Machine Words in Isabelle/HOL

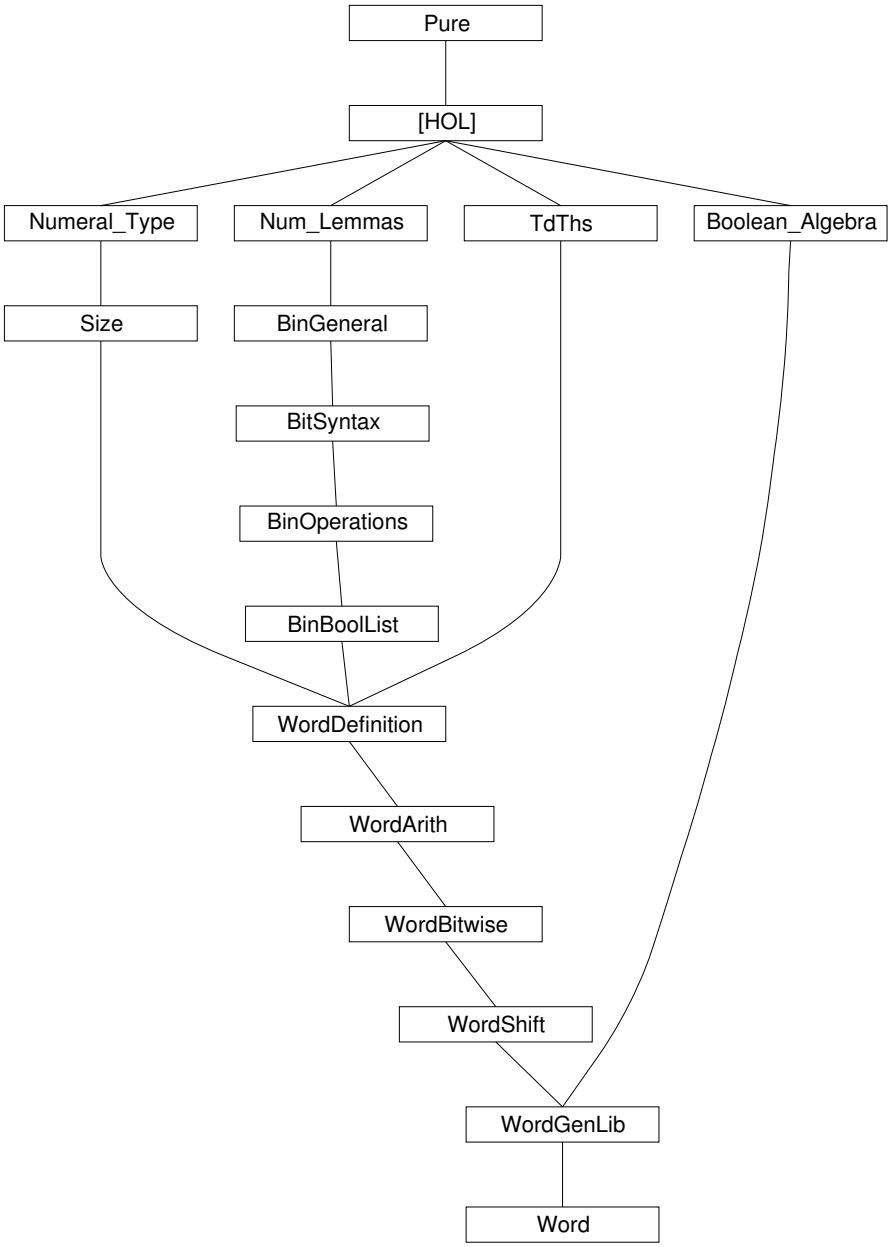
Jeremy Dawson, Paul Graunke, Brian Huffman, Gerwin Klein, and John Matthews

June 21, 2010

Abstract

A formalisation of generic, fixed size machine words in Isabelle/HOL. An earlier version of this formalisation is described in [1].

Contents


Pure
[HOL]
Numeral_Type
Num_Lemmas
TdThs
Boolean_Algebra
Size
BinGeneral
BitSyntax
BinOperations
BinBoolList
WordDefinition
WordArith
WordBitwise
WordShift
WordGenLib
Word

1 Numeral-Type: Numeral Syntax for Types

theory *Numeral-Type*
imports *Main*
begin

1.1 Preliminary lemmas

lemma (**in** *type-definition*) *univ*:
  $UNIV = Abs\ `\ A$
proof
  show $Abs\ `\ A \subseteq UNIV$ **by** (*rule subset-UNIV*)
  show $UNIV \subseteq Abs\ `\ A$
  proof
    fix $x :: 'b$
    have $x = Abs\ (Rep\ x)$ **by** (*rule Rep-inverse* [*symmetric*])
    moreover have $Rep\ x \in A$ **by** (*rule Rep*)
    ultimately show $x \in Abs\ `\ A$ **by** (*rule image-eqI*)
  qed
qed

lemma (**in** *type-definition*) *card*: $card\ (UNIV :: 'b\ set) = card\ A$
  by (*simp add*: *univ card-image inj-on-def Abs-inject*)

1.2 Cardinalities of types

syntax *-type-card* :: *type* => *nat* ((*1CARD/(1'(-'))*))

translations $CARD('t) => CONST\ card\ (CONST\ UNIV :: 't\ set)$

typed-print-translation ⟪
let
  fun card-univ-tr' show-sorts - [*Const* (@{*const-syntax UNIV*}, *Type*(-, [*T*, -]))] =
    Syntax.const @{*syntax-const -type-card*} $ *Syntax.term-of-typ show-sorts T*;
in [(@{*const-syntax card*}, *card-univ-tr'*)]
end
⟫

lemma *card-unit* [*simp*]: $CARD(unit) = 1$
  unfolding *UNIV-unit* **by** *simp*

lemma *card-bool* [*simp*]: $CARD(bool) = 2$
  unfolding *UNIV-bool* **by** *simp*

lemma *card-prod* [*simp*]: $CARD('a \times 'b) = CARD('a::finite) * CARD('b::finite)$
  unfolding *UNIV-Times-UNIV* [*symmetric*] **by** (*simp only*: *card-cartesian-product*)

lemma *card-sum* [*simp*]: $CARD('a + 'b) = CARD('a::finite) + CARD('b::finite)$
  unfolding *UNIV-Plus-UNIV* [*symmetric*] **by** (*simp only*: *finite card-Plus*)

lemma *card-option* [*simp*]: *CARD*('*a option*) = *Suc CARD*('*a*::*finite*)
unfolding *UNIV-option-conv*
apply (*subgoal-tac* (*None*::'*a option*) ∉ *range Some*)
apply (*simp add*: *card-image*)
apply *fast*
done

lemma *card-set* [*simp*]: *CARD*('*a set*) = *2* ^ *CARD*('*a*::*finite*)
unfolding *Pow-UNIV* [*symmetric*]
by (*simp only*: *card-Pow finite numeral-2-eq-2*)

lemma *card-nat* [*simp*]: *CARD*(*nat*) = *0*
by (*simp add*: *infinite-UNIV-nat card-eq-0-iff*)

1.3 Classes with at least 1 and 2

Class finite already captures "at least 1"

lemma *zero-less-card-finite* [*simp*]: *0* < *CARD*('*a*::*finite*)
unfolding *neq0-conv* [*symmetric*] **by** *simp*

lemma *one-le-card-finite* [*simp*]: *Suc 0* ≤ *CARD*('*a*::*finite*)
by (*simp add*: *less-Suc-eq-le* [*symmetric*])

Class for cardinality "at least 2"

class *card2* = *finite* +
assumes *two-le-card*: *2* ≤ *CARD*('*a*)

lemma *one-less-card*: *Suc 0* < *CARD*('*a*::*card2*)
using *two-le-card* [**where** '*a*='*a*] **by** *simp*

lemma *one-less-int-card*: *1* < *int CARD*('*a*::*card2*)
using *one-less-card* [**where** '*a*='*a*] **by** *simp*

1.4 Numeral Types

typedef (**open**) *num0* = *UNIV* :: *nat set* **..**
typedef (**open**) *num1* = *UNIV* :: *unit set* **..**

typedef (**open**) '*a bit0* = {*0* ..< *2* ∗ *int CARD*('*a*::*finite*)}
proof
show *0* ∈ {*0* ..< *2* ∗ *int CARD*('*a*)}
by *simp*
qed

typedef (**open**) '*a bit1* = {*0* ..< *1* + *2* ∗ *int CARD*('*a*::*finite*)}
proof
show *0* ∈ {*0* ..< *1* + *2* ∗ *int CARD*('*a*)}
by *simp*

qed

lemma *card-num0* [*simp*]: *CARD* (*num0*) = *0*
unfolding *type-definition.card* [*OF type-definition-num0*]
by *simp*

lemma *card-num1* [*simp*]: *CARD*(*num1*) = *1*
unfolding *type-definition.card* [*OF type-definition-num1*]
by (*simp only*: *card-unit*)

lemma *card-bit0* [*simp*]: *CARD*('*a bit0*) = *2* ∗ *CARD*('*a*::*finite*)
unfolding *type-definition.card* [*OF type-definition-bit0*]
by *simp*

lemma *card-bit1* [*simp*]: *CARD*('*a bit1*) = *Suc* (*2* ∗ *CARD*('*a*::*finite*))
unfolding *type-definition.card* [*OF type-definition-bit1*]
by *simp*

instance *num1* :: *finite*
proof
show *finite* (*UNIV*::*num1 set*)
unfolding *type-definition.univ* [*OF type-definition-num1*]
using *finite* **by** (*rule finite-imageI*)
qed

instance *bit0* :: (*finite*) *card2*
proof
show *finite* (*UNIV*::'*a bit0 set*)
unfolding *type-definition.univ* [*OF type-definition-bit0*]
by *simp*
show *2* ≤ *CARD*('*a bit0*)
by *simp*
qed

instance *bit1* :: (*finite*) *card2*
proof
show *finite* (*UNIV*::'*a bit1 set*)
unfolding *type-definition.univ* [*OF type-definition-bit1*]
by *simp*
show *2* ≤ *CARD*('*a bit1*)
by *simp*
qed

1.5 Locale for modular arithmetic subtypes

locale *mod-type* =
fixes *n* :: *int*
and *Rep* :: '*a*::{*zero*,*one*,*plus*,*times*,*uminus*,*minus*} ⇒ *int*
and *Abs* :: *int* ⇒ '*a*::{*zero*,*one*,*plus*,*times*,*uminus*,*minus*}

assumes *type*: *type-definition Rep Abs* $\{0..<n\}$
and *size1*: $1 < n$
and *zero-def*: $0 = Abs\ 0$
and *one-def*: $1 = Abs\ 1$
and *add-def*: $x + y = Abs\ ((Rep\ x + Rep\ y)\ mod\ n)$
and *mult-def*: $x * y = Abs\ ((Rep\ x * Rep\ y)\ mod\ n)$
and *diff-def*: $x - y = Abs\ ((Rep\ x - Rep\ y)\ mod\ n)$
and *minus-def*: $- x = Abs\ ((-\ Rep\ x)\ mod\ n)$
begin

lemma *size0*: $0 < n$
using *size1* **by** *simp*

lemmas *definitions* =
zero-def one-def add-def mult-def minus-def diff-def

lemma *Rep-less-n*: $Rep\ x < n$
by (*rule type-definition.Rep* [*OF type*, *simplified*, *THEN conjunct2*])

lemma *Rep-le-n*: $Rep\ x \leq n$
by (*rule Rep-less-n* [*THEN order-less-imp-le*])

lemma *Rep-inject-sym*: $x = y \longleftrightarrow Rep\ x = Rep\ y$
by (*rule type-definition.Rep-inject* [*OF type*, *symmetric*])

lemma *Rep-inverse*: $Abs\ (Rep\ x) = x$
by (*rule type-definition.Rep-inverse* [*OF type*])

lemma *Abs-inverse*: $m \in \{0..<n\} \Longrightarrow Rep\ (Abs\ m) = m$
by (*rule type-definition.Abs-inverse* [*OF type*])

lemma *Rep-Abs-mod*: $Rep\ (Abs\ (m\ mod\ n)) = m\ mod\ n$
by (*simp add*: *Abs-inverse pos-mod-conj* [*OF size0*])

lemma *Rep-Abs-0*: $Rep\ (Abs\ 0) = 0$
by (*simp add*: *Abs-inverse size0*)

lemma *Rep-0*: $Rep\ 0 = 0$
by (*simp add*: *zero-def Rep-Abs-0*)

lemma *Rep-Abs-1*: $Rep\ (Abs\ 1) = 1$
by (*simp add*: *Abs-inverse size1*)

lemma *Rep-1*: $Rep\ 1 = 1$
by (*simp add*: *one-def Rep-Abs-1*)

lemma *Rep-mod*: $Rep\ x\ mod\ n = Rep\ x$
apply (*rule-tac x*=*x* **in** *type-definition.Abs-cases* [*OF type*])
apply (*simp add*: *type-definition.Abs-inverse* [*OF type*])

apply (*simp add*: *mod-pos-pos-trivial*)
done

lemmas *Rep-simps* =
  Rep-inject-sym Rep-inverse Rep-Abs-mod Rep-mod Rep-Abs-0 Rep-Abs-1

lemma *comm-ring-1*: *OFCLASS*('*a*, *comm-ring-1-class*)
apply (*intro-classes*, *unfold definitions*)
apply (*simp-all add*: *Rep-simps zmod-simps field-simps*)
done

end

locale *mod-ring* = *mod-type* +
  constrains *n* :: *int*
  and *Rep* :: '*a*::{*number-ring*} ⇒ *int*
  and *Abs* :: *int* ⇒ '*a*::{*number-ring*}
begin

lemma *of-nat-eq*: *of-nat k* = *Abs* (*int k mod n*)
apply (*induct k*)
apply (*simp add*: *zero-def*)
apply (*simp add*: *Rep-simps add-def one-def zmod-simps add-ac*)
done

lemma *of-int-eq*: *of-int z* = *Abs* (*z mod n*)
apply (*cases z rule*: *int-diff-cases*)
apply (*simp add*: *Rep-simps of-nat-eq diff-def zmod-simps*)
done

lemma *Rep-number-of*:
  Rep (*number-of w*) = *number-of w mod n*
by (*simp add*: *number-of-eq of-int-eq Rep-Abs-mod*)

lemma *iszero-number-of*:
  iszero (*number-of w*::'*a*) ⟷ *number-of w mod n* = *0*
by (*simp add*: *Rep-simps number-of-eq of-int-eq iszero-def zero-def*)

lemma *cases*:
  assumes *1*: ⋀*z*. ⟦(*x*::'*a*) = *of-int z*; *0* ≤ *z*; *z* < *n*⟧ ⟹ *P*
  shows *P*
apply (*cases x rule*: *type-definition.Abs-cases* [*OF type*])
apply (*rule-tac z*=*y* **in** *1*)
apply (*simp-all add*: *of-int-eq mod-pos-pos-trivial*)
done

lemma *induct*:
  (⋀*z*. ⟦*0* ≤ *z*; *z* < *n*⟧ ⟹ *P* (*of-int z*)) ⟹ *P* (*x*::'*a*)
by (*cases x rule*: *cases*) *simp*

end

1.6 Number ring instances

Unfortunately a number ring instance is not possible for *num1*, since 0 and 1 are not distinct.

instantiation *num1* :: {*comm-ring*,*comm-monoid-mult*,*number*}
begin

lemma *num1-eq-iff*: (*x*::*num1*) = (*y*::*num1*) ⟷ *True*
by (*induct x*, *induct y*) *simp*

instance proof
qed (*simp-all add*: *num1-eq-iff*)

end

instantiation
bit0 **and** *bit1* :: (*finite*) {*zero*,*one*,*plus*,*times*,*uminus*,*minus*}
begin

definition *Abs-bit0'* :: *int* ⇒ *'a bit0* **where**
Abs-bit0' x = *Abs-bit0* (*x mod int CARD*(*'a bit0*))

definition *Abs-bit1'* :: *int* ⇒ *'a bit1* **where**
Abs-bit1' x = *Abs-bit1* (*x mod int CARD*(*'a bit1*))

definition *0* = *Abs-bit0 0*
definition *1* = *Abs-bit0 1*
definition *x* + *y* = *Abs-bit0'* (*Rep-bit0 x* + *Rep-bit0 y*)
definition *x* ∗ *y* = *Abs-bit0'* (*Rep-bit0 x* ∗ *Rep-bit0 y*)
definition *x* − *y* = *Abs-bit0'* (*Rep-bit0 x* − *Rep-bit0 y*)
definition − *x* = *Abs-bit0'* (− *Rep-bit0 x*)

definition *0* = *Abs-bit1 0*
definition *1* = *Abs-bit1 1*
definition *x* + *y* = *Abs-bit1'* (*Rep-bit1 x* + *Rep-bit1 y*)
definition *x* ∗ *y* = *Abs-bit1'* (*Rep-bit1 x* ∗ *Rep-bit1 y*)
definition *x* − *y* = *Abs-bit1'* (*Rep-bit1 x* − *Rep-bit1 y*)
definition − *x* = *Abs-bit1'* (− *Rep-bit1 x*)

instance ..

end

interpretation *bit0*:
mod-type int CARD(*'a*::*finite bit0*)
Rep-bit0 :: *'a*::*finite bit0* ⇒ *int*

Abs-bit0 :: *int* ⇒ *'a*::*finite bit0*
apply (*rule mod-type.intro*)
apply (*simp add*: *int-mult type-definition-bit0*)
apply (*rule one-less-int-card*)
apply (*rule zero-bit0-def*)
apply (*rule one-bit0-def*)
apply (*rule plus-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule times-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule minus-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule uminus-bit0-def* [*unfolded Abs-bit0'-def*])
done

interpretation *bit1*:
mod-type int CARD('*a*::*finite bit1*)
Rep-bit1 :: *'a*::*finite bit1* ⇒ *int*
Abs-bit1 :: *int* ⇒ *'a*::*finite bit1*
apply (*rule mod-type.intro*)
apply (*simp add*: *int-mult type-definition-bit1*)
apply (*rule one-less-int-card*)
apply (*rule zero-bit1-def*)
apply (*rule one-bit1-def*)
apply (*rule plus-bit1-def* [*unfolded Abs-bit1'-def*])
apply (*rule times-bit1-def* [*unfolded Abs-bit1'-def*])
apply (*rule minus-bit1-def* [*unfolded Abs-bit1'-def*])
apply (*rule uminus-bit1-def* [*unfolded Abs-bit1'-def*])
done

instance *bit0* :: (*finite*) *comm-ring-1*
by (*rule bit0.comm-ring-1*)+

instance *bit1* :: (*finite*) *comm-ring-1*
by (*rule bit1.comm-ring-1*)+

instantiation *bit0* **and** *bit1* :: (*finite*) *number-ring*
begin

definition (*number-of w* :: - *bit0*) = *of-int w*

definition (*number-of w* :: - *bit1*) = *of-int w*

instance proof
qed (*rule number-of-bit0-def number-of-bit1-def*)+

end

interpretation *bit0*:
mod-ring int CARD('*a*::*finite bit0*)
Rep-bit0 :: *'a*::*finite bit0* ⇒ *int*
Abs-bit0 :: *int* ⇒ *'a*::*finite bit0*

..

interpretation *bit1*:
mod-ring int CARD('a::finite bit1)
Rep-bit1 :: 'a::finite bit1 ⇒ int
Abs-bit1 :: int ⇒ 'a::finite bit1
..

