a*::*len0*))
⟨*proof*⟩

lemma *sint-sbintrunc'*:
sint (*word-of-int bin* :: *'a word*) =
(*sbintrunc* (*len-of TYPE* (*'a* :: *len*) − *1*) *bin*)
⟨*proof*⟩

lemma *uint-sint*:
uint w = *bintrunc* (*len-of TYPE*(*'a*)) (*sint* (*w* :: *'a* :: *len word*))
⟨*proof*⟩

lemma *bintr-uint'*:
n >= *size w* ==> *bintrunc n* (*uint w*) = *uint w*
⟨*proof*⟩

lemma *wi-bintr'*:
wb = *word-of-int bin* ==> *n* >= *size wb* ==>
word-of-int (*bintrunc n bin*) = *wb*
⟨*proof*⟩

lemmas *bintr-uint* = *bintr-uint'* [*unfolded word-size*]
lemmas *wi-bintr* = *wi-bintr'* [*unfolded word-size*]

lemma *td-ext-sbin*:
td-ext (*sint* :: *'a word* => *int*) *word-of-int* (*sints* (*len-of TYPE*(*'a*::*len*)))
(*sbintrunc* (*len-of TYPE*(*'a*) − *1*))

⟨*proof*⟩

lemmas *td-ext-sint* = *td-ext-sbin*
[*simplified len-gt-0 no-sbintr-alt2 Suc-pred'* [*symmetric*]]

interpretation *word-sint*:
td-ext sint ::'*a*::*len word* => *int*
word-of-int
sints (*len-of TYPE*('*a*::*len*))
%*w*. (*w* + *2*^(*len-of TYPE*('*a*::*len*) − *1*)) *mod* *2*^*len-of TYPE*('*a*::*len*) −
2 ^ (*len-of TYPE*('*a*::*len*) − *1*)
⟨*proof*⟩

interpretation *word-sbin*:
td-ext sint ::'*a*::*len word* => *int*
word-of-int
sints (*len-of TYPE*('*a*::*len*))
sbintrunc (*len-of TYPE*('*a*::*len*) − *1*)
⟨*proof*⟩

lemmas *int-word-sint* = *td-ext-sint* [*THEN td-ext.eq-norm*, *standard*]

lemmas *td-sint* = *word-sint.td*

lemma *word-number-of-alt*: *number-of b* == *word-of-int* (*number-of b*)
⟨*proof*⟩

lemma *word-no-wi*: *number-of* = *word-of-int*
⟨*proof*⟩

lemma *to-bl-def'*:
(*to-bl* :: '*a* :: *len0 word* => *bool list*) =
bin-to-bl (*len-of TYPE*('*a*)) *o uint*
⟨*proof*⟩

lemmas *word-reverse-no-def* [*simp*] = *word-reverse-def* [*of number-of w*, *standard*]

lemmas *uints-mod* = *uints-def* [*unfolded no-bintr-alt1*]

lemma *uint-bintrunc*: *uint* (*number-of bin* :: '*a word*) =
number-of (*bintrunc* (*len-of TYPE* ('*a* :: *len0*)) *bin*)
⟨*proof*⟩

lemma *sint-sbintrunc*: *sint* (*number-of bin* :: '*a word*) =
number-of (*sbintrunc* (*len-of TYPE* ('*a* :: *len*) − *1*) *bin*)
⟨*proof*⟩

lemma *unat-bintrunc*:

unat (*number-of bin* :: *'a* :: *len0 word*) =
number-of (*bintrunc* (*len-of TYPE*(*'a*)) *bin*)
⟨*proof*⟩

declare
uint-bintrunc [*simp*]
sint-sbintrunc [*simp*]
unat-bintrunc [*simp*]

lemma *size-0-eq*: *size* (*w* :: *'a* :: *len0 word*) = *0* ==> *v* = *w*
⟨*proof*⟩

lemmas *uint-lem* = *word-uint.Rep* [*unfolded uints-num mem-Collect-eq*]
lemmas *sint-lem* = *word-sint.Rep* [*unfolded sints-num mem-Collect-eq*]
lemmas *uint-ge-0* [*iff*] = *uint-lem* [*THEN conjunct1*, *standard*]
lemmas *uint-lt2p* [*iff*] = *uint-lem* [*THEN conjunct2*, *standard*]
lemmas *sint-ge* = *sint-lem* [*THEN conjunct1*, *standard*]
lemmas *sint-lt* = *sint-lem* [*THEN conjunct2*, *standard*]

lemma *sign-uint-Pls* [*simp*]:
bin-sign (*uint x*) = *Int.Pls*
⟨*proof*⟩

lemmas *uint-m2p-neg* = *iffD2* [*OF diff-less-0-iff-less uint-lt2p*, *standard*]
lemmas *uint-m2p-not-non-neg* =
iffD2 [*OF linorder-not-le uint-m2p-neg*, *standard*]

lemma *lt2p-lem*:
len-of TYPE(*'a*) <= *n* ==> *uint* (*w* :: *'a* :: *len0 word*) < *2* ^ *n*
⟨*proof*⟩

lemmas *uint-le-0-iff* [*simp*] =
uint-ge-0 [*THEN leD*, *THEN linorder-antisym-conv1*, *standard*]

lemma *uint-nat*: *uint w* == *int* (*unat w*)
⟨*proof*⟩

lemma *uint-number-of*:
uint (*number-of b* :: *'a* :: *len0 word*) = *number-of b mod 2* ^ *len-of TYPE*(*'a*)
⟨*proof*⟩

lemma *unat-number-of*:
bin-sign b = *Int.Pls* ==>
unat (*number-of b*::*'a*::*len0 word*) = *number-of b mod 2* ^ *len-of TYPE* (*'a*)
⟨*proof*⟩

lemma *sint-number-of*: *sint* (*number-of b* :: *'a* :: *len word*) = (*number-of b* +
2 ^ (*len-of TYPE*(*'a*) − *1*)) *mod 2* ^ *len-of TYPE*(*'a*) −

2 ^ (len-of TYPE('a) - 1)
⟨*proof*⟩

lemma *word-of-int-bin* [*simp*] :
(word-of-int (number-of bin) :: 'a :: len0 word) = (number-of bin)
⟨*proof*⟩

lemma *word-int-case-wi*:
word-int-case f (word-of-int i :: 'b word) =
f (i mod 2 ^ len-of TYPE('b::len0))
⟨*proof*⟩

lemma *word-int-split*:
P (word-int-case f x) =
(ALL i. x = (word-of-int i :: 'b :: len0 word) &
0 <= i & i < 2 ^ len-of TYPE('b) --> P (f i))
⟨*proof*⟩

lemma *word-int-split-asm*:
P (word-int-case f x) =
(~ (EX n. x = (word-of-int n :: 'b::len0 word) &
0 <= n & n < 2 ^ len-of TYPE('b::len0) & ~ P (f n)))
⟨*proof*⟩

lemmas *uint-range' =*
word-uint.Rep [*unfolded uints-num mem-Collect-eq*, *standard*]
lemmas *sint-range' = word-sint.Rep* [*unfolded One-nat-def*
sints-num mem-Collect-eq, *standard*]

lemma *uint-range-size*: *0 <= uint w & uint w < 2 ^ size w*
⟨*proof*⟩

lemma *sint-range-size*:
- (2 ^ (size w - Suc 0)) <= sint w & sint w < 2 ^ (size w - Suc 0)
⟨*proof*⟩

lemmas *sint-above-size = sint-range-size*
[*THEN conjunct2*, *THEN* [*2*] *xtr8*, *folded One-nat-def*, *standard*]

lemmas *sint-below-size = sint-range-size*
[*THEN conjunct1*, *THEN* [*2*] *order-trans*, *folded One-nat-def*, *standard*]

lemma *test-bit-eq-iff*: *(test-bit (u::'a::len0 word) = test-bit v) = (u = v)*
⟨*proof*⟩

lemma *test-bit-size* [*rule-format*] : *(w::'a::len0 word) !! n --> n < size w*
⟨*proof*⟩

lemma *word-eqI* [*rule-format*] :

fixes $u :: 'a::len0\ word$
shows $(ALL\ n.\ n < size\ u\ --> u\ !!\ n = v\ !!\ n) ==> u = v$
⟨*proof*⟩

lemmas *word-eqD* = *test-bit-eq-iff* [*THEN iffD2*, *THEN fun-cong*, *standard*]

lemma *test-bit-bin'*: $w\ !!\ n = (n < size\ w\ \&\ bin\text{-}nth\ (uint\ w)\ n)$
⟨*proof*⟩

lemmas *test-bit-bin* = *test-bit-bin'* [*unfolded word-size*]

lemma *bin-nth-uint-imp'*: $bin\text{-}nth\ (uint\ w)\ n\ --> n < size\ w$
⟨*proof*⟩

lemma *bin-nth-sint'*:
$n >= size\ w\ --> bin\text{-}nth\ (sint\ w)\ n = bin\text{-}nth\ (sint\ w)\ (size\ w - 1)$
⟨*proof*⟩

lemmas *bin-nth-uint-imp* = *bin-nth-uint-imp'* [*rule-format*, *unfolded word-size*]
lemmas *bin-nth-sint* = *bin-nth-sint'* [*rule-format*, *unfolded word-size*]

lemma *td-bl*:
type-definition $(to\text{-}bl :: 'a::len0\ word => bool\ list)$
of-bl
$\{bl.\ length\ bl = len\text{-}of\ TYPE('a)\}$
⟨*proof*⟩

interpretation *word-bl*:
type-definition to-bl $:: 'a::len0\ word => bool\ list$
of-bl
$\{bl.\ length\ bl = len\text{-}of\ TYPE('a::len0)\}$
⟨*proof*⟩

lemma *word-size-bl*: $size\ w == size\ (to\text{-}bl\ w)$
⟨*proof*⟩

lemma *to-bl-use-of-bl*:
$(to\text{-}bl\ w = bl) = (w = of\text{-}bl\ bl \wedge length\ bl = length\ (to\text{-}bl\ w))$
⟨*proof*⟩

lemma *to-bl-word-rev*: $to\text{-}bl\ (word\text{-}reverse\ w) = rev\ (to\text{-}bl\ w)$
⟨*proof*⟩

lemma *word-rev-rev* [*simp*] : $word\text{-}reverse\ (word\text{-}reverse\ w) = w$
⟨*proof*⟩

lemma *word-rev-gal*: $word\text{-}reverse\ w = u ==> word\text{-}reverse\ u = w$
⟨*proof*⟩

lemmas *word-rev-gal'* = *sym* [*THEN word-rev-gal*, *symmetric*, *standard*]

lemmas *length-bl-gt-0* [*iff*] = *xtr1* [*OF word-bl.Rep' len-gt-0*, *standard*]
lemmas *bl-not-Nil* [*iff*] =
length-bl-gt-0 [*THEN length-greater-0-conv* [*THEN iffD1*], *standard*]
lemmas *length-bl-neq-0* [*iff*] = *length-bl-gt-0* [*THEN gr-implies-not0*]

lemma *hd-bl-sign-sint*: *hd* (*to-bl w*) = (*bin-sign* (*sint w*) = *Int.Min*)
⟨*proof*⟩

lemma *of-bl-drop'*:
lend = *length bl* − *len-of TYPE* ('*a* :: *len0*) ==>
of-bl (*drop lend bl*) = (*of-bl bl* :: '*a word*)
⟨*proof*⟩

lemmas *of-bl-no* = *of-bl-def* [*folded word-number-of-def*]

lemma *test-bit-of-bl*:
(*of-bl bl*::'*a*::*len0 word*) !! *n* = (*rev bl* ! *n* ∧ *n* < *len-of TYPE*('*a*) ∧ *n* < *length bl*)
⟨*proof*⟩

lemma *no-of-bl*:
(*number-of bin* ::'*a*::*len0 word*) = *of-bl* (*bin-to-bl* (*len-of TYPE* ('*a*)) *bin*)
⟨*proof*⟩

lemma *uint-bl*: *to-bl w* == *bin-to-bl* (*size w*) (*uint w*)
⟨*proof*⟩

lemma *to-bl-bin*: *bl-to-bin* (*to-bl w*) = *uint w*
⟨*proof*⟩

lemma *to-bl-of-bin*:
to-bl (*word-of-int bin*::'*a*::*len0 word*) = *bin-to-bl* (*len-of TYPE*('*a*)) *bin*
⟨*proof*⟩

lemmas *to-bl-no-bin* [*simp*] = *to-bl-of-bin* [*folded word-number-of-def*]

lemma *to-bl-to-bin* [*simp*] : *bl-to-bin* (*to-bl w*) = *uint w*
⟨*proof*⟩

lemmas *uint-bl-bin* [*simp*] = *trans* [*OF bin-bl-bin word-ubin.norm-Rep*, *standard*]

lemmas *num-AB-u* [*simp*] = *word-uint.Rep-inverse*
[*unfolded o-def word-number-of-def* [*symmetric*], *standard*]
lemmas *num-AB-s* [*simp*] = *word-sint.Rep-inverse*
[*unfolded o-def word-number-of-def* [*symmetric*], *standard*]

lemma *uints-unats*: *uints n = int ‘ unats n*
⟨*proof*⟩

lemma *unats-uints*: *unats n = nat ‘ uints n*
⟨*proof*⟩

lemmas *bintr-num = word-ubin.norm-eq-iff*
[*symmetric*, *folded word-number-of-def*, *standard*]
lemmas *sbintr-num = word-sbin.norm-eq-iff*
[*symmetric*, *folded word-number-of-def*, *standard*]

lemmas *num-of-bintr = word-ubin.Abs-norm* [*folded word-number-of-def*, *standard*]
lemmas *num-of-sbintr = word-sbin.Abs-norm* [*folded word-number-of-def*, *standard*]

lemma *num-of-bintr'*:
bintrunc (*len-of TYPE*('*a* :: *len0*)) *a = b ==>*
number-of a = (*number-of b* :: '*a word*)
⟨*proof*⟩

lemma *num-of-sbintr'*:
sbintrunc (*len-of TYPE*('*a* :: *len*) − *1*) *a = b ==>*
number-of a = (*number-of b* :: '*a word*)
⟨*proof*⟩

lemmas *num-abs-bintr = sym* [*THEN trans*,
OF num-of-bintr word-number-of-def, *standard*]
lemmas *num-abs-sbintr = sym* [*THEN trans*,
OF num-of-sbintr word-number-of-def, *standard*]

lemma *ucast-id*: *ucast w = w*
⟨*proof*⟩

lemma *scast-id*: *scast w = w*
⟨*proof*⟩

lemma *ucast-bl*: *ucast w == of-bl* (*to-bl w*)
⟨*proof*⟩

lemma *nth-ucast*:
(*ucast w*::'*a*::*len0 word*) !! *n* = (*w* !! *n* & *n* < *len-of TYPE*('*a*))
⟨*proof*⟩

lemma *ucast-bintr* [*simp*]:
ucast (*number-of w* ::'*a*::*len0 word*) =
number-of (*bintrunc* (*len-of TYPE*('*a*)) *w*)
⟨*proof*⟩

lemma *scast-sbintr* [*simp*]:
scast (*number-of w* ::'*a*::*len word*) =
number-of (*sbintrunc* (*len-of TYPE*('*a*) − *Suc 0*) *w*)
⟨*proof*⟩

lemmas *source-size* = *source-size-def* [*unfolded Let-def word-size*]
lemmas *target-size* = *target-size-def* [*unfolded Let-def word-size*]
lemmas *is-down* = *is-down-def* [*unfolded source-size target-size*]
lemmas *is-up* = *is-up-def* [*unfolded source-size target-size*]

lemmas *is-up-down* = *trans* [*OF is-up is-down* [*symmetric*], *standard*]

lemma *down-cast-same'*: *uc* = *ucast* ==> *is-down uc* ==> *uc* = *scast*
⟨*proof*⟩

lemma *word-rev-tf'*:
r = *to-bl* (*of-bl bl*) ==> *r* = *rev* (*takefill False* (*length r*) (*rev bl*))
⟨*proof*⟩

lemmas *word-rev-tf* = *refl* [*THEN word-rev-tf'*, *unfolded word-bl.Rep'*, *standard*]

lemmas *word-rep-drop* = *word-rev-tf* [*simplified takefill-alt*,
simplified, *simplified rev-take*, *simplified*]

lemma *to-bl-ucast*:
to-bl (*ucast* (*w*::'*b*::*len0 word*) ::'*a*::*len0 word*) =
replicate (*len-of TYPE*('*a*) − *len-of TYPE*('*b*)) *False* @
drop (*len-of TYPE*('*b*) − *len-of TYPE*('*a*)) (*to-bl w*)
⟨*proof*⟩

lemma *ucast-up-app'*:
uc = *ucast* ==> *source-size uc* + *n* = *target-size uc* ==>
to-bl (*uc w*) = *replicate n False* @ (*to-bl w*)
⟨*proof*⟩

lemma *ucast-down-drop'*:
uc = *ucast* ==> *source-size uc* = *target-size uc* + *n* ==>
to-bl (*uc w*) = *drop n* (*to-bl w*)
⟨*proof*⟩

lemma *scast-down-drop'*:
sc = *scast* ==> *source-size sc* = *target-size sc* + *n* ==>

to-bl (*sc w*) = *drop n* (*to-bl w*)
⟨*proof*⟩

lemma *sint-up-scast'*:
sc = *scast* ==> *is-up sc* ==> *sint* (*sc w*) = *sint w*
⟨*proof*⟩

lemma *uint-up-ucast'*:
uc = *ucast* ==> *is-up uc* ==> *uint* (*uc w*) = *uint w*
⟨*proof*⟩

lemmas *down-cast-same* = *refl* [*THEN down-cast-same'*]
lemmas *ucast-up-app* = *refl* [*THEN ucast-up-app'*]
lemmas *ucast-down-drop* = *refl* [*THEN ucast-down-drop'*]
lemmas *scast-down-drop* = *refl* [*THEN scast-down-drop'*]
lemmas *uint-up-ucast* = *refl* [*THEN uint-up-ucast'*]
lemmas *sint-up-scast* = *refl* [*THEN sint-up-scast'*]

lemma *ucast-up-ucast'*: *uc* = *ucast* ==> *is-up uc* ==> *ucast* (*uc w*) = *ucast w*
⟨*proof*⟩

lemma *scast-up-scast'*: *sc* = *scast* ==> *is-up sc* ==> *scast* (*sc w*) = *scast w*
⟨*proof*⟩

lemma *ucast-of-bl-up'*:
w = *of-bl bl* ==> *size bl* <= *size w* ==> *ucast w* = *of-bl bl*
⟨*proof*⟩

lemmas *ucast-up-ucast* = *refl* [*THEN ucast-up-ucast'*]
lemmas *scast-up-scast* = *refl* [*THEN scast-up-scast'*]
lemmas *ucast-of-bl-up* = *refl* [*THEN ucast-of-bl-up'*]

lemmas *ucast-up-ucast-id* = *trans* [*OF ucast-up-ucast ucast-id*]
lemmas *scast-up-scast-id* = *trans* [*OF scast-up-scast scast-id*]

lemmas *isduu* = *is-up-down* [**where** *c* = *ucast*, *THEN iffD2*]
lemmas *isdus* = *is-up-down* [**where** *c* = *scast*, *THEN iffD2*]
lemmas *ucast-down-ucast-id* = *isduu* [*THEN ucast-up-ucast-id*]
lemmas *scast-down-scast-id* = *isdus* [*THEN ucast-up-ucast-id*]

lemma *up-ucast-surj*:
is-up (*ucast* :: *'b*::*len0 word* => *'a*::*len0 word*) ==>
surj (*ucast* :: *'a word* => *'b word*)
⟨*proof*⟩

lemma *up-scast-surj*:
is-up (*scast* :: *'b*::*len word* => *'a*::*len word*) ==>
surj (*scast* :: *'a word* => *'b word*)
⟨*proof*⟩

lemma *down-scast-inj*:
is-down (*scast* :: *'b*::*len word* => *'a*::*len word*) ==>
inj-on (*ucast* :: *'a word* => *'b word*) *A*
⟨*proof*⟩

lemma *down-ucast-inj*:
is-down (*ucast* :: *'b*::*len0 word* => *'a*::*len0 word*) ==>
inj-on (*ucast* :: *'a word* => *'b word*) *A*
⟨*proof*⟩

lemma *of-bl-append-same*: *of-bl* (*X* @ *to-bl w*) = *w*
⟨*proof*⟩

lemma *ucast-down-no'*:
uc = *ucast* ==> *is-down uc* ==> *uc* (*number-of bin*) = *number-of bin*
⟨*proof*⟩

lemmas *ucast-down-no* = *ucast-down-no'* [*OF refl*]

lemma *ucast-down-bl'*: *uc* = *ucast* ==> *is-down uc* ==> *uc* (*of-bl bl*) = *of-bl bl*
⟨*proof*⟩

lemmas *ucast-down-bl* = *ucast-down-bl'* [*OF refl*]

lemmas *slice-def'* = *slice-def* [*unfolded word-size*]
lemmas *test-bit-def'* = *word-test-bit-def* [*THEN fun-cong*]

lemmas *word-log-defs* = *word-and-def word-or-def word-xor-def word-not-def*
lemmas *word-log-bin-defs* = *word-log-defs*

Executable equality

instantiation *word* :: ({*len0*}) *eq*
begin

definition *eq-word* :: *'a word* ⇒ *'a word* ⇒ *bool* **where**
eq-word k l ⟷ *HOL.eq* (*uint k*) (*uint l*)

instance ⟨*proof*⟩

end

end

11 WordArith: Word Arithmetic

theory *WordArith*
imports *WordDefinition*

begin

lemma *word-less-alt*: $(a < b) = (uint\ a < uint\ b)$
⟨*proof*⟩

lemma *signed-linorder*: *class.linorder word-sle word-sless*
⟨*proof*⟩

interpretation *signed*: *linorder word-sle word-sless*
⟨*proof*⟩

lemmas *word-arith-wis* =
word-add-def word-mult-def word-minus-def
word-succ-def word-pred-def word-0-wi word-1-wi

lemma *udvdI*:
$0 \leq n \Longrightarrow uint\ b = n * uint\ a \Longrightarrow a\ udvd\ b$
⟨*proof*⟩

lemmas *word-div-no* [*simp*] =
word-div-def [*of number-of a number-of b*, *standard*]

lemmas *word-mod-no* [*simp*] =
word-mod-def [*of number-of a number-of b*, *standard*]

lemmas *word-less-no* [*simp*] =
word-less-def [*of number-of a number-of b*, *standard*]

lemmas *word-le-no* [*simp*] =
word-le-def [*of number-of a number-of b*, *standard*]

lemmas *word-sless-no* [*simp*] =
word-sless-def [*of number-of a number-of b*, *standard*]

lemmas *word-sle-no* [*simp*] =
word-sle-def [*of number-of a number-of b*, *standard*]

lemmas *word-0-wi-Pls* = *word-0-wi* [*folded Pls-def*]
lemmas *word-0-no* = *word-0-wi-Pls* [*folded word-no-wi*]

lemma *int-one-bin*: $(1 :: int) == (Int.Pls\ BIT\ bit.B1)$
⟨*proof*⟩

lemma *word-1-no*:
$(1 :: 'a :: len0\ word) == number\text{-}of\ (Int.Pls\ BIT\ bit.B1)$
⟨*proof*⟩

lemma *word-m1-wi*: $-1 == word\text{-}of\text{-}int\ -1$

⟨*proof*⟩

lemma *word-m1-wi-Min*: $-1 = word\text{-}of\text{-}int\ Int.Min$
⟨*proof*⟩

lemma *word-0-bl*: $of\text{-}bl\ [] = 0$
⟨*proof*⟩

lemma *word-1-bl*: $of\text{-}bl\ [True] = 1$
⟨*proof*⟩

lemma *uint-0* [*simp*] : $(uint\ 0 = 0)$
⟨*proof*⟩

lemma *of-bl-0* [*simp*] : $of\text{-}bl\ (replicate\ n\ False) = 0$
⟨*proof*⟩

lemma *to-bl-0*:
$to\text{-}bl\ (0::'a::len0\ word) = replicate\ (len\text{-}of\ TYPE('a))\ False$
⟨*proof*⟩

lemma *uint-0-iff*: $(uint\ x = 0) = (x = 0)$
⟨*proof*⟩

lemma *unat-0-iff*: $(unat\ x = 0) = (x = 0)$
⟨*proof*⟩

lemma *unat-0* [*simp*]: $unat\ 0 = 0$
⟨*proof*⟩

lemma *size-0-same'*: $size\ w = 0 ==> w = (v :: 'a :: len0\ word)$
⟨*proof*⟩

lemmas $size\text{-}0\text{-}same = size\text{-}0\text{-}same'$ [*folded word-size*]

lemmas *unat-eq-0* = *unat-0-iff*
lemmas *unat-eq-zero* = *unat-0-iff*

lemma *unat-gt-0*: $(0 < unat\ x) = (x\ ^{\sim}= 0)$
⟨*proof*⟩

lemma *ucast-0* [*simp*] : $ucast\ 0 = 0$
⟨*proof*⟩

lemma *sint-0* [*simp*] : $sint\ 0 = 0$
⟨*proof*⟩

lemma *scast-0* [*simp*] : $scast\ 0 = 0$
⟨*proof*⟩

lemma *sint-n1* [*simp*] : *sint −1 = −1*
⟨*proof*⟩

lemma *scast-n1* [*simp*] : *scast −1 = −1*
⟨*proof*⟩

lemma *uint-1* [*simp*] : *uint* (*1 :: 'a :: len word*) = *1*
⟨*proof*⟩

lemma *unat-1* [*simp*] : *unat* (*1 :: 'a :: len word*) = *1*
⟨*proof*⟩

lemma *ucast-1* [*simp*] : *ucast* (*1 :: 'a :: len word*) = *1*
⟨*proof*⟩

lemmas *arths* =
bintr-ariths [*THEN word-ubin.norm-eq-iff* [*THEN iffD1*],
folded word-ubin.eq-norm, *standard*]

lemma *wi-homs*:
shows
wi-hom-add: *word-of-int a + word-of-int b = word-of-int* (*a + b*) **and**
wi-hom-mult: *word-of-int a * word-of-int b = word-of-int* (*a * b*) **and**
wi-hom-neg: *− word-of-int a = word-of-int* (*− a*) **and**
wi-hom-succ: *word-succ* (*word-of-int a*) = *word-of-int* (*Int.succ a*) **and**
wi-hom-pred: *word-pred* (*word-of-int a*) = *word-of-int* (*Int.pred a*)
⟨*proof*⟩

lemmas *wi-hom-syms* = *wi-homs* [*symmetric*]

lemma *word-sub-def*: *a − b == a + −* (*b :: 'a :: len0 word*)
⟨*proof*⟩

lemmas *word-diff-minus* = *word-sub-def* [*THEN meta-eq-to-obj-eq*, *standard*]

lemma *word-of-int-sub-hom*:
(*word-of-int a*) *− word-of-int b = word-of-int* (*a − b*)
⟨*proof*⟩

lemmas *new-word-of-int-homs* =
word-of-int-sub-hom wi-homs word-0-wi word-1-wi

lemmas *new-word-of-int-hom-syms* = *new-word-of-int-homs* [*symmetric*, *standard*]

lemmas *word-of-int-hom-syms* =
new-word-of-int-hom-syms [*unfolded succ-def pred-def*]

lemmas *word-of-int-homs* =
new-word-of-int-homs [*unfolded succ-def pred-def*]

lemmas *word-of-int-add-hom* = *word-of-int-homs* (*2*)
lemmas *word-of-int-mult-hom* = *word-of-int-homs* (*3*)
lemmas *word-of-int-minus-hom* = *word-of-int-homs* (*4*)
lemmas *word-of-int-succ-hom* = *word-of-int-homs* (*5*)
lemmas *word-of-int-pred-hom* = *word-of-int-homs* (*6*)
lemmas *word-of-int-0-hom* = *word-of-int-homs* (*7*)
lemmas *word-of-int-1-hom* = *word-of-int-homs* (*8*)

lemmas *word-arith-alts* =
word-sub-wi [*unfolded succ-def pred-def*, *standard*]
word-arith-wis [*unfolded succ-def pred-def*, *standard*]

lemmas *word-sub-alt* = *word-arith-alts* (*1*)
lemmas *word-add-alt* = *word-arith-alts* (*2*)
lemmas *word-mult-alt* = *word-arith-alts* (*3*)
lemmas *word-minus-alt* = *word-arith-alts* (*4*)
lemmas *word-succ-alt* = *word-arith-alts* (*5*)
lemmas *word-pred-alt* = *word-arith-alts* (*6*)
lemmas *word-0-alt* = *word-arith-alts* (*7*)
lemmas *word-1-alt* = *word-arith-alts* (*8*)

11.1 Transferring goals from words to ints

lemma *word-ths*:
shows
word-succ-p1: *word-succ a* = *a* + *1* **and**
word-pred-m1: *word-pred a* = *a* − *1* **and**
word-pred-succ: *word-pred* (*word-succ a*) = *a* **and**
word-succ-pred: *word-succ* (*word-pred a*) = *a* **and**
word-mult-succ: *word-succ a* ∗ *b* = *b* + *a* ∗ *b*
⟨*proof*⟩

lemmas *uint-cong* = *arg-cong* [**where** *f* = *uint*]

lemmas *uint-word-ariths* =
word-arith-alts [*THEN trans* [*OF uint-cong int-word-uint*], *standard*]

lemmas *uint-word-arith-bintrs* = *uint-word-ariths* [*folded bintrunc-mod2p*]

lemmas *sint-word-ariths* = *uint-word-arith-bintrs*
[*THEN uint-sint* [*symmetric*, *THEN trans*],
unfolded uint-sint bintr-arith1s bintr-ariths

len-gt-0 [*THEN bin-sbin-eq-iff'*] *word-sbin.norm-Rep*, *standard*]

lemmas *uint-div-alt* = *word-div-def*
[*THEN trans* [*OF uint-cong int-word-uint*], *standard*]
lemmas *uint-mod-alt* = *word-mod-def*
[*THEN trans* [*OF uint-cong int-word-uint*], *standard*]

lemma *word-pred-0-n1*: *word-pred 0* = *word-of-int* −*1*
⟨*proof*⟩

lemma *word-pred-0-Min*: *word-pred 0* = *word-of-int Int.Min*
⟨*proof*⟩

lemma *word-m1-Min*: − *1* = *word-of-int Int.Min*
⟨*proof*⟩

lemma *succ-pred-no* [*simp*]:
word-succ (*number-of bin*) = *number-of* (*Int.succ bin*) &
word-pred (*number-of bin*) = *number-of* (*Int.pred bin*)
⟨*proof*⟩

lemma *word-sp-01* [*simp*] :
word-succ −*1* = *0* & *word-succ 0* = *1* & *word-pred 0* = −*1* & *word-pred 1* = *0*
⟨*proof*⟩

lemma *word-of-int-Ex*:
∃ *y*. *x* = *word-of-int y*
⟨*proof*⟩

lemma *word-arith-eqs*:
fixes *a* :: *'a*::*len0 word*
fixes *b* :: *'a*::*len0 word*
shows
word-add-0: *0* + *a* = *a* **and**
word-add-0-right: *a* + *0* = *a* **and**
word-mult-1: *1* ∗ *a* = *a* **and**
word-mult-1-right: *a* ∗ *1* = *a* **and**
word-add-commute: *a* + *b* = *b* + *a* **and**
word-add-assoc: *a* + *b* + *c* = *a* + (*b* + *c*) **and**
word-add-left-commute: *a* + (*b* + *c*) = *b* + (*a* + *c*) **and**
word-mult-commute: *a* ∗ *b* = *b* ∗ *a* **and**
word-mult-assoc: *a* ∗ *b* ∗ *c* = *a* ∗ (*b* ∗ *c*) **and**
word-mult-left-commute: *a* ∗ (*b* ∗ *c*) = *b* ∗ (*a* ∗ *c*) **and**
word-left-distrib: (*a* + *b*) ∗ *c* = *a* ∗ *c* + *b* ∗ *c* **and**
word-right-distrib: *a* ∗ (*b* + *c*) = *a* ∗ *b* + *a* ∗ *c* **and**
word-left-minus: − *a* + *a* = *0* **and**
word-diff-0-right: *a* − *0* = *a* **and**
word-diff-self: *a* − *a* = *0*

⟨*proof*⟩

lemmas *word-add-ac = word-add-commute word-add-assoc word-add-left-commute*
lemmas *word-mult-ac = word-mult-commute word-mult-assoc word-mult-left-commute*

lemmas *word-plus-ac0 = word-add-0 word-add-0-right word-add-ac*
lemmas *word-times-ac1 = word-mult-1 word-mult-1-right word-mult-ac*

11.2 Order on fixed-length words

lemma *word-order-trans*: $x <= y ==> y <= z ==> x <= (z :: 'a :: len0\ word)$
⟨*proof*⟩

lemma *word-order-refl*: $z <= (z :: 'a :: len0\ word)$
⟨*proof*⟩

lemma *word-order-antisym*: $x <= y ==> y <= x ==> x = (y :: 'a :: len0\ word)$
⟨*proof*⟩

lemma *word-order-linear*:
$y <= x \mid x <= (y :: 'a :: len0\ word)$
⟨*proof*⟩

lemma *word-zero-le* [*simp*] :
$0 <= (y :: 'a :: len0\ word)$
⟨*proof*⟩

instance *word* :: (*len0*) *semigroup-add*
⟨*proof*⟩

instance *word* :: (*len0*) *linorder*
⟨*proof*⟩

instance *word* :: (*len0*) *ring*
⟨*proof*⟩

lemma *word-m1-ge* [*simp*] : *word-pred 0 >= y*
⟨*proof*⟩

lemmas *word-n1-ge* [*simp*] = *word-m1-ge* [*simplified word-sp-01*]

lemmas *word-not-simps* [*simp*] =
word-zero-le [*THEN leD*] *word-m1-ge* [*THEN leD*] *word-n1-ge* [*THEN leD*]

lemma *word-gt-0*: $0 < y = (0 \sim= (y :: 'a :: len0\ word))$
⟨*proof*⟩

lemmas *word-gt-0-no* [*simp*] = *word-gt-0* [*of number-of y, standard*]

lemma *word-sless-alt*: $(a <s b) == (sint\ a < sint\ b)$
⟨*proof*⟩

lemma *word-le-nat-alt*: $(a <= b) = (unat\ a <= unat\ b)$
⟨*proof*⟩

lemma *word-less-nat-alt*: $(a < b) = (unat\ a < unat\ b)$
⟨*proof*⟩

lemma *wi-less*:
$(word\text{-}of\text{-}int\ n < (word\text{-}of\text{-}int\ m :: 'a :: len0\ word)) =$
$(n\ mod\ 2\ \hat{}\ len\text{-}of\ TYPE('a) < m\ mod\ 2\ \hat{}\ len\text{-}of\ TYPE('a))$
⟨*proof*⟩

lemma *wi-le*:
$(word\text{-}of\text{-}int\ n <= (word\text{-}of\text{-}int\ m :: 'a :: len0\ word)) =$
$(n\ mod\ 2\ \hat{}\ len\text{-}of\ TYPE('a) <= m\ mod\ 2\ \hat{}\ len\text{-}of\ TYPE('a))$
⟨*proof*⟩

lemma *udvd-nat-alt*: $a\ udvd\ b = (EX\ n>=0.\ unat\ b = n * unat\ a)$
⟨*proof*⟩

lemma *udvd-iff-dvd*: $x\ udvd\ y <-> unat\ x\ dvd\ unat\ y$
⟨*proof*⟩

lemmas *unat-mono* = *word-less-nat-alt* [*THEN iffD1*, *standard*]

lemma *word-zero-neq-one*: $0 < len\text{-}of\ TYPE\ ('a :: len0) ==> (0 :: 'a\ word)\ ~= 1$
⟨*proof*⟩

lemmas *lenw1-zero-neq-one* = *len-gt-0* [*THEN word-zero-neq-one*]

lemma *no-no* [*simp*] : *number-of* (*number-of b*) = *number-of b*
⟨*proof*⟩

lemma *unat-minus-one*: $x\ ~= 0 ==> unat\ (x - 1) = unat\ x - 1$
⟨*proof*⟩

lemma *measure-unat*: $p\ ~= 0 ==> unat\ (p - 1) < unat\ p$
⟨*proof*⟩

lemmas *uint-add-ge0* [*simp*] =
add-nonneg-nonneg [*OF uint-ge-0 uint-ge-0*, *standard*]
lemmas *uint-mult-ge0* [*simp*] =
mult-nonneg-nonneg [*OF uint-ge-0 uint-ge-0*, *standard*]

lemma *uint-sub-lt2p* [*simp*]:
$uint\ (x :: 'a :: len0\ word) - uint\ (y :: 'b :: len0\ word) <$