Set up cases, induction, and arithmetic

lemmas *bit0-cases* [*case-names of-int*, *cases type*: *bit0*] = *bit0.cases*
lemmas *bit1-cases* [*case-names of-int*, *cases type*: *bit1*] = *bit1.cases*

lemmas *bit0-induct* [*case-names of-int*, *induct type*: *bit0*] = *bit0.induct*
lemmas *bit1-induct* [*case-names of-int*, *induct type*: *bit1*] = *bit1.induct*

lemmas *bit0-iszero-number-of* [*simp*] = *bit0.iszero-number-of*
lemmas *bit1-iszero-number-of* [*simp*] = *bit1.iszero-number-of*

1.7 Syntax

syntax
-NumeralType :: num-const => type (-)
-NumeralType0 :: type (*0*)
-NumeralType1 :: type (*1*)

translations
(*type*) *1* == (*type*) *num1*
(*type*) *0* == (*type*) *num0*

parse-translation ⟪
let
fun mk-bintype n =
let
fun mk-bit 0 = Syntax.const @{*type-syntax bit0*}
| *mk-bit 1 = Syntax.const* @{*type-syntax bit1*};
fun bin-of n =
if n = 1 then Syntax.const @{*type-syntax num1*}
else if n = 0 then Syntax.const @{*type-syntax num0*}
else if n = ~1 then raise TERM (*negative type numeral*, [])
else
let val (*q*, *r*) = *Integer.div-mod n 2*;
in mk-bit r $ *bin-of q end*;
in bin-of n end;

fun numeral-tr (∗*-NumeralType*∗) [*Const* (*str*, -)] =
mk-bintype (*the* (*Int.fromString str*))
| *numeral-tr* (∗*-NumeralType*∗) *ts = raise TERM* (*numeral-tr*, *ts*);

in [(@{*syntax-const -NumeralType*}, *numeral-tr*)] *end*;

⟩⟩

print-translation ⟪⟪
let
fun int-of [] = *0*
| *int-of* (*b* :: *bs*) = *b* + *2* ∗ *int-of bs*;

fun bin-of (*Const* (@{*type-syntax num0*}, -)) = []
| *bin-of* (*Const* (@{*type-syntax num1*}, -)) = [*1*]
| *bin-of* (*Const* (@{*type-syntax bit0*}, -) $ *bs*) = *0* :: *bin-of bs*
| *bin-of* (*Const* (@{*type-syntax bit1*}, -) $ *bs*) = *1* :: *bin-of bs*
| *bin-of t* = *raise TERM* (*bin-of*, [*t*]);

fun bit-tr' *b* [*t*] =
let
val rev-digs = *b* :: *bin-of t handle TERM* - => *raise Match*
val i = *int-of rev-digs*;
val num = *string-of-int* (*abs i*);
in
Syntax.const @{*syntax-const -NumeralType*} $ *Syntax.free num*
end
| *bit-tr'* *b* - = *raise Match*;
in [(@{*type-syntax bit0*}, *bit-tr' 0*), (@{*type-syntax bit1*}, *bit-tr' 1*)] *end*;
⟫⟫

1.8 Examples

lemma *CARD*(*0*) = *0* **by** *simp*
lemma *CARD*(*17*) = *17* **by** *simp*
lemma *8* ∗ *11* ^ *3* − *6* = (*2*::*5*) **by** *simp*

end

2 Size: The len classes

theory *Size*
imports *Numeral-Type*
begin

The aim of this is to allow any type as index type, but to provide a default instantiation for numeral types. This independence requires some duplication with the definitions in *Numeral-Type*.

class *len0* =
fixes *len-of* :: *'a itself* ⇒ *nat*

Some theorems are only true on words with length greater 0.

class *len* = *len0* +

assumes *len-gt-0* [*iff*]: *0 < len-of TYPE ('a)*

instantiation *num0* **and** *num1* :: *len0*
begin

definition
len-num0: *len-of (x::num0 itself) = 0*

definition
len-num1: *len-of (x::num1 itself) = 1*

instance ..

end

instantiation *bit0* **and** *bit1* :: (*len0*) *len0*
begin

definition
len-bit0: *len-of (x::'a::len0 bit0 itself) = 2 * len-of TYPE ('a)*

definition
len-bit1: *len-of (x::'a::len0 bit1 itself) = 2 * len-of TYPE ('a) + 1*

instance ..

end

lemmas *len-of-numeral-defs* [*simp*] = *len-num0 len-num1 len-bit0 len-bit1*

instance *num1* :: *len* **by** (*intro-classes*) *simp*
instance *bit0* :: (*len*) *len* **by** (*intro-classes*) *simp*
instance *bit1* :: (*len0*) *len* **by** (*intro-classes*) *simp*

— Examples:
lemma *len-of TYPE(17) = 17* **by** *simp*
lemma *len-of TYPE(0) = 0* **by** *simp*

— not simplified:
lemma *len-of TYPE('a::len0) = x*
oops

end

3 Num-Lemmas: Useful Numerical Lemmas

theory *Num-Lemmas*

imports *Main Parity*
begin

lemma *contentsI*: $y = \{x\} ==> contents\ y = x$
  unfolding *contents-def* **by** *auto* — FIXME move

lemmas *split-split* = *prod.split* [*unfolded prod-case-split*]
lemmas *split-split-asm* = *prod.split-asm* [*unfolded prod-case-split*]
lemmas *split-splits* = *split-split split-split-asm*

lemmas *funpow-0* = *funpow.simps*(*1*)
lemmas *funpow-Suc* = *funpow.simps*(*2*)

lemma *nonemptyE*: $S \sim= \{\} ==> (!!x.\ x : S ==> R) ==> R$ **by** *auto*

lemma *gt-or-eq-0*: $0 < y \vee 0 = (y::nat)$ **by** *arith*

declare *iszero-0* [*iff*]

lemmas *xtr1* = *xtrans*(*1*)
lemmas *xtr2* = *xtrans*(*2*)
lemmas *xtr3* = *xtrans*(*3*)
lemmas *xtr4* = *xtrans*(*4*)
lemmas *xtr5* = *xtrans*(*5*)
lemmas *xtr6* = *xtrans*(*6*)
lemmas *xtr7* = *xtrans*(*7*)
lemmas *xtr8* = *xtrans*(*8*)

lemmas *nat-simps* = *diff-add-inverse2 diff-add-inverse*
lemmas *nat-iffs* = *le-add1 le-add2*

lemma *sum-imp-diff*: $j = k + i ==> j - i = (k :: nat)$ **by** *arith*

lemma *nobm1*:
  $0 < (number\text{-}of\ w :: nat) ==>$
   $number\text{-}of\ w - (1 :: nat) = number\text{-}of\ (Int.pred\ w)$
  apply (*unfold nat-number-of-def One-nat-def nat-1* [*symmetric*] *pred-def*)
  apply (*simp add*: *number-of-eq nat-diff-distrib* [*symmetric*])
  done

lemma *zless2*: $0 < (2 :: int)$ **by** *arith*

lemmas *zless2p* [*simp*] = *zless2* [*THEN zero-less-power*]
lemmas *zle2p* [*simp*] = *zless2p* [*THEN order-less-imp-le*]

lemmas *pos-mod-sign2* = *zless2* [*THEN pos-mod-sign* [**where** *b* = *2*::*int*]]
lemmas *pos-mod-bound2* = *zless2* [*THEN pos-mod-bound* [**where** *b* = *2*::*int*]]

— the inverse(s) of *number-of*

lemma *nmod2*: *n mod (2::int) = 0 | n mod 2 = 1* **by** *arith*

lemma *emep1*:
even n ==> even d ==> 0 <= d ==> (n + 1) mod (d :: int) = (n mod d) + 1
apply (*simp add*: *add-commute*)
apply (*safe dest!*: *even-equiv-def* [*THEN iffD1*])
apply (*subst pos-zmod-mult-2*)
apply *arith*
apply (*simp add*: *mod-mult-mult1*)
done

lemmas *eme1p = emep1* [*simplified add-commute*]

lemma *le-diff-eq'*: *(a ≤ c − b) = (b + a ≤ (c::int))* **by** *arith*

lemma *less-diff-eq'*: *(a < c − b) = (b + a < (c::int))* **by** *arith*

lemma *diff-le-eq'*: *(a − b ≤ c) = (a ≤ b + (c::int))* **by** *arith*

lemma *diff-less-eq'*: *(a − b < c) = (a < b + (c::int))* **by** *arith*

lemmas *m1mod2k = zless2p* [*THEN zmod-minus1*]
lemmas *m1mod22k = mult-pos-pos* [*OF zless2 zless2p*, *THEN zmod-minus1*]
lemmas *p1mod22k' = zless2p* [*THEN order-less-imp-le*, *THEN pos-zmod-mult-2*]
lemmas *z1pmod2' = zero-le-one* [*THEN pos-zmod-mult-2*, *simplified*]
lemmas *z1pdiv2' = zero-le-one* [*THEN pos-zdiv-mult-2*, *simplified*]

lemma *p1mod22k*:
*(2 * b + 1) mod (2 * 2 ^ n) = 2 * (b mod 2 ^ n) + (1::int)*
by (*simp add*: *p1mod22k' add-commute*)

lemma *z1pmod2*:
*(2 * b + 1) mod 2 = (1::int)* **by** *arith*

lemma *z1pdiv2*:
*(2 * b + 1) div 2 = (b::int)* **by** *arith*

lemmas *zdiv-le-dividend = xtr3* [*OF div-by-1* [*symmetric*] *zdiv-mono2*,
simplified int-one-le-iff-zero-less, *simplified*, *standard*]

lemma *axxbyy*:
a + m + m = b + n + n ==> (a = 0 | a = 1) ==> (b = 0 | b = 1) ==>
a = b & m = (n :: int) **by** *arith*

lemma *axxmod2*:
(1 + x + x) mod 2 = (1 :: int) & (0 + x + x) mod 2 = (0 :: int) **by** *arith*

lemma *axxdiv2*:

$(1 + x + x)\ div\ 2 = (x :: int)\ \&\ (0 + x + x)\ div\ 2 = (x :: int)$ **by** *arith*

lemmas *iszero-minus* = *trans* [*THEN trans*,
OF iszero-def neg-equal-0-iff-equal iszero-def [*symmetric*], *standard*]

lemmas *zadd-diff-inverse* = *trans* [*OF diff-add-cancel* [*symmetric*] *add-commute*,
standard]

lemmas *add-diff-cancel2* = *add-commute* [*THEN diff-eq-eq* [*THEN iffD2*], *standard*]

lemma *zmod-uminus*: $-((a :: int)\ mod\ b)\ mod\ b = -a\ mod\ b$
by (*simp add* : *zmod-zminus1-eq-if*)

lemma *zmod-zsub-distrib*: $((a::int) - b)\ mod\ c = (a\ mod\ c - b\ mod\ c)\ mod\ c$
apply (*unfold diff-int-def*)
apply (*rule trans* [*OF - mod-add-eq* [*symmetric*]])
apply (*simp add*: *zmod-uminus mod-add-eq* [*symmetric*])
done

lemma *zmod-zsub-right-eq*: $((a::int) - b)\ mod\ c = (a - b\ mod\ c)\ mod\ c$
apply (*unfold diff-int-def*)
apply (*rule trans* [*OF - mod-add-right-eq* [*symmetric*]])
apply (*simp add* : *zmod-uminus mod-add-right-eq* [*symmetric*])
done

lemma *zmod-zsub-left-eq*: $((a::int) - b)\ mod\ c = (a\ mod\ c - b)\ mod\ c$
by (*rule mod-add-left-eq* [**where** $b = -b$, *simplified diff-int-def* [*symmetric*]])

lemma *zmod-zsub-self* [*simp*]:
$((b :: int) - a)\ mod\ a = b\ mod\ a$
by (*simp add*: *zmod-zsub-right-eq*)

lemma *zmod-zmult1-eq-rev*:
$b * a\ mod\ c = b\ mod\ c * a\ mod\ (c::int)$
apply (*simp add*: *mult-commute*)
apply (*subst zmod-zmult1-eq*)
apply *simp*
done

lemmas *rdmods* [*symmetric*] = *zmod-uminus* [*symmetric*]
zmod-zsub-left-eq zmod-zsub-right-eq mod-add-left-eq
mod-add-right-eq zmod-zmult1-eq zmod-zmult1-eq-rev

lemma *mod-plus-right*:
$((a + x)\ mod\ m = (b + x)\ mod\ m) = (a\ mod\ m = b\ mod\ (m :: nat))$
apply (*induct x*)
apply (*simp-all add*: *mod-Suc*)
apply *arith*

done

lemma *nat-minus-mod*: $(n - n \bmod m) \bmod m = (0 :: nat)$
by (*induct n*) (*simp-all add* : *mod-Suc*)

lemmas *nat-minus-mod-plus-right* = *trans* [*OF nat-minus-mod mod-0* [*symmetric*], *THEN mod-plus-right* [*THEN iffD2*], *standard*, *simplified*]

lemmas *push-mods'* = *mod-add-eq* [*standard*]
mod-mult-eq [*standard*] *zmod-zsub-distrib* [*standard*]
zmod-uminus [*symmetric*, *standard*]

lemmas *push-mods* = *push-mods'* [*THEN eq-reflection*, *standard*]
lemmas *pull-mods* = *push-mods* [*symmetric*] *rdmods* [*THEN eq-reflection*, *standard*]
lemmas *mod-simps* =
mod-mult-self2-is-0 [*THEN eq-reflection*]
mod-mult-self1-is-0 [*THEN eq-reflection*]
mod-mod-trivial [*THEN eq-reflection*]

lemma *nat-mod-eq*:
$!!b.\ b < n ==> a \bmod n = b \bmod n ==> a \bmod n = (b :: nat)$
by (*induct a*) *auto*

lemmas *nat-mod-eq'* = *refl* [*THEN* [*2*] *nat-mod-eq*]

lemma *nat-mod-lem*:
$(0 :: nat) < n ==> b < n = (b \bmod n = b)$
apply *safe*
apply (*erule nat-mod-eq'*)
apply (*erule subst*)
apply (*erule mod-less-divisor*)
done

lemma *mod-nat-add*:
$(x :: nat) < z ==> y < z ==>$
$(x + y) \bmod z = (if\ x + y < z\ then\ x + y\ else\ x + y - z)$
apply (*rule nat-mod-eq*)
apply *auto*
apply (*rule trans*)
apply (*rule le-mod-geq*)
apply *simp*
apply (*rule nat-mod-eq'*)
apply *arith*
done

lemma *mod-nat-sub*:
$(x :: nat) < z ==> (x - y) \bmod z = x - y$
by (*rule nat-mod-eq'*) *arith*

lemma *int-mod-lem*:
(*0* :: *int*) < *n* ==> (*0* <= *b* & *b* < *n*) = (*b mod n* = *b*)
apply *safe*
apply (*erule* (*1*) *mod-pos-pos-trivial*)
apply (*erule-tac* [!] *subst*)
apply *auto*
done

lemma *int-mod-eq*:
(*0* :: *int*) <= *b* ==> *b* < *n* ==> *a mod n* = *b mod n* ==> *a mod n* = *b*
by *clarsimp* (*rule mod-pos-pos-trivial*)

lemmas *int-mod-eq'* = *refl* [*THEN* [*3*] *int-mod-eq*]

lemma *int-mod-le*: *0* <= *a* ==> *0* < (*n* :: *int*) ==> *a mod n* <= *a*
apply (*cases a* < *n*)
apply (*auto dest*: *mod-pos-pos-trivial pos-mod-bound* [**where** *a*=*a*])
done

lemma *int-mod-le'*: *0* <= *b* − *n* ==> *0* < (*n* :: *int*) ==> *b mod n* <= *b* − *n*
by (*rule int-mod-le* [**where** *a* = *b* − *n* **and** *n* = *n*, *simplified*])

lemma *int-mod-ge*: *a* < *n* ==> *0* < (*n* :: *int*) ==> *a* <= *a mod n*
apply (*cases 0* <= *a*)
apply (*drule* (*1*) *mod-pos-pos-trivial*)
apply *simp*
apply (*rule order-trans* [*OF - pos-mod-sign*])
apply *simp*
apply *assumption*
done

lemma *int-mod-ge'*: *b* < *0* ==> *0* < (*n* :: *int*) ==> *b* + *n* <= *b mod n*
by (*rule int-mod-ge* [**where** *a* = *b* + *n* **and** *n* = *n*, *simplified*])

lemma *mod-add-if-z*:
(*x* :: *int*) < *z* ==> *y* < *z* ==> *0* <= *y* ==> *0* <= *x* ==> *0* <= *z* ==>
(*x* + *y*) *mod z* = (*if x* + *y* < *z then x* + *y else x* + *y* − *z*)
by (*auto intro*: *int-mod-eq*)

lemma *mod-sub-if-z*:
(*x* :: *int*) < *z* ==> *y* < *z* ==> *0* <= *y* ==> *0* <= *x* ==> *0* <= *z* ==>
(*x* − *y*) *mod z* = (*if y* <= *x then x* − *y else x* − *y* + *z*)
by (*auto intro*: *int-mod-eq*)

lemmas *zmde* = *zmod-zdiv-equality* [*THEN diff-eq-eq* [*THEN iffD2*], *symmetric*]
lemmas *mcl* = *mult-cancel-left* [*THEN iffD1*, *THEN make-pos-rule*]

lemma *zdiv-mult-self*: $m \neq (0 :: int) \Longrightarrow (a + m * n)\ div\ m = a\ div\ m + n$
apply (*rule mcl*)
prefer *2*
apply (*erule asm-rl*)
apply (*simp add*: *zmde ring-distribs*)
done

lemma *eqne*: *equiv A r* $\Longrightarrow$ *X* : *A* // *r* $\Longrightarrow$ *X* $\neq$ {}
unfolding *equiv-def refl-on-def quotient-def Image-def* **by** *auto*

lemmas *Rep-Integ-ne* = *Integ.Rep-Integ*
[*THEN equiv-intrel* [*THEN eqne*, *simplified Integ-def* [*symmetric*]], *standard*]

lemmas *riq* = *Integ.Rep-Integ* [*simplified Integ-def*]
lemmas *intrel-refl* = *refl* [*THEN equiv-intrel-iff* [*THEN iffD1*], *standard*]
lemmas *Rep-Integ-equiv* = *quotient-eq-iff*
[*OF equiv-intrel riq riq*, *simplified Integ.Rep-Integ-inject*, *standard*]
lemmas *Rep-Integ-same* =
Rep-Integ-equiv [*THEN intrel-refl* [*THEN rev-iffD2*], *standard*]

lemma *RI-int*: (*a*, *0*) : *Rep-Integ* (*int a*)
unfolding *int-def* **by** *auto*

lemmas *RI-intrel* [*simp*] = *UNIV-I* [*THEN quotientI*,
THEN Integ.Abs-Integ-inverse [*simplified Integ-def*], *standard*]

lemma *RI-minus*: (*a*, *b*) : *Rep-Integ x* $\Longrightarrow$ (*b*, *a*) : *Rep-Integ* (− *x*)
apply (*rule-tac z*=*x* **in** *eq-Abs-Integ*)
apply (*clarsimp simp*: *minus*)
done

lemma *RI-add*:
(*a*, *b*) : *Rep-Integ x* $\Longrightarrow$ (*c*, *d*) : *Rep-Integ y* $\Longrightarrow$
(*a* + *c*, *b* + *d*) : *Rep-Integ* (*x* + *y*)
apply (*rule-tac z*=*x* **in** *eq-Abs-Integ*)
apply (*rule-tac z*=*y* **in** *eq-Abs-Integ*)
apply (*clarsimp simp*: *add*)
done

lemma *mem-same*: *a* : *S* $\Longrightarrow$ *a* = *b* $\Longrightarrow$ *b* : *S*
by *fast*

lemma *RI-eq-diff'*: (*a*, *b*) : *Rep-Integ* (*int a* − *int b*)
apply (*unfold diff-def*)
apply (*rule mem-same*)
apply (*rule RI-minus RI-add RI-int*)+
apply *simp*

done

lemma *RI-eq-diff*: $((a,\ b) : \mathit{Rep\text{-}Integ}\ x) = (\mathit{int}\ a - \mathit{int}\ b = x)$
apply *safe*
apply (*rule Rep-Integ-same*)
prefer *2*
apply (*erule asm-rl*)
apply (*rule RI-eq-diff'*)+
done

lemma *mod-power-lem*:
$a > 1 \Longrightarrow a \hat{}\ n \bmod a \hat{}\ m = (\mathit{if}\ m <= n\ \mathit{then}\ 0\ \mathit{else}\ (a :: \mathit{int}) \hat{}\ n)$
apply *clarsimp*
apply *safe*
apply (*simp add*: *dvd-eq-mod-eq-0* [*symmetric*])
apply (*drule le-iff-add* [*THEN iffD1*])
apply (*force simp*: *zpower-zadd-distrib*)
apply (*rule mod-pos-pos-trivial*)
apply (*simp*)
apply (*rule power-strict-increasing*)
apply *auto*
done

lemma *min-pm* [*simp*]: $\mathit{min}\ a\ b + (a - b) = (a :: \mathit{nat})$ **by** *arith*

lemmas *min-pm1* [*simp*] = *trans* [*OF add-commute min-pm*]

lemma *rev-min-pm* [*simp*]: $\mathit{min}\ b\ a + (a - b) = (a{::}\mathit{nat})$ **by** *arith*

lemmas *rev-min-pm1* [*simp*] = *trans* [*OF add-commute rev-min-pm*]

lemma *pl-pl-rels*:
$a + b = c + d \Longrightarrow$
$a >= c\ \&\ b <= d \mid a <= c\ \&\ b >= (d :: \mathit{nat})$ **by** *arith*

lemmas *pl-pl-rels'* = *add-commute* [*THEN* [*2*] *trans*, *THEN pl-pl-rels*]

lemma *minus-eq*: $(m - k = m) = (k = 0 \mid m = (0 :: \mathit{nat}))$ **by** *arith*

lemma *pl-pl-mm*: $(a :: \mathit{nat}) + b = c + d \Longrightarrow a - c = d - b$ **by** *arith*

lemmas *pl-pl-mm'* = *add-commute* [*THEN* [*2*] *trans*, *THEN pl-pl-mm*]

lemma *min-minus* [*simp*] : $\mathit{min}\ m\ (m - k) = (m - k :: \mathit{nat})$ **by** *arith*

lemmas *min-minus'* [*simp*] = *trans* [*OF min-max.inf-commute min-minus*]

lemma *nat-no-eq-iff*:
$(\mathit{number\text{-}of}\ b :: \mathit{int}) >= 0 \Longrightarrow (\mathit{number\text{-}of}\ c :: \mathit{int}) >= 0 \Longrightarrow$

```
  (number-of b = (number-of c :: nat)) = (b = c)
 apply (unfold nat-number-of-def)
 apply safe
 apply (drule (2) eq-nat-nat-iff [THEN iffD1])
 apply (simp add: number-of-eq)
 done

lemmas dme = box-equals [OF div-mod-equality add-0-right add-0-right]
lemmas dtle = xtr3 [OF dme [symmetric] le-add1]
lemmas th2 = order-trans [OF order-refl [THEN [2] mult-le-mono] dtle]

lemma td-gal:
  0 < c ==> (a >= b * c) = (a div c >= (b :: nat))
 apply safe
  apply (erule (1) xtr4 [OF div-le-mono div-mult-self-is-m])
 apply (erule th2)
 done

lemmas td-gal-lt = td-gal [simplified not-less [symmetric], simplified]

lemma div-mult-le: (a :: nat) div b * b <= a
 apply (cases b)
  prefer 2
  apply (rule order-refl [THEN th2])
 apply auto
 done

lemmas sdl = split-div-lemma [THEN iffD1, symmetric]

lemma given-quot: f > (0 :: nat) ==> (f * l + (f - 1)) div f = l
 by (rule sdl, assumption) (simp (no-asm))

lemma given-quot-alt: f > (0 :: nat) ==> (l * f + f - Suc 0) div f = l
 apply (frule given-quot)
 apply (rule trans)
  prefer 2
  apply (erule asm-rl)
 apply (rule-tac f=%n. n div f in arg-cong)
 apply (simp add : mult-ac)
 done

lemma diff-mod-le: (a::nat) < d ==> b dvd d ==> a - a mod b <= d - b
 apply (unfold dvd-def)
 apply clarify
 apply (case-tac k)
  apply clarsimp
 apply clarify
 apply (cases b > 0)
  apply (drule mult-commute [THEN xtr1])
```

apply (*frule* (*1*) *td-gal-lt* [*THEN iffD1*])
apply (*clarsimp simp*: *le-simps*)
apply (*rule mult-div-cancel* [*THEN* [*2*] *xtr4*])
apply (*rule mult-mono*)
apply *auto*
done

lemma *less-le-mult'*:
$w * c < b * c ==> 0 \leq c ==> (w + 1) * c \leq b * (c::int)$
apply (*rule mult-right-mono*)
apply (*rule zless-imp-add1-zle*)
apply (*erule* (*1*) *mult-right-less-imp-less*)
apply *assumption*
done

lemmas *less-le-mult* = *less-le-mult'* [*simplified left-distrib*, *simplified*]

lemmas *less-le-mult-minus* = *iffD2* [*OF le-diff-eq less-le-mult*,
simplified left-diff-distrib, *standard*]

lemma *lrlem'*:
assumes *d*: $(i::nat) \leq j \vee m < j'$
assumes *R1*: $i * k \leq j * k \Longrightarrow R$
assumes *R2*: $Suc\ m * k' \leq j' * k' \Longrightarrow R$
shows *R* **using** *d*
apply *safe*
apply (*rule R1*, *erule mult-le-mono1*)
apply (*rule R2*, *erule Suc-le-eq* [*THEN iffD2* [*THEN mult-le-mono1*]])
done

lemma *lrlem*: $(0::nat) < sc ==>$
$(sc - n + (n + lb * n) <= m * n) = (sc + lb * n <= m * n)$
apply *safe*
apply *arith*
apply (*case-tac* $sc >= n$)
apply *arith*
apply (*insert linorder-le-less-linear* [*of m lb*])
apply (*erule-tac* $k=n$ **and** $k'=n$ **in** *lrlem'*)
apply *arith*
apply *simp*
done

lemma *gen-minus*: $0 < n ==> f\ n = f\ (Suc\ (n - 1))$
by *auto*

lemma *mpl-lem*: $j <= (i :: nat) ==> k < j ==> i - j + k < i$ **by** *arith*

lemma *nonneg-mod-div*:
$0 <= a ==> 0 <= b ==> 0 <= (a\ mod\ b :: int)\ \&\ 0 <= a\ div\ b$

apply (*cases b = 0*, *clarsimp*)
apply (*auto intro*: *pos-imp-zdiv-nonneg-iff* [*THEN iffD2*])
done

end

4 BinGeneral: Basic Definitions for Binary Integers

theory *BinGeneral*
imports *Num-Lemmas*
begin

4.1 Further properties of numerals

datatype *bit* = *B0* | *B1*

definition
Bit :: *int* ⇒ *bit* ⇒ *int* (**infixl** *BIT* 90) **where**
k BIT b = (*case b of B0* ⇒ *0* | *B1* ⇒ *1*) + *k* + *k*

lemma *BIT-B0-eq-Bit0* [*simp*]: *w BIT B0* = *Int.Bit0 w*
unfolding *Bit-def Bit0-def* **by** *simp*

lemma *BIT-B1-eq-Bit1* [*simp*]: *w BIT B1* = *Int.Bit1 w*
unfolding *Bit-def Bit1-def* **by** *simp*

lemmas *BIT-simps* = *BIT-B0-eq-Bit0 BIT-B1-eq-Bit1*

hide-const (**open**) *B0 B1*

lemma *Min-ne-Pls* [*iff*]:
Int.Min ~= *Int.Pls*
unfolding *Min-def Pls-def* **by** *auto*

lemmas *Pls-ne-Min* [*iff*] = *Min-ne-Pls* [*symmetric*]

lemmas *PlsMin-defs* [*intro*!] =
Pls-def Min-def Pls-def [*symmetric*] *Min-def* [*symmetric*]

lemmas *PlsMin-simps* [*simp*] = *PlsMin-defs* [*THEN Eq-TrueI*]

lemma *number-of-False-cong*:
False ⟹ *number-of x* = *number-of y*
by (*rule FalseE*)

lemma *BIT-eq*: *u BIT b = v BIT c ==> u = v & b = c*
apply (*unfold Bit-def*)
apply (*simp* (*no-asm-use*) *split*: *bit.split-asm*)
apply *simp-all*
apply (*drule-tac f=even* **in** *arg-cong*, *clarsimp*)+
done

lemmas *BIT-eqE* [*elim*!] = *BIT-eq* [*THEN conjE*, *standard*]

lemma *BIT-eq-iff* [*simp*]:
(*u BIT b = v BIT c*) = (*u = v ∧ b = c*)
by (*rule iffI*) *auto*

lemmas *BIT-eqI* [*intro*!] = *conjI* [*THEN BIT-eq-iff* [*THEN iffD2*]]

lemma *less-Bits*:
(*v BIT b < w BIT c*) = (*v < w | v <= w & b = bit.B0 & c = bit.B1*)
unfolding *Bit-def* **by** (*auto split*: *bit.split*)

lemma *le-Bits*:
(*v BIT b <= w BIT c*) = (*v < w | v <= w & (b ~= bit.B1 | c ~= bit.B0*))
unfolding *Bit-def* **by** (*auto split*: *bit.split*)

lemma *no-no* [*simp*]: *number-of* (*number-of i*) = *i*
unfolding *number-of-eq* **by** *simp*

lemma *Bit-B0*:
k BIT bit.B0 = k + k
by (*unfold Bit-def*) *simp*

lemma *Bit-B1*:
k BIT bit.B1 = k + k + 1
by (*unfold Bit-def*) *simp*

lemma *Bit-B0-2t*: *k BIT bit.B0 = 2 * k*
by (*rule trans*, *rule Bit-B0*) *simp*

lemma *Bit-B1-2t*: *k BIT bit.B1 = 2 * k + 1*
by (*rule trans*, *rule Bit-B1*) *simp*

lemma *B-mod-2'*:
X = 2 ==> (w BIT bit.B1) mod X = 1 & (w BIT bit.B0) mod X = 0
apply (*simp* (*no-asm*) *only*: *Bit-B0 Bit-B1*)
apply (*simp add*: *z1pmod2*)
done

lemma *B1-mod-2* [*simp*]: (*Int.Bit1 w*) *mod 2 = 1*
unfolding *numeral-simps number-of-is-id* **by** (*simp add*: *z1pmod2*)

lemma *B0-mod-2* [*simp*]: (*Int.Bit0 w*) *mod 2 = 0*
unfolding *numeral-simps number-of-is-id* **by** *simp*

lemma *neB1E* [*elim*!]:
assumes *ne*: *y ≠ bit.B1*
assumes *y*: *y = bit.B0 ⟹ P*
shows *P*
apply (*rule y*)
apply (*cases y rule*: *bit.exhaust*, *simp*)
apply (*simp add*: *ne*)
done

lemma *bin-ex-rl*: *EX w b. w BIT b = bin*
apply (*unfold Bit-def*)
apply (*cases even bin*)
apply (*clarsimp simp*: *even-equiv-def*)
apply (*auto simp*: *odd-equiv-def split*: *bit.split*)
done

lemma *bin-exhaust*:
assumes *Q*: *⋀x b. bin = x BIT b ⟹ Q*
shows *Q*
apply (*insert bin-ex-rl* [*of bin*])
apply (*erule exE*)+
apply (*rule Q*)
apply *force*
done

4.2 Destructors for binary integers

definition *bin-rl* :: *int ⇒ int × bit* **where**
[*code del*]: *bin-rl w = (THE (r, l). w = r BIT l)*

lemma *bin-rl-char*: (*bin-rl w = (r, l)*) = (*r BIT l = w*)
apply (*unfold bin-rl-def*)
apply *safe*
apply (*cases w rule*: *bin-exhaust*)
apply *auto*
done

definition
bin-rest-def [*code del*]: *bin-rest w = fst* (*bin-rl w*)

definition
bin-last-def [*code del*] : *bin-last w = snd* (*bin-rl w*)

primrec *bin-nth* **where**
Z: *bin-nth w 0 =* (*bin-last w = bit.B1*)

| *Suc*: *bin-nth w* (*Suc n*) = *bin-nth* (*bin-rest w*) *n*

lemma *bin-rl*: *bin-rl w* = (*bin-rest w*, *bin-last w*)
unfolding *bin-rest-def bin-last-def* **by** *auto*

lemma *bin-rl-simps* [*simp*]:
bin-rl Int.Pls = (*Int.Pls*, *bit.B0*)
bin-rl Int.Min = (*Int.Min*, *bit.B1*)
bin-rl (*Int.Bit0 r*) = (*r*, *bit.B0*)
bin-rl (*Int.Bit1 r*) = (*r*, *bit.B1*)
bin-rl (*r BIT b*) = (*r*, *b*)
unfolding *bin-rl-char* **by** *simp-all*

declare *bin-rl-simps*(*1*−*4*) [*code*]

lemmas *bin-rl-simp* [*simp*] = *iffD1* [*OF bin-rl-char bin-rl*]

lemma *bin-abs-lem*:
bin = (*w BIT b*) ==> ~ *bin* = *Int.Min* −−> ~ *bin* = *Int.Pls* −−>
nat (*abs w*) < *nat* (*abs bin*)
apply (*clarsimp simp add*: *bin-rl-char*)
apply (*unfold Pls-def Min-def Bit-def*)
apply (*cases b*)
apply (*clarsimp*, *arith*)
apply (*clarsimp*, *arith*)
done

lemma *bin-induct*:
assumes *PPls*: *P Int.Pls*
and *PMin*: *P Int.Min*
and *PBit*: !!*bin bit*. *P bin* ==> *P* (*bin BIT bit*)
shows *P bin*
apply (*rule-tac P*=*P* **and** *a*=*bin* **and** *f1*=*nat o abs*
in *wf-measure* [*THEN wf-induct*])
apply (*simp add*: *measure-def inv-image-def*)
apply (*case-tac x rule*: *bin-exhaust*)
apply (*frule bin-abs-lem*)
apply (*auto simp add* : *PPls PMin PBit*)
done

lemma *numeral-induct*:
assumes *Pls*: *P Int.Pls*
assumes *Min*: *P Int.Min*
assumes *Bit0*: ⋀*w*. ⟦*P w*; *w* ≠ *Int.Pls*⟧ ⟹ *P* (*Int.Bit0 w*)
assumes *Bit1*: ⋀*w*. ⟦*P w*; *w* ≠ *Int.Min*⟧ ⟹ *P* (*Int.Bit1 w*)
shows *P x*
apply (*induct x rule*: *bin-induct*)
apply (*rule Pls*)
apply (*rule Min*)

apply (*case-tac bit*)
apply (*case-tac bin = Int.Pls*)
apply *simp*
apply (*simp add*: *Bit0*)
apply (*case-tac bin = Int.Min*)
apply *simp*
apply (*simp add*: *Bit1*)
done

lemma *bin-rest-simps* [*simp*]:
bin-rest Int.Pls = Int.Pls
bin-rest Int.Min = Int.Min
bin-rest (*Int.Bit0 w*) = *w*
bin-rest (*Int.Bit1 w*) = *w*
bin-rest (*w BIT b*) = *w*
unfolding *bin-rest-def* **by** *auto*

declare *bin-rest-simps*(*1−4*) [*code*]

lemma *bin-last-simps* [*simp*]:
bin-last Int.Pls = bit.B0
bin-last Int.Min = bit.B1
bin-last (*Int.Bit0 w*) = *bit.B0*
bin-last (*Int.Bit1 w*) = *bit.B1*
bin-last (*w BIT b*) = *b*
unfolding *bin-last-def* **by** *auto*

declare *bin-last-simps*(*1−4*) [*code*]

lemma *bin-r-l-extras* [*simp*]:
bin-last 0 = bit.B0
bin-last (*− 1*) = *bit.B1*
bin-last −1 = bit.B1
bin-last 1 = bit.B1
bin-rest 1 = 0
bin-rest 0 = 0
bin-rest (*− 1*) = *− 1*
bin-rest −1 = −1
apply (*unfold number-of-Min*)
apply (*unfold Pls-def* [*symmetric*] *Min-def* [*symmetric*])
apply (*unfold numeral-1-eq-1* [*symmetric*])
apply (*auto simp*: *number-of-eq*)
done

lemma *bin-last-mod*:
bin-last w = (*if w mod 2 = 0 then bit.B0 else bit.B1*)
apply (*case-tac w rule*: *bin-exhaust*)
apply (*case-tac b*)
apply *auto*

done

lemma *bin-rest-div*:
bin-rest w = w div 2
apply (*case-tac w rule*: *bin-exhaust*)
apply (*rule trans*)
apply *clarsimp*
apply (*rule refl*)
apply (*drule trans*)
apply (*rule Bit-def*)
apply (*simp add*: *z1pdiv2 split*: *bit.split*)
done

lemma *Bit-div2* [*simp*]: *(w BIT b) div 2 = w*
unfolding *bin-rest-div* [*symmetric*] **by** *auto*

lemma *Bit0-div2* [*simp*]: *(Int.Bit0 w) div 2 = w*
using *Bit-div2* [**where** *b=bit.B0*] **by** *simp*

lemma *Bit1-div2* [*simp*]: *(Int.Bit1 w) div 2 = w*
using *Bit-div2* [**where** *b=bit.B1*] **by** *simp*

lemma *bin-nth-lem* [*rule-format*]:
ALL y. bin-nth x = bin-nth y −−> x = y
apply (*induct x rule*: *bin-induct*)
apply *safe*
apply (*erule rev-mp*)
apply (*induct-tac y rule*: *bin-induct*)
apply (*safe del*: *subset-antisym*)
apply (*drule-tac x=0* **in** *fun-cong*, *force*)
apply (*erule notE*, *rule ext*,
drule-tac x=Suc x **in** *fun-cong*, *force*)
apply (*drule-tac x=0* **in** *fun-cong*, *force*)
apply (*erule rev-mp*)
apply (*induct-tac y rule*: *bin-induct*)
apply (*safe del*: *subset-antisym*)
apply (*drule-tac x=0* **in** *fun-cong*, *force*)
apply (*erule notE*, *rule ext*,
drule-tac x=Suc x **in** *fun-cong*, *force*)
apply (*drule-tac x=0* **in** *fun-cong*, *force*)
apply (*case-tac y rule*: *bin-exhaust*)
apply *clarify*
apply (*erule allE*)
apply (*erule impE*)
prefer *2*
apply (*erule BIT-eqI*)
apply (*drule-tac x=0* **in** *fun-cong*, *force*)
apply (*rule ext*)
apply (*drule-tac x=Suc ?x* **in** *fun-cong*, *force*)

done

lemma *bin-nth-eq-iff*: $(bin\text{-}nth\ x = bin\text{-}nth\ y) = (x = y)$
by (*auto elim*: *bin-nth-lem*)

lemmas *bin-eqI* = *ext* [*THEN bin-nth-eq-iff* [*THEN iffD1*], *standard*]

lemma *bin-nth-Pls* [*simp*]: $\sim bin\text{-}nth\ Int.Pls\ n$
by (*induct n*) *auto*

lemma *bin-nth-Min* [*simp*]: *bin-nth Int.Min n*
by (*induct n*) *auto*

lemma *bin-nth-0-BIT*: $bin\text{-}nth\ (w\ BIT\ b)\ 0 = (b = bit.B1)$
by *auto*

lemma *bin-nth-Suc-BIT*: $bin\text{-}nth\ (w\ BIT\ b)\ (Suc\ n) = bin\text{-}nth\ w\ n$
by *auto*

lemma *bin-nth-minus* [*simp*]: $0 < n \Longrightarrow bin\text{-}nth\ (w\ BIT\ b)\ n = bin\text{-}nth\ w\ (n - 1)$
by (*cases n*) *auto*

lemma *bin-nth-minus-Bit0* [*simp*]:
$0 < n \Longrightarrow bin\text{-}nth\ (Int.Bit0\ w)\ n = bin\text{-}nth\ w\ (n - 1)$
using *bin-nth-minus* [**where** *b*=*bit.B0*] **by** *simp*

lemma *bin-nth-minus-Bit1* [*simp*]:
$0 < n \Longrightarrow bin\text{-}nth\ (Int.Bit1\ w)\ n = bin\text{-}nth\ w\ (n - 1)$
using *bin-nth-minus* [**where** *b*=*bit.B1*] **by** *simp*

lemmas *bin-nth-0* = *bin-nth.simps*(*1*)
lemmas *bin-nth-Suc* = *bin-nth.simps*(*2*)

lemmas *bin-nth-simps* =
bin-nth-0 bin-nth-Suc bin-nth-Pls bin-nth-Min bin-nth-minus
bin-nth-minus-Bit0 bin-nth-minus-Bit1

4.3 Recursion combinator for binary integers

lemma *brlem*: $(bin = Int.Min) = (-\ bin + Int.pred\ 0 = 0)$
unfolding *Min-def pred-def* **by** *arith*

function
$bin\text{-}rec :: 'a \Rightarrow 'a \Rightarrow (int \Rightarrow bit \Rightarrow 'a \Rightarrow 'a) \Rightarrow int \Rightarrow 'a$
where
bin-rec f1 f2 f3 bin = (*if bin* = *Int.Pls then f1*
else if bin = *Int.Min then f2*
else case bin-rl bin of (*w*, *b*) => *f3 w b* (*bin-rec f1 f2 f3 w*))

by *pat-completeness auto*

termination
apply (*relation measure* (*nat o abs o snd o snd o snd*))
apply *simp*
apply (*simp add*: *Pls-def brlem*)
apply (*clarsimp simp*: *bin-rl-char pred-def*)
apply (*frule thin-rl* [*THEN refl* [*THEN bin-abs-lem* [*rule-format*]]])
apply (*unfold Pls-def Min-def number-of-eq*)
prefer *2*
apply (*erule asm-rl*)
apply *auto*
done

declare *bin-rec.simps* [*simp del*]

lemma *bin-rec-PM*:
f = *bin-rec f1 f2 f3* ==> *f Int.Pls* = *f1* & *f Int.Min* = *f2*
by (*auto simp add*: *bin-rec.simps*)

lemma *bin-rec-Pls*: *bin-rec f1 f2 f3 Int.Pls* = *f1*
by (*simp add*: *bin-rec.simps*)

lemma *bin-rec-Min*: *bin-rec f1 f2 f3 Int.Min* = *f2*
by (*simp add*: *bin-rec.simps*)

lemma *bin-rec-Bit0*:
f3 Int.Pls bit.B0 f1 = *f1* ⟹
bin-rec f1 f2 f3 (*Int.Bit0 w*) = *f3 w bit.B0* (*bin-rec f1 f2 f3 w*)
by (*simp add*: *bin-rec-Pls bin-rec.simps* [*of - - - Int.Bit0 w*])

lemma *bin-rec-Bit1*:
f3 Int.Min bit.B1 f2 = *f2* ⟹
bin-rec f1 f2 f3 (*Int.Bit1 w*) = *f3 w bit.B1* (*bin-rec f1 f2 f3 w*)
by (*simp add*: *bin-rec-Min bin-rec.simps* [*of - - - Int.Bit1 w*])

lemma *bin-rec-Bit*:
f = *bin-rec f1 f2 f3* ==> *f3 Int.Pls bit.B0 f1* = *f1* ==>
f3 Int.Min bit.B1 f2 = *f2* ==> *f* (*w BIT b*) = *f3 w b* (*f w*)
by (*cases b*, *simp add*: *bin-rec-Bit0*, *simp add*: *bin-rec-Bit1*)

lemmas *bin-rec-simps* = *refl* [*THEN bin-rec-Bit*] *bin-rec-Pls bin-rec-Min*
bin-rec-Bit0 bin-rec-Bit1

4.4 Truncating binary integers

definition
bin-sign-def [*code del*] : *bin-sign* = *bin-rec Int.Pls Int.Min* (%*w b s*. *s*)

lemma *bin-sign-simps* [*simp*]:
bin-sign Int.Pls = Int.Pls
bin-sign Int.Min = Int.Min
bin-sign (Int.Bit0 w) = bin-sign w
bin-sign (Int.Bit1 w) = bin-sign w
bin-sign (w BIT b) = bin-sign w
unfolding *bin-sign-def* **by** (*auto simp*: *bin-rec-simps*)

declare *bin-sign-simps*(*1−4*) [*code*]

lemma *bin-sign-rest* [*simp*]:
bin-sign (*bin-rest w*) = (*bin-sign w*)
by (*cases w rule*: *bin-exhaust*) *auto*

consts
bintrunc :: *nat* => *int* => *int*
primrec
Z : *bintrunc 0 bin = Int.Pls*
Suc : *bintrunc* (*Suc n*) *bin* = *bintrunc n* (*bin-rest bin*) *BIT* (*bin-last bin*)

consts
sbintrunc :: *nat* => *int* => *int*
primrec
Z : *sbintrunc 0 bin* =
(*case bin-last bin of bit.B1* => *Int.Min* | *bit.B0* => *Int.Pls*)
Suc : *sbintrunc* (*Suc n*) *bin* = *sbintrunc n* (*bin-rest bin*) *BIT* (*bin-last bin*)

lemma *sign-bintr*:
!!*w*. *bin-sign* (*bintrunc n w*) = *Int.Pls*
by (*induct n*) *auto*

lemma *bintrunc-mod2p*:
!!*w*. *bintrunc n w* = (*w mod 2* ^ *n* :: *int*)
apply (*induct n*, *clarsimp*)
apply (*simp add*: *bin-last-mod bin-rest-div Bit-def zmod-zmult2-eq*
cong: *number-of-False-cong*)
done

lemma *sbintrunc-mod2p*:
!!*w*. *sbintrunc n w* = ((*w* + *2* ^ *n*) *mod 2* ^ (*Suc n*) − *2* ^ *n* :: *int*)
apply (*induct n*)
apply *clarsimp*
apply (*subst mod-add-left-eq*)
apply (*simp add*: *bin-last-mod*)
apply (*simp add*: *number-of-eq*)
apply *clarsimp*
apply (*simp add*: *bin-last-mod bin-rest-div Bit-def*
cong: *number-of-False-cong*)
apply (*clarsimp simp*: *mod-mult-mult1* [*symmetric*]

zmod-zdiv-equality [*THEN diff-eq-eq* [*THEN iffD2* [*THEN sym*]]])
apply (*rule trans* [*symmetric*, *OF - emep1*])
apply *auto*
apply (*auto simp*: *even-def*)
done