2 ^ len-of TYPE('a)
⟨*proof*⟩

11.3 Conditions for the addition (etc) of two words to overflow

lemma *uint-add-lem*:
(uint x + uint y < 2 ^ len-of TYPE('a)) =
(uint (x + y :: 'a :: len0 word) = uint x + uint y)
⟨*proof*⟩

lemma *uint-mult-lem*:
*(uint x * uint y < 2 ^ len-of TYPE('a)) =*
*(uint (x * y :: 'a :: len0 word) = uint x * uint y)*
⟨*proof*⟩

lemma *uint-sub-lem*:
(uint x >= uint y) = (uint (x - y) = uint x - uint y)
⟨*proof*⟩

lemma *uint-add-le*: *uint (x + y) <= uint x + uint y*
⟨*proof*⟩

lemma *uint-sub-ge*: *uint (x - y) >= uint x - uint y*
⟨*proof*⟩

lemmas *uint-sub-if' =*
trans [OF uint-word-ariths(1) mod-sub-if-z, simplified, standard]
lemmas *uint-plus-if' =*
trans [OF uint-word-ariths(2) mod-add-if-z, simplified, standard]

11.4 Definition of uint_arith

lemma *word-of-int-inverse*:
word-of-int r = a ==> 0 <= r ==> r < 2 ^ len-of TYPE('a) ==>
uint (a::'a::len0 word) = r
⟨*proof*⟩

lemma *uint-split*:
fixes *x::'a::len0 word*
shows *P (uint x) =*
(ALL i. word-of-int i = x & 0 <= i & i < 2^len-of TYPE('a) --> P i)
⟨*proof*⟩

lemma *uint-split-asm*:
fixes *x::'a::len0 word*
shows *P (uint x) =*
(~(EX i. word-of-int i = x & 0 <= i & i < 2^len-of TYPE('a) & ~ P i))
⟨*proof*⟩

lemmas *uint-splits = uint-split uint-split-asm*

lemmas *uint-arith-simps =*
word-le-def word-less-alt
word-uint.Rep-inject [*symmetric*]
uint-sub-if' uint-plus-if'

lemma *power-False-cong*: *False ==> a ^ b = c ^ d*
⟨*proof*⟩

⟨*ML*⟩

11.5 More on overflows and monotonicity

lemma *no-plus-overflow-uint-size*:
((x :: 'a :: len0 word) <= x + y) = (uint x + uint y < 2 ^ size x)
⟨*proof*⟩

lemmas *no-olen-add = no-plus-overflow-uint-size* [*unfolded word-size*]

lemma *no-ulen-sub*: *((x :: 'a :: len0 word) >= x - y) = (uint y <= uint x)*
⟨*proof*⟩

lemma *no-olen-add'*:
fixes *x :: 'a::len0 word*
shows *(x ≤ y + x) = (uint y + uint x < 2 ^ len-of TYPE('a))*
⟨*proof*⟩

lemmas *olen-add-eqv = trans* [*OF no-olen-add no-olen-add'* [*symmetric*], *standard*]

lemmas *uint-plus-simple-iff = trans* [*OF no-olen-add uint-add-lem*, *standard*]
lemmas *uint-plus-simple = uint-plus-simple-iff* [*THEN iffD1*, *standard*]
lemmas *uint-minus-simple-iff = trans* [*OF no-ulen-sub uint-sub-lem*, *standard*]
lemmas *uint-minus-simple-alt = uint-sub-lem* [*folded word-le-def*]
lemmas *word-sub-le-iff = no-ulen-sub* [*folded word-le-def*]
lemmas *word-sub-le = word-sub-le-iff* [*THEN iffD2*, *standard*]

lemma *word-less-sub1*:
(x :: 'a :: len word) ~= 0 ==> (1 < x) = (0 < x - 1)
⟨*proof*⟩

lemma *word-le-sub1*:
(x :: 'a :: len word) ~= 0 ==> (1 <= x) = (0 <= x - 1)
⟨*proof*⟩

lemma *sub-wrap-lt*:
$((x :: 'a :: len0\ word) < x - z) = (x < z)$
⟨*proof*⟩

lemma *sub-wrap*:
$((x :: 'a :: len0\ word) <= x - z) = (z = 0 \mid x < z)$
⟨*proof*⟩

lemma *plus-minus-not-NULL-ab*:
$(x :: 'a :: len0\ word) <= ab - c ==> c <= ab ==> c \sim= 0 ==> x + c \sim= 0$
⟨*proof*⟩

lemma *plus-minus-no-overflow-ab*:
$(x :: 'a :: len0\ word) <= ab - c ==> c <= ab ==> x <= x + c$
⟨*proof*⟩

lemma *le-minus'*:
$(a :: 'a :: len0\ word) + c <= b ==> a <= a + c ==> c <= b - a$
⟨*proof*⟩

lemma *le-plus'*:
$(a :: 'a :: len0\ word) <= b ==> c <= b - a ==> a + c <= b$
⟨*proof*⟩

lemmas *le-plus* = *le-plus'* [*rotated*]

lemmas *le-minus* = *leD* [*THEN thin-rl*, *THEN le-minus'*, *standard*]

lemma *word-plus-mono-right*:
$(y :: 'a :: len0\ word) <= z ==> x <= x + z ==> x + y <= x + z$
⟨*proof*⟩

lemma *word-less-minus-cancel*:
$y - x < z - x ==> x <= z ==> (y :: 'a :: len0\ word) < z$
⟨*proof*⟩

lemma *word-less-minus-mono-left*:
$(y :: 'a :: len0\ word) < z ==> x <= y ==> y - x < z - x$
⟨*proof*⟩

lemma *word-less-minus-mono*:
$a < c ==> d < b ==> a - b < a ==> c - d < c$
$==> a - b < c - (d::'a::len\ word)$
⟨*proof*⟩

lemma *word-le-minus-cancel*:
$y - x <= z - x ==> x <= z ==> (y :: 'a :: len0\ word) <= z$
⟨*proof*⟩

lemma *word-le-minus-mono-left*:
$(y :: 'a :: len0\ word) <= z ==> x <= y ==> y - x <= z - x$
⟨*proof*⟩

lemma *word-le-minus-mono*:
$a <= c ==> d <= b ==> a - b <= a ==> c - d <= c$
$==> a - b <= c - (d::'a::len\ word)$
⟨*proof*⟩

lemma *plus-le-left-cancel-wrap*:
$(x :: 'a :: len0\ word) + y' < x ==> x + y < x ==> (x + y' < x + y) = (y' < y)$
⟨*proof*⟩

lemma *plus-le-left-cancel-nowrap*:
$(x :: 'a :: len0\ word) <= x + y' ==> x <= x + y ==>$
$(x + y' < x + y) = (y' < y)$
⟨*proof*⟩

lemma *word-plus-mono-right2*:
$(a :: 'a :: len0\ word) <= a + b ==> c <= b ==> a <= a + c$
⟨*proof*⟩

lemma *word-less-add-right*:
$(x :: 'a :: len0\ word) < y - z ==> z <= y ==> x + z < y$
⟨*proof*⟩

lemma *word-less-sub-right*:
$(x :: 'a :: len0\ word) < y + z ==> y <= x ==> x - y < z$
⟨*proof*⟩

lemma *word-le-plus-either*:
$(x :: 'a :: len0\ word) <= y \mid x <= z ==> y <= y + z ==> x <= y + z$
⟨*proof*⟩

lemma *word-less-nowrapI*:
$(x :: 'a :: len0\ word) < z - k ==> k <= z ==> 0 < k ==> x < x + k$
⟨*proof*⟩

lemma *inc-le*: $(i :: 'a :: len\ word) < m ==> i + 1 <= m$
⟨*proof*⟩

lemma *inc-i*:
$(1 :: 'a :: len\ word) <= i ==> i < m ==> 1 <= (i + 1)\ \&\ i + 1 <= m$
⟨*proof*⟩

lemma *udvd-incr-lem*:
$up < uq ==> up = ua + n * uint\ K ==>$
$uq = ua + n' * uint\ K ==> up + uint\ K <= uq$

⟨*proof*⟩

lemma *udvd-incr'*:
*p < q ==> uint p = ua + n * uint K ==>*
*uint q = ua + n' * uint K ==> p + K <= q*
⟨*proof*⟩

lemma *udvd-decr'*:
*p < q ==> uint p = ua + n * uint K ==>*
*uint q = ua + n' * uint K ==> p <= q - K*
⟨*proof*⟩

lemmas *udvd-incr-lem0 = udvd-incr-lem* [**where** *ua=0*, *simplified*]
lemmas *udvd-incr0 = udvd-incr'* [**where** *ua=0*, *simplified*]
lemmas *udvd-decr0 = udvd-decr'* [**where** *ua=0*, *simplified*]

lemma *udvd-minus-le'*:
xy < k ==> z udvd xy ==> z udvd k ==> xy <= k - z
⟨*proof*⟩

⟨*ML*⟩

lemma *udvd-incr2-K*:
p < a + s ==> a <= a + s ==> K udvd s ==> K udvd p - a ==> a <= p ==>
0 < K ==> p <= p + K & p + K <= a + s
⟨*proof*⟩

⟨*ML*⟩

lemma *word-succ-rbl*:
to-bl w = bl ==> to-bl (word-succ w) = (rev (rbl-succ (rev bl)))
⟨*proof*⟩

lemma *word-pred-rbl*:
to-bl w = bl ==> to-bl (word-pred w) = (rev (rbl-pred (rev bl)))
⟨*proof*⟩

lemma *word-add-rbl*:
to-bl v = vbl ==> to-bl w = wbl ==>
to-bl (v + w) = (rev (rbl-add (rev vbl) (rev wbl)))
⟨*proof*⟩

lemma *word-mult-rbl*:
to-bl v = vbl ==> to-bl w = wbl ==>
*to-bl (v * w) = (rev (rbl-mult (rev vbl) (rev wbl)))*
⟨*proof*⟩

lemma *rtb-rbl-ariths*:
rev (*to-bl w*) = *ys* ⟹ *rev* (*to-bl* (*word-succ w*)) = *rbl-succ ys*

rev (*to-bl w*) = *ys* ⟹ *rev* (*to-bl* (*word-pred w*)) = *rbl-pred ys*

[| *rev* (*to-bl v*) = *ys*; *rev* (*to-bl w*) = *xs* |]
==> *rev* (*to-bl* (*v* * *w*)) = *rbl-mult ys xs*

[| *rev* (*to-bl v*) = *ys*; *rev* (*to-bl w*) = *xs* |]
==> *rev* (*to-bl* (*v* + *w*)) = *rbl-add ys xs*
⟨*proof*⟩

11.6 Arithmetic type class instantiations

instance *word* :: (*len0*) *comm-monoid-add* ⟨*proof*⟩

instance *word* :: (*len0*) *comm-monoid-mult*
⟨*proof*⟩

instance *word* :: (*len0*) *comm-semiring*
⟨*proof*⟩

instance *word* :: (*len0*) *ab-group-add* ⟨*proof*⟩

instance *word* :: (*len0*) *comm-ring* ⟨*proof*⟩

instance *word* :: (*len*) *comm-semiring-1*
⟨*proof*⟩

instance *word* :: (*len*) *comm-ring-1* ⟨*proof*⟩

instance *word* :: (*len0*) *comm-semiring-0* ⟨*proof*⟩

instance *word* :: (*len0*) *order* ⟨*proof*⟩

lemma *zero-bintrunc*:
iszero (*number-of x* :: *'a* :: *len word*) =
(*bintrunc* (*len-of TYPE*(*'a*)) *x* = *Int.Pls*)
⟨*proof*⟩

lemmas *word-le-0-iff* [*simp*] =
word-zero-le [*THEN leD*, *THEN linorder-antisym-conv1*]

lemma *word-of-nat*: *of-nat n* = *word-of-int* (*int n*)
⟨*proof*⟩

lemma *word-of-int*: *of-int* = *word-of-int*
⟨*proof*⟩

lemma *word-of-int-nat*:
0 <= x ==> word-of-int x = of-nat (nat x)
⟨*proof*⟩

lemma *word-number-of-eq*:
number-of w = (of-int w :: 'a :: len word)
⟨*proof*⟩

instance *word* :: (*len*) *number-ring*
⟨*proof*⟩

lemma *iszero-word-no* [*simp*] :
iszero (number-of bin :: 'a :: len word) =
iszero (number-of (bintrunc (len-of TYPE('a)) bin) :: int)
⟨*proof*⟩

11.7 Word and nat

lemma *td-ext-unat'*:
n = len-of TYPE ('a :: len) ==>
td-ext (unat :: 'a word => nat) of-nat
(unats n) (%i. i mod 2 ^ n)
⟨*proof*⟩

lemmas *td-ext-unat = refl* [*THEN td-ext-unat'*]
lemmas *unat-of-nat = td-ext-unat* [*THEN td-ext.eq-norm, standard*]

interpretation *word-unat*:
td-ext unat::'a::len word => nat
of-nat
unats (len-of TYPE('a::len))
%i. i mod 2 ^ len-of TYPE('a::len)
⟨*proof*⟩

lemmas *td-unat = word-unat.td-thm*

lemmas *unat-lt2p* [*iff*] = *word-unat.Rep* [*unfolded unats-def mem-Collect-eq*]

lemma *unat-le*: *y <= unat (z :: 'a :: len word) ==> y : unats (len-of TYPE ('a))*
⟨*proof*⟩

lemma *word-nchotomy*:
ALL w. EX n. (w :: 'a :: len word) = of-nat n & n < 2 ^ len-of TYPE ('a)
⟨*proof*⟩

lemma *of-nat-eq*:
fixes *w :: 'a::len word*

shows (*of-nat n = w*) = (∃ *q*. *n = unat w + q * 2 ^ len-of TYPE('a)*)
⟨*proof*⟩

lemma *of-nat-eq-size*:
(*of-nat n = w*) = (*EX q*. *n = unat w + q * 2 ^ size w*)
⟨*proof*⟩

lemma *of-nat-0*:
(*of-nat m = (0::'a::len word)*) = (∃ *q*. *m = q * 2 ^ len-of TYPE('a)*)
⟨*proof*⟩

lemmas *of-nat-2p = mult-1* [*symmetric*, *THEN iffD2* [*OF of-nat-0 exI*]]

lemma *of-nat-gt-0*: *of-nat k ~= 0 ==> 0 < k*
⟨*proof*⟩

lemma *of-nat-neq-0*:
0 < k ==> k < 2 ^ len-of TYPE ('a :: len) ==> of-nat k ~= (0 :: 'a word)
⟨*proof*⟩

lemma *Abs-fnat-hom-add*:
of-nat a + of-nat b = of-nat (a + b)
⟨*proof*⟩

lemma *Abs-fnat-hom-mult*:
*of-nat a * of-nat b = (of-nat (a * b) :: 'a :: len word)*
⟨*proof*⟩

lemma *Abs-fnat-hom-Suc*:
word-succ (of-nat a) = of-nat (Suc a)
⟨*proof*⟩

lemma *Abs-fnat-hom-0*: *(0::'a::len word) = of-nat 0*
⟨*proof*⟩

lemma *Abs-fnat-hom-1*: *(1::'a::len word) = of-nat (Suc 0)*
⟨*proof*⟩

lemmas *Abs-fnat-homs =*
Abs-fnat-hom-add Abs-fnat-hom-mult Abs-fnat-hom-Suc
Abs-fnat-hom-0 Abs-fnat-hom-1

lemma *word-arith-nat-add*:
a + b = of-nat (unat a + unat b)
⟨*proof*⟩

lemma *word-arith-nat-mult*:
*a * b = of-nat (unat a * unat b)*
⟨*proof*⟩

lemma *word-arith-nat-Suc*:
word-succ a = of-nat (Suc (unat a))
⟨*proof*⟩

lemma *word-arith-nat-div*:
a div b = of-nat (unat a div unat b)
⟨*proof*⟩

lemma *word-arith-nat-mod*:
a mod b = of-nat (unat a mod unat b)
⟨*proof*⟩

lemmas *word-arith-nat-defs =*
word-arith-nat-add word-arith-nat-mult
word-arith-nat-Suc Abs-fnat-hom-0
Abs-fnat-hom-1 word-arith-nat-div
word-arith-nat-mod

lemmas *unat-cong = arg-cong* [**where** *f = unat*]

lemmas *unat-word-ariths = word-arith-nat-defs*
[*THEN trans* [*OF unat-cong unat-of-nat*], *standard*]

lemmas *word-sub-less-iff = word-sub-le-iff*
[*simplified linorder-not-less* [*symmetric*], *simplified*]

lemma *unat-add-lem*:
(unat x + unat y < 2 ^ len-of TYPE('a)) =
(unat (x + y :: 'a :: len word) = unat x + unat y)
⟨*proof*⟩

lemma *unat-mult-lem*:
*(unat x * unat y < 2 ^ len-of TYPE('a)) =*
*(unat (x * y :: 'a :: len word) = unat x * unat y)*
⟨*proof*⟩

lemmas *unat-plus-if' =*
trans [*OF unat-word-ariths(1) mod-nat-add*, *simplified*, *standard*]

lemma *le-no-overflow*:
x <= b ==> a <= a + b ==> x <= a + (b :: 'a :: len0 word)
⟨*proof*⟩

lemmas *un-ui-le = trans*
[*OF word-le-nat-alt* [*symmetric*]
word-le-def,
standard]

lemma *unat-sub-if-size*:
unat $(x - y) =$ (*if unat* $y <= unat\ x$
then unat $x - unat\ y$
else unat $x + 2\ \hat{}\ size\ x - unat\ y$)
⟨*proof*⟩

lemmas *unat-sub-if'* = *unat-sub-if-size* [*unfolded word-size*]

lemma *unat-div*: *unat* $((x :: 'a :: len\ word)\ div\ y) = unat\ x\ div\ unat\ y$
⟨*proof*⟩

lemma *unat-mod*: *unat* $((x :: 'a :: len\ word)\ mod\ y) = unat\ x\ mod\ unat\ y$
⟨*proof*⟩

lemma *uint-div*: *uint* $((x :: 'a :: len\ word)\ div\ y) = uint\ x\ div\ uint\ y$
⟨*proof*⟩

lemma *uint-mod*: *uint* $((x :: 'a :: len\ word)\ mod\ y) = uint\ x\ mod\ uint\ y$
⟨*proof*⟩