4.5 Simplifications for (s)bintrunc

lemma *bit-bool*:
(*b* = (*b'* = *bit.B1*)) = (*b'* = (*if b then bit.B1 else bit.B0*))
by (*cases b'*) *auto*

lemmas *bit-bool1* [*simp*] = *refl* [*THEN bit-bool* [*THEN iffD1*], *symmetric*]

lemma *bin-sign-lem*:
!!*bin*. (*bin-sign* (*sbintrunc n bin*) = *Int.Min*) = *bin-nth bin n*
apply (*induct n*)
apply (*case-tac bin rule*: *bin-exhaust*, *case-tac b*, *auto*)+
done

lemma *nth-bintr*:
!!*w m*. *bin-nth* (*bintrunc m w*) *n* = (*n* < *m* & *bin-nth w n*)
apply (*induct n*)
apply (*case-tac m*, *auto*)[*1*]
apply (*case-tac m*, *auto*)[*1*]
done

lemma *nth-sbintr*:
!!*w m*. *bin-nth* (*sbintrunc m w*) *n* =
(*if n* < *m then bin-nth w n else bin-nth w m*)
apply (*induct n*)
apply (*case-tac m*, *simp-all split*: *bit.splits*)[*1*]
apply (*case-tac m*, *simp-all split*: *bit.splits*)[*1*]
done

lemma *bin-nth-Bit*:
bin-nth (*w BIT b*) *n* = (*n* = *0* & *b* = *bit.B1* | (*EX m*. *n* = *Suc m* & *bin-nth w m*))
by (*cases n*) *auto*

lemma *bin-nth-Bit0*:
bin-nth (*Int.Bit0 w*) *n* = (*EX m*. *n* = *Suc m* & *bin-nth w m*)
using *bin-nth-Bit* [**where** *b*=*bit.B0*] **by** *simp*

lemma *bin-nth-Bit1*:
bin-nth (*Int.Bit1 w*) *n* = (*n* = *0* | (*EX m*. *n* = *Suc m* & *bin-nth w m*))
using *bin-nth-Bit* [**where** *b*=*bit.B1*] **by** *simp*

lemma *bintrunc-bintrunc-l*:

n <= m ==> (bintrunc m (bintrunc n w) = bintrunc n w)
by (*rule bin-eqI*) (*auto simp add* : *nth-bintr*)

lemma *sbintrunc-sbintrunc-l*:
n <= m ==> (sbintrunc m (sbintrunc n w) = sbintrunc n w)
by (*rule bin-eqI*) (*auto simp*: *nth-sbintr*)

lemma *bintrunc-bintrunc-ge*:
n <= m ==> (bintrunc n (bintrunc m w) = bintrunc n w)
by (*rule bin-eqI*) (*auto simp*: *nth-bintr*)

lemma *bintrunc-bintrunc-min* [*simp*]:
bintrunc m (bintrunc n w) = bintrunc (min m n) w
apply (*rule bin-eqI*)
apply (*auto simp*: *nth-bintr*)
done

lemma *sbintrunc-sbintrunc-min* [*simp*]:
sbintrunc m (sbintrunc n w) = sbintrunc (min m n) w
apply (*rule bin-eqI*)
apply (*auto simp*: *nth-sbintr min-max.inf-absorb1 min-max.inf-absorb2*)
done

lemmas *bintrunc-Pls* =
bintrunc.Suc [**where** *bin=Int.Pls*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *bintrunc-Min* [*simp*] =
bintrunc.Suc [**where** *bin=Int.Min*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *bintrunc-BIT* [*simp*] =
bintrunc.Suc [**where** *bin=w BIT b*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemma *bintrunc-Bit0* [*simp*]:
bintrunc (Suc n) (Int.Bit0 w) = Int.Bit0 (bintrunc n w)
using *bintrunc-BIT* [**where** *b=bit.B0*] **by** *simp*

lemma *bintrunc-Bit1* [*simp*]:
bintrunc (Suc n) (Int.Bit1 w) = Int.Bit1 (bintrunc n w)
using *bintrunc-BIT* [**where** *b=bit.B1*] **by** *simp*

lemmas *bintrunc-Sucs = bintrunc-Pls bintrunc-Min bintrunc-BIT*
bintrunc-Bit0 bintrunc-Bit1

lemmas *sbintrunc-Suc-Pls* =
sbintrunc.Suc [**where** *bin=Int.Pls*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *sbintrunc-Suc-Min* =
sbintrunc.Suc [**where** *bin*=*Int.Min*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemmas *sbintrunc-Suc-BIT* [*simp*] =
sbintrunc.Suc [**where** *bin*=*w BIT b*, *simplified bin-last-simps bin-rest-simps*, *standard*]

lemma *sbintrunc-Suc-Bit0* [*simp*]:
sbintrunc (*Suc n*) (*Int.Bit0 w*) = *Int.Bit0* (*sbintrunc n w*)
using *sbintrunc-Suc-BIT* [**where** *b*=*bit.B0*] **by** *simp*

lemma *sbintrunc-Suc-Bit1* [*simp*]:
sbintrunc (*Suc n*) (*Int.Bit1 w*) = *Int.Bit1* (*sbintrunc n w*)
using *sbintrunc-Suc-BIT* [**where** *b*=*bit.B1*] **by** *simp*

lemmas *sbintrunc-Sucs* = *sbintrunc-Suc-Pls sbintrunc-Suc-Min sbintrunc-Suc-BIT*
sbintrunc-Suc-Bit0 sbintrunc-Suc-Bit1

lemmas *sbintrunc-Pls* =
sbintrunc.Z [**where** *bin*=*Int.Pls*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemmas *sbintrunc-Min* =
sbintrunc.Z [**where** *bin*=*Int.Min*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemmas *sbintrunc-0-BIT-B0* [*simp*] =
sbintrunc.Z [**where** *bin*=*w BIT bit.B0*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemmas *sbintrunc-0-BIT-B1* [*simp*] =
sbintrunc.Z [**where** *bin*=*w BIT bit.B1*,
simplified bin-last-simps bin-rest-simps bit.simps, *standard*]

lemma *sbintrunc-0-Bit0* [*simp*]: *sbintrunc 0* (*Int.Bit0 w*) = *Int.Pls*
using *sbintrunc-0-BIT-B0* **by** *simp*

lemma *sbintrunc-0-Bit1* [*simp*]: *sbintrunc 0* (*Int.Bit1 w*) = *Int.Min*
using *sbintrunc-0-BIT-B1* **by** *simp*

lemmas *sbintrunc-0-simps* =
sbintrunc-Pls sbintrunc-Min sbintrunc-0-BIT-B0 sbintrunc-0-BIT-B1
sbintrunc-0-Bit0 sbintrunc-0-Bit1

lemmas *bintrunc-simps* = *bintrunc.Z bintrunc-Sucs*
lemmas *sbintrunc-simps* = *sbintrunc-0-simps sbintrunc-Sucs*

lemma *bintrunc-minus*:
0 < n ==> bintrunc (Suc (n − 1)) w = bintrunc n w
by *auto*

lemma *sbintrunc-minus*:
0 < n ==> sbintrunc (Suc (n − 1)) w = sbintrunc n w
by *auto*

lemmas *bintrunc-minus-simps =*
bintrunc-Sucs [THEN [2] bintrunc-minus [symmetric, THEN trans], standard]
lemmas *sbintrunc-minus-simps =*
sbintrunc-Sucs [THEN [2] sbintrunc-minus [symmetric, THEN trans], standard]

lemma *bintrunc-n-Pls [simp]*:
bintrunc n Int.Pls = Int.Pls
by *(induct n) auto*

lemma *sbintrunc-n-PM [simp]*:
sbintrunc n Int.Pls = Int.Pls
sbintrunc n Int.Min = Int.Min
by *(induct n) auto*

lemmas *thobini1 = arg-cong [***where** *f = %w. w BIT b, standard]*

lemmas *bintrunc-BIT-I = trans [OF bintrunc-BIT thobini1]*
lemmas *bintrunc-Min-I = trans [OF bintrunc-Min thobini1]*

lemmas *bmsts = bintrunc-minus-simps(1−3) [THEN thobini1 [THEN [2] trans], standard]*
lemmas *bintrunc-Pls-minus-I = bmsts(1)*
lemmas *bintrunc-Min-minus-I = bmsts(2)*
lemmas *bintrunc-BIT-minus-I = bmsts(3)*

lemma *bintrunc-0-Min*: *bintrunc 0 Int.Min = Int.Pls*
by *auto*
lemma *bintrunc-0-BIT*: *bintrunc 0 (w BIT b) = Int.Pls*
by *auto*

lemma *bintrunc-Suc-lem*:
bintrunc (Suc n) x = y ==> m = Suc n ==> bintrunc m x = y
by *auto*

lemmas *bintrunc-Suc-Ialts =*
bintrunc-Min-I [THEN bintrunc-Suc-lem, standard]
bintrunc-BIT-I [THEN bintrunc-Suc-lem, standard]

lemmas *sbintrunc-BIT-I = trans [OF sbintrunc-Suc-BIT thobini1]*

lemmas *sbintrunc-Suc-Is =*

sbintrunc-Sucs(*1−3*) [*THEN thobini1* [*THEN* [*2*] *trans*], *standard*]

lemmas *sbintrunc-Suc-minus-Is* =
sbintrunc-minus-simps(*1−3*) [*THEN thobini1* [*THEN* [*2*] *trans*], *standard*]

lemma *sbintrunc-Suc-lem*:
sbintrunc (*Suc n*) *x* = *y* ==> *m* = *Suc n* ==> *sbintrunc m x* = *y*
by *auto*

lemmas *sbintrunc-Suc-Ialts* =
sbintrunc-Suc-Is [*THEN sbintrunc-Suc-lem*, *standard*]

lemma *sbintrunc-bintrunc-lt*:
m > *n* ==> *sbintrunc n* (*bintrunc m w*) = *sbintrunc n w*
by (*rule bin-eqI*) (*auto simp*: *nth-sbintr nth-bintr*)

lemma *bintrunc-sbintrunc-le*:
m <= *Suc n* ==> *bintrunc m* (*sbintrunc n w*) = *bintrunc m w*
apply (*rule bin-eqI*)
apply (*auto simp*: *nth-sbintr nth-bintr*)
apply (*subgoal-tac x=n*, *safe*, *arith+*)[*1*]
apply (*subgoal-tac x=n*, *safe*, *arith+*)[*1*]
done

lemmas *bintrunc-sbintrunc* [*simp*] = *order-refl* [*THEN bintrunc-sbintrunc-le*]
lemmas *sbintrunc-bintrunc* [*simp*] = *lessI* [*THEN sbintrunc-bintrunc-lt*]
lemmas *bintrunc-bintrunc* [*simp*] = *order-refl* [*THEN bintrunc-bintrunc-l*]
lemmas *sbintrunc-sbintrunc* [*simp*] = *order-refl* [*THEN sbintrunc-sbintrunc-l*]

lemma *bintrunc-sbintrunc'* [*simp*]:
0 < *n* ⟹ *bintrunc n* (*sbintrunc* (*n* − *1*) *w*) = *bintrunc n w*
by (*cases n*) (*auto simp del*: *bintrunc.Suc*)

lemma *sbintrunc-bintrunc'* [*simp*]:
0 < *n* ⟹ *sbintrunc* (*n* − *1*) (*bintrunc n w*) = *sbintrunc* (*n* − *1*) *w*
by (*cases n*) (*auto simp del*: *bintrunc.Suc*)

lemma *bin-sbin-eq-iff*:
bintrunc (*Suc n*) *x* = *bintrunc* (*Suc n*) *y* <->
sbintrunc n x = *sbintrunc n y*
apply (*rule iffI*)
apply (*rule box-equals* [*OF - sbintrunc-bintrunc sbintrunc-bintrunc*])
apply *simp*
apply (*rule box-equals* [*OF - bintrunc-sbintrunc bintrunc-sbintrunc*])
apply *simp*
done

lemma *bin-sbin-eq-iff'*:
0 < *n* ⟹ *bintrunc n x* = *bintrunc n y* <->

sbintrunc (*n* − *1*) *x* = *sbintrunc* (*n* − *1*) *y*
by (*cases n*) (*simp-all add*: *bin-sbin-eq-iff del*: *bintrunc.Suc*)

lemmas *bintrunc-sbintruncS0* [*simp*] = *bintrunc-sbintrunc'* [*unfolded One-nat-def*]
lemmas *sbintrunc-bintruncS0* [*simp*] = *sbintrunc-bintrunc'* [*unfolded One-nat-def*]

lemmas *bintrunc-bintrunc-l'* = *le-add1* [*THEN bintrunc-bintrunc-l*]
lemmas *sbintrunc-sbintrunc-l'* = *le-add1* [*THEN sbintrunc-sbintrunc-l*]

lemmas *nat-non0-gr* =
trans [*OF iszero-def* [*THEN Not-eq-iff* [*THEN iffD2*]] *refl*, *standard*]

lemmas *bintrunc-pred-simps* [*simp*] =
bintrunc-minus-simps [*of number-of bin*, *simplified nobm1*, *standard*]

lemmas *sbintrunc-pred-simps* [*simp*] =
sbintrunc-minus-simps [*of number-of bin*, *simplified nobm1*, *standard*]

lemma *no-bintr-alt*:
number-of (*bintrunc n w*) = *w mod 2* ^ *n*
by (*simp add*: *number-of-eq bintrunc-mod2p*)

lemma *no-bintr-alt1*: *bintrunc n* = (%*w*. *w mod 2* ^ *n* :: *int*)
by (*rule ext*) (*rule bintrunc-mod2p*)

lemma *range-bintrunc*: *range* (*bintrunc n*) = {*i*. *0* <= *i* & *i* < *2* ^ *n*}
apply (*unfold no-bintr-alt1*)
apply (*auto simp add*: *image-iff*)
apply (*rule exI*)
apply (*auto intro*: *int-mod-lem* [*THEN iffD1*, *symmetric*])
done

lemma *no-bintr*:
number-of (*bintrunc n w*) = (*number-of w mod 2* ^ *n* :: *int*)
by (*simp add* : *bintrunc-mod2p number-of-eq*)

lemma *no-sbintr-alt2*:
sbintrunc n = (%*w*. (*w* + *2* ^ *n*) *mod 2* ^ *Suc n* − *2* ^ *n* :: *int*)
by (*rule ext*) (*simp add* : *sbintrunc-mod2p*)

lemma *no-sbintr*:
number-of (*sbintrunc n w*) =
((*number-of w* + *2* ^ *n*) *mod 2* ^ *Suc n* − *2* ^ *n* :: *int*)
by (*simp add* : *no-sbintr-alt2 number-of-eq*)

lemma *range-sbintrunc*:
range (*sbintrunc n*) = {*i*. − (*2* ^ *n*) <= *i* & *i* < *2* ^ *n*}

apply (*unfold no-sbintr-alt2*)
apply (*auto simp add*: *image-iff eq-diff-eq*)
apply (*rule exI*)
apply (*auto intro*: *int-mod-lem* [*THEN iffD1*, *symmetric*])
done

lemma *sb-inc-lem*:
(*a*::*int*) + 2^*k* < 0 ⟹ *a* + 2^*k* + 2^(*Suc k*) <= (*a* + 2^*k*) *mod* 2^(*Suc k*)
apply (*erule int-mod-ge'* [**where** *n* = 2 ^ (*Suc k*) **and** *b* = *a* + 2 ^ *k*, *simplified zless2p*])
apply (*rule TrueI*)
done

lemma *sb-inc-lem'*:
(*a*::*int*) < − (2^*k*) ⟹ *a* + 2^*k* + 2^(*Suc k*) <= (*a* + 2^*k*) *mod* 2^(*Suc k*)
by (*rule sb-inc-lem*) *simp*

lemma *sbintrunc-inc*:
x < − (2^*n*) ==> *x* + 2^(*Suc n*) <= *sbintrunc n x*
unfolding *no-sbintr-alt2* **by** (*drule sb-inc-lem'*) *simp*

lemma *sb-dec-lem*:
(*0*::*int*) <= − (2^*k*) + *a* ==> (*a* + 2^*k*) *mod* (2 * 2 ^ *k*) <= − (2 ^ *k*) + *a*
by (*rule int-mod-le'* [**where** *n* = 2 ^ (*Suc k*) **and** *b* = *a* + 2 ^ *k*,
simplified zless2p, *OF - TrueI*, *simplified*])

lemma *sb-dec-lem'*:
(*2*::*int*) ^ *k* <= *a* ==> (*a* + 2 ^ *k*) *mod* (2 * 2 ^ *k*) <= − (2 ^ *k*) + *a*
by (*rule iffD1* [*OF diff-le-eq'*, *THEN sb-dec-lem*, *simplified*])

lemma *sbintrunc-dec*:
x >= (2 ^ *n*) ==> *x* − 2 ^ (*Suc n*) >= *sbintrunc n x*
unfolding *no-sbintr-alt2* **by** (*drule sb-dec-lem'*) *simp*

lemmas *zmod-uminus'* = *zmod-uminus* [**where** *b*=*c*, *standard*]
lemmas *zpower-zmod'* = *zpower-zmod* [**where** *m*=*c* **and** *y*=*k*, *standard*]

lemmas *brdmod1s'* [*symmetric*] =
mod-add-left-eq mod-add-right-eq
zmod-zsub-left-eq zmod-zsub-right-eq
zmod-zmult1-eq zmod-zmult1-eq-rev

lemmas *brdmods'* [*symmetric*] =
zpower-zmod' [*symmetric*]
trans [*OF mod-add-left-eq mod-add-right-eq*]
trans [*OF zmod-zsub-left-eq zmod-zsub-right-eq*]
trans [*OF zmod-zmult1-eq zmod-zmult1-eq-rev*]
zmod-uminus' [*symmetric*]
mod-add-left-eq [**where** *b* = *1*::*int*]

zmod-zsub-left-eq [**where** *b = 1*]

lemmas *bintr-arith1s =*
brdmod1s' [**where** *c=2^n::int*, *folded pred-def succ-def bintrunc-mod2p*, *standard*]
lemmas *bintr-ariths =*
brdmods' [**where** *c=2^n::int*, *folded pred-def succ-def bintrunc-mod2p*, *standard*]

lemmas *m2pths = pos-mod-sign pos-mod-bound* [*OF zless2p*, *standard*]

lemma *bintr-ge0*: *(0 :: int) <= number-of (bintrunc n w)*
by (*simp add : no-bintr m2pths*)

lemma *bintr-lt2p*: *number-of (bintrunc n w) < (2 ^ n :: int)*
by (*simp add : no-bintr m2pths*)

lemma *bintr-Min*:
number-of (bintrunc n Int.Min) = (2 ^ n :: int) - 1
by (*simp add : no-bintr m1mod2k*)

lemma *sbintr-ge*: *(- (2 ^ n) :: int) <= number-of (sbintrunc n w)*
by (*simp add : no-sbintr m2pths*)

lemma *sbintr-lt*: *number-of (sbintrunc n w) < (2 ^ n :: int)*
by (*simp add : no-sbintr m2pths*)

lemma *bintrunc-Suc*:
bintrunc (Suc n) bin = bintrunc n (bin-rest bin) BIT bin-last bin
by (*case-tac bin rule*: *bin-exhaust*) *auto*

lemma *sign-Pls-ge-0*:
(bin-sign bin = Int.Pls) = (number-of bin >= (0 :: int))
by (*induct bin rule*: *numeral-induct*) *auto*

lemma *sign-Min-lt-0*:
(bin-sign bin = Int.Min) = (number-of bin < (0 :: int))
by (*induct bin rule*: *numeral-induct*) *auto*

lemmas *sign-Min-neg = trans* [*OF sign-Min-lt-0 neg-def* [*symmetric*]]

lemma *bin-rest-trunc*:
!!bin. (bin-rest (bintrunc n bin)) = bintrunc (n - 1) (bin-rest bin)
by (*induct n*) *auto*

lemma *bin-rest-power-trunc* [*rule-format*] :
(bin-rest ^^ k) (bintrunc n bin) =
bintrunc (n - k) ((bin-rest ^^ k) bin)
by (*induct k*) (*auto simp*: *bin-rest-trunc*)

lemma *bin-rest-trunc-i*:

bintrunc n (bin-rest bin) = bin-rest (bintrunc (Suc n) bin)
by *auto*

lemma *bin-rest-strunc*:
!!bin. bin-rest (sbintrunc (Suc n) bin) = sbintrunc n (bin-rest bin)
by *(induct n) auto*

lemma *bintrunc-rest [simp]*:
!!bin. bintrunc n (bin-rest (bintrunc n bin)) = bin-rest (bintrunc n bin)
apply *(induct n, simp)*
apply *(case-tac bin rule: bin-exhaust)*
apply *(auto simp: bintrunc-bintrunc-l)*
done

lemma *sbintrunc-rest [simp]*:
!!bin. sbintrunc n (bin-rest (sbintrunc n bin)) = bin-rest (sbintrunc n bin)
apply *(induct n, simp)*
apply *(case-tac bin rule: bin-exhaust)*
apply *(auto simp: bintrunc-bintrunc-l split: bit.splits)*
done

lemma *bintrunc-rest'*:
bintrunc n o bin-rest o bintrunc n = bin-rest o bintrunc n
by *(rule ext) auto*

lemma *sbintrunc-rest'* :
sbintrunc n o bin-rest o sbintrunc n = bin-rest o sbintrunc n
by *(rule ext) auto*

lemma *rco-lem*:
f o g o f = g o f ==> f o (g o f) ^^ n = g ^^ n o f
apply *(rule ext)*
apply *(induct-tac n)*
apply *(simp-all (no-asm))*
apply *(drule fun-cong)*
apply *(unfold o-def)*
apply *(erule trans)*
apply *simp*
done

lemma *rco-alt*: *(f o g) ^^ n o f = f o (g o f) ^^ n*
apply *(rule ext)*
apply *(induct n)*
apply *(simp-all add: o-def)*
done

lemmas *rco-bintr = bintrunc-rest'*
[THEN rco-lem [THEN fun-cong], unfolded o-def]
lemmas *rco-sbintr = sbintrunc-rest'*

[*THEN rco-lem* [*THEN fun-cong*], *unfolded o-def*]

4.6 Splitting and concatenation

primrec *bin-split* :: *nat* ⇒ *int* ⇒ *int* × *int* **where**
Z: *bin-split 0 w* = (*w*, *Int.Pls*)
| *Suc*: *bin-split* (*Suc n*) *w* = (*let* (*w1*, *w2*) = *bin-split n* (*bin-rest w*)
in (*w1*, *w2 BIT bin-last w*))

primrec *bin-cat* :: *int* ⇒ *nat* ⇒ *int* ⇒ *int* **where**
Z: *bin-cat w 0 v* = *w*
| *Suc*: *bin-cat w* (*Suc n*) *v* = *bin-cat w n* (*bin-rest v*) *BIT bin-last v*

4.7 Miscellaneous lemmas

lemma *funpow-minus-simp*:
0 < *n* ⟹ *f* ^^ *n* = *f* ∘ *f* ^^ (*n* − *1*)
by (*cases n*) *simp-all*

lemmas *funpow-pred-simp* [*simp*] =
funpow-minus-simp [*of number-of bin*, *simplified nobm1*, *standard*]

lemmas *replicate-minus-simp* =
trans [*OF gen-minus* [**where** *f* = %*n*. *replicate n x*] *replicate.replicate-Suc*,
standard]

lemmas *replicate-pred-simp* [*simp*] =
replicate-minus-simp [*of number-of bin*, *simplified nobm1*, *standard*]

lemmas *power-Suc-no* [*simp*] = *power-Suc* [*of number-of a*, *standard*]

lemmas *power-minus-simp* =
trans [*OF gen-minus* [**where** *f* = *power f*] *power-Suc*, *standard*]

lemmas *power-pred-simp* =
power-minus-simp [*of number-of bin*, *simplified nobm1*, *standard*]
lemmas *power-pred-simp-no* [*simp*] = *power-pred-simp* [**where** *f*= *number-of f*,
standard]

lemma *list-exhaust-size-gt0*:
assumes *y*: ⋀*a list*. *y* = *a* # *list* ⟹ *P*
shows *0* < *length y* ⟹ *P*
apply (*cases y*, *simp*)
apply (*rule y*)
apply *fastsimp*
done

lemma *list-exhaust-size-eq0*:
assumes *y*: *y* = [] ⟹ *P*
shows *length y* = *0* ⟹ *P*

apply (*cases y*)
apply (*rule y*, *simp*)
apply *simp*
done

lemma *size-Cons-lem-eq*:
y = xa # list ==> size y = Suc k ==> size list = k
by *auto*

lemma *size-Cons-lem-eq-bin*:
y = xa # list ==> size y = number-of (Int.succ k) ==>
size list = number-of k
by (*auto simp*: *pred-def succ-def split add* : *split-if-asm*)

lemmas *ls-splits* =
prod.split split-split prod.split-asm split-split-asm split-if-asm

lemma *not-B1-is-B0*: *y ≠ bit.B1 ⟹ y = bit.B0*
by (*cases y*) *auto*

lemma *B1-ass-B0*:
assumes *y*: *y = bit.B0 ⟹ y = bit.B1*
shows *y = bit.B1*
apply (*rule classical*)
apply (*drule not-B1-is-B0*)
apply (*erule y*)
done

— simplifications for specific word lengths
lemmas *n2s-ths* [*THEN eq-reflection*] = *add-2-eq-Suc add-2-eq-Suc'*

lemmas *s2n-ths* = *n2s-ths* [*symmetric*]

end

5 BitSyntax: Syntactic classes for bitwise operations

theory *BitSyntax*
imports *BinGeneral*
begin

class *bit* =
fixes *bitNOT* :: *'a ⇒ 'a* (*NOT - [70] 71*)
and *bitAND* :: *'a ⇒ 'a ⇒ 'a* (**infixr** *AND 64*)
and *bitOR* :: *'a ⇒ 'a ⇒ 'a* (**infixr** *OR 59*)
and *bitXOR* :: *'a ⇒ 'a ⇒ 'a* (**infixr** *XOR 59*)

We want the bitwise operations to bind slightly weaker than + and −, but $^{\sim\sim}$ to bind slightly stronger than $*$.

Testing and shifting operations.

class *bits* = *bit* +
  fixes *test-bit* :: *'a* ⇒ *nat* ⇒ *bool* (**infixl** !! *100*)
    and *lsb* :: *'a* ⇒ *bool*
    and *set-bit* :: *'a* ⇒ *nat* ⇒ *bool* ⇒ *'a*
    and *set-bits* :: (*nat* ⇒ *bool*) ⇒ *'a* (**binder** *BITS* *10*)
    and *shiftl* :: *'a* ⇒ *nat* ⇒ *'a* (**infixl** << *55*)
    and *shiftr* :: *'a* ⇒ *nat* ⇒ *'a* (**infixl** >> *55*)

class *bitss* = *bits* +
  fixes *msb* :: *'a* ⇒ *bool*

5.1 Bitwise operations on *bit*

instantiation *bit* :: *bit*
begin

primrec *bitNOT-bit* **where**
  NOT bit.B0 = *bit.B1*
  | *NOT bit.B1* = *bit.B0*

primrec *bitAND-bit* **where**
  bit.B0 AND y = *bit.B0*
  | *bit.B1 AND y* = *y*

primrec *bitOR-bit* **where**
  bit.B0 OR y = *y*
  | *bit.B1 OR y* = *bit.B1*

primrec *bitXOR-bit* **where**
  bit.B0 XOR y = *y*
  | *bit.B1 XOR y* = *NOT y*

instance **..**

end

lemmas *bit-simps* =
  bitNOT-bit.simps bitAND-bit.simps bitOR-bit.simps bitXOR-bit.simps

lemma *bit-extra-simps* [*simp*]:
  x AND bit.B0 = *bit.B0*
  x AND bit.B1 = *x*
  x OR bit.B1 = *bit.B1*
  x OR bit.B0 = *x*
  x XOR bit.B1 = *NOT x*

x XOR bit.B0 = x
by (*cases x*, *auto*)+

lemma *bit-ops-comm*:
(*x*::*bit*) *AND y = y AND x*
(*x*::*bit*) *OR y = y OR x*
(*x*::*bit*) *XOR y = y XOR x*
by (*cases y*, *auto*)+

lemma *bit-ops-same* [*simp*]:
(*x*::*bit*) *AND x = x*
(*x*::*bit*) *OR x = x*
(*x*::*bit*) *XOR x = bit.B0*
by (*cases x*, *auto*)+

lemma *bit-not-not* [*simp*]: *NOT* (*NOT* (*x*::*bit*)) = *x*
by (*cases x*) *auto*

end

6 BinOperations: Bitwise Operations on Binary Integers

theory *BinOperations*
imports *BinGeneral BitSyntax*
begin

6.1 Logical operations

bit-wise logical operations on the int type

instantiation *int* :: *bit*
begin

definition
int-not-def [*code del*]: *bitNOT = bin-rec Int.Min Int.Pls*
(λ*w b s*. *s BIT* (*NOT b*))

definition
int-and-def [*code del*]: *bitAND = bin-rec* (λ*x*. *Int.Pls*) (λ*y*. *y*)
(λ*w b s y*. *s* (*bin-rest y*) *BIT* (*b AND bin-last y*))

definition
int-or-def [*code del*]: *bitOR = bin-rec* (λ*x*. *x*) (λ*y*. *Int.Min*)
(λ*w b s y*. *s* (*bin-rest y*) *BIT* (*b OR bin-last y*))

definition
int-xor-def [*code del*]: *bitXOR = bin-rec* (λ*x*. *x*) *bitNOT*

```
  (λw b s y. s (bin-rest y) BIT (b XOR bin-last y))

instance ..

end

lemma int-not-simps [simp]:
  NOT Int.Pls = Int.Min
  NOT Int.Min = Int.Pls
  NOT (Int.Bit0 w) = Int.Bit1 (NOT w)
  NOT (Int.Bit1 w) = Int.Bit0 (NOT w)
  NOT (w BIT b) = (NOT w) BIT (NOT b)
  unfolding int-not-def by (simp-all add: bin-rec-simps)

declare int-not-simps(1-4) [code]

lemma int-xor-Pls [simp, code]:
  Int.Pls XOR x = x
  unfolding int-xor-def by (simp add: bin-rec-PM)

lemma int-xor-Min [simp, code]:
  Int.Min XOR x = NOT x
  unfolding int-xor-def by (simp add: bin-rec-PM)

lemma int-xor-Bits [simp]:
  (x BIT b) XOR (y BIT c) = (x XOR y) BIT (b XOR c)
  apply (unfold int-xor-def)
  apply (rule bin-rec-simps (1) [THEN fun-cong, THEN trans])
    apply (rule ext, simp)
   prefer 2
   apply simp
  apply (rule ext)
  apply (simp add: int-not-simps [symmetric])
  done

lemma int-xor-Bits2 [simp, code]:
  (Int.Bit0 x) XOR (Int.Bit0 y) = Int.Bit0 (x XOR y)
  (Int.Bit0 x) XOR (Int.Bit1 y) = Int.Bit1 (x XOR y)
  (Int.Bit1 x) XOR (Int.Bit0 y) = Int.Bit1 (x XOR y)
  (Int.Bit1 x) XOR (Int.Bit1 y) = Int.Bit0 (x XOR y)
  unfolding BIT-simps [symmetric] int-xor-Bits by simp-all

lemma int-xor-x-simps':
  w XOR (Int.Pls BIT bit.B0) = w
  w XOR (Int.Min BIT bit.B1) = NOT w
  apply (induct w rule: bin-induct)
       apply simp-all[4]
   apply (unfold int-xor-Bits)
   apply clarsimp+
```

done

lemma *int-xor-extra-simps* [*simp*, *code*]:
w XOR Int.Pls = w
w XOR Int.Min = NOT w
using *int-xor-x-simps'* **by** *simp-all*

lemma *int-or-Pls* [*simp*, *code*]:
Int.Pls OR x = x
by (*unfold int-or-def*) (*simp add*: *bin-rec-PM*)

lemma *int-or-Min* [*simp*, *code*]:
Int.Min OR x = Int.Min
by (*unfold int-or-def*) (*simp add*: *bin-rec-PM*)

lemma *int-or-Bits* [*simp*]:
(x BIT b) OR (y BIT c) = (x OR y) BIT (b OR c)
unfolding *int-or-def* **by** (*simp add*: *bin-rec-simps*)

lemma *int-or-Bits2* [*simp*, *code*]:
(Int.Bit0 x) OR (Int.Bit0 y) = Int.Bit0 (x OR y)
(Int.Bit0 x) OR (Int.Bit1 y) = Int.Bit1 (x OR y)
(Int.Bit1 x) OR (Int.Bit0 y) = Int.Bit1 (x OR y)
(Int.Bit1 x) OR (Int.Bit1 y) = Int.Bit1 (x OR y)
unfolding *BIT-simps* [*symmetric*] *int-or-Bits* **by** *simp-all*

lemma *int-or-x-simps'*:
w OR (Int.Pls BIT bit.B0) = w
w OR (Int.Min BIT bit.B1) = Int.Min
apply (*induct w rule*: *bin-induct*)
apply *simp-all*[*4*]
apply (*unfold int-or-Bits*)
apply *clarsimp+*
done

lemma *int-or-extra-simps* [*simp*, *code*]:
w OR Int.Pls = w
w OR Int.Min = Int.Min
using *int-or-x-simps'* **by** *simp-all*

lemma *int-and-Pls* [*simp*, *code*]:
Int.Pls AND x = Int.Pls
unfolding *int-and-def* **by** (*simp add*: *bin-rec-PM*)

lemma *int-and-Min* [*simp*, *code*]:
Int.Min AND x = x
unfolding *int-and-def* **by** (*simp add*: *bin-rec-PM*)

lemma *int-and-Bits* [*simp*]:

$(x\ BIT\ b)\ AND\ (y\ BIT\ c) = (x\ AND\ y)\ BIT\ (b\ AND\ c)$
unfolding *int-and-def* **by** (*simp add*: *bin-rec-simps*)

lemma *int-and-Bits2* [*simp*, *code*]:
$(Int.Bit0\ x)\ AND\ (Int.Bit0\ y) = Int.Bit0\ (x\ AND\ y)$
$(Int.Bit0\ x)\ AND\ (Int.Bit1\ y) = Int.Bit0\ (x\ AND\ y)$
$(Int.Bit1\ x)\ AND\ (Int.Bit0\ y) = Int.Bit0\ (x\ AND\ y)$
$(Int.Bit1\ x)\ AND\ (Int.Bit1\ y) = Int.Bit1\ (x\ AND\ y)$
unfolding *BIT-simps* [*symmetric*] *int-and-Bits* **by** *simp-all*

lemma *int-and-x-simps'*:
$w\ AND\ (Int.Pls\ BIT\ bit.B0) = Int.Pls$
$w\ AND\ (Int.Min\ BIT\ bit.B1) = w$
apply (*induct w rule*: *bin-induct*)
apply *simp-all*[*4*]
apply (*unfold int-and-Bits*)
apply *clarsimp*+
done

lemma *int-and-extra-simps* [*simp*, *code*]:
$w\ AND\ Int.Pls = Int.Pls$
$w\ AND\ Int.Min = w$
using *int-and-x-simps'* **by** *simp-all*

lemma *bin-ops-comm*:
shows
int-and-comm: $!!y{::}int.\ x\ AND\ y = y\ AND\ x$ **and**
int-or-comm: $!!y{::}int.\ x\ OR\ y = y\ OR\ x$ **and**
int-xor-comm: $!!y{::}int.\ x\ XOR\ y = y\ XOR\ x$
apply (*induct x rule*: *bin-induct*)
apply *simp-all*[*6*]
apply (*case-tac y rule*: *bin-exhaust*, *simp add*: *bit-ops-comm*)+
done

lemma *bin-ops-same* [*simp*]:
$(x{::}int)\ AND\ x = x$
$(x{::}int)\ OR\ x = x$
$(x{::}int)\ XOR\ x = Int.Pls$
by (*induct x rule*: *bin-induct*) *auto*

lemma *int-not-not* [*simp*]: $NOT\ (NOT\ (x{::}int)) = x$
by (*induct x rule*: *bin-induct*) *auto*

lemmas *bin-log-esimps* =
int-and-extra-simps *int-or-extra-simps* *int-xor-extra-simps*
int-and-Pls *int-and-Min* *int-or-Pls* *int-or-Min* *int-xor-Pls* *int-xor-Min*

lemma *bbw-ao-absorb*:
!!*y*::*int*. *x AND* (*y OR x*) = *x* & *x OR* (*y AND x*) = *x*
apply (*induct x rule*: *bin-induct*)
apply *auto*
apply (*case-tac* [!] *y rule*: *bin-exhaust*)
apply *auto*
apply (*case-tac* [!] *bit*)
apply *auto*
done

lemma *bbw-ao-absorbs-other*:
x AND (*x OR y*) = *x* ∧ (*y AND x*) *OR x* = (*x*::*int*)
(*y OR x*) *AND x* = *x* ∧ *x OR* (*x AND y*) = (*x*::*int*)
(*x OR y*) *AND x* = *x* ∧ (*x AND y*) *OR x* = (*x*::*int*)
apply (*auto simp*: *bbw-ao-absorb int-or-comm*)
apply (*subst int-or-comm*)
apply (*simp add*: *bbw-ao-absorb*)
apply (*subst int-and-comm*)
apply (*subst int-or-comm*)
apply (*simp add*: *bbw-ao-absorb*)
apply (*subst int-and-comm*)
apply (*simp add*: *bbw-ao-absorb*)
done

lemmas *bbw-ao-absorbs* [*simp*] = *bbw-ao-absorb bbw-ao-absorbs-other*

lemma *int-xor-not*:
!!*y*::*int*. (*NOT x*) *XOR y* = *NOT* (*x XOR y*) &
x XOR (*NOT y*) = *NOT* (*x XOR y*)
apply (*induct x rule*: *bin-induct*)
apply *auto*
apply (*case-tac y rule*: *bin-exhaust*, *auto*,
case-tac b, *auto*)+
done

lemma *bbw-assocs′*:
!!*y z*::*int*. (*x AND y*) *AND z* = *x AND* (*y AND z*) &
(*x OR y*) *OR z* = *x OR* (*y OR z*) &
(*x XOR y*) *XOR z* = *x XOR* (*y XOR z*)
apply (*induct x rule*: *bin-induct*)
apply (*auto simp*: *int-xor-not*)
apply (*case-tac* [!] *y rule*: *bin-exhaust*)
apply (*case-tac* [!] *z rule*: *bin-exhaust*)
apply (*case-tac* [!] *bit*)
apply (*case-tac* [!] *b*)
apply (*auto simp del*: *BIT-simps*)
done

lemma *int-and-assoc*:
(*x AND y*) *AND* (*z*::*int*) = *x AND* (*y AND z*)
by (*simp add*: *bbw-assocs'*)

lemma *int-or-assoc*:
(*x OR y*) *OR* (*z*::*int*) = *x OR* (*y OR z*)
by (*simp add*: *bbw-assocs'*)

lemma *int-xor-assoc*:
(*x XOR y*) *XOR* (*z*::*int*) = *x XOR* (*y XOR z*)
by (*simp add*: *bbw-assocs'*)

lemmas *bbw-assocs* = *int-and-assoc int-or-assoc int-xor-assoc*

lemma *bbw-lcs* [*simp*]:
(*y*::*int*) *AND* (*x AND z*) = *x AND* (*y AND z*)
(*y*::*int*) *OR* (*x OR z*) = *x OR* (*y OR z*)
(*y*::*int*) *XOR* (*x XOR z*) = *x XOR* (*y XOR z*)
apply (*auto simp*: *bbw-assocs* [*symmetric*])
apply (*auto simp*: *bin-ops-comm*)
done

lemma *bbw-not-dist*:
!!*y*::*int*. *NOT* (*x OR y*) = (*NOT x*) *AND* (*NOT y*)
!!*y*::*int*. *NOT* (*x AND y*) = (*NOT x*) *OR* (*NOT y*)
apply (*induct x rule*: *bin-induct*)
apply *auto*
apply (*case-tac* [!] *y rule*: *bin-exhaust*)
apply (*case-tac* [!] *bit*, *auto simp del*: *BIT-simps*)
done

lemma *bbw-oa-dist*:
!!*y z*::*int*. (*x AND y*) *OR z* =
(*x OR z*) *AND* (*y OR z*)
apply (*induct x rule*: *bin-induct*)
apply *auto*
apply (*case-tac y rule*: *bin-exhaust*)
apply (*case-tac z rule*: *bin-exhaust*)
apply (*case-tac ba*, *auto simp del*: *BIT-simps*)
done

lemma *bbw-ao-dist*:
!!*y z*::*int*. (*x OR y*) *AND z* =
(*x AND z*) *OR* (*y AND z*)
apply (*induct x rule*: *bin-induct*)
apply *auto*
apply (*case-tac y rule*: *bin-exhaust*)
apply (*case-tac z rule*: *bin-exhaust*)
apply (*case-tac ba*, *auto simp del*: *BIT-simps*)

done

lemma *plus-and-or* [*rule-format*]:
ALL y::int. (x AND y) + (x OR y) = x + y
apply (*induct x rule*: *bin-induct*)
apply *clarsimp*
apply *clarsimp*
apply *clarsimp*
apply (*case-tac y rule*: *bin-exhaust*)
apply *clarsimp*
apply (*unfold Bit-def*)
apply *clarsimp*
apply (*erule-tac x = x* **in** *allE*)
apply (*simp split*: *bit.split*)
done

lemma *le-int-or*:
!!x. bin-sign y = Int.Pls ==> x <= x OR y
apply (*induct y rule*: *bin-induct*)
apply *clarsimp*
apply *clarsimp*
apply (*case-tac x rule*: *bin-exhaust*)
apply (*case-tac b*)
apply (*case-tac* [!] *bit*)
apply (*auto simp*: *less-eq-int-code*)
done

lemmas *int-and-le =*
xtr3 [*OF bbw-ao-absorbs* (*2*) [*THEN conjunct2*, *symmetric*] *le-int-or*]

lemma *bin-nth-ops*:
!!x y. bin-nth (x AND y) n = (bin-nth x n & bin-nth y n)
!!x y. bin-nth (x OR y) n = (bin-nth x n | bin-nth y n)
!!x y. bin-nth (x XOR y) n = (bin-nth x n ~= bin-nth y n)
!!x. bin-nth (NOT x) n = (~ bin-nth x n)
apply (*induct n*)
apply *safe*
apply (*case-tac* [!] *x rule*: *bin-exhaust*)
apply (*simp-all del*: *BIT-simps*)
apply (*case-tac* [!] *y rule*: *bin-exhaust*)
apply (*simp-all del*: *BIT-simps*)
apply (*auto dest*: *not-B1-is-B0 intro*: *B1-ass-B0*)
done

lemma *bin-add-not*: *x + NOT x = Int.Min*

```
 apply (induct x rule: bin-induct)
  apply clarsimp
 apply clarsimp
apply (case-tac bit, auto)
done


lemma bin-trunc-ao:
 !!x y. (bintrunc n x) AND (bintrunc n y) = bintrunc n (x AND y)
 !!x y. (bintrunc n x) OR (bintrunc n y) = bintrunc n (x OR y)
 apply (induct n)
    apply auto
    apply (case-tac [!] x rule: bin-exhaust)
    apply (case-tac [!] y rule: bin-exhaust)
    apply auto
 done

lemma bin-trunc-xor:
 !!x y. bintrunc n (bintrunc n x XOR bintrunc n y) =
        bintrunc n (x XOR y)
 apply (induct n)
  apply auto
  apply (case-tac [!] x rule: bin-exhaust)
  apply (case-tac [!] y rule: bin-exhaust)
  apply auto
 done

lemma bin-trunc-not:
 !!x. bintrunc n (NOT (bintrunc n x)) = bintrunc n (NOT x)
 apply (induct n)
  apply auto
  apply (case-tac [!] x rule: bin-exhaust)
  apply auto
 done


lemma bintr-bintr-i:
 x = bintrunc n y ==> bintrunc n x = bintrunc n y
 by auto

lemmas bin-trunc-and = bin-trunc-ao(1) [THEN bintr-bintr-i]
lemmas bin-trunc-or = bin-trunc-ao(2) [THEN bintr-bintr-i]
```