11.8 Definition of unat_arith tactic

lemma *unat-split*:
fixes *x*::'*a*::*len word*
shows *P* (*unat x*) =
(*ALL n. of-nat n = x & n < 2^len-of TYPE*('*a*) −−> *P n*)
⟨*proof*⟩

lemma *unat-split-asm*:
fixes *x*::'*a*::*len word*
shows *P* (*unat x*) =
(~(*EX n. of-nat n = x & n < 2^len-of TYPE*('*a*) & ~ *P n*))
⟨*proof*⟩

lemmas *of-nat-inverse* =
word-unat.Abs-inverse' [*rotated*, *unfolded unats-def*, *simplified*]

lemmas *unat-splits* = *unat-split unat-split-asm*

lemmas *unat-arith-simps* =
word-le-nat-alt word-less-nat-alt
word-unat.Rep-inject [*symmetric*]
unat-sub-if' unat-plus-if' unat-div unat-mod

⟨*ML*⟩

lemma *no-plus-overflow-unat-size*:
$((x :: 'a :: len\ word) <= x + y) = (unat\ x + unat\ y < 2\ \hat{}\ size\ x)$

⟨*proof*⟩

lemma *unat-sub*: $b <= a ==> unat\ (a - b) = unat\ a - unat\ (b :: 'a :: len\ word)$
⟨*proof*⟩

lemmas *no-olen-add-nat* = *no-plus-overflow-unat-size* [*unfolded word-size*]

lemmas *unat-plus-simple* = *trans* [*OF no-olen-add-nat unat-add-lem*, *standard*]

lemma *word-div-mult*:
$(0 :: 'a :: len\ word) < y ==> unat\ x * unat\ y < 2\ \hat{}\ len\text{-}of\ TYPE('a) ==>$
$x * y\ div\ y = x$
⟨*proof*⟩

lemma *div-lt'*: $(i :: 'a :: len\ word) <= k\ div\ x ==>$
$unat\ i * unat\ x < 2\ \hat{}\ len\text{-}of\ TYPE('a)$
⟨*proof*⟩

lemmas *div-lt''* = *order-less-imp-le* [*THEN div-lt'*]

lemma *div-lt-mult*: $(i :: 'a :: len\ word) < k\ div\ x ==> 0 < x ==> i * x < k$
⟨*proof*⟩

lemma *div-le-mult*:
$(i :: 'a :: len\ word) <= k\ div\ x ==> 0 < x ==> i * x <= k$
⟨*proof*⟩

lemma *div-lt-uint'*:
$(i :: 'a :: len\ word) <= k\ div\ x ==> uint\ i * uint\ x < 2\ \hat{}\ len\text{-}of\ TYPE('a)$
⟨*proof*⟩

lemmas *div-lt-uint''* = *order-less-imp-le* [*THEN div-lt-uint'*]

lemma *word-le-exists'*:
$(x :: 'a :: len0\ word) <= y ==>$
$(EX\ z.\ y = x + z\ \&\ uint\ x + uint\ z < 2\ \hat{}\ len\text{-}of\ TYPE('a))$
⟨*proof*⟩

lemmas *plus-minus-not-NULL* = *order-less-imp-le* [*THEN plus-minus-not-NULL-ab*]

lemmas *plus-minus-no-overflow* =
order-less-imp-le [*THEN plus-minus-no-overflow-ab*]

lemmas *mcs* = *word-less-minus-cancel word-less-minus-mono-left*
word-le-minus-cancel word-le-minus-mono-left

lemmas *word-l-diffs* = *mcs* [**where** $y = w + x$, *unfolded add-diff-cancel*, *standard*]
lemmas *word-diff-ls* = *mcs* [**where** $z = w + x$, *unfolded add-diff-cancel*, *standard*]
lemmas *word-plus-mcs* = *word-diff-ls*

[**where** $y = v + x$, *unfolded add-diff-cancel*, *standard*]

lemmas *le-unat-uoi* = *unat-le* [*THEN word-unat.Abs-inverse*]

lemmas *thd* = *refl* [*THEN* [*2*] *split-div-lemma* [*THEN iffD2*], *THEN conjunct1*]

lemma *thd1*:
$a\ div\ b * b \leq (a::nat)$
⟨*proof*⟩

lemmas *uno-simps* [*THEN le-unat-uoi*, *standard*] =
mod-le-divisor div-le-dividend thd1

lemma *word-mod-div-equality*:
$(n\ div\ b) * b + (n\ mod\ b) = (n :: 'a :: len\ word)$
⟨*proof*⟩

lemma *word-div-mult-le*: $a\ div\ b * b <= (a::'a::len\ word)$
⟨*proof*⟩

lemma *word-mod-less-divisor*: $0 < n ==> m\ mod\ n < (n :: 'a :: len\ word)$
⟨*proof*⟩

lemma *word-of-int-power-hom*:
$word\text{-}of\text{-}int\ a\ \hat{}\ n = (word\text{-}of\text{-}int\ (a\ \hat{}\ n) :: 'a :: len\ word)$
⟨*proof*⟩

lemma *word-arith-power-alt*:
$a\ \hat{}\ n = (word\text{-}of\text{-}int\ (uint\ a\ \hat{}\ n) :: 'a :: len\ word)$
⟨*proof*⟩

lemma *of-bl-length-less*:
$length\ x = k ==> k < len\text{-}of\ TYPE('a) ==> (of\text{-}bl\ x :: 'a :: len\ word) < 2\ \hat{}\ k$
⟨*proof*⟩

11.9 Cardinality, finiteness of set of words

lemmas *card-lessThan'* = *card-lessThan* [*unfolded lessThan-def*]

lemmas *card-eq* = *word-unat.Abs-inj-on* [*THEN card-image*,
unfolded word-unat.image, *unfolded unats-def*, *standard*]

lemmas *card-word* = *trans* [*OF card-eq card-lessThan'*, *standard*]

lemma *finite-word-UNIV*: *finite* $(UNIV :: 'a :: len\ word\ set)$
⟨*proof*⟩

lemma *card-word-size*:
$card\ (UNIV :: 'a :: len\ word\ set) = (2\ \hat{}\ size\ (x :: 'a\ word))$

⟨*proof*⟩

end

12 WordBitwise: Bitwise Operations on Words

theory *WordBitwise*
imports *WordArith*
begin

lemmas *bin-log-bintrs = bin-trunc-not bin-trunc-xor bin-trunc-and bin-trunc-or*

lemmas *wils1 = bin-log-bintrs* [*THEN word-ubin.norm-eq-iff* [*THEN iffD1*],
folded word-ubin.eq-norm, *THEN eq-reflection*, *standard*]

lemmas *word-log-binary-defs =*
word-and-def word-or-def word-xor-def

lemmas *word-no-log-defs* [*simp*] =
word-not-def [**where** *a=number-of a*,
unfolded word-no-wi wils1, *folded word-no-wi*, *standard*]
word-log-binary-defs [**where** *a=number-of a* **and** *b=number-of b*,
unfolded word-no-wi wils1, *folded word-no-wi*, *standard*]

lemmas *word-wi-log-defs = word-no-log-defs* [*unfolded word-no-wi*]

lemma *uint-or*: *uint* (*x OR y*) = (*uint x*) *OR* (*uint y*)
⟨*proof*⟩

lemma *uint-and*: *uint* (*x AND y*) = (*uint x*) *AND* (*uint y*)
⟨*proof*⟩

lemma *word-ops-nth-size*:
n < *size* (*x*::'*a*::*len0 word*) ==>
(*x OR y*) !! *n* = (*x* !! *n* | *y* !! *n*) &
(*x AND y*) !! *n* = (*x* !! *n* & *y* !! *n*) &
(*x XOR y*) !! *n* = (*x* !! *n* ~= *y* !! *n*) &
(*NOT x*) !! *n* = (~ *x* !! *n*)
⟨*proof*⟩

lemma *word-ao-nth*:
fixes *x* :: '*a*::*len0 word*
shows (*x OR y*) !! *n* = (*x* !! *n* | *y* !! *n*) &
(*x AND y*) !! *n* = (*x* !! *n* & *y* !! *n*)

⟨*proof*⟩

lemmas *bwsimps* =
word-of-int-homs(*2*)
word-0-wi-Pls
word-m1-wi-Min
word-wi-log-defs

lemma *word-bw-assocs*:
fixes *x* :: *'a*::*len0 word*
shows
(*x AND y*) *AND z* = *x AND y AND z*
(*x OR y*) *OR z* = *x OR y OR z*
(*x XOR y*) *XOR z* = *x XOR y XOR z*
⟨*proof*⟩

lemma *word-bw-comms*:
fixes *x* :: *'a*::*len0 word*
shows
x AND y = *y AND x*
x OR y = *y OR x*
x XOR y = *y XOR x*
⟨*proof*⟩

lemma *word-bw-lcs*:
fixes *x* :: *'a*::*len0 word*
shows
y AND x AND z = *x AND y AND z*
y OR x OR z = *x OR y OR z*
y XOR x XOR z = *x XOR y XOR z*
⟨*proof*⟩

lemma *word-log-esimps* [*simp*]:
fixes *x* :: *'a*::*len0 word*
shows
x AND 0 = *0*
x AND −1 = *x*
x OR 0 = *x*
x OR −1 = *−1*
x XOR 0 = *x*
x XOR −1 = *NOT x*
0 AND x = *0*
−1 AND x = *x*
0 OR x = *x*
−1 OR x = *−1*
0 XOR x = *x*
−1 XOR x = *NOT x*

⟨*proof*⟩

lemma *word-not-dist*:
fixes x :: *'a::len0 word*
shows
$NOT\ (x\ OR\ y) = NOT\ x\ AND\ NOT\ y$
$NOT\ (x\ AND\ y) = NOT\ x\ OR\ NOT\ y$
⟨*proof*⟩

lemma *word-bw-same*:
fixes x :: *'a::len0 word*
shows
$x\ AND\ x = x$
$x\ OR\ x = x$
$x\ XOR\ x = 0$
⟨*proof*⟩

lemma *word-ao-absorbs* [*simp*]:
fixes x :: *'a::len0 word*
shows
$x\ AND\ (y\ OR\ x) = x$
$x\ OR\ y\ AND\ x = x$
$x\ AND\ (x\ OR\ y) = x$
$y\ AND\ x\ OR\ x = x$
$(y\ OR\ x)\ AND\ x = x$
$x\ OR\ x\ AND\ y = x$
$(x\ OR\ y)\ AND\ x = x$
$x\ AND\ y\ OR\ x = x$
⟨*proof*⟩

lemma *word-not-not* [*simp*]:
$NOT\ NOT\ (x::'a::len0\ word) = x$
⟨*proof*⟩

lemma *word-ao-dist*:
fixes x :: *'a::len0 word*
shows $(x\ OR\ y)\ AND\ z = x\ AND\ z\ OR\ y\ AND\ z$
⟨*proof*⟩

lemma *word-oa-dist*:
fixes x :: *'a::len0 word*
shows $x\ AND\ y\ OR\ z = (x\ OR\ z)\ AND\ (y\ OR\ z)$
⟨*proof*⟩

lemma *word-add-not* [*simp*]:
fixes x :: *'a::len0 word*
shows $x + NOT\ x = -1$
⟨*proof*⟩

lemma *word-plus-and-or* [*simp*]:
fixes $x :: 'a::len0\ word$
shows $(x\ AND\ y) + (x\ OR\ y) = x + y$
⟨*proof*⟩

lemma *leoa*:
fixes $x :: 'a::len0\ word$
shows $(w = (x\ OR\ y)) ==> (y = (w\ AND\ y))$ ⟨*proof*⟩
lemma *leao*:
fixes $x' :: 'a::len0\ word$
shows $(w' = (x'\ AND\ y')) ==> (x' = (x'\ OR\ w'))$ ⟨*proof*⟩

lemmas *word-ao-equiv* = *leao* [*COMP leoa* [*COMP iffI*]]

lemma *le-word-or2*: $x <= x\ OR\ (y::'a::len0\ word)$
⟨*proof*⟩

lemmas *le-word-or1* = *xtr3* [*OF word-bw-comms* (*2*) *le-word-or2*, *standard*]
lemmas *word-and-le1* =
xtr3 [*OF word-ao-absorbs* (*4*) [*symmetric*] *le-word-or2*, *standard*]
lemmas *word-and-le2* =
xtr3 [*OF word-ao-absorbs* (*8*) [*symmetric*] *le-word-or2*, *standard*]

lemma *bl-word-not*: *to-bl* $(NOT\ w) = map\ Not\ (to\text{-}bl\ w)$
⟨*proof*⟩

lemma *bl-word-xor*: *to-bl* $(v\ XOR\ w) = map2\ op\ \sim= (to\text{-}bl\ v)\ (to\text{-}bl\ w)$
⟨*proof*⟩

lemma *bl-word-or*: *to-bl* $(v\ OR\ w) = map2\ op\ |\ (to\text{-}bl\ v)\ (to\text{-}bl\ w)$
⟨*proof*⟩

lemma *bl-word-and*: *to-bl* $(v\ AND\ w) = map2\ op\ \&\ (to\text{-}bl\ v)\ (to\text{-}bl\ w)$
⟨*proof*⟩

lemma *word-lsb-alt*: *lsb* $(w::'a::len0\ word) = test\text{-}bit\ w\ 0$
⟨*proof*⟩

lemma *word-lsb-1-0*: *lsb* $(1::'a::len\ word)\ \&\ \sim\ lsb\ (0::'b::len0\ word)$
⟨*proof*⟩

lemma *word-lsb-last*: *lsb* $(w::'a::len\ word) = last\ (to\text{-}bl\ w)$
⟨*proof*⟩

lemma *word-lsb-int*: *lsb* $w = (uint\ w\ mod\ 2 = 1)$
⟨*proof*⟩

lemma *word-msb-sint*: *msb* $w = (sint\ w < 0)$
⟨*proof*⟩

lemma *word-msb-no'*:
w = number-of bin ==> msb (w::'a::len word) = bin-nth bin (size w − 1)
⟨*proof*⟩

lemmas *word-msb-no = refl [THEN word-msb-no', unfolded word-size]*

lemma *word-msb-nth'*: *msb (w::'a::len word) = bin-nth (uint w) (size w − 1)*
⟨*proof*⟩

lemmas *word-msb-nth = word-msb-nth' [unfolded word-size]*

lemma *word-msb-alt*: *msb (w::'a::len word) = hd (to-bl w)*
⟨*proof*⟩

lemma *word-set-nth*:
set-bit w n (test-bit w n) = (w::'a::len0 word)
⟨*proof*⟩

lemma *bin-nth-uint'*:
bin-nth (uint w) n = (rev (bin-to-bl (size w) (uint w)) ! n & n < size w)
⟨*proof*⟩

lemmas *bin-nth-uint = bin-nth-uint' [unfolded word-size]*

lemma *test-bit-bl*: *w !! n = (rev (to-bl w) ! n & n < size w)*
⟨*proof*⟩

lemma *to-bl-nth*: *n < size w ==> to-bl w ! n = w !! (size w − Suc n)*
⟨*proof*⟩

lemma *test-bit-set*:
fixes *w :: 'a::len0 word*
shows *(set-bit w n x) !! n = (n < size w & x)*
⟨*proof*⟩

lemma *test-bit-set-gen*:
fixes *w :: 'a::len0 word*
shows *test-bit (set-bit w n x) m =*
(if m = n then n < size w & x else test-bit w m)
⟨*proof*⟩

lemma *of-bl-rep-False*: *of-bl (replicate n False @ bs) = of-bl bs*
⟨*proof*⟩

lemma *msb-nth'*:
fixes *w :: 'a::len word*
shows *msb w = w !! (size w − 1)*
⟨*proof*⟩

lemmas *msb-nth = msb-nth'* [*unfolded word-size*]