6.2 Setting and clearing bits

```
primrec
 bin-sc :: nat => bit => int => int
where
 Z: bin-sc 0 b w = bin-rest w BIT b
```

| *Suc*: *bin-sc* (*Suc n*) *b w* = *bin-sc n b* (*bin-rest w*) *BIT bin-last w*

lemma *bin-nth-sc* [*simp*]:
!!*w*. *bin-nth* (*bin-sc n b w*) *n* = (*b* = *bit.B1*)
by (*induct n*) *auto*

lemma *bin-sc-sc-same* [*simp*]:
!!*w*. *bin-sc n c* (*bin-sc n b w*) = *bin-sc n c w*
by (*induct n*) *auto*

lemma *bin-sc-sc-diff*:
!!*w m*. *m* ~= *n* ==>
bin-sc m c (*bin-sc n b w*) = *bin-sc n b* (*bin-sc m c w*)
apply (*induct n*)
apply (*case-tac* [!] *m*)
apply *auto*
done

lemma *bin-nth-sc-gen*:
!!*w m*. *bin-nth* (*bin-sc n b w*) *m* = (*if m* = *n then b* = *bit.B1 else bin-nth w m*)
by (*induct n*) (*case-tac* [!] *m*, *auto*)

lemma *bin-sc-nth* [*simp*]:
!!*w*. (*bin-sc n* (*If* (*bin-nth w n*) *bit.B1 bit.B0*) *w*) = *w*
by (*induct n*) *auto*

lemma *bin-sign-sc* [*simp*]:
!!*w*. *bin-sign* (*bin-sc n b w*) = *bin-sign w*
by (*induct n*) *auto*

lemma *bin-sc-bintr* [*simp*]:
!!*w m*. *bintrunc m* (*bin-sc n x* (*bintrunc m* (*w*))) = *bintrunc m* (*bin-sc n x w*)
apply (*induct n*)
apply (*case-tac* [!] *w rule*: *bin-exhaust*)
apply (*case-tac* [!] *m*, *auto*)
done

lemma *bin-clr-le*:
!!*w*. *bin-sc n bit.B0 w* <= *w*
apply (*induct n*)
apply (*case-tac* [!] *w rule*: *bin-exhaust*)
apply (*auto simp del*: *BIT-simps*)
apply (*unfold Bit-def*)
apply (*simp-all split*: *bit.split*)
done

lemma *bin-set-ge*:

```
 !!w. bin-sc n bit.B1 w >= w
 apply (induct n)
  apply (case-tac [!] w rule: bin-exhaust)
  apply (auto simp del: BIT-simps)
  apply (unfold Bit-def)
  apply (simp-all split: bit.split)
 done

lemma bintr-bin-clr-le:
 !!w m. bintrunc n (bin-sc m bit.B0 w) <= bintrunc n w
 apply (induct n)
  apply simp
 apply (case-tac w rule: bin-exhaust)
 apply (case-tac m)
  apply (auto simp del: BIT-simps)
  apply (unfold Bit-def)
  apply (simp-all split: bit.split)
 done

lemma bintr-bin-set-ge:
 !!w m. bintrunc n (bin-sc m bit.B1 w) >= bintrunc n w
 apply (induct n)
  apply simp
 apply (case-tac w rule: bin-exhaust)
 apply (case-tac m)
  apply (auto simp del: BIT-simps)
  apply (unfold Bit-def)
  apply (simp-all split: bit.split)
 done

lemma bin-sc-FP [simp]: bin-sc n bit.B0 Int.Pls = Int.Pls
 by (induct n) auto

lemma bin-sc-TM [simp]: bin-sc n bit.B1 Int.Min = Int.Min
 by (induct n) auto

lemmas bin-sc-simps = bin-sc.Z bin-sc.Suc bin-sc-TM bin-sc-FP

lemma bin-sc-minus:
 0 < n ==> bin-sc (Suc (n - 1)) b w = bin-sc n b w
 by auto

lemmas bin-sc-Suc-minus =
 trans [OF bin-sc-minus [symmetric] bin-sc.Suc, standard]

lemmas bin-sc-Suc-pred [simp] =
 bin-sc-Suc-minus [of number-of bin, simplified nobm1, standard]
```

6.3 Operations on lists of booleans

primrec *bl-to-bin-aux* :: *bool list* ⇒ *int* ⇒ *int* **where**
Nil: *bl-to-bin-aux* [] *w* = *w*
| *Cons*: *bl-to-bin-aux* (*b* # *bs*) *w* =
bl-to-bin-aux bs (*w BIT* (*if b then bit.B1 else bit.B0*))

definition *bl-to-bin* :: *bool list* ⇒ *int* **where**
bl-to-bin-def : *bl-to-bin bs* = *bl-to-bin-aux bs Int.Pls*

primrec *bin-to-bl-aux* :: *nat* ⇒ *int* ⇒ *bool list* ⇒ *bool list* **where**
Z: *bin-to-bl-aux 0 w bl* = *bl*
| *Suc*: *bin-to-bl-aux* (*Suc n*) *w bl* =
bin-to-bl-aux n (*bin-rest w*) ((*bin-last w* = *bit.B1*) # *bl*)

definition *bin-to-bl* :: *nat* ⇒ *int* ⇒ *bool list* **where**
bin-to-bl-def : *bin-to-bl n w* = *bin-to-bl-aux n w* []

primrec *bl-of-nth* :: *nat* ⇒ (*nat* ⇒ *bool*) ⇒ *bool list* **where**
Suc: *bl-of-nth* (*Suc n*) *f* = *f n* # *bl-of-nth n f*
| *Z*: *bl-of-nth 0 f* = []

primrec *takefill* :: *'a* ⇒ *nat* ⇒ *'a list* ⇒ *'a list* **where**
Z: *takefill fill 0 xs* = []
| *Suc*: *takefill fill* (*Suc n*) *xs* = (
case xs of [] => *fill* # *takefill fill n xs*
| *y* # *ys* => *y* # *takefill fill n ys*)

definition *map2* :: (*'a* ⇒ *'b* ⇒ *'c*) ⇒ *'a list* ⇒ *'b list* ⇒ *'c list* **where**
map2 f as bs = *map* (*split f*) (*zip as bs*)

6.4 Splitting and concatenation

definition *bin-rcat* :: *nat* ⇒ *int list* ⇒ *int* **where**
bin-rcat n = *foldl* (%*u v*. *bin-cat u n v*) *Int.Pls*

fun *bin-rsplit-aux* :: *nat* ⇒ *nat* ⇒ *int* ⇒ *int list* ⇒ *int list* **where**
bin-rsplit-aux n m c bs =
(*if m* = *0* | *n* = *0 then bs else*
let (*a*, *b*) = *bin-split n c*
in bin-rsplit-aux n (*m* − *n*) *a* (*b* # *bs*))

definition *bin-rsplit* :: *nat* ⇒ *nat* × *int* ⇒ *int list* **where**
bin-rsplit n w = *bin-rsplit-aux n* (*fst w*) (*snd w*) []

fun *bin-rsplitl-aux* :: *nat* ⇒ *nat* ⇒ *int* ⇒ *int list* ⇒ *int list* **where**
bin-rsplitl-aux n m c bs =
(*if m* = *0* | *n* = *0 then bs else*
let (*a*, *b*) = *bin-split* (*min m n*) *c*
in bin-rsplitl-aux n (*m* − *n*) *a* (*b* # *bs*))

definition *bin-rsplitl* :: *nat* ⇒ *nat* × *int* ⇒ *int list* **where**
bin-rsplitl n w = *bin-rsplitl-aux n* (*fst w*) (*snd w*) []

declare *bin-rsplit-aux.simps* [*simp del*]
declare *bin-rsplitl-aux.simps* [*simp del*]

lemma *bin-sign-cat*:
!!*y*. *bin-sign* (*bin-cat x n y*) = *bin-sign x*
by (*induct n*) *auto*

lemma *bin-cat-Suc-Bit*:
bin-cat w (*Suc n*) (*v BIT b*) = *bin-cat w n v BIT b*
by *auto*

lemma *bin-nth-cat*:
!!*n y*. *bin-nth* (*bin-cat x k y*) *n* =
(*if n* < *k then bin-nth y n else bin-nth x* (*n* − *k*))
apply (*induct k*)
apply *clarsimp*
apply (*case-tac n*, *auto*)
done

lemma *bin-nth-split*:
!!*b c*. *bin-split n c* = (*a*, *b*) ==>
(*ALL k*. *bin-nth a k* = *bin-nth c* (*n* + *k*)) &
(*ALL k*. *bin-nth b k* = (*k* < *n* & *bin-nth c k*))
apply (*induct n*)
apply *clarsimp*
apply (*clarsimp simp*: *Let-def split*: *ls-splits*)
apply (*case-tac k*)
apply *auto*
done

lemma *bin-cat-assoc*:
!!*z*. *bin-cat* (*bin-cat x m y*) *n z* = *bin-cat x* (*m* + *n*) (*bin-cat y n z*)
by (*induct n*) *auto*

lemma *bin-cat-assoc-sym*: !!*z m*.
bin-cat x m (*bin-cat y n z*) = *bin-cat* (*bin-cat x* (*m* − *n*) *y*) (*min m n*) *z*
apply (*induct n*, *clarsimp*)
apply (*case-tac m*, *auto*)
done

lemma *bin-cat-Pls* [*simp*]:
!!*w*. *bin-cat Int.Pls n w* = *bintrunc n w*
by (*induct n*) *auto*

lemma *bintr-cat1*:

!!*b*. *bintrunc* (*k* + *n*) (*bin-cat a n b*) = *bin-cat* (*bintrunc k a*) *n b*
by (*induct n*) *auto*

lemma *bintr-cat*: *bintrunc m* (*bin-cat a n b*) =
bin-cat (*bintrunc* (*m* − *n*) *a*) *n* (*bintrunc* (*min m n*) *b*)
by (*rule bin-eqI*) (*auto simp*: *bin-nth-cat nth-bintr*)

lemma *bintr-cat-same* [*simp*]:
bintrunc n (*bin-cat a n b*) = *bintrunc n b*
by (*auto simp add* : *bintr-cat*)

lemma *cat-bintr* [*simp*]:
!!*b*. *bin-cat a n* (*bintrunc n b*) = *bin-cat a n b*
by (*induct n*) *auto*

lemma *split-bintrunc*:
!!*b c*. *bin-split n c* = (*a*, *b*) ==> *b* = *bintrunc n c*
by (*induct n*) (*auto simp*: *Let-def split*: *ls-splits*)

lemma *bin-cat-split*:
!!*v w*. *bin-split n w* = (*u*, *v*) ==> *w* = *bin-cat u n v*
by (*induct n*) (*auto simp*: *Let-def split*: *ls-splits*)

lemma *bin-split-cat*:
!!*w*. *bin-split n* (*bin-cat v n w*) = (*v*, *bintrunc n w*)
by (*induct n*) *auto*

lemma *bin-split-Pls* [*simp*]:
bin-split n Int.Pls = (*Int.Pls*, *Int.Pls*)
by (*induct n*) (*auto simp*: *Let-def split*: *ls-splits*)

lemma *bin-split-Min* [*simp*]:
bin-split n Int.Min = (*Int.Min*, *bintrunc n Int.Min*)
by (*induct n*) (*auto simp*: *Let-def split*: *ls-splits*)

lemma *bin-split-trunc*:
!!*m b c*. *bin-split* (*min m n*) *c* = (*a*, *b*) ==>
bin-split n (*bintrunc m c*) = (*bintrunc* (*m* − *n*) *a*, *b*)
apply (*induct n*, *clarsimp*)
apply (*simp add*: *bin-rest-trunc Let-def split*: *ls-splits*)
apply (*case-tac m*)
apply (*auto simp*: *Let-def split*: *ls-splits*)
done

lemma *bin-split-trunc1*:
!!*m b c*. *bin-split n c* = (*a*, *b*) ==>
bin-split n (*bintrunc m c*) = (*bintrunc* (*m* − *n*) *a*, *bintrunc m b*)
apply (*induct n*, *clarsimp*)
apply (*simp add*: *bin-rest-trunc Let-def split*: *ls-splits*)

apply (*case-tac m*)
apply (*auto simp*: *Let-def split*: *ls-splits*)
done

lemma *bin-cat-num*:
!!*b*. *bin-cat a n b* = *a* ∗ *2* ^ *n* + *bintrunc n b*
apply (*induct n*, *clarsimp*)
apply (*simp add*: *Bit-def cong*: *number-of-False-cong*)
done

lemma *bin-split-num*:
!!*b*. *bin-split n b* = (*b div 2* ^ *n*, *b mod 2* ^ *n*)
apply (*induct n*, *clarsimp*)
apply (*simp add*: *bin-rest-div zdiv-zmult2-eq*)
apply (*case-tac b rule*: *bin-exhaust*)
apply *simp*
apply (*simp add*: *Bit-def mod-mult-mult1 p1mod22k*
split: *bit.split*
cong: *number-of-False-cong*)
done

6.5 Miscellaneous lemmas

lemma *nth-2p-bin*:
!!*m*. *bin-nth* (*2* ^ *n*) *m* = (*m* = *n*)
apply (*induct n*)
apply *clarsimp*
apply *safe*
apply (*case-tac m*)
apply (*auto simp*: *trans* [*OF numeral-1-eq-1* [*symmetric*] *number-of-eq*])
apply (*case-tac m*)
apply (*auto simp*: *Bit-B0-2t* [*symmetric*])
done

lemma *ex-eq-or*:
(*EX m*. *n* = *Suc m* & (*m* = *k* | *P m*)) = (*n* = *Suc k* | (*EX m*. *n* = *Suc m* & *P m*))
by *auto*

end

7 BinBoolList: Bool lists and integers

theory *BinBoolList*
imports *BinOperations*

begin

7.1 Arithmetic in terms of bool lists

primrec *rbl-succ* :: *bool list* => *bool list* **where**
Nil: *rbl-succ Nil* = *Nil*
| *Cons*: *rbl-succ* (*x* # *xs*) = (*if x then False* # *rbl-succ xs else True* # *xs*)

primrec *rbl-pred* :: *bool list* => *bool list* **where**
Nil: *rbl-pred Nil* = *Nil*
| *Cons*: *rbl-pred* (*x* # *xs*) = (*if x then False* # *xs else True* # *rbl-pred xs*)

primrec *rbl-add* :: *bool list* => *bool list* => *bool list* **where**

Nil: *rbl-add Nil x* = *Nil*
| *Cons*: *rbl-add* (*y* # *ys*) *x* = (*let ws* = *rbl-add ys* (*tl x*) *in*
(*y* ~= *hd x*) # (*if hd x* & *y then rbl-succ ws else ws*))

primrec *rbl-mult* :: *bool list* => *bool list* => *bool list* **where**

Nil: *rbl-mult Nil x* = *Nil*
| *Cons*: *rbl-mult* (*y* # *ys*) *x* = (*let ws* = *False* # *rbl-mult ys x in*
if y then rbl-add ws x else ws)

lemma *butlast-power*:
(*butlast* ^^ *n*) *bl* = *take* (*length bl* − *n*) *bl*
by (*induct n*) (*auto simp*: *butlast-take*)

lemma *bin-to-bl-aux-Pls-minus-simp* [*simp*]:
0 < *n* ==> *bin-to-bl-aux n Int.Pls bl* =
bin-to-bl-aux (*n* − *1*) *Int.Pls* (*False* # *bl*)
by (*cases n*) *auto*

lemma *bin-to-bl-aux-Min-minus-simp* [*simp*]:
0 < *n* ==> *bin-to-bl-aux n Int.Min bl* =
bin-to-bl-aux (*n* − *1*) *Int.Min* (*True* # *bl*)
by (*cases n*) *auto*

lemma *bin-to-bl-aux-Bit-minus-simp* [*simp*]:
0 < *n* ==> *bin-to-bl-aux n* (*w BIT b*) *bl* =
bin-to-bl-aux (*n* − *1*) *w* ((*b* = *bit.B1*) # *bl*)
by (*cases n*) *auto*

lemma *bin-to-bl-aux-Bit0-minus-simp* [*simp*]:
0 < *n* ==> *bin-to-bl-aux n* (*Int.Bit0 w*) *bl* =
bin-to-bl-aux (*n* − *1*) *w* (*False* # *bl*)
by (*cases n*) *auto*

lemma *bin-to-bl-aux-Bit1-minus-simp* [*simp*]:

0 < n ==> bin-to-bl-aux n (Int.Bit1 w) bl =
bin-to-bl-aux (n − 1) w (True # bl)
by (*cases n*) *auto*

lemma *bl-to-bin-aux-append*:
bl-to-bin-aux (bs @ cs) w = bl-to-bin-aux cs (bl-to-bin-aux bs w)
by (*induct bs arbitrary*: *w*) *auto*

lemma *bin-to-bl-aux-append*:
bin-to-bl-aux n w bs @ cs = bin-to-bl-aux n w (bs @ cs)
by (*induct n arbitrary*: *w bs*) *auto*

lemma *bl-to-bin-append*:
bl-to-bin (bs @ cs) = bl-to-bin-aux cs (bl-to-bin bs)
unfolding *bl-to-bin-def* **by** (*rule bl-to-bin-aux-append*)

lemma *bin-to-bl-aux-alt*:
bin-to-bl-aux n w bs = bin-to-bl n w @ bs
unfolding *bin-to-bl-def* **by** (*simp add* : *bin-to-bl-aux-append*)

lemma *bin-to-bl-0*: *bin-to-bl 0 bs = []*
unfolding *bin-to-bl-def* **by** *auto*

lemma *size-bin-to-bl-aux*:
size (bin-to-bl-aux n w bs) = n + length bs
by (*induct n arbitrary*: *w bs*) *auto*

lemma *size-bin-to-bl*: *size (bin-to-bl n w) = n*
unfolding *bin-to-bl-def* **by** (*simp add* : *size-bin-to-bl-aux*)

lemma *bin-bl-bin'*:
bl-to-bin (bin-to-bl-aux n w bs) =
bl-to-bin-aux bs (bintrunc n w)
by (*induct n arbitrary*: *w bs*) (*auto simp add* : *bl-to-bin-def*)

lemma *bin-bl-bin*: *bl-to-bin (bin-to-bl n w) = bintrunc n w*
unfolding *bin-to-bl-def bin-bl-bin'* **by** *auto*

lemma *bl-bin-bl'*:
bin-to-bl (n + length bs) (bl-to-bin-aux bs w) =
bin-to-bl-aux n w bs
apply (*induct bs arbitrary*: *w n*)
apply *auto*
apply (*simp-all only* : *add-Suc* [*symmetric*])
apply (*auto simp add* : *bin-to-bl-def*)
done

lemma *bl-bin-bl*: *bin-to-bl* (*length bs*) (*bl-to-bin bs*) = *bs*
unfolding *bl-to-bin-def*
apply (*rule box-equals*)
apply (*rule bl-bin-bl'*)
prefer *2*
apply (*rule bin-to-bl-aux.Z*)
apply *simp*
done

declare
bin-to-bl-0 [*simp*]
size-bin-to-bl [*simp*]
bin-bl-bin [*simp*]
bl-bin-bl [*simp*]

lemma *bl-to-bin-inj*:
bl-to-bin bs = *bl-to-bin cs* ==> *length bs* = *length cs* ==> *bs* = *cs*
apply (*rule-tac box-equals*)
defer
apply (*rule bl-bin-bl*)
apply (*rule bl-bin-bl*)
apply *simp*
done

lemma *bl-to-bin-False*: *bl-to-bin* (*False* # *bl*) = *bl-to-bin bl*
unfolding *bl-to-bin-def* **by** *auto*

lemma *bl-to-bin-Nil*: *bl-to-bin* [] = *Int.Pls*
unfolding *bl-to-bin-def* **by** *auto*

lemma *bin-to-bl-Pls-aux*:
bin-to-bl-aux n Int.Pls bl = *replicate n False* @ *bl*
by (*induct n arbitrary*: *bl*) (*auto simp*: *replicate-app-Cons-same*)

lemma *bin-to-bl-Pls*: *bin-to-bl n Int.Pls* = *replicate n False*
unfolding *bin-to-bl-def* **by** (*simp add* : *bin-to-bl-Pls-aux*)

lemma *bin-to-bl-Min-aux* [*rule-format*] :
ALL bl. bin-to-bl-aux n Int.Min bl = *replicate n True* @ *bl*
by (*induct n*) (*auto simp*: *replicate-app-Cons-same*)

lemma *bin-to-bl-Min*: *bin-to-bl n Int.Min* = *replicate n True*
unfolding *bin-to-bl-def* **by** (*simp add* : *bin-to-bl-Min-aux*)

lemma *bl-to-bin-rep-F*:
bl-to-bin (*replicate n False* @ *bl*) = *bl-to-bin bl*
apply (*simp add*: *bin-to-bl-Pls-aux* [*symmetric*] *bin-bl-bin'*)
apply (*simp add*: *bl-to-bin-def*)
done

lemma *bin-to-bl-trunc*:
n <= m ==> bin-to-bl n (bintrunc m w) = bin-to-bl n w
by (*auto intro*: *bl-to-bin-inj*)

declare
bin-to-bl-trunc [*simp*]
bl-to-bin-False [*simp*]
bl-to-bin-Nil [*simp*]

lemma *bin-to-bl-aux-bintr* [*rule-format*] :
ALL m bin bl. bin-to-bl-aux n (bintrunc m bin) bl =
replicate (n − m) False @ bin-to-bl-aux (min n m) bin bl
apply (*induct n*)
apply *clarsimp*
apply *clarsimp*
apply (*case-tac m*)
apply (*clarsimp simp*: *bin-to-bl-Pls-aux*)
apply (*erule thin-rl*)
apply (*induct-tac n*)
apply *auto*
done

lemmas *bin-to-bl-bintr* =
bin-to-bl-aux-bintr [**where** *bl* = [], *folded bin-to-bl-def*]

lemma *bl-to-bin-rep-False*: *bl-to-bin (replicate n False) = Int.Pls*
by (*induct n*) *auto*

lemma *len-bin-to-bl-aux*:
length (bin-to-bl-aux n w bs) = n + length bs
by (*induct n arbitrary*: *w bs*) *auto*

lemma *len-bin-to-bl* [*simp*]: *length (bin-to-bl n w) = n*
unfolding *bin-to-bl-def len-bin-to-bl-aux* **by** *auto*

lemma *sign-bl-bin'*:
bin-sign (bl-to-bin-aux bs w) = bin-sign w
by (*induct bs arbitrary*: *w*) *auto*

lemma *sign-bl-bin*: *bin-sign (bl-to-bin bs) = Int.Pls*
unfolding *bl-to-bin-def* **by** (*simp add* : *sign-bl-bin'*)

lemma *bl-sbin-sign-aux*:
hd (bin-to-bl-aux (Suc n) w bs) =
(bin-sign (sbintrunc n w) = Int.Min)
apply (*induct n arbitrary*: *w bs*)
apply *clarsimp*
apply (*cases w rule*: *bin-exhaust*)