lemmas *msb0 = len-gt-0* [*THEN diff-Suc-less*, *THEN*
word-ops-nth-size [*unfolded word-size*], *standard*]
lemmas *msb1 = msb0* [**where** *i = 0*]
lemmas *word-ops-msb = msb1* [*unfolded msb-nth* [*symmetric*, *unfolded One-nat-def*]]

lemmas *lsb0 = len-gt-0* [*THEN word-ops-nth-size* [*unfolded word-size*], *standard*]
lemmas *word-ops-lsb = lsb0* [*unfolded word-lsb-alt*]

lemma *td-ext-nth'*:
n = size (*w*::'*a*::*len0 word*) ==> *ofn = set-bits* ==> [*w*, *ofn g*] = *l* ==>
td-ext test-bit ofn {*f*. *ALL i*. *f i* −−> *i* < *n*} (%*h i*. *h i* & *i* < *n*)
⟨*proof*⟩

lemmas *td-ext-nth = td-ext-nth'* [*OF refl refl refl*, *unfolded word-size*]

interpretation *test-bit*:
td-ext op !! :: '*a*::*len0 word* => *nat* => *bool*
set-bits
{*f*. ∀ *i*. *f i* ⟶ *i* < *len-of TYPE*('*a*::*len0*)}
(λ*h i*. *h i* ∧ *i* < *len-of TYPE*('*a*::*len0*))
⟨*proof*⟩

declare *test-bit.Rep'* [*simp del*]
declare *test-bit.Rep'* [*rule del*]

lemmas *td-nth = test-bit.td-thm*

lemma *word-set-set-same*:
fixes *w* :: '*a*::*len0 word*
shows *set-bit* (*set-bit w n x*) *n y = set-bit w n y*
⟨*proof*⟩

lemma *word-set-set-diff*:
fixes *w* :: '*a*::*len0 word*
assumes *m* ~= *n*
shows *set-bit* (*set-bit w m x*) *n y = set-bit* (*set-bit w n y*) *m x*
⟨*proof*⟩

lemma *test-bit-no'*:
fixes *w* :: '*a*::*len0 word*
shows *w = number-of bin* ==> *test-bit w n* = (*n* < *size w* & *bin-nth bin n*)
⟨*proof*⟩

lemmas *test-bit-no* =
refl [*THEN test-bit-no'*, *unfolded word-size*, *THEN eq-reflection*, *standard*]

lemma *nth-0*: $\sim$ (*0*::*'a*::*len0 word*) !! *n*
⟨*proof*⟩

lemma *nth-sint*:
fixes *w* :: *'a*::*len word*
defines *l* ≡ *len-of TYPE* (*'a*)
shows *bin-nth* (*sint w*) *n* = (*if n* < *l* − *1 then w* !! *n else w* !! (*l* − *1*))
⟨*proof*⟩

lemma *word-lsb-no*:
lsb (*number-of bin* :: *'a* :: *len word*) = (*bin-last bin* = *bit.B1*)
⟨*proof*⟩

lemma *word-set-no*:
set-bit (*number-of bin*::*'a*::*len0 word*) *n b* =
number-of (*bin-sc n* (*if b then bit.B1 else bit.B0*) *bin*)
⟨*proof*⟩

lemmas *setBit-no* = *setBit-def* [*THEN trans* [*OF meta-eq-to-obj-eq word-set-no*],
simplified if-simps, *THEN eq-reflection*, *standard*]
lemmas *clearBit-no* = *clearBit-def* [*THEN trans* [*OF meta-eq-to-obj-eq word-set-no*],
simplified if-simps, *THEN eq-reflection*, *standard*]

lemma *to-bl-n1*:
to-bl (−*1*::*'a*::*len0 word*) = *replicate* (*len-of TYPE* (*'a*)) *True*
⟨*proof*⟩

lemma *word-msb-n1*: *msb* (−*1*::*'a*::*len word*)
⟨*proof*⟩

declare *word-set-set-same* [*simp*] *word-set-nth* [*simp*]
test-bit-no [*simp*] *word-set-no* [*simp*] *nth-0* [*simp*]
setBit-no [*simp*] *clearBit-no* [*simp*]
word-lsb-no [*simp*] *word-msb-no* [*simp*] *word-msb-n1* [*simp*] *word-lsb-1-0* [*simp*]

lemma *word-set-nth-iff*:
(*set-bit w n b* = *w*) = (*w* !! *n* = *b* | *n* >= *size* (*w*::*'a*::*len0 word*))
⟨*proof*⟩

lemma *test-bit-2p'*:
w = *word-of-int* (*2* ^ *n*) ==>
w !! *m* = (*m* = *n* & *m* < *size* (*w* :: *'a* :: *len word*))
⟨*proof*⟩

lemmas *test-bit-2p* = *refl* [*THEN test-bit-2p'*, *unfolded word-size*]

lemma *nth-w2p*:
((*2*::*'a*::*len word*) ^ *n*) !! *m* ⟷ *m* = *n* ∧ *m* < *len-of TYPE*(*'a*::*len*)
⟨*proof*⟩

lemma *uint-2p*:
(*0*::*'a*::*len word*) < *2* ^ *n* ==> *uint* (*2* ^ *n*::*'a*::*len word*) = *2* ^ *n*
⟨*proof*⟩

lemma *word-of-int-2p*: (*word-of-int* (*2* ^ *n*) :: *'a* :: *len word*) = *2* ^ *n*
⟨*proof*⟩

lemma *bang-is-le*: *x* !! *m* ==> *2* ^ *m* <= (*x* :: *'a* :: *len word*)
⟨*proof*⟩

lemma *word-clr-le*:
fixes *w* :: *'a*::*len0 word*
shows *w* >= *set-bit w n False*
⟨*proof*⟩

lemma *word-set-ge*:
fixes *w* :: *'a*::*len word*
shows *w* <= *set-bit w n True*
⟨*proof*⟩

end

13 WordShift: Shifting, Rotating, and Splitting Words

theory *WordShift*
imports *WordBitwise*
begin

13.1 Bit shifting

lemma *shiftl1-number* [*simp*] :
shiftl1 (*number-of w*) = *number-of* (*w BIT bit.B0*)
⟨*proof*⟩

lemma *shiftl1-0* [*simp*] : *shiftl1 0* = *0*
⟨*proof*⟩

lemmas *shiftl1-def-u* = *shiftl1-def* [*folded word-number-of-def*]

lemma *shiftl1-def-s*: *shiftl1 w* = *number-of* (*sint w BIT bit.B0*)
⟨*proof*⟩

lemma *shiftr1-0* [*simp*] : *shiftr1 0* = *0*
⟨*proof*⟩

lemma *sshiftr1-0* [*simp*] : *sshiftr1 0 = 0*
⟨*proof*⟩

lemma *sshiftr1-n1* [*simp*] : *sshiftr1 −1 = −1*
⟨*proof*⟩

lemma *shiftl-0* [*simp*] : *(0::'a::len0 word) << n = 0*
⟨*proof*⟩

lemma *shiftr-0* [*simp*] : *(0::'a::len0 word) >> n = 0*
⟨*proof*⟩

lemma *sshiftr-0* [*simp*] : *0 >>> n = 0*
⟨*proof*⟩

lemma *sshiftr-n1* [*simp*] : *−1 >>> n = −1*
⟨*proof*⟩

lemma *nth-shiftl1*: *shiftl1 w !! n = (n < size w & n > 0 & w !! (n − 1))*
⟨*proof*⟩

lemma *nth-shiftl'* [*rule-format*]:
ALL n. ((w::'a::len0 word) << m) !! n = (n < size w & n >= m & w !! (n − m))
⟨*proof*⟩

lemmas *nth-shiftl = nth-shiftl'* [*unfolded word-size*]

lemma *nth-shiftr1*: *shiftr1 w !! n = w !! Suc n*
⟨*proof*⟩

lemma *nth-shiftr*:
⋀n. ((w::'a::len0 word) >> m) !! n = w !! (n + m)
⟨*proof*⟩

lemma *uint-shiftr1*: *uint (shiftr1 w) = bin-rest (uint w)*
⟨*proof*⟩

lemma *nth-sshiftr1*:
sshiftr1 w !! n = (if n = size w − 1 then w !! n else w !! Suc n)
⟨*proof*⟩

lemma *nth-sshiftr* [*rule-format*] :
ALL n. sshiftr w m !! n = (n < size w &
(if n + m >= size w then w !! (size w − 1) else w !! (n + m)))
⟨*proof*⟩

lemma *shiftr1-div-2*: *uint* (*shiftr1 w*) = *uint w div 2*
⟨*proof*⟩

lemma *sshiftr1-div-2*: *sint* (*sshiftr1 w*) = *sint w div 2*
⟨*proof*⟩

lemma *shiftr-div-2n*: *uint* (*shiftr w n*) = *uint w div 2* ^ *n*
⟨*proof*⟩

lemma *sshiftr-div-2n*: *sint* (*sshiftr w n*) = *sint w div 2* ^ *n*
⟨*proof*⟩

13.1.1 shift functions in terms of lists of bools

lemmas *bshiftr1-no-bin* [*simp*] =
bshiftr1-def [**where** *w*=*number-of w*, *unfolded to-bl-no-bin*, *standard*]

lemma *bshiftr1-bl*: *to-bl* (*bshiftr1 b w*) = *b* # *butlast* (*to-bl w*)
⟨*proof*⟩

lemma *shiftl1-of-bl*: *shiftl1* (*of-bl bl*) = *of-bl* (*bl* @ [*False*])
⟨*proof*⟩

lemma *shiftl1-bl*: *shiftl1* (*w*::'*a*::*len0 word*) = *of-bl* (*to-bl w* @ [*False*])
⟨*proof*⟩

lemma *bl-shiftl1*:
to-bl (*shiftl1* (*w* :: '*a* :: *len word*)) = *tl* (*to-bl w*) @ [*False*]
⟨*proof*⟩

lemma *shiftr1-bl*: *shiftr1 w* = *of-bl* (*butlast* (*to-bl w*))
⟨*proof*⟩

lemma *bl-shiftr1*:
to-bl (*shiftr1* (*w* :: '*a* :: *len word*)) = *False* # *butlast* (*to-bl w*)
⟨*proof*⟩

lemma *shiftl1-rev*:
shiftl1 (*w* :: '*a* :: *len word*) = *word-reverse* (*shiftr1* (*word-reverse w*))
⟨*proof*⟩

lemma *shiftl-rev*:
shiftl (*w* :: '*a* :: *len word*) *n* = *word-reverse* (*shiftr* (*word-reverse w*) *n*)
⟨*proof*⟩

lemmas *rev-shiftl* =

shiftl-rev [**where** *w* = *word-reverse w*, *simplified*, *standard*]

lemmas *shiftr-rev* = *rev-shiftl* [*THEN word-rev-gal'*, *standard*]
lemmas *rev-shiftr* = *shiftl-rev* [*THEN word-rev-gal'*, *standard*]

lemma *bl-sshiftr1*:
to-bl (*sshiftr1* (*w* :: 'a :: *len word*)) = *hd* (*to-bl w*) # *butlast* (*to-bl w*)
⟨*proof*⟩

lemma *drop-shiftr*:
drop n (*to-bl* ((*w* :: 'a :: *len word*) >> *n*)) = *take* (*size w* − *n*) (*to-bl w*)
⟨*proof*⟩

lemma *drop-sshiftr*:
drop n (*to-bl* ((*w* :: 'a :: *len word*) >>> *n*)) = *take* (*size w* − *n*) (*to-bl w*)
⟨*proof*⟩

lemma *take-shiftr* [*rule-format*] :
n <= *size* (*w* :: 'a :: *len word*) −−> *take n* (*to-bl* (*w* >> *n*)) =
replicate n False
⟨*proof*⟩

lemma *take-sshiftr'* [*rule-format*] :
n <= *size* (*w* :: 'a :: *len word*) −−> *hd* (*to-bl* (*w* >>> *n*)) = *hd* (*to-bl w*) &
take n (*to-bl* (*w* >>> *n*)) = *replicate n* (*hd* (*to-bl w*))
⟨*proof*⟩

lemmas *hd-sshiftr* = *take-sshiftr'* [*THEN conjunct1*, *standard*]
lemmas *take-sshiftr* = *take-sshiftr'* [*THEN conjunct2*, *standard*]

lemma *atd-lem*: *take n xs* = *t* ==> *drop n xs* = *d* ==> *xs* = *t* @ *d*
⟨*proof*⟩

lemmas *bl-shiftr* = *atd-lem* [*OF take-shiftr drop-shiftr*]
lemmas *bl-sshiftr* = *atd-lem* [*OF take-sshiftr drop-sshiftr*]

lemma *shiftl-of-bl*: *of-bl bl* << *n* = *of-bl* (*bl* @ *replicate n False*)
⟨*proof*⟩

lemma *shiftl-bl*:
(*w*::'*a*::*len0 word*) << (*n*::*nat*) = *of-bl* (*to-bl w* @ *replicate n False*)
⟨*proof*⟩

lemmas *shiftl-number* [*simp*] = *shiftl-def* [**where** *w*=*number-of w*, *standard*]

lemma *bl-shiftl*:
to-bl (*w* << *n*) = *drop n* (*to-bl w*) @ *replicate* (*min* (*size w*) *n*) *False*
⟨*proof*⟩

lemma *shiftl-zero-size*:
fixes $x :: 'a{::}len0\ word$
shows $size\ x <= n ==> x << n = 0$
⟨*proof*⟩

lemma *shiftl1-2t*: $shiftl1\ (w :: 'a :: len\ word) = 2 * w$
⟨*proof*⟩

lemma *shiftl1-p*: $shiftl1\ (w :: 'a :: len\ word) = w + w$
⟨*proof*⟩

lemma *shiftl-t2n*: $shiftl\ (w :: 'a :: len\ word)\ n = 2\ \hat{}\ n * w$
⟨*proof*⟩

lemma *shiftr1-bintr* [*simp*]:
$(shiftr1\ (number\text{-}of\ w) :: 'a :: len0\ word) =$
$number\text{-}of\ (bin\text{-}rest\ (bintrunc\ (len\text{-}of\ TYPE\ ('a))\ w))$
⟨*proof*⟩

lemma *sshiftr1-sbintr* [*simp*] :
$(sshiftr1\ (number\text{-}of\ w) :: 'a :: len\ word) =$
$number\text{-}of\ (bin\text{-}rest\ (sbintrunc\ (len\text{-}of\ TYPE\ ('a) - 1)\ w))$
⟨*proof*⟩

lemma *shiftr-no'*:
$w = number\text{-}of\ bin ==>$
$(w::'a::len0\ word) >> n = number\text{-}of\ ((bin\text{-}rest\ \hat{}\hat{}\ n)\ (bintrunc\ (size\ w)\ bin))$
⟨*proof*⟩

lemma *sshiftr-no'*:
$w = number\text{-}of\ bin ==> w >>> n = number\text{-}of\ ((bin\text{-}rest\ \hat{}\hat{}\ n)$
$(sbintrunc\ (size\ w - 1)\ bin))$
⟨*proof*⟩

lemmas *sshiftr-no* [*simp*] =
sshiftr-no' [**where** $w = number\text{-}of\ w$, *OF refl*, *unfolded word-size*, *standard*]

lemmas *shiftr-no* [*simp*] =
shiftr-no' [**where** $w = number\text{-}of\ w$, *OF refl*, *unfolded word-size*, *standard*]

lemma *shiftr1-bl-of'*:
$us = shiftr1\ (of\text{-}bl\ bl) ==> length\ bl <= size\ us ==>$
$us = of\text{-}bl\ (butlast\ bl)$
⟨*proof*⟩

lemmas *shiftr1-bl-of* = *refl* [*THEN shiftr1-bl-of'*, *unfolded word-size*]

lemma *shiftr-bl-of'* [*rule-format*]:
us = of-bl bl >> n ==> length bl <= size us -->
us = of-bl (take (length bl − n) bl)
⟨*proof*⟩

lemmas *shiftr-bl-of = refl* [*THEN shiftr-bl-of'*, *unfolded word-size*]

lemmas *shiftr-bl = word-bl.Rep'* [*THEN eq-imp-le*, *THEN shiftr-bl-of*, *simplified word-size*, *simplified*, *THEN eq-reflection*, *standard*]

lemma *msb-shift'*: *msb (w::'a::len word) <-> (w >> (size w − 1)) ~= 0*
⟨*proof*⟩

lemmas *msb-shift = msb-shift'* [*unfolded word-size*]

lemma *align-lem-or* [*rule-format*] :
ALL x m. length x = n + m --> length y = n + m -->
drop m x = replicate n False --> take m y = replicate m False -->
map2 op | x y = take m x @ drop m y
⟨*proof*⟩

lemma *align-lem-and* [*rule-format*] :
ALL x m. length x = n + m --> length y = n + m -->
drop m x = replicate n False --> take m y = replicate m False -->
map2 op & x y = replicate (n + m) False
⟨*proof*⟩

lemma *aligned-bl-add-size'*:
size x − n = m ==> n <= size x ==> drop m (to-bl x) = replicate n False ==>
take m (to-bl y) = replicate m False ==>
to-bl (x + y) = take m (to-bl x) @ drop m (to-bl y)
⟨*proof*⟩

lemmas *aligned-bl-add-size = refl* [*THEN aligned-bl-add-size'*]