apply (*simp split add* : *bit.split*)
apply *clarsimp*
done

lemma *bl-sbin-sign*:
hd (*bin-to-bl* (*Suc n*) *w*) = (*bin-sign* (*sbintrunc n w*) = *Int.Min*)
unfolding *bin-to-bl-def* **by** (*rule bl-sbin-sign-aux*)

lemma *bin-nth-of-bl-aux* [*rule-format*]:
∀ *w*. *bin-nth* (*bl-to-bin-aux bl w*) *n* =
(*n* < *size bl* & *rev bl* ! *n* | *n* >= *length bl* & *bin-nth w* (*n* − *size bl*))
apply (*induct-tac bl*)
apply *clarsimp*
apply *clarsimp*
apply (*cut-tac x*=*n* **and** *y*=*size list* **in** *linorder-less-linear*)
apply (*erule disjE*, *simp add*: *nth-append*)+
apply *auto*
done

lemma *bin-nth-of-bl*: *bin-nth* (*bl-to-bin bl*) *n* = (*n* < *length bl* & *rev bl* ! *n*)
unfolding *bl-to-bin-def* **by** (*simp add* : *bin-nth-of-bl-aux*)

lemma *bin-nth-bl* [*rule-format*] : *ALL m w*. *n* < *m* −−>
bin-nth w n = *nth* (*rev* (*bin-to-bl m w*)) *n*
apply (*induct n*)
apply *clarsimp*
apply (*case-tac m*, *clarsimp*)
apply (*clarsimp simp*: *bin-to-bl-def*)
apply (*simp add*: *bin-to-bl-aux-alt*)
apply *clarsimp*
apply (*case-tac m*, *clarsimp*)
apply (*clarsimp simp*: *bin-to-bl-def*)
apply (*simp add*: *bin-to-bl-aux-alt*)
done

lemma *nth-rev* [*rule-format*] :
n < *length xs* −−> *rev xs* ! *n* = *xs* ! (*length xs* − *1* − *n*)
apply (*induct-tac xs*)
apply *simp*
apply (*clarsimp simp add* : *nth-append nth.simps split add* : *nat.split*)
apply (*rule-tac f* = %*n*. *list* ! *n* **in** *arg-cong*)
apply *arith*
done

lemmas *nth-rev-alt* = *nth-rev* [**where** *xs* = *rev ys*, *simplified*, *standard*]

lemma *nth-bin-to-bl-aux* [*rule-format*] :
ALL w n bl. *n* < *m* + *length bl* −−> (*bin-to-bl-aux m w bl*) ! *n* =
(*if n* < *m then bin-nth w* (*m* − *1* − *n*) *else bl* ! (*n* − *m*))

apply (*induct m*)
apply *clarsimp*
apply *clarsimp*
apply (*case-tac w rule*: *bin-exhaust*)
apply *clarsimp*
apply (*case-tac n − m*)
apply *arith*
apply *simp*
apply (*rule-tac f* = %*n*. *bl* ! *n* **in** *arg-cong*)
apply *arith*
done

lemma *nth-bin-to-bl*: *n* < *m* ==> (*bin-to-bl m w*) ! *n* = *bin-nth w* (*m* − *Suc n*)
unfolding *bin-to-bl-def* **by** (*simp add* : *nth-bin-to-bl-aux*)

lemma *bl-to-bin-lt2p-aux* [*rule-format*]:
∀ *w*. *bl-to-bin-aux bs w* < (*w* + *1*) ∗ (*2* ^ *length bs*)
apply (*induct bs*)
apply *clarsimp*
apply *clarsimp*
apply *safe*
apply (*erule allE*, *erule xtr8* [*rotated*],
simp add: *numeral-simps algebra-simps cong add* : *number-of-False-cong*)+
done

lemma *bl-to-bin-lt2p*: *bl-to-bin bs* < (*2* ^ *length bs*)
apply (*unfold bl-to-bin-def*)
apply (*rule xtr1*)
prefer *2*
apply (*rule bl-to-bin-lt2p-aux*)
apply *simp*
done

lemma *bl-to-bin-ge2p-aux* [*rule-format*] :
∀ *w*. *bl-to-bin-aux bs w* >= *w* ∗ (*2* ^ *length bs*)
apply (*induct bs*)
apply *clarsimp*
apply *clarsimp*
apply *safe*
apply (*erule allE*, *erule preorder-class.order-trans* [*rotated*],
simp add: *numeral-simps algebra-simps cong add* : *number-of-False-cong*)+
done

lemma *bl-to-bin-ge0*: *bl-to-bin bs* >= *0*
apply (*unfold bl-to-bin-def*)
apply (*rule xtr4*)
apply (*rule bl-to-bin-ge2p-aux*)
apply *simp*
done

lemma *butlast-rest-bin*:
butlast (*bin-to-bl n w*) = *bin-to-bl* (*n* − *1*) (*bin-rest w*)
apply (*unfold bin-to-bl-def*)
apply (*cases w rule*: *bin-exhaust*)
apply (*cases n*, *clarsimp*)
apply *clarsimp*
apply (*auto simp add*: *bin-to-bl-aux-alt*)
done

lemmas *butlast-bin-rest* = *butlast-rest-bin*
[**where** *w*=*bl-to-bin bl* **and** *n*=*length bl*, *simplified*, *standard*]

lemma *butlast-rest-bl2bin-aux*:
bl ~= [] ⟹
bl-to-bin-aux (*butlast bl*) *w* = *bin-rest* (*bl-to-bin-aux bl w*)
by (*induct bl arbitrary*: *w*) *auto*

lemma *butlast-rest-bl2bin*:
bl-to-bin (*butlast bl*) = *bin-rest* (*bl-to-bin bl*)
apply (*unfold bl-to-bin-def*)
apply (*cases bl*)
apply (*auto simp add*: *butlast-rest-bl2bin-aux*)
done

lemma *trunc-bl2bin-aux* [*rule-format*]:
ALL w. *bintrunc m* (*bl-to-bin-aux bl w*) =
bl-to-bin-aux (*drop* (*length bl* − *m*) *bl*) (*bintrunc* (*m* − *length bl*) *w*)
apply (*induct-tac bl*)
apply *clarsimp*
apply *clarsimp*
apply *safe*
apply (*case-tac m* − *size list*)
apply (*simp add* : *diff-is-0-eq* [*THEN iffD1*, *THEN Suc-diff-le*])
apply *simp*
apply (*rule-tac f* = %*nat*. *bl-to-bin-aux list* (*Int.Bit1* (*bintrunc nat w*))
in *arg-cong*)
apply *simp*
apply (*case-tac m* − *size list*)
apply (*simp add*: *diff-is-0-eq* [*THEN iffD1*, *THEN Suc-diff-le*])
apply *simp*
apply (*rule-tac f* = %*nat*. *bl-to-bin-aux list* (*Int.Bit0* (*bintrunc nat w*))
in *arg-cong*)
apply *simp*
done

lemma *trunc-bl2bin*:
bintrunc m (*bl-to-bin bl*) = *bl-to-bin* (*drop* (*length bl* − *m*) *bl*)
unfolding *bl-to-bin-def* **by** (*simp add* : *trunc-bl2bin-aux*)

lemmas *trunc-bl2bin-len* [*simp*] =
trunc-bl2bin [*of length bl bl*, *simplified*, *standard*]

lemma *bl2bin-drop*:
bl-to-bin (*drop k bl*) = *bintrunc* (*length bl* − *k*) (*bl-to-bin bl*)
apply (*rule trans*)
prefer *2*
apply (*rule trunc-bl2bin* [*symmetric*])
apply (*cases k* <= *length bl*)
apply *auto*
done

lemma *nth-rest-power-bin* [*rule-format*] :
ALL n. *bin-nth* ((*bin-rest* ^^ *k*) *w*) *n* = *bin-nth w* (*n* + *k*)
apply (*induct k*, *clarsimp*)
apply *clarsimp*
apply (*simp only*: *bin-nth.Suc* [*symmetric*] *add-Suc*)
done

lemma *take-rest-power-bin*:
m <= *n* ==> *take m* (*bin-to-bl n w*) = *bin-to-bl m* ((*bin-rest* ^^ (*n* − *m*)) *w*)
apply (*rule nth-equalityI*)
apply *simp*
apply (*clarsimp simp add*: *nth-bin-to-bl nth-rest-power-bin*)
done

lemma *hd-butlast*: *size xs* > *1* ==> *hd* (*butlast xs*) = *hd xs*
by (*cases xs*) *auto*

lemma *last-bin-last'*:
size xs > *0* ⟹ *last xs* = (*bin-last* (*bl-to-bin-aux xs w*) = *bit.B1*)
by (*induct xs arbitrary*: *w*) *auto*

lemma *last-bin-last*:
size xs > *0* ==> *last xs* = (*bin-last* (*bl-to-bin xs*) = *bit.B1*)
unfolding *bl-to-bin-def* **by** (*erule last-bin-last'*)

lemma *bin-last-last*:
bin-last w = (*if last* (*bin-to-bl* (*Suc n*) *w*) *then bit.B1 else bit.B0*)
apply (*unfold bin-to-bl-def*)
apply *simp*
apply (*auto simp add*: *bin-to-bl-aux-alt*)
done

lemma *map2-Nil* [*simp*]: *map2 f* [] *ys* = []
unfolding *map2-def* **by** *auto*

lemma *map2-Cons* [*simp*]:
map2 f (*x* # *xs*) (*y* # *ys*) = *f x y* # *map2 f xs ys*
unfolding *map2-def* **by** *auto*

lemma *bl-xor-aux-bin* [*rule-format*] : *ALL v w bs cs*.
map2 (%*x y*. *x* ~= *y*) (*bin-to-bl-aux n v bs*) (*bin-to-bl-aux n w cs*) =
bin-to-bl-aux n (*v XOR w*) (*map2* (%*x y*. *x* ~= *y*) *bs cs*)
apply (*induct-tac n*)
apply *safe*
apply *simp*
apply (*case-tac v rule*: *bin-exhaust*)
apply (*case-tac w rule*: *bin-exhaust*)
apply *clarsimp*
apply (*case-tac b*)
apply (*case-tac ba*, *safe*, *simp-all*)+
done

lemma *bl-or-aux-bin* [*rule-format*] : *ALL v w bs cs*.
map2 (*op* |) (*bin-to-bl-aux n v bs*) (*bin-to-bl-aux n w cs*) =
bin-to-bl-aux n (*v OR w*) (*map2* (*op* |) *bs cs*)
apply (*induct-tac n*)
apply *safe*
apply *simp*
apply (*case-tac v rule*: *bin-exhaust*)
apply (*case-tac w rule*: *bin-exhaust*)
apply *clarsimp*
apply (*case-tac b*)
apply (*case-tac ba*, *safe*, *simp-all*)+
done

lemma *bl-and-aux-bin* [*rule-format*] : *ALL v w bs cs*.
map2 (*op* &) (*bin-to-bl-aux n v bs*) (*bin-to-bl-aux n w cs*) =
bin-to-bl-aux n (*v AND w*) (*map2* (*op* &) *bs cs*)
apply (*induct-tac n*)
apply *safe*
apply *simp*
apply (*case-tac v rule*: *bin-exhaust*)
apply (*case-tac w rule*: *bin-exhaust*)
apply *clarsimp*
apply (*case-tac b*)
apply (*case-tac ba*, *safe*, *simp-all*)+
done

lemma *bl-not-aux-bin* [*rule-format*] :
ALL w cs. *map Not* (*bin-to-bl-aux n w cs*) =
bin-to-bl-aux n (*NOT w*) (*map Not cs*)
apply (*induct n*)
apply *clarsimp*

apply *clarsimp*
apply (*case-tac w rule*: *bin-exhaust*)
apply (*case-tac b*)
apply *auto*
done

lemmas *bl-not-bin* = *bl-not-aux-bin*
[**where** *cs* = [], *unfolded bin-to-bl-def* [*symmetric*] *map.simps*]

lemmas *bl-and-bin* = *bl-and-aux-bin* [**where** *bs*=[] **and** *cs*=[],
unfolded map2-Nil, *folded bin-to-bl-def*]

lemmas *bl-or-bin* = *bl-or-aux-bin* [**where** *bs*=[] **and** *cs*=[],
unfolded map2-Nil, *folded bin-to-bl-def*]

lemmas *bl-xor-bin* = *bl-xor-aux-bin* [**where** *bs*=[] **and** *cs*=[],
unfolded map2-Nil, *folded bin-to-bl-def*]

lemma *drop-bin2bl-aux* [*rule-format*] :
ALL m bin bs. *drop m* (*bin-to-bl-aux n bin bs*) =
bin-to-bl-aux (*n* − *m*) *bin* (*drop* (*m* − *n*) *bs*)
apply (*induct n*, *clarsimp*)
apply *clarsimp*
apply (*case-tac bin rule*: *bin-exhaust*)
apply (*case-tac m* <= *n*, *simp*)
apply (*case-tac m* − *n*, *simp*)
apply *simp*
apply (*rule-tac f* = %*nat*. *drop nat bs* **in** *arg-cong*)
apply *simp*
done

lemma *drop-bin2bl*: *drop m* (*bin-to-bl n bin*) = *bin-to-bl* (*n* − *m*) *bin*
unfolding *bin-to-bl-def* **by** (*simp add* : *drop-bin2bl-aux*)

lemma *take-bin2bl-lem1* [*rule-format*] :
ALL w bs. *take m* (*bin-to-bl-aux m w bs*) = *bin-to-bl m w*
apply (*induct m*, *clarsimp*)
apply *clarsimp*
apply (*simp add*: *bin-to-bl-aux-alt*)
apply (*simp add*: *bin-to-bl-def*)
apply (*simp add*: *bin-to-bl-aux-alt*)
done

lemma *take-bin2bl-lem* [*rule-format*] :
ALL w bs. *take m* (*bin-to-bl-aux* (*m* + *n*) *w bs*) =
take m (*bin-to-bl* (*m* + *n*) *w*)
apply (*induct n*)
apply *clarify*
apply (*simp-all* (*no-asm*) *add*: *bin-to-bl-def take-bin2bl-lem1*)

apply *simp*
done

lemma *bin-split-take* [*rule-format*] :
ALL b c. bin-split n c = (a, b) −−>
bin-to-bl m a = take m (bin-to-bl (m + n) c)
apply (*induct n*)
apply *clarsimp*
apply (*clarsimp simp*: *Let-def split*: *ls-splits*)
apply (*simp add*: *bin-to-bl-def*)
apply (*simp add*: *take-bin2bl-lem*)
done

lemma *bin-split-take1*:
k = m + n ==> bin-split n c = (a, b) ==>
bin-to-bl m a = take m (bin-to-bl k c)
by (*auto elim*: *bin-split-take*)

lemma *nth-takefill* [*rule-format*] : *ALL m l. m < n −−>*
takefill fill n l ! m = (if m < length l then l ! m else fill)
apply (*induct n*, *clarsimp*)
apply *clarsimp*
apply (*case-tac m*)
apply (*simp split*: *list.split*)
apply *clarsimp*
apply (*erule allE*)+
apply (*erule* (*1*) *impE*)
apply (*simp split*: *list.split*)
done

lemma *takefill-alt* [*rule-format*] :
ALL l. takefill fill n l = take n l @ replicate (n − length l) fill
by (*induct n*) (*auto split*: *list.split*)

lemma *takefill-replicate* [*simp*]:
takefill fill n (replicate m fill) = replicate n fill
by (*simp add* : *takefill-alt replicate-add* [*symmetric*])

lemma *takefill-le′* [*rule-format*] :
ALL l n. n = m + k −−> takefill x m (takefill x n l) = takefill x m l
by (*induct m*) (*auto split*: *list.split*)

lemma *length-takefill* [*simp*]: *length (takefill fill n l) = n*
by (*simp add* : *takefill-alt*)

lemma *take-takefill′*:
!!w n. n = k + m ==> take k (takefill fill n w) = takefill fill k w
by (*induct k*) (*auto split add* : *list.split*)

lemma *drop-takefill*:
!!w. *drop k* (*takefill fill* (*m* + *k*) *w*) = *takefill fill m* (*drop k w*)
by (*induct k*) (*auto split add* : *list.split*)

lemma *takefill-le* [*simp*]:
$m \leq n \Longrightarrow$ *takefill x m* (*takefill x n l*) = *takefill x m l*
by (*auto simp*: *le-iff-add takefill-le'*)

lemma *take-takefill* [*simp*]:
$m \leq n \Longrightarrow$ *take m* (*takefill fill n w*) = *takefill fill m w*
by (*auto simp*: *le-iff-add take-takefill'*)

lemma *takefill-append*:
takefill fill (*m* + *length xs*) (*xs* @ *w*) = *xs* @ (*takefill fill m w*)
by (*induct xs*) *auto*

lemma *takefill-same'*:
l = *length xs* ==> *takefill fill l xs* = *xs*
by *clarify* (*induct xs*, *auto*)

lemmas *takefill-same* [*simp*] = *takefill-same'* [*OF refl*]

lemma *takefill-bintrunc*:
takefill False n bl = *rev* (*bin-to-bl n* (*bl-to-bin* (*rev bl*)))
apply (*rule nth-equalityI*)
apply *simp*
apply (*clarsimp simp*: *nth-takefill nth-rev nth-bin-to-bl bin-nth-of-bl*)
done

lemma *bl-bin-bl-rtf*:
bin-to-bl n (*bl-to-bin bl*) = *rev* (*takefill False n* (*rev bl*))
by (*simp add* : *takefill-bintrunc*)

lemmas *bl-bin-bl-rep-drop* =
bl-bin-bl-rtf [*simplified takefill-alt*,
simplified, *simplified rev-take*, *simplified*]

lemma *tf-rev*:
n + *k* = *m* + *length bl* ==> *takefill x m* (*rev* (*takefill y n bl*)) =
rev (*takefill y m* (*rev* (*takefill x k* (*rev bl*))))
apply (*rule nth-equalityI*)
apply (*auto simp add*: *nth-takefill nth-rev*)
apply (*rule-tac f* = %*n*. *bl* ! *n* **in** *arg-cong*)
apply *arith*
done

lemma *takefill-minus*:
0 < *n* ==> *takefill fill* (*Suc* (*n* − *1*)) *w* = *takefill fill n w*
by *auto*

lemmas *takefill-Suc-cases* =
list.cases [*THEN takefill.Suc* [*THEN trans*], *standard*]

lemmas *takefill-Suc-Nil* = *takefill-Suc-cases* (*1*)
lemmas *takefill-Suc-Cons* = *takefill-Suc-cases* (*2*)

lemmas *takefill-minus-simps* = *takefill-Suc-cases* [*THEN* [*2*]
takefill-minus [*symmetric*, *THEN trans*], *standard*]

lemmas *takefill-pred-simps* [*simp*] =
takefill-minus-simps [**where** *n*=*number-of bin*, *simplified nobm1*, *standard*]

lemma *bl-to-bin-aux-cat*:
!!*nv v*. *bl-to-bin-aux bs* (*bin-cat w nv v*) =
bin-cat w (*nv* + *length bs*) (*bl-to-bin-aux bs v*)
apply (*induct bs*)
apply *simp*
apply (*simp add*: *bin-cat-Suc-Bit* [*symmetric*] *del*: *bin-cat.simps*)
done

lemma *bin-to-bl-aux-cat*:
!!*w bs*. *bin-to-bl-aux* (*nv* + *nw*) (*bin-cat v nw w*) *bs* =
bin-to-bl-aux nv v (*bin-to-bl-aux nw w bs*)
by (*induct nw*) *auto*

lemmas *bl-to-bin-aux-alt* =
bl-to-bin-aux-cat [**where** *nv* = *0* **and** *v* = *Int.Pls*,
simplified bl-to-bin-def [*symmetric*], *simplified*]

lemmas *bin-to-bl-cat* =
bin-to-bl-aux-cat [**where** *bs* = [], *folded bin-to-bl-def*]

lemmas *bl-to-bin-aux-app-cat* =
trans [*OF bl-to-bin-aux-append bl-to-bin-aux-alt*]

lemmas *bin-to-bl-aux-cat-app* =
trans [*OF bin-to-bl-aux-cat bin-to-bl-aux-alt*]

lemmas *bl-to-bin-app-cat* = *bl-to-bin-aux-app-cat*
[**where** *w* = *Int.Pls*, *folded bl-to-bin-def*]

lemmas *bin-to-bl-cat-app* = *bin-to-bl-aux-cat-app*
[**where** *bs* = [], *folded bin-to-bl-def*]

lemma *bl-to-bin-app-cat-alt*:

bin-cat (*bl-to-bin cs*) *n w* = *bl-to-bin* (*cs* @ *bin-to-bl n w*)
by (*simp add* : *bl-to-bin-app-cat*)

lemma *mask-lem*: (*bl-to-bin* (*True* # *replicate n False*)) =
Int.succ (*bl-to-bin* (*replicate n True*))
apply (*unfold bl-to-bin-def*)
apply (*induct n*)
apply *simp*
apply (*simp only*: *Suc-eq-plus1 replicate-add*
append-Cons [*symmetric*] *bl-to-bin-aux-append*)
apply *simp*
done

lemma *length-bl-of-nth* [*simp*]: *length* (*bl-of-nth n f*) = *n*
by (*induct n*) *auto*

lemma *nth-bl-of-nth* [*simp*]:
m < *n* ⟹ *rev* (*bl-of-nth n f*) ! *m* = *f m*
apply (*induct n*)
apply *simp*
apply (*clarsimp simp add* : *nth-append*)
apply (*rule-tac f* = *f* **in** *arg-cong*)
apply *simp*
done

lemma *bl-of-nth-inj*:
(!!*k*. *k* < *n* ==> *f k* = *g k*) ==> *bl-of-nth n f* = *bl-of-nth n g*
by (*induct n*) *auto*

lemma *bl-of-nth-nth-le* [*rule-format*] : *ALL xs*.
length xs >= *n* −−> *bl-of-nth n* (*nth* (*rev xs*)) = *drop* (*length xs* − *n*) *xs*
apply (*induct n*, *clarsimp*)
apply *clarsimp*
apply (*rule trans* [*OF - hd-Cons-tl*])
apply (*frule Suc-le-lessD*)
apply (*simp add*: *nth-rev trans* [*OF drop-Suc drop-tl*, *symmetric*])
apply (*subst hd-drop-conv-nth*)
apply *force*
apply *simp-all*
apply (*rule-tac f* = %*n*. *drop n xs* **in** *arg-cong*)
apply *simp*
done