13.1.2 Mask

lemma *nth-mask'*: *m = mask n ==> test-bit m i = (i < n & i < size m)*
⟨*proof*⟩

lemmas *nth-mask* [*simp*] *= refl* [*THEN nth-mask'*]

lemma *mask-bl*: *mask n = of-bl (replicate n True)*
⟨*proof*⟩

lemma *mask-bin*: *mask n = number-of (bintrunc n Int.Min)*
⟨*proof*⟩

lemma *and-mask-bintr*: *w AND mask n = number-of (bintrunc n (uint w))*
⟨*proof*⟩

lemma *and-mask-no*: *number-of i AND mask n = number-of (bintrunc n i)*
⟨*proof*⟩

lemmas *and-mask-wi = and-mask-no* [*unfolded word-number-of-def*]

lemma *bl-and-mask*:
to-bl (w AND mask n :: 'a :: len word) =
replicate (len-of TYPE('a) − n) False @
drop (len-of TYPE('a) − n) (to-bl w)
⟨*proof*⟩

lemmas *and-mask-mod-2p =*
and-mask-bintr [*unfolded word-number-of-alt no-bintr-alt*]

lemma *and-mask-lt-2p*: *uint (w AND mask n) < 2 ^ n*
⟨*proof*⟩

lemmas *eq-mod-iff = trans* [*symmetric*, *OF int-mod-lem eq-sym-conv*]

lemma *mask-eq-iff*: *(w AND mask n) = w <-> uint w < 2 ^ n*
⟨*proof*⟩

lemma *and-mask-dvd*: *2 ^ n dvd uint w = (w AND mask n = 0)*
⟨*proof*⟩

lemma *and-mask-dvd-nat*: *2 ^ n dvd unat w = (w AND mask n = 0)*
⟨*proof*⟩

lemma *word-2p-lem*:
n < size w ==> w < 2 ^ n = (uint (w :: 'a :: len word) < 2 ^ n)
⟨*proof*⟩

lemma *less-mask-eq*: *x < 2 ^ n ==> x AND mask n = (x :: 'a :: len word)*
⟨*proof*⟩

lemmas *mask-eq-iff-w2p =*
trans [*OF mask-eq-iff word-2p-lem* [*symmetric*], *standard*]

lemmas *and-mask-less' =*
iffD2 [*OF word-2p-lem and-mask-lt-2p*, *simplified word-size*, *standard*]

lemma *and-mask-less-size*: *n < size x ==> x AND mask n < 2^n*
⟨*proof*⟩

lemma *word-mod-2p-is-mask'*:
c = 2 ^ n ==> c > 0 ==> x mod c = (x :: 'a :: len word) AND mask n

⟨*proof*⟩

lemmas *word-mod-2p-is-mask* = *refl* [*THEN word-mod-2p-is-mask'*]

lemma *mask-eqs*:
(a AND mask n) + b AND mask n = a + b AND mask n
a + (b AND mask n) AND mask n = a + b AND mask n
(a AND mask n) − b AND mask n = a − b AND mask n
a − (b AND mask n) AND mask n = a − b AND mask n
*a * (b AND mask n) AND mask n = a * b AND mask n*
*(b AND mask n) * a AND mask n = b * a AND mask n*
(a AND mask n) + (b AND mask n) AND mask n = a + b AND mask n
(a AND mask n) − (b AND mask n) AND mask n = a − b AND mask n
*(a AND mask n) * (b AND mask n) AND mask n = a * b AND mask n*
− (a AND mask n) AND mask n = − a AND mask n
word-succ (a AND mask n) AND mask n = word-succ a AND mask n
word-pred (a AND mask n) AND mask n = word-pred a AND mask n
⟨*proof*⟩

lemma *mask-power-eq*:
(x AND mask n) ^ k AND mask n = x ^ k AND mask n
⟨*proof*⟩

13.1.3 Revcast

lemmas *revcast-def'* = *revcast-def* [*simplified*]
lemmas *revcast-def''* = *revcast-def'* [*simplified word-size*]
lemmas *revcast-no-def* [*simp*] =
revcast-def' [**where** *w=number-of w*, *unfolded word-size*, *standard*]

lemma *to-bl-revcast*:
to-bl (revcast w :: 'a :: len0 word) =
takefill False (len-of TYPE ('a)) (to-bl w)
⟨*proof*⟩

lemma *revcast-rev-ucast'*:
cs = [rc, uc] ==> rc = revcast (word-reverse w) ==> uc = ucast w ==>
rc = word-reverse uc
⟨*proof*⟩

lemmas *revcast-rev-ucast* = *revcast-rev-ucast'* [*OF refl refl refl*]

lemmas *revcast-ucast* = *revcast-rev-ucast*
[**where** *w = word-reverse w*, *simplified word-rev-rev*, *standard*]

lemmas *ucast-revcast* = *revcast-rev-ucast* [*THEN word-rev-gal'*, *standard*]
lemmas *ucast-rev-revcast* = *revcast-ucast* [*THEN word-rev-gal'*, *standard*]

— linking revcast and cast via shift

lemmas *wsst-TYs = source-size target-size word-size*

lemma *revcast-down-uu'*:
rc = revcast ==> source-size rc = target-size rc + n ==>
rc (w :: 'a :: len word) = ucast (w >> n)
⟨*proof*⟩

lemma *revcast-down-us'*:
rc = revcast ==> source-size rc = target-size rc + n ==>
rc (w :: 'a :: len word) = ucast (w >>> n)
⟨*proof*⟩

lemma *revcast-down-su'*:
rc = revcast ==> source-size rc = target-size rc + n ==>
rc (w :: 'a :: len word) = scast (w >> n)
⟨*proof*⟩

lemma *revcast-down-ss'*:
rc = revcast ==> source-size rc = target-size rc + n ==>
rc (w :: 'a :: len word) = scast (w >>> n)
⟨*proof*⟩

lemmas *revcast-down-uu = refl [THEN revcast-down-uu']*
lemmas *revcast-down-us = refl [THEN revcast-down-us']*
lemmas *revcast-down-su = refl [THEN revcast-down-su']*
lemmas *revcast-down-ss = refl [THEN revcast-down-ss']*

lemma *cast-down-rev*:
uc = ucast ==> source-size uc = target-size uc + n ==>
uc w = revcast ((w :: 'a :: len word) << n)
⟨*proof*⟩

lemma *revcast-up'*:
rc = revcast ==> source-size rc + n = target-size rc ==>
rc w = (ucast w :: 'a :: len word) << n
⟨*proof*⟩

lemmas *revcast-up = refl [THEN revcast-up']*

lemmas *rc1 = revcast-up [THEN*
revcast-rev-ucast [symmetric, THEN trans, THEN word-rev-gal, symmetric]]
lemmas *rc2 = revcast-down-uu [THEN*
revcast-rev-ucast [symmetric, THEN trans, THEN word-rev-gal, symmetric]]

lemmas *ucast-up =*
rc1 [simplified rev-shiftr [symmetric] revcast-ucast [symmetric]]
lemmas *ucast-down =*

rc2 [*simplified rev-shiftr revcast-ucast* [*symmetric*]]

13.1.4 Slices

lemmas *slice1-no-bin* [*simp*] =
slice1-def [**where** *w*=*number-of w*, *unfolded to-bl-no-bin*, *standard*]

lemmas *slice-no-bin* [*simp*] =
trans [*OF slice-def* [*THEN meta-eq-to-obj-eq*]
slice1-no-bin [*THEN meta-eq-to-obj-eq*],
unfolded word-size, *standard*]

lemma *slice1-0* [*simp*] : *slice1 n 0 = 0*
⟨*proof*⟩

lemma *slice-0* [*simp*] : *slice n 0 = 0*
⟨*proof*⟩

lemma *slice-take'*: *slice n w = of-bl* (*take* (*size w − n*) (*to-bl w*))
⟨*proof*⟩

lemmas *slice-take = slice-take'* [*unfolded word-size*]

— shiftr to a word of the same size is just slice, slice is just shiftr then ucast
lemmas *shiftr-slice = trans*
[*OF shiftr-bl* [*THEN meta-eq-to-obj-eq*] *slice-take* [*symmetric*], *standard*]

lemma *slice-shiftr*: *slice n w = ucast* (*w >> n*)
⟨*proof*⟩

lemma *nth-slice*:
(*slice n w* :: *'a* :: *len0 word*) !! *m* =
(*w* !! (*m* + *n*) & *m* < *len-of TYPE* (*'a*))
⟨*proof*⟩

lemma *slice1-down-alt'*:
sl = slice1 n w ==> fs = size sl ==> fs + k = n ==>
to-bl sl = takefill False fs (*drop k* (*to-bl w*))
⟨*proof*⟩

lemma *slice1-up-alt'*:
sl = slice1 n w ==> fs = size sl ==> fs = n + k ==>
to-bl sl = takefill False fs (*replicate k False* @ (*to-bl w*))
⟨*proof*⟩

lemmas *sd1 = slice1-down-alt'* [*OF refl refl*, *unfolded word-size*]
lemmas *su1 = slice1-up-alt'* [*OF refl refl*, *unfolded word-size*]
lemmas *slice1-down-alt = le-add-diff-inverse* [*THEN sd1*]
lemmas *slice1-up-alts* =

le-add-diff-inverse [*symmetric*, *THEN su1*]
le-add-diff-inverse2 [*symmetric*, *THEN su1*]

lemma *ucast-slice1*: *ucast w = slice1 (size w) w*
⟨*proof*⟩

lemma *ucast-slice*: *ucast w = slice 0 w*
⟨*proof*⟩

lemmas *slice-id = trans* [*OF ucast-slice* [*symmetric*] *ucast-id*]

lemma *revcast-slice1'*:
rc = revcast w ==> slice1 (size rc) w = rc
⟨*proof*⟩

lemmas *revcast-slice1 = refl* [*THEN revcast-slice1'*]

lemma *slice1-tf-tf'*:
to-bl (slice1 n w :: 'a :: len0 word) =
rev (takefill False (len-of TYPE('a)) (rev (takefill False n (to-bl w))))
⟨*proof*⟩

lemmas *slice1-tf-tf = slice1-tf-tf'*
[*THEN word-bl.Rep-inverse'*, *symmetric*, *standard*]

lemma *rev-slice1*:
n + k = len-of TYPE('a) + len-of TYPE('b) ⟹
slice1 n (word-reverse w :: 'b :: len0 word) =
word-reverse (slice1 k w :: 'a :: len0 word)
⟨*proof*⟩

lemma *rev-slice'*:
res = slice n (word-reverse w) ==> n + k + size res = size w ==>
res = word-reverse (slice k w)
⟨*proof*⟩

lemmas *rev-slice = refl* [*THEN rev-slice'*, *unfolded word-size*]

lemmas *sym-notr =*
not-iff [*THEN iffD2*, *THEN not-sym*, *THEN not-iff* [*THEN iffD1*]]

— problem posed by TPHOLs referee: criterion for overflow of addition of signed integers

lemma *sofl-test*:
(sint (x :: 'a :: len word) + sint y = sint (x + y)) =
((((x+y) XOR x) AND ((x+y) XOR y)) >> (size x − 1) = 0)
⟨*proof*⟩

13.2 Split and cat

lemmas *word-split-bin'* = *word-split-def* [*THEN meta-eq-to-obj-eq*, *standard*]
lemmas *word-cat-bin'* = *word-cat-def* [*THEN meta-eq-to-obj-eq*, *standard*]

lemma *word-rsplit-no*:
(*word-rsplit* (*number-of bin* :: *'b* :: *len0 word*) :: *'a word list*) =
map number-of (*bin-rsplit* (*len-of TYPE*(*'a* :: *len*))
(*len-of TYPE*(*'b*), *bintrunc* (*len-of TYPE*(*'b*)) *bin*))
⟨*proof*⟩

lemmas *word-rsplit-no-cl* [*simp*] = *word-rsplit-no*
[*unfolded bin-rsplitl-def bin-rsplit-l* [*symmetric*]]

lemma *test-bit-cat*:
wc = *word-cat a b* ==> *wc* !! *n* = (*n* < *size wc* &
(*if n* < *size b then b* !! *n else a* !! (*n* − *size b*)))
⟨*proof*⟩

lemma *word-cat-bl*: *word-cat a b* = *of-bl* (*to-bl a* @ *to-bl b*)
⟨*proof*⟩

lemma *of-bl-append*:
(*of-bl* (*xs* @ *ys*) :: *'a* :: *len word*) = *of-bl xs* ∗ *2^*(*length ys*) + *of-bl ys*
⟨*proof*⟩

lemma *of-bl-False* [*simp*]:
of-bl (*False*#*xs*) = *of-bl xs*
⟨*proof*⟩

lemma *of-bl-True*:
(*of-bl* (*True*#*xs*)::*'a*::*len word*) = *2^length xs* + *of-bl xs*
⟨*proof*⟩

lemma *of-bl-Cons*:
of-bl (*x*#*xs*) = *of-bool x* ∗ *2^length xs* + *of-bl xs*
⟨*proof*⟩

lemma *split-uint-lem*: *bin-split n* (*uint* (*w* :: *'a* :: *len0 word*)) = (*a*, *b*) ==>
a = *bintrunc* (*len-of TYPE*(*'a*) − *n*) *a* & *b* = *bintrunc* (*len-of TYPE*(*'a*)) *b*
⟨*proof*⟩

lemma *word-split-bl'*:
std = *size c* − *size b* ==> (*word-split c* = (*a*, *b*)) ==>
(*a* = *of-bl* (*take std* (*to-bl c*)) & *b* = *of-bl* (*drop std* (*to-bl c*)))
⟨*proof*⟩

lemma *word-split-bl*: *std* = *size c* − *size b* ==>
(*a* = *of-bl* (*take std* (*to-bl c*)) & *b* = *of-bl* (*drop std* (*to-bl c*))) <−>
word-split c = (*a*, *b*)

⟨*proof*⟩

lemma *word-split-bl-eq*:
(*word-split* (*c*::'*a*::*len word*) :: ('*c* :: *len0 word* ∗ '*d* :: *len0 word*)) =
(*of-bl* (*take* (*len-of TYPE*('*a*::*len*) − *len-of TYPE*('*d*::*len0*)) (*to-bl c*)),
of-bl (*drop* (*len-of TYPE*('*a*) − *len-of TYPE*('*d*)) (*to-bl c*)))
⟨*proof*⟩
lemma *test-bit-split'*:
word-split c = (*a*, *b*) −−> (*ALL n m*. *b* !! *n* = (*n* < *size b* & *c* !! *n*) &
a !! *m* = (*m* < *size a* & *c* !! (*m* + *size b*)))
⟨*proof*⟩

lemmas *test-bit-split* =
test-bit-split' [*THEN mp*, *simplified all-simps*, *standard*]

lemma *test-bit-split-eq*: *word-split c* = (*a*, *b*) <−>
((*ALL n*::*nat*. *b* !! *n* = (*n* < *size b* & *c* !! *n*)) &
(*ALL m*::*nat*. *a* !! *m* = (*m* < *size a* & *c* !! (*m* + *size b*))))
⟨*proof*⟩

lemma *word-cat-id*: *word-cat a b* = *b*
⟨*proof*⟩
lemma *word-cat-hom*:
len-of TYPE('*a*::*len0*) <= *len-of TYPE*('*b*::*len0*) + *len-of TYPE* ('*c*::*len0*)
==>
(*word-cat* (*word-of-int w* :: '*b word*) (*b* :: '*c word*) :: '*a word*) =
word-of-int (*bin-cat w* (*size b*) (*uint b*))
⟨*proof*⟩

lemma *word-cat-split-alt*:
size w <= *size u* + *size v* ==> *word-split w* = (*u*, *v*) ==> *word-cat u v* = *w*
⟨*proof*⟩

lemmas *word-cat-split-size* =
sym [*THEN* [*2*] *word-cat-split-alt* [*symmetric*], *standard*]

13.2.1 Split and slice

lemma *split-slices*:
word-split w = (*u*, *v*) ==> *u* = *slice* (*size v*) *w* & *v* = *slice 0 w*
⟨*proof*⟩

lemma *slice-cat1'*:
wc = *word-cat a b* ==> *size wc* >= *size a* + *size b* ==> *slice* (*size b*) *wc* = *a*
⟨*proof*⟩

lemmas *slice-cat1* = *refl* [*THEN slice-cat1'*]
lemmas *slice-cat2* = *trans* [*OF slice-id word-cat-id*]

lemma *cat-slices*:
a = slice n c ==> b = slice 0 c ==> n = size b ==>
size a + size b >= size c ==> word-cat a b = c
⟨*proof*⟩

lemma *word-split-cat-alt*:
w = word-cat u v ==> size u + size v <= size w ==> word-split w = (u, v)
⟨*proof*⟩

lemmas *word-cat-bl-no-bin* [*simp*] =
word-cat-bl [**where** *a=number-of a*
and *b=number-of b*,
unfolded to-bl-no-bin, *standard*]

lemmas *word-split-bl-no-bin* [*simp*] =
word-split-bl-eq [**where** *c=number-of c*, *unfolded to-bl-no-bin*, *standard*]

— this odd result arises from the fact that the statement of the result implies that the decoded words are of the same type, and therefore of the same length, as the original word

lemma *word-rsplit-same*: *word-rsplit w = [w]*
⟨*proof*⟩

lemma *word-rsplit-empty-iff-size*:
(word-rsplit w = []) = (size w = 0)
⟨*proof*⟩

lemma *test-bit-rsplit*:
sw = word-rsplit w ==> m < size (hd sw :: 'a :: len word) ==>
*k < length sw ==> (rev sw ! k) !! m = (w !! (k * size (hd sw) + m))*
⟨*proof*⟩

lemma *word-rcat-bl*: *word-rcat wl == of-bl (concat (map to-bl wl))*
⟨*proof*⟩

lemma *size-rcat-lem'*:
*size (concat (map to-bl wl)) = length wl * size (hd wl)*
⟨*proof*⟩

lemmas *size-rcat-lem = size-rcat-lem'* [*unfolded word-size*]

lemmas *td-gal-lt-len = len-gt-0* [*THEN td-gal-lt*, *standard*]

lemma *nth-rcat-lem'* [*rule-format*] :
*sw = size (hd wl :: 'a :: len word) ==> (ALL n. n < size wl * sw -->*
rev (concat (map to-bl wl)) ! n =
rev (to-bl (rev wl ! (n div sw))) ! (n mod sw))
⟨*proof*⟩

lemmas *nth-rcat-lem* = *refl* [*THEN nth-rcat-lem'*, *unfolded word-size*]

lemma *test-bit-rcat*:
sw = *size* (*hd wl* :: *'a* :: *len word*) ==> *rc* = *word-rcat wl* ==> *rc* !! *n* =
(*n* < *size rc* & *n div sw* < *size wl* & (*rev wl*) ! (*n div sw*) !! (*n mod sw*))
⟨*proof*⟩

lemma *foldl-eq-foldr* [*rule-format*] :
ALL x. *foldl op* + *x xs* = *foldr op* + (*x* # *xs*) (*0* :: *'a* :: *comm-monoid-add*)
⟨*proof*⟩

lemmas *test-bit-cong* = *arg-cong* [**where** *f* = *test-bit*, *THEN fun-cong*]

lemmas *test-bit-rsplit-alt* =
trans [*OF nth-rev-alt* [*THEN test-bit-cong*]
test-bit-rsplit [*OF refl asm-rl diff-Suc-less*]]

— lazy way of expressing that u and v, and su and sv, have same types
lemma *word-rsplit-len-indep'*:
[*u*,*v*] = *p* ==> [*su*,*sv*] = *q* ==> *word-rsplit u* = *su* ==>
word-rsplit v = *sv* ==> *length su* = *length sv*
⟨*proof*⟩

lemmas *word-rsplit-len-indep* = *word-rsplit-len-indep'* [*OF refl refl refl refl*]

lemma *length-word-rsplit-size*:
n = *len-of TYPE* (*'a* :: *len*) ==>
(*length* (*word-rsplit w* :: *'a word list*) <= *m*) = (*size w* <= *m* * *n*)
⟨*proof*⟩

lemmas *length-word-rsplit-lt-size* =
length-word-rsplit-size [*unfolded Not-eq-iff linorder-not-less* [*symmetric*]]

lemma *length-word-rsplit-exp-size*:
n = *len-of TYPE* (*'a* :: *len*) ==>
length (*word-rsplit w* :: *'a word list*) = (*size w* + *n* − *1*) *div n*
⟨*proof*⟩

lemma *length-word-rsplit-even-size*:
n = *len-of TYPE* (*'a* :: *len*) ==> *size w* = *m* * *n* ==>
length (*word-rsplit w* :: *'a word list*) = *m*
⟨*proof*⟩

lemmas *length-word-rsplit-exp-size'* = *refl* [*THEN length-word-rsplit-exp-size*]

lemmas *tdle* = *iffD2* [*OF split-div-lemma refl*, *THEN conjunct1*]
lemmas *dtle* = *xtr4* [*OF tdle mult-commute*]

lemma *word-rcat-rsplit*: *word-rcat* (*word-rsplit w*) = *w*
⟨*proof*⟩

lemma *size-word-rsplit-rcat-size'*:
word-rcat (*ws* :: 'a :: *len word list*) = *frcw* ==>
size frcw = *length ws* * *len-of TYPE* ('a) ==>
size (*hd* [*word-rsplit frcw*, *ws*]) = *size ws*
⟨*proof*⟩

lemmas *size-word-rsplit-rcat-size* =
size-word-rsplit-rcat-size' [*simplified*]

lemma *msrevs*:
fixes *n*::*nat*
shows 0 < *n* ⟹ (*k* * *n* + *m*) *div n* = *m div n* + *k*
and (*k* * *n* + *m*) *mod n* = *m mod n*
⟨*proof*⟩

lemma *word-rsplit-rcat-size'*:
word-rcat (*ws* :: 'a :: *len word list*) = *frcw* ==>
size frcw = *length ws* * *len-of TYPE* ('a) ==> *word-rsplit frcw* = *ws*
⟨*proof*⟩

lemmas *word-rsplit-rcat-size* = *refl* [*THEN word-rsplit-rcat-size'*]

13.3 Rotation

lemmas *rotater-0'* [*simp*] = *rotater-def* [**where** *n* = 0, *simplified*]

lemmas *word-rot-defs* = *word-roti-def word-rotr-def word-rotl-def*

lemma *rotate-eq-mod*:
m mod length xs = *n mod length xs* ==> *rotate m xs* = *rotate n xs*
⟨*proof*⟩

lemmas *rotate-eqs* [*standard*] =
trans [*OF rotate0* [*THEN fun-cong*] *id-apply*]
rotate-rotate [*symmetric*]
rotate-id
rotate-conv-mod
rotate-eq-mod

13.3.1 Rotation of list to right

lemma *rotate1-rl'*: *rotater1* (*l* @ [*a*]) = *a* # *l*
⟨*proof*⟩

lemma *rotate1-rl* [*simp*] : *rotater1* (*rotate1 l*) = *l*
⟨*proof*⟩

lemma *rotate1-lr* [*simp*] : *rotate1* (*rotater1 l*) = *l*
⟨*proof*⟩

lemma *rotater1-rev'*: *rotater1* (*rev xs*) = *rev* (*rotate1 xs*)
⟨*proof*⟩

lemma *rotater-rev'*: *rotater n* (*rev xs*) = *rev* (*rotate n xs*)
⟨*proof*⟩

lemmas *rotater-rev* = *rotater-rev'* [**where** *xs* = *rev ys*, *simplified*, *standard*]

lemma *rotater-drop-take*:
rotater n xs =
drop (*length xs* − *n mod length xs*) *xs* @
take (*length xs* − *n mod length xs*) *xs*
⟨*proof*⟩

lemma *rotater-Suc* [*simp*] :
rotater (*Suc n*) *xs* = *rotater1* (*rotater n xs*)
⟨*proof*⟩

lemma *rotate-inv-plus* [*rule-format*] :
ALL k. *k* = *m* + *n* −−> *rotater k* (*rotate n xs*) = *rotater m xs* &
rotate k (*rotater n xs*) = *rotate m xs* &
rotater n (*rotate k xs*) = *rotate m xs* &
rotate n (*rotater k xs*) = *rotater m xs*
⟨*proof*⟩

lemmas *rotate-inv-rel* = *le-add-diff-inverse2* [*symmetric*, *THEN rotate-inv-plus*]

lemmas *rotate-inv-eq* = *order-refl* [*THEN rotate-inv-rel*, *simplified*]

lemmas *rotate-lr* [*simp*] = *rotate-inv-eq* [*THEN conjunct1*, *standard*]
lemmas *rotate-rl* [*simp*] =
rotate-inv-eq [*THEN conjunct2*, *THEN conjunct1*, *standard*]

lemma *rotate-gal*: (*rotater n xs* = *ys*) = (*rotate n ys* = *xs*)
⟨*proof*⟩

lemma *rotate-gal'*: (*ys* = *rotater n xs*) = (*xs* = *rotate n ys*)
⟨*proof*⟩

lemma *length-rotater* [*simp*]:
length (*rotater n xs*) = *length xs*
⟨*proof*⟩

lemmas *rrs0* = *rotate-eqs* [*THEN restrict-to-left*,
simplified rotate-gal [*symmetric*] *rotate-gal'* [*symmetric*], *standard*]

lemmas *rrs1 = rrs0 [THEN refl [THEN rev-iffD1]]*
lemmas *rotater-eqs = rrs1 [simplified length-rotater, standard]*
lemmas *rotater-0 = rotater-eqs (1)*
lemmas *rotater-add = rotater-eqs (2)*

13.3.2 map, map2, commuting with rotate(r)

lemma *last-map*: *xs ~= [] ==> last (map f xs) = f (last xs)*
⟨*proof*⟩

lemma *butlast-map*:
xs ~= [] ==> butlast (map f xs) = map f (butlast xs)
⟨*proof*⟩

lemma *rotater1-map*: *rotater1 (map f xs) = map f (rotater1 xs)*
⟨*proof*⟩

lemma *rotater-map*:
rotater n (map f xs) = map f (rotater n xs)
⟨*proof*⟩

lemma *but-last-zip [rule-format]* :
ALL ys. length xs = length ys --> xs ~= [] -->
last (zip xs ys) = (last xs, last ys) &
butlast (zip xs ys) = zip (butlast xs) (butlast ys)
⟨*proof*⟩

lemma *but-last-map2 [rule-format]* :
ALL ys. length xs = length ys --> xs ~= [] -->
last (map2 f xs ys) = f (last xs) (last ys) &
butlast (map2 f xs ys) = map2 f (butlast xs) (butlast ys)
⟨*proof*⟩

lemma *rotater1-zip*:
length xs = length ys ==>
rotater1 (zip xs ys) = zip (rotater1 xs) (rotater1 ys)
⟨*proof*⟩

lemma *rotater1-map2*:
length xs = length ys ==>
rotater1 (map2 f xs ys) = map2 f (rotater1 xs) (rotater1 ys)
⟨*proof*⟩

lemmas *lrth =*
box-equals [OF asm-rl length-rotater [symmetric]
length-rotater [symmetric],
THEN rotater1-map2]

lemma *rotater-map2*:

length xs = length ys ==>
rotater n (map2 f xs ys) = map2 f (rotater n xs) (rotater n ys)
⟨*proof*⟩

lemma *rotate1-map2*:
length xs = length ys ==>
rotate1 (map2 f xs ys) = map2 f (rotate1 xs) (rotate1 ys)
⟨*proof*⟩

lemmas *lth = box-equals* [*OF asm-rl length-rotate* [*symmetric*]
length-rotate [*symmetric*], *THEN rotate1-map2*]

lemma *rotate-map2*:
length xs = length ys ==>
rotate n (map2 f xs ys) = map2 f (rotate n xs) (rotate n ys)
⟨*proof*⟩

lemma *to-bl-rotl*:
to-bl (word-rotl n w) = rotate n (to-bl w)
⟨*proof*⟩

lemmas *blrs0 = rotate-eqs* [*THEN to-bl-rotl* [*THEN trans*]]

lemmas *word-rotl-eqs =*
blrs0 [*simplified word-bl.Rep' word-bl.Rep-inject to-bl-rotl* [*symmetric*]]

lemma *to-bl-rotr*:
to-bl (word-rotr n w) = rotater n (to-bl w)
⟨*proof*⟩

lemmas *brrs0 = rotater-eqs* [*THEN to-bl-rotr* [*THEN trans*]]

lemmas *word-rotr-eqs =*
brrs0 [*simplified word-bl.Rep' word-bl.Rep-inject to-bl-rotr* [*symmetric*]]

declare *word-rotr-eqs* (*1*) [*simp*]
declare *word-rotl-eqs* (*1*) [*simp*]

lemma
word-rot-rl [*simp*]:
word-rotl k (word-rotr k v) = v **and**
word-rot-lr [*simp*]:
word-rotr k (word-rotl k v) = v
⟨*proof*⟩

lemma
word-rot-gal:
(word-rotr n v = w) = (word-rotl n w = v) **and**
word-rot-gal':

$(w = \textit{word-rotr}\ n\ v) = (v = \textit{word-rotl}\ n\ w)$
⟨*proof*⟩

lemma *word-rotr-rev*:
$\textit{word-rotr}\ n\ w = \textit{word-reverse}\ (\textit{word-rotl}\ n\ (\textit{word-reverse}\ w))$
⟨*proof*⟩

lemma *word-roti-0* [*simp*]: $\textit{word-roti}\ 0\ w = w$
⟨*proof*⟩

lemmas *abl-cong* = *arg-cong* [**where** *f* = *of-bl*]

lemma *word-roti-add*:
$\textit{word-roti}\ (m + n)\ w = \textit{word-roti}\ m\ (\textit{word-roti}\ n\ w)$
⟨*proof*⟩

lemma *word-roti-conv-mod'*: $\textit{word-roti}\ n\ w = \textit{word-roti}\ (n\ \textit{mod}\ \textit{int}\ (\textit{size}\ w))\ w$
⟨*proof*⟩

lemmas *word-roti-conv-mod* = *word-roti-conv-mod'* [*unfolded word-size*]

13.3.3 Word rotation commutes with bit-wise operations

locale *word-rotate*

context *word-rotate*
begin

lemmas *word-rot-defs'* = *to-bl-rotl to-bl-rotr*

lemmas *blwl-syms* [*symmetric*] = *bl-word-not bl-word-and bl-word-or bl-word-xor*

lemmas *lbl-lbl* = *trans* [*OF word-bl.Rep' word-bl.Rep'* [*symmetric*]]

lemmas *ths-map2* [*OF lbl-lbl*] = *rotate-map2 rotater-map2*

lemmas *ths-map* [**where** *xs* = *to-bl v*, *standard*] = *rotate-map rotater-map*

lemmas *th1s* [*simplified word-rot-defs'* [*symmetric*]] = *ths-map2 ths-map*

lemma *word-rot-logs*:
$\textit{word-rotl}\ n\ (\textit{NOT}\ v) = \textit{NOT word-rotl}\ n\ v$
$\textit{word-rotr}\ n\ (\textit{NOT}\ v) = \textit{NOT word-rotr}\ n\ v$
$\textit{word-rotl}\ n\ (x\ \textit{AND}\ y) = \textit{word-rotl}\ n\ x\ \textit{AND word-rotl}\ n\ y$
$\textit{word-rotr}\ n\ (x\ \textit{AND}\ y) = \textit{word-rotr}\ n\ x\ \textit{AND word-rotr}\ n\ y$
$\textit{word-rotl}\ n\ (x\ \textit{OR}\ y) = \textit{word-rotl}\ n\ x\ \textit{OR word-rotl}\ n\ y$
$\textit{word-rotr}\ n\ (x\ \textit{OR}\ y) = \textit{word-rotr}\ n\ x\ \textit{OR word-rotr}\ n\ y$
$\textit{word-rotl}\ n\ (x\ \textit{XOR}\ y) = \textit{word-rotl}\ n\ x\ \textit{XOR word-rotl}\ n\ y$
$\textit{word-rotr}\ n\ (x\ \textit{XOR}\ y) = \textit{word-rotr}\ n\ x\ \textit{XOR word-rotr}\ n\ y$

⟨*proof*⟩
end

lemmas *word-rot-logs* = *word-rotate.word-rot-logs*

lemmas *bl-word-rotl-dt* = *trans* [*OF to-bl-rotl rotate-drop-take*,
simplified word-bl.Rep', *standard*]

lemmas *bl-word-rotr-dt* = *trans* [*OF to-bl-rotr rotater-drop-take*,
simplified word-bl.Rep', *standard*]

lemma *bl-word-roti-dt'*:
n = *nat* ((− *i*) *mod int* (*size* (*w* :: 'a :: *len word*))) ==>
to-bl (*word-roti i w*) = *drop n* (*to-bl w*) @ *take n* (*to-bl w*)
⟨*proof*⟩

lemmas *bl-word-roti-dt* = *bl-word-roti-dt'* [*unfolded word-size*]

lemmas *word-rotl-dt* = *bl-word-rotl-dt*
[*THEN word-bl.Rep-inverse'* [*symmetric*], *standard*]
lemmas *word-rotr-dt* = *bl-word-rotr-dt*
[*THEN word-bl.Rep-inverse'* [*symmetric*], *standard*]
lemmas *word-roti-dt* = *bl-word-roti-dt*
[*THEN word-bl.Rep-inverse'* [*symmetric*], *standard*]

lemma *word-rotx-0* [*simp*] : *word-rotr i 0* = *0* & *word-rotl i 0* = *0*
⟨*proof*⟩

lemma *word-roti-0'* [*simp*] : *word-roti n 0* = *0*
⟨*proof*⟩

lemmas *word-rotr-dt-no-bin'* [*simp*] =
word-rotr-dt [**where** *w*=*number-of w*, *unfolded to-bl-no-bin*, *standard*]

lemmas *word-rotl-dt-no-bin'* [*simp*] =
word-rotl-dt [**where** *w*=*number-of w*, *unfolded to-bl-no-bin*, *standard*]

declare *word-roti-def* [*simp*]

end

14 Boolean-Algebra: Boolean Algebras

theory *Boolean-Algebra*
imports *Main*
begin

locale *boolean* =
  fixes *conj* :: *'a* ⇒ *'a* ⇒ *'a* (**infixr** ⊓ *70*)
  fixes *disj* :: *'a* ⇒ *'a* ⇒ *'a* (**infixr** ⊔ *65*)
  fixes *compl* :: *'a* ⇒ *'a* (∼ - [*81*] *80*)
  fixes *zero* :: *'a* (**0**)
  fixes *one* :: *'a* (**1**)
  assumes *conj-assoc*: (*x* ⊓ *y*) ⊓ *z* = *x* ⊓ (*y* ⊓ *z*)
  assumes *disj-assoc*: (*x* ⊔ *y*) ⊔ *z* = *x* ⊔ (*y* ⊔ *z*)
  assumes *conj-commute*: *x* ⊓ *y* = *y* ⊓ *x*
  assumes *disj-commute*: *x* ⊔ *y* = *y* ⊔ *x*
  assumes *conj-disj-distrib*: *x* ⊓ (*y* ⊔ *z*) = (*x* ⊓ *y*) ⊔ (*x* ⊓ *z*)
  assumes *disj-conj-distrib*: *x* ⊔ (*y* ⊓ *z*) = (*x* ⊔ *y*) ⊓ (*x* ⊔ *z*)
  assumes *conj-one-right* [*simp*]: *x* ⊓ **1** = *x*
  assumes *disj-zero-right* [*simp*]: *x* ⊔ **0** = *x*
  assumes *conj-cancel-right* [*simp*]: *x* ⊓ ∼ *x* = **0**
  assumes *disj-cancel-right* [*simp*]: *x* ⊔ ∼ *x* = **1**

sublocale *boolean* < *conj*!: *abel-semigroup conj* ⟨*proof*⟩

sublocale *boolean* < *disj*!: *abel-semigroup disj* ⟨*proof*⟩

context *boolean*
begin

lemmas *conj-left-commute* = *conj.left-commute*

lemmas *disj-left-commute* = *disj.left-commute*

lemmas *conj-ac* = *conj.assoc conj.commute conj.left-commute*
lemmas *disj-ac* = *disj.assoc disj.commute disj.left-commute*

lemma *dual*: *boolean disj conj compl one zero*
⟨*proof*⟩

14.1 Complement

lemma *complement-unique*:
  assumes *1*: *a* ⊓ *x* = **0**
  assumes *2*: *a* ⊔ *x* = **1**
  assumes *3*: *a* ⊓ *y* = **0**
  assumes *4*: *a* ⊔ *y* = **1**
  shows *x* = *y*
⟨*proof*⟩

lemma *compl-unique*: ⟦*x* ⊓ *y* = **0**; *x* ⊔ *y* = **1**⟧ ⟹ ∼ *x* = *y*
⟨*proof*⟩

lemma *double-compl* [*simp*]: ∼ (∼ *x*) = *x*
⟨*proof*⟩

lemma *compl-eq-compl-iff* [*simp*]: $(\sim x = \sim y) = (x = y)$
⟨*proof*⟩

14.2 Conjunction

lemma *conj-absorb* [*simp*]: $x \sqcap x = x$
⟨*proof*⟩

lemma *conj-zero-right* [*simp*]: $x \sqcap \mathbf{0} = \mathbf{0}$
⟨*proof*⟩

lemma *compl-one* [*simp*]: $\sim \mathbf{1} = \mathbf{0}$
⟨*proof*⟩

lemma *conj-zero-left* [*simp*]: $\mathbf{0} \sqcap x = \mathbf{0}$
⟨*proof*⟩

lemma *conj-one-left* [*simp*]: $\mathbf{1} \sqcap x = x$
⟨*proof*⟩

lemma *conj-cancel-left* [*simp*]: $\sim x \sqcap x = \mathbf{0}$
⟨*proof*⟩

lemma *conj-left-absorb* [*simp*]: $x \sqcap (x \sqcap y) = x \sqcap y$
⟨*proof*⟩

lemma *conj-disj-distrib2*:
$(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
⟨*proof*⟩

lemmas *conj-disj-distribs* =
conj-disj-distrib conj-disj-distrib2

14.3 Disjunction

lemma *disj-absorb* [*simp*]: $x \sqcup x = x$
⟨*proof*⟩

lemma *disj-one-right* [*simp*]: $x \sqcup \mathbf{1} = \mathbf{1}$
⟨*proof*⟩

lemma *compl-zero* [*simp*]: $\sim \mathbf{0} = \mathbf{1}$
⟨*proof*⟩

lemma *disj-zero-left* [*simp*]: $\mathbf{0} \sqcup x = x$
⟨*proof*⟩

lemma *disj-one-left* [*simp*]: $\mathbf{1} \sqcup x = \mathbf{1}$
⟨*proof*⟩

lemma *disj-cancel-left* [*simp*]: $\sim x \sqcup x = \mathbf{1}$
⟨*proof*⟩

lemma *disj-left-absorb* [*simp*]: $x \sqcup (x \sqcup y) = x \sqcup y$
⟨*proof*⟩

lemma *disj-conj-distrib2*:
$(y \sqcap z) \sqcup x = (y \sqcup x) \sqcap (z \sqcup x)$
⟨*proof*⟩

lemmas *disj-conj-distribs* =
disj-conj-distrib disj-conj-distrib2

14.4 De Morgan's Laws

lemma *de-Morgan-conj* [*simp*]: $\sim (x \sqcap y) = \sim x \sqcup \sim y$
⟨*proof*⟩

lemma *de-Morgan-disj* [*simp*]: $\sim (x \sqcup y) = \sim x \sqcap \sim y$
⟨*proof*⟩

end

14.5 Symmetric Difference

locale *boolean-xor* = *boolean* +
fixes *xor* :: *'a* => *'a* => *'a* (**infixr** $\oplus$ *65*)
assumes *xor-def*: $x \oplus y = (x \sqcap \sim y) \sqcup (\sim x \sqcap y)$

sublocale *boolean-xor* < *xor*!: *abel-semigroup xor* ⟨*proof*⟩

context *boolean-xor*
begin

lemmas *xor-assoc* = *xor.assoc*
lemmas *xor-commute* = *xor.commute*
lemmas *xor-left-commute* = *xor.left-commute*

lemmas *xor-ac* = *xor.assoc xor.commute xor.left-commute*

lemma *xor-def2*:
$x \oplus y = (x \sqcup y) \sqcap (\sim x \sqcup \sim y)$
⟨*proof*⟩

lemma *xor-zero-right* [*simp*]: $x \oplus \mathbf{0} = x$
⟨*proof*⟩

lemma *xor-zero-left* [*simp*]: $\mathbf{0} \oplus x = x$
⟨*proof*⟩

lemma *xor-one-right* [*simp*]: $x \oplus \mathbf{1} = \sim x$
⟨*proof*⟩

lemma *xor-one-left* [*simp*]: $\mathbf{1} \oplus x = \sim x$
⟨*proof*⟩

lemma *xor-self* [*simp*]: $x \oplus x = \mathbf{0}$
⟨*proof*⟩

lemma *xor-left-self* [*simp*]: $x \oplus (x \oplus y) = y$
⟨*proof*⟩

lemma *xor-compl-left* [*simp*]: $\sim x \oplus y = \sim (x \oplus y)$
⟨*proof*⟩

lemma *xor-compl-right* [*simp*]: $x \oplus \sim y = \sim (x \oplus y)$
⟨*proof*⟩

lemma *xor-cancel-right*: $x \oplus \sim x = \mathbf{1}$
⟨*proof*⟩

lemma *xor-cancel-left*: $\sim x \oplus x = \mathbf{1}$
⟨*proof*⟩

lemma *conj-xor-distrib*: $x \sqcap (y \oplus z) = (x \sqcap y) \oplus (x \sqcap z)$
⟨*proof*⟩

lemma *conj-xor-distrib2*:
$(y \oplus z) \sqcap x = (y \sqcap x) \oplus (z \sqcap x)$
⟨*proof*⟩

lemmas *conj-xor-distribs* =
conj-xor-distrib conj-xor-distrib2

end

end

15 WordGenLib: Miscellaneous Library for Words

theory *WordGenLib*
imports *WordShift Boolean-Algebra*
begin

declare *of-nat-2p* [*simp*]

lemma *word-int-cases*:

⟦⋀n. ⟦(x ::'a::len0 word) = word-of-int n; 0 ≤ n; n < 2^len-of TYPE('a)⟧ ⟹ P⟧
⟹ P
⟨proof⟩

lemma *word-nat-cases* [*cases type*: *word*]:
⟦⋀n. ⟦(x ::'a::len word) = of-nat n; n < 2^len-of TYPE('a)⟧ ⟹ P⟧
⟹ P
⟨proof⟩

lemma *max-word-eq*:
(max-word::'a::len word) = 2^len-of TYPE('a) − 1
⟨proof⟩

lemma *max-word-max* [*simp*,*intro*!]:
n ≤ max-word
⟨proof⟩

lemma *word-of-int-2p-len*:
word-of-int (2 ^ len-of TYPE('a)) = (0::'a::len0 word)
⟨proof⟩

lemma *word-pow-0*:
(2::'a::len word) ^ len-of TYPE('a) = 0
⟨proof⟩

lemma *max-word-wrap*: x + 1 = 0 ⟹ x = max-word
⟨proof⟩

lemma *max-word-minus*:
max-word = (−1::'a::len word)
⟨proof⟩

lemma *max-word-bl* [*simp*]:
to-bl (max-word::'a::len word) = replicate (len-of TYPE('a)) True
⟨proof⟩

lemma *max-test-bit* [*simp*]:
(max-word::'a::len word) !! n = (n < len-of TYPE('a))
⟨proof⟩

lemma *word-and-max* [*simp*]:
x AND max-word = x
⟨proof⟩

lemma *word-or-max* [*simp*]:
x OR max-word = max-word
⟨proof⟩

lemma *word-ao-dist2*:
x AND (y OR z) = x AND y OR x AND (z::'a::len0 word)
⟨*proof*⟩

lemma *word-oa-dist2*:
x OR y AND z = (x OR y) AND (x OR (z::'a::len0 word))
⟨*proof*⟩

lemma *word-and-not* [*simp*]:
x AND NOT x = (0::'a::len0 word)
⟨*proof*⟩

lemma *word-or-not* [*simp*]:
x OR NOT x = max-word
⟨*proof*⟩

lemma *word-boolean*:
boolean (op AND) (op OR) bitNOT 0 max-word
⟨*proof*⟩

interpretation *word-bool-alg*:
boolean op AND op OR bitNOT 0 max-word
⟨*proof*⟩

lemma *word-xor-and-or*:
x XOR y = x AND NOT y OR NOT x AND (y::'a::len0 word)
⟨*proof*⟩

interpretation *word-bool-alg*:
boolean-xor op AND op OR bitNOT 0 max-word op XOR
⟨*proof*⟩

lemma *shiftr-0* [*iff*]:
(x::'a::len0 word) >> 0 = x
⟨*proof*⟩

lemma *shiftl-0* [*simp*]:
(x :: 'a :: len word) << 0 = x
⟨*proof*⟩

lemma *shiftl-1* [*simp*]:
(1::'a::len word) << n = 2^n
⟨*proof*⟩

lemma *uint-lt-0* [*simp*]:
uint x < 0 = False
⟨*proof*⟩

lemma *shiftr1-1* [*simp*]:

shiftr1 (1::'a::len word) = 0
⟨*proof*⟩

lemma *shiftr-1*[*simp*]:
(1::'a::len word) >> n = (if n = 0 then 1 else 0)
⟨*proof*⟩

lemma *word-less-1* [*simp*]:
((x::'a::len word) < 1) = (x = 0)
⟨*proof*⟩

lemma *to-bl-mask*:
to-bl (mask n :: 'a::len word) =
replicate (len-of TYPE('a) − n) False @
replicate (min (len-of TYPE('a)) n) True
⟨*proof*⟩

lemma *map-replicate-True*:
n = length xs ==>
map (λ(x,y). x & y) (zip xs (replicate n True)) = xs
⟨*proof*⟩

lemma *map-replicate-False*:
n = length xs ==> map (λ(x,y). x & y)
(zip xs (replicate n False)) = replicate n False
⟨*proof*⟩

lemma *bl-and-mask*:
fixes *w :: 'a::len word*
fixes *n*
defines *n' ≡ len-of TYPE('a) − n*
shows *to-bl (w AND mask n) = replicate n' False @ drop n' (to-bl w)*
⟨*proof*⟩

lemma *drop-rev-takefill*:
length xs ≤ n ==>
drop (n − length xs) (rev (takefill False n (rev xs))) = xs
⟨*proof*⟩

lemma *map-nth-0* [*simp*]:
map (op !! (0::'a::len0 word)) xs = replicate (length xs) False
⟨*proof*⟩

lemma *uint-plus-if-size*:
uint (x + y) =
(if uint x + uint y < 2^size x then
uint x + uint y
else
uint x + uint y − 2^size x)

⟨*proof*⟩

lemma *unat-plus-if-size*:
unat (*x* + (*y*::′*a*::*len word*)) =
(*if unat x* + *unat y* < 2^*size x then*
unat x + *unat y*
else
unat x + *unat y* − 2^*size x*)
⟨*proof*⟩

lemma *word-neq-0-conv* [*simp*]:
fixes *w* :: ′*a* :: *len word*
shows (*w* ≠ *0*) = (*0* < *w*)
⟨*proof*⟩

lemma *max-lt*:
unat (*max a b div c*) = *unat* (*max a b*) *div unat* (*c*:: ′*a* :: *len word*)
⟨*proof*⟩

lemma *uint-sub-if-size*:
uint (*x* − *y*) =
(*if uint y* ≤ *uint x then*
uint x − *uint y*
else
uint x − *uint y* + 2^*size x*)
⟨*proof*⟩

lemma *unat-sub-simple*:
x ≤ *y* ==> *unat* (*y* − *x*) = *unat y* − *unat x*
⟨*proof*⟩

lemmas *unat-sub* = *unat-sub-simple*

lemma *word-less-sub1*:
fixes *x* :: ′*a* :: *len word*
shows *x* ≠ *0* ==> *1* < *x* = (*0* < *x* − *1*)
⟨*proof*⟩

lemma *word-le-sub1*:
fixes *x* :: ′*a* :: *len word*
shows *x* ≠ *0* ==> *1* ≤ *x* = (*0* ≤ *x* − *1*)
⟨*proof*⟩

lemmas *word-less-sub1-numberof* [*simp*] =
word-less-sub1 [*of number-of w*, *standard*]
lemmas *word-le-sub1-numberof* [*simp*] =
word-le-sub1 [*of number-of w*, *standard*]

lemma *word-of-int-minus*:

word-of-int (*2^len-of TYPE*('*a*) − *i*) = (*word-of-int* (−*i*)::'*a*::*len word*)
⟨*proof*⟩

lemmas *word-of-int-inj* =
word-uint.Abs-inject [*unfolded uints-num*, *simplified*]

lemma *word-le-less-eq*:
(*x* ::'*z*::*len word*) ≤ *y* = (*x* = *y* ∨ *x* < *y*)
⟨*proof*⟩

lemma *mod-plus-cong*:
assumes *1*: (*b*::*int*) = *b'*
and *2*: *x mod b'* = *x' mod b'*
and *3*: *y mod b'* = *y' mod b'*
and *4*: *x'* + *y'* = *z'*
shows (*x* + *y*) *mod b* = *z' mod b'*
⟨*proof*⟩

lemma *mod-minus-cong*:
assumes *1*: (*b*::*int*) = *b'*
and *2*: *x mod b'* = *x' mod b'*
and *3*: *y mod b'* = *y' mod b'*
and *4*: *x'* − *y'* = *z'*
shows (*x* − *y*) *mod b* = *z' mod b'*
⟨*proof*⟩

lemma *word-induct-less*:
⟦*P* (*0*::'*a*::*len word*); ⋀*n*. ⟦*n* < *m*; *P n*⟧ ⟹ *P* (*1* + *n*)⟧ ⟹ *P m*
⟨*proof*⟩

lemma *word-induct*:
⟦*P* (*0*::'*a*::*len word*); ⋀*n*. *P n* ⟹ *P* (*1* + *n*)⟧ ⟹ *P m*
⟨*proof*⟩

lemma *word-induct2* [*induct type*]:
⟦*P 0*; ⋀*n*. ⟦*1* + *n* ≠ *0*; *P n*⟧ ⟹ *P* (*1* + *n*)⟧ ⟹ *P* (*n*::'*b*::*len word*)
⟨*proof*⟩

definition *word-rec* :: '*a* ⇒ ('*b*::*len word* ⇒ '*a* ⇒ '*a*) ⇒ '*b word* ⇒ '*a* **where**
word-rec forZero forSuc n ≡ *nat-rec forZero* (*forSuc* ∘ *of-nat*) (*unat n*)

lemma *word-rec-0*: *word-rec z s 0* = *z*
⟨*proof*⟩

lemma *word-rec-Suc*:
1 + *n* ≠ (*0*::'*a*::*len word*) ⟹ *word-rec z s* (*1* + *n*) = *s n* (*word-rec z s n*)
⟨*proof*⟩

lemma *word-rec-Pred*:

$n \neq 0 \Longrightarrow$ *word-rec z s n* $=$ *s* $(n - 1)$ $($*word-rec z s* $(n - 1))$
⟨*proof*⟩

lemma *word-rec-in*:
f $($*word-rec z* $(\lambda$-. *f*$)$ *n*$)$ $=$ *word-rec* $($*f z*$)$ $(\lambda$-. *f*$)$ *n*
⟨*proof*⟩

lemma *word-rec-in2*:
f n $($*word-rec z f n*$)$ $=$ *word-rec* $($*f 0 z*$)$ $($*f* $\circ$ *op* $+$ *1*$)$ *n*
⟨*proof*⟩

lemma *word-rec-twice*:
$m \leq n \Longrightarrow$ *word-rec z f n* $=$ *word-rec* $($*word-rec z f* $(n - m))$ $($*f* $\circ$ *op* $+$ $(n - m))$ *m*
⟨*proof*⟩

lemma *word-rec-id*: *word-rec z* $(\lambda$-. *id*$)$ *n* $=$ *z*
⟨*proof*⟩

lemma *word-rec-id-eq*: $\forall\, m < n.$ *f m* $=$ *id* $\Longrightarrow$ *word-rec z f n* $=$ *z*
⟨*proof*⟩

lemma *word-rec-max*:
$\forall\, m \geq n.$ $m \neq -1 \longrightarrow$ *f m* $=$ *id* $\Longrightarrow$ *word-rec z f* -1 $=$ *word-rec z f n*
⟨*proof*⟩

lemma *unatSuc*:
1 $+$ *n* $\neq$ $(0::'a::len\ word) \Longrightarrow$ *unat* $(1 + n)$ $=$ *Suc* $($*unat n*$)$
⟨*proof*⟩

lemmas *word-no-1* [*simp*] $=$ *word-1-no* [*symmetric*, *unfolded BIT-simps*]
lemmas *word-no-0* [*simp*] $=$ *word-0-no* [*symmetric*]

declare *word-0-bl* [*simp*]
declare *bin-to-bl-def* [*simp*]
declare *to-bl-0* [*simp*]
declare *of-bl-True* [*simp*]

end

16 Word: Word Library interface

theory *Word*
imports *WordGenLib*
uses ~~/*src/HOL/Tools/SMT/smt-word.ML*
begin

⟨*ML*⟩

see *Examples/WordExamples.thy* for examples

end

References

[1] Jeremy Dawson. Isabelle theories for machine words. In Michael Goldsmith and Bill Roscoe, editors, *Seventh International Workshop on Automated Verification of Critical Systems (AVOCS'07)*, Electronic Notes in Theoretical Computer Science, page 15, Oxford, September 2007. Elsevier. to appear.