lemmas *bl-of-nth-nth* [*simp*] = *order-refl* [*THEN bl-of-nth-nth-le*, *simplified*]

lemma *size-rbl-pred*: *length* (*rbl-pred bl*) = *length bl*
by (*induct bl*) *auto*

lemma *size-rbl-succ*: *length* (*rbl-succ bl*) = *length bl*
by (*induct bl*) *auto*

lemma *size-rbl-add*:
!!*cl*. *length* (*rbl-add bl cl*) = *length bl*
by (*induct bl*) (*auto simp*: *Let-def size-rbl-succ*)

lemma *size-rbl-mult*:
!!*cl*. *length* (*rbl-mult bl cl*) = *length bl*
by (*induct bl*) (*auto simp add* : *Let-def size-rbl-add*)

lemmas *rbl-sizes* [*simp*] =
size-rbl-pred size-rbl-succ size-rbl-add size-rbl-mult

lemmas *rbl-Nils* =
rbl-pred.Nil rbl-succ.Nil rbl-add.Nil rbl-mult.Nil

lemma *rbl-pred*:
!!*bin*. *rbl-pred* (*rev* (*bin-to-bl n bin*)) = *rev* (*bin-to-bl n* (*Int.pred bin*))
apply (*induct n*, *simp*)
apply (*unfold bin-to-bl-def*)
apply *clarsimp*
apply (*case-tac bin rule*: *bin-exhaust*)
apply (*case-tac b*)
apply (*clarsimp simp*: *bin-to-bl-aux-alt*)+
done

lemma *rbl-succ*:
!!*bin*. *rbl-succ* (*rev* (*bin-to-bl n bin*)) = *rev* (*bin-to-bl n* (*Int.succ bin*))
apply (*induct n*, *simp*)
apply (*unfold bin-to-bl-def*)
apply *clarsimp*
apply (*case-tac bin rule*: *bin-exhaust*)
apply (*case-tac b*)
apply (*clarsimp simp*: *bin-to-bl-aux-alt*)+
done

lemma *rbl-add*:
!!*bina binb*. *rbl-add* (*rev* (*bin-to-bl n bina*)) (*rev* (*bin-to-bl n binb*)) =
rev (*bin-to-bl n* (*bina* + *binb*))
apply (*induct n*, *simp*)
apply (*unfold bin-to-bl-def*)
apply *clarsimp*
apply (*case-tac bina rule*: *bin-exhaust*)
apply (*case-tac binb rule*: *bin-exhaust*)
apply (*case-tac b*)
apply (*case-tac* [!] *ba*)
apply (*auto simp*: *rbl-succ succ-def bin-to-bl-aux-alt Let-def add-ac*)
done

lemma *rbl-add-app2*:
!!*blb*. *length blb* >= *length bla* ==>
rbl-add bla (*blb* @ *blc*) = *rbl-add bla blb*
apply (*induct bla*, *simp*)
apply *clarsimp*
apply (*case-tac blb*, *clarsimp*)
apply (*clarsimp simp*: *Let-def*)
done

lemma *rbl-add-take2*:
!!*blb*. *length blb* >= *length bla* ==>
rbl-add bla (*take* (*length bla*) *blb*) = *rbl-add bla blb*
apply (*induct bla*, *simp*)
apply *clarsimp*
apply (*case-tac blb*, *clarsimp*)
apply (*clarsimp simp*: *Let-def*)
done

lemma *rbl-add-long*:
m >= *n* ==> *rbl-add* (*rev* (*bin-to-bl n bina*)) (*rev* (*bin-to-bl m binb*)) =
rev (*bin-to-bl n* (*bina* + *binb*))
apply (*rule box-equals* [*OF* - *rbl-add-take2 rbl-add*])
apply (*rule-tac f* = *rbl-add* (*rev* (*bin-to-bl n bina*)) **in** *arg-cong*)
apply (*rule rev-swap* [*THEN iffD1*])
apply (*simp add*: *rev-take drop-bin2bl*)
apply *simp*
done

lemma *rbl-mult-app2*:
!!*blb*. *length blb* >= *length bla* ==>
rbl-mult bla (*blb* @ *blc*) = *rbl-mult bla blb*
apply (*induct bla*, *simp*)
apply *clarsimp*
apply (*case-tac blb*, *clarsimp*)
apply (*clarsimp simp*: *Let-def rbl-add-app2*)
done

lemma *rbl-mult-take2*:
length blb >= *length bla* ==>
rbl-mult bla (*take* (*length bla*) *blb*) = *rbl-mult bla blb*
apply (*rule trans*)
apply (*rule rbl-mult-app2* [*symmetric*])
apply *simp*
apply (*rule-tac f* = *rbl-mult bla* **in** *arg-cong*)
apply (*rule append-take-drop-id*)
done

lemma *rbl-mult-gt1*:

```
  m >= length bl ==> rbl-mult bl (rev (bin-to-bl m binb)) =
    rbl-mult bl (rev (bin-to-bl (length bl) binb))
  apply (rule trans)
   apply (rule rbl-mult-take2 [symmetric])
   apply simp-all
  apply (rule-tac f = rbl-mult bl in arg-cong)
  apply (rule rev-swap [THEN iffD1])
  apply (simp add: rev-take drop-bin2bl)
  done

lemma rbl-mult-gt:
  m > n ==> rbl-mult (rev (bin-to-bl n bina)) (rev (bin-to-bl m binb)) =
    rbl-mult (rev (bin-to-bl n bina)) (rev (bin-to-bl n binb))
  by (auto intro: trans [OF rbl-mult-gt1])

lemmas rbl-mult-Suc = lessI [THEN rbl-mult-gt]

lemma rbbl-Cons:
  b # rev (bin-to-bl n x) = rev (bin-to-bl (Suc n) (x BIT If b bit.B1 bit.B0))
  apply (unfold bin-to-bl-def)
  apply simp
  apply (simp add: bin-to-bl-aux-alt)
  done

lemma rbl-mult: !!bina binb.
    rbl-mult (rev (bin-to-bl n bina)) (rev (bin-to-bl n binb)) =
    rev (bin-to-bl n (bina * binb))
  apply (induct n)
   apply simp
  apply (unfold bin-to-bl-def)
  apply clarsimp
  apply (case-tac bina rule: bin-exhaust)
  apply (case-tac binb rule: bin-exhaust)
  apply (case-tac b)
   apply (case-tac [!] ba)
     apply (auto simp: bin-to-bl-aux-alt Let-def)
     apply (auto simp: rbbl-Cons rbl-mult-Suc rbl-add)
  done

lemma rbl-add-split:
  P (rbl-add (y # ys) (x # xs)) =
    (ALL ws. length ws = length ys --> ws = rbl-add ys xs -->
    (y --> ((x --> P (False # rbl-succ ws)) & (~ x -->  P (True # ws)))) &
    (~ y --> P (x # ws)))
  apply (auto simp add: Let-def)
   apply (case-tac [!] y)
     apply auto
  done
```

lemma *rbl-mult-split*:
P (rbl-mult (y # ys) xs) =
(ALL ws. length ws = Suc (length ys) −−> ws = False # rbl-mult ys xs −−>

(y −−> P (rbl-add ws xs)) & (~ y −−> P ws))
by *(clarsimp simp add : Let-def)*

lemma *and-len*: *xs = ys ==> xs = ys & length xs = length ys*
by *auto*

lemma *size-if*: *size (if p then xs else ys) = (if p then size xs else size ys)*
by *auto*

lemma *tl-if*: *tl (if p then xs else ys) = (if p then tl xs else tl ys)*
by *auto*

lemma *hd-if*: *hd (if p then xs else ys) = (if p then hd xs else hd ys)*
by *auto*

lemma *if-Not-x*: *(if p then ~ x else x) = (p = (~ x))*
by *auto*

lemma *if-x-Not*: *(if p then x else ~ x) = (p = x)*
by *auto*

lemma *if-same-and*: *(If p x y & If p u v) = (if p then x & u else y & v)*
by *auto*

lemma *if-same-eq*: *(If p x y = (If p u v)) = (if p then x = (u) else y = (v))*
by *auto*

lemma *if-same-eq-not*:
(If p x y = (~ If p u v)) = (if p then x = (~u) else y = (~v))
by *auto*

lemma *if-Cons*: *(if p then x # xs else y # ys) = If p x y # If p xs ys*
by *auto*

lemma *if-single*:
(if xc then [xab] else [an]) = [if xc then xab else an]
by *auto*

lemma *if-bool-simps*:
If p True y = (p | y) & If p False y = (~p & y) &
If p y True = (p −−> y) & If p y False = (p & y)
by *auto*

lemmas *if-simps = if-x-Not if-Not-x if-cancel if-True if-False if-bool-simps*

lemmas *seqr* = *eq-reflection* [**where** *x* = *size w*, *standard*]

lemmas *tl-Nil* = *tl.simps* (*1*)
lemmas *tl-Cons* = *tl.simps* (*2*)

7.2 Repeated splitting or concatenation

lemma *sclem*:
size (*concat* (*map* (*bin-to-bl n*) *xs*)) = *length xs* ∗ *n*
by (*induct xs*) *auto*

lemma *bin-cat-foldl-lem* [*rule-format*] :
ALL x. *foldl* (%*u*. *bin-cat u n*) *x xs* =
bin-cat x (*size xs* ∗ *n*) (*foldl* (%*u*. *bin-cat u n*) *y xs*)
apply (*induct xs*)
apply *simp*
apply *clarify*
apply (*simp* (*no-asm*))
apply (*frule asm-rl*)
apply (*drule spec*)
apply (*erule trans*)
apply (*drule-tac x* = *bin-cat y n a* **in** *spec*)
apply (*simp add* : *bin-cat-assoc-sym min-max.inf-absorb2*)
done

lemma *bin-rcat-bl*:
(*bin-rcat n wl*) = *bl-to-bin* (*concat* (*map* (*bin-to-bl n*) *wl*))
apply (*unfold bin-rcat-def*)
apply (*rule sym*)
apply (*induct wl*)
apply (*auto simp add* : *bl-to-bin-append*)
apply (*simp add* : *bl-to-bin-aux-alt sclem*)
apply (*simp add* : *bin-cat-foldl-lem* [*symmetric*])
done

lemmas *bin-rsplit-aux-simps* = *bin-rsplit-aux.simps bin-rsplitl-aux.simps*
lemmas *rsplit-aux-simps* = *bin-rsplit-aux-simps*

lemmas *th-if-simp1* = *split-if* [**where** *P* = *op* = *l*,
THEN iffD1, *THEN conjunct1*, *THEN mp*, *standard*]
lemmas *th-if-simp2* = *split-if* [**where** *P* = *op* = *l*,
THEN iffD1, *THEN conjunct2*, *THEN mp*, *standard*]

lemmas *rsplit-aux-simp1s* = *rsplit-aux-simps* [*THEN th-if-simp1*]

lemmas *rsplit-aux-simp2ls* = *rsplit-aux-simps* [*THEN th-if-simp2*]

lemmas *bin-rsplit-aux-simp2s* [*simp*] = *rsplit-aux-simp2ls* [*unfolded Let-def*]

lemmas *rbscl* = *bin-rsplit-aux-simp2s* (*2*)

lemmas *rsplit-aux-0-simps* [*simp*] =
rsplit-aux-simp1s [*OF disjI1*] *rsplit-aux-simp1s* [*OF disjI2*]

lemma *bin-rsplit-aux-append*:
bin-rsplit-aux n m c (*bs* @ *cs*) = *bin-rsplit-aux n m c bs* @ *cs*
apply (*induct n m c bs rule*: *bin-rsplit-aux.induct*)
apply (*subst bin-rsplit-aux.simps*)
apply (*subst bin-rsplit-aux.simps*)
apply (*clarsimp split*: *ls-splits*)
apply *auto*
done

lemma *bin-rsplitl-aux-append*:
bin-rsplitl-aux n m c (*bs* @ *cs*) = *bin-rsplitl-aux n m c bs* @ *cs*
apply (*induct n m c bs rule*: *bin-rsplitl-aux.induct*)
apply (*subst bin-rsplitl-aux.simps*)
apply (*subst bin-rsplitl-aux.simps*)
apply (*clarsimp split*: *ls-splits*)
apply *auto*
done

lemmas *rsplit-aux-apps* [**where** *bs* = []] =
bin-rsplit-aux-append bin-rsplitl-aux-append

lemmas *rsplit-def-auxs* = *bin-rsplit-def bin-rsplitl-def*

lemmas *rsplit-aux-alts* = *rsplit-aux-apps*
[*unfolded append-Nil rsplit-def-auxs* [*symmetric*]]

lemma *bin-split-minus*: *0* < *n* ==> *bin-split* (*Suc* (*n* − *1*)) *w* = *bin-split n w*
by *auto*

lemmas *bin-split-minus-simp* =
bin-split.Suc [*THEN* [*2*] *bin-split-minus* [*symmetric*, *THEN trans*], *standard*]

lemma *bin-split-pred-simp* [*simp*]:
(*0*::*nat*) < *number-of bin* ⟹
bin-split (*number-of bin*) *w* =
(*let* (*w1*, *w2*) = *bin-split* (*number-of* (*Int.pred bin*)) (*bin-rest w*)
in (*w1*, *w2 BIT bin-last w*))
by (*simp only*: *nobm1 bin-split-minus-simp*)

declare *bin-split-pred-simp* [*simp*]

lemma *bin-rsplit-aux-simp-alt*:
bin-rsplit-aux n m c bs =
(*if m* = *0* ∨ *n* = *0*

```
   then bs
   else let (a, b) = bin-split n c in bin-rsplit n (m − n, a) @ b # bs)
  unfolding bin-rsplit-aux.simps [of n m c bs]
  apply simp
  apply (subst rsplit-aux-alts)
  apply (simp add: bin-rsplit-def)
  done

lemmas bin-rsplit-simp-alt =
  trans [OF bin-rsplit-def
           bin-rsplit-aux-simp-alt, standard]

lemmas bthrs = bin-rsplit-simp-alt [THEN [2] trans]

lemma bin-rsplit-size-sign' [rule-format] :
  n > 0 ==> (ALL nw w. rev sw = bin-rsplit n (nw, w) −−>
    (ALL v: set sw. bintrunc n v = v))
  apply (induct sw)
   apply clarsimp
  apply clarsimp
  apply (drule bthrs)
  apply (simp (no-asm-use) add: Let-def split: ls-splits)
  apply clarify
  apply (erule impE, rule exI, erule exI)
  apply (drule split-bintrunc)
  apply simp
  done

lemmas bin-rsplit-size-sign = bin-rsplit-size-sign' [OF asm-rl
  rev-rev-ident [THEN trans] set-rev [THEN equalityD2 [THEN subsetD]],
  standard]

lemma bin-nth-rsplit [rule-format] :
  n > 0 ==> m < n ==> (ALL w k nw. rev sw = bin-rsplit n (nw, w) −−>
      k < size sw −−> bin-nth (sw ! k) m = bin-nth w (k * n + m))
  apply (induct sw)
   apply clarsimp
  apply clarsimp
  apply (drule bthrs)
  apply (simp (no-asm-use) add: Let-def split: ls-splits)
  apply clarify
  apply (erule allE, erule impE, erule exI)
  apply (case-tac k)
   apply clarsimp
   prefer 2
   apply clarsimp
   apply (erule allE)
   apply (erule (1) impE)
   apply (drule bin-nth-split, erule conjE, erule allE,
```

erule trans, *simp add* : *add-ac*)+
done

lemma *bin-rsplit-all*:
0 < nw ==> nw <= n ==> bin-rsplit n (*nw*, *w*) = [*bintrunc n w*]
unfolding *bin-rsplit-def*
by (*clarsimp dest*!: *split-bintrunc simp*: *rsplit-aux-simp2ls split*: *ls-splits*)

lemma *bin-rsplit-l* [*rule-format*] :
ALL bin. bin-rsplitl n (*m*, *bin*) = *bin-rsplit n* (*m*, *bintrunc m bin*)
apply (*rule-tac a* = *m* **in** *wf-less-than* [*THEN wf-induct*])
apply (*simp* (*no-asm*) *add* : *bin-rsplitl-def bin-rsplit-def*)
apply (*rule allI*)
apply (*subst bin-rsplitl-aux.simps*)
apply (*subst bin-rsplit-aux.simps*)
apply (*clarsimp simp*: *Let-def split*: *ls-splits*)
apply (*drule bin-split-trunc*)
apply (*drule sym* [*THEN trans*], *assumption*)
apply (*subst rsplit-aux-alts*(*1*))
apply (*subst rsplit-aux-alts*(*2*))
apply *clarsimp*
unfolding *bin-rsplit-def bin-rsplitl-def*
apply *simp*
done

lemma *bin-rsplit-rcat* [*rule-format*] :
n > 0 --> bin-rsplit n (*n * size ws*, *bin-rcat n ws*) = *map* (*bintrunc n*) *ws*
apply (*unfold bin-rsplit-def bin-rcat-def*)
apply (*rule-tac xs* = *ws* **in** *rev-induct*)
apply *clarsimp*
apply *clarsimp*
apply (*subst rsplit-aux-alts*)
unfolding *bin-split-cat*
apply *simp*
done

lemma *bin-rsplit-aux-len-le* [*rule-format*] :
∀ *ws m*. *n* ≠ *0* ⟶ *ws* = *bin-rsplit-aux n nw w bs* ⟶
length ws ≤ *m* ⟷ *nw* + *length bs* * *n* ≤ *m* * *n*
apply (*induct n nw w bs rule*: *bin-rsplit-aux.induct*)
apply (*subst bin-rsplit-aux.simps*)
apply (*simp add*: *lrlem Let-def split*: *ls-splits*)
done

lemma *bin-rsplit-len-le*:
n ≠ *0 --> ws = bin-rsplit n* (*nw*, *w*) --> (*length ws <= m*) = (*nw <= m * n*)
unfolding *bin-rsplit-def* **by** (*clarsimp simp add* : *bin-rsplit-aux-len-le*)

lemma *bin-rsplit-aux-len* [*rule-format*] :
n≠0 −−> length (*bin-rsplit-aux n nw w cs*) =
(*nw* + *n* − *1*) *div n* + *length cs*
apply (*induct n nw w cs rule*: *bin-rsplit-aux.induct*)
apply (*subst bin-rsplit-aux.simps*)
apply (*clarsimp simp*: *Let-def split*: *ls-splits*)
apply (*erule thin-rl*)
apply (*case-tac m*)
apply *simp*
apply (*case-tac m <= n*)
apply *auto*
done

lemma *bin-rsplit-len*:
n≠0 ==> length (*bin-rsplit n* (*nw*, *w*)) = (*nw* + *n* − *1*) *div n*
unfolding *bin-rsplit-def* **by** (*clarsimp simp add* : *bin-rsplit-aux-len*)

lemma *bin-rsplit-aux-len-indep*:
n ≠ *0* ⟹ *length bs* = *length cs* ⟹
length (*bin-rsplit-aux n nw v bs*) =
length (*bin-rsplit-aux n nw w cs*)
proof (*induct n nw w cs arbitrary*: *v bs rule*: *bin-rsplit-aux.induct*)
case (*1 n m w cs v bs*) **show** *?case*
proof (*cases m* = *0*)
case *True* **then show** *?thesis* **using** ‹*length bs* = *length cs*› **by** *simp*
next
case *False*
from *1.hyps* ‹*m* ≠ *0*› ‹*n* ≠ *0*› **have** *hyp*: ⋀*v bs*. *length bs* = *Suc* (*length cs*)
⟹
length (*bin-rsplit-aux n* (*m* − *n*) *v bs*) =
length (*bin-rsplit-aux n* (*m* − *n*) (*fst* (*bin-split n w*)) (*snd* (*bin-split n w*) #
cs))
by *auto*
show *?thesis* **using** ‹*length bs* = *length cs*› ‹*n* ≠ *0*›
by (*auto simp add*: *bin-rsplit-aux-simp-alt Let-def bin-rsplit-len*
split: *ls-splits*)
qed
qed

lemma *bin-rsplit-len-indep*:
n≠0 ==> length (*bin-rsplit n* (*nw*, *v*)) = *length* (*bin-rsplit n* (*nw*, *w*))
apply (*unfold bin-rsplit-def*)
apply (*simp* (*no-asm*))
apply (*erule bin-rsplit-aux-len-indep*)
apply (*rule refl*)
done

end

8 TdThs: Type Definition Theorems

theory *TdThs*
imports *Main*
begin

9 More lemmas about normal type definitions

lemma
tdD1: *type-definition Rep Abs A* $\Longrightarrow \forall x.$ *Rep x* $\in$ *A* **and**
tdD2: *type-definition Rep Abs A* $\Longrightarrow \forall x.$ *Abs* (*Rep x*) = *x* **and**
tdD3: *type-definition Rep Abs A* $\Longrightarrow \forall y.$ *y* $\in$ *A* $\longrightarrow$ *Rep* (*Abs y*) = *y*
by (*auto simp*: *type-definition-def*)

lemma *td-nat-int*:
type-definition int nat (*Collect* (*op* <= *0*))
unfolding *type-definition-def* **by** *auto*

context *type-definition*
begin

lemmas *Rep'* [*iff*] = *Rep* [*simplified*]

declare *Rep-inverse* [*simp*] *Rep-inject* [*simp*]

lemma *Abs-eqD*: *Abs x* = *Abs y* ==> *x* $\in$ *A* ==> *y* $\in$ *A* ==> *x* = *y*
by (*simp add*: *Abs-inject*)

lemma *Abs-inverse'*:
r : *A* ==> *Abs r* = *a* ==> *Rep a* = *r*
by (*safe elim*!: *Abs-inverse*)

lemma *Rep-comp-inverse*:
Rep o f = *g* ==> *Abs o g* = *f*
using *Rep-inverse* **by** (*auto intro*: *ext*)

lemma *Rep-eqD* [*elim*!]: *Rep x* = *Rep y* ==> *x* = *y*
by *simp*

lemma *Rep-inverse'*: *Rep a* = *r* ==> *Abs r* = *a*
by (*safe intro*!: *Rep-inverse*)

lemma *comp-Abs-inverse*:
f o Abs = *g* ==> *g o Rep* = *f*
using *Rep-inverse* **by** (*auto intro*: *ext*)

lemma *set-Rep*:
A = *range Rep*
proof (*rule set-ext*)

fix *x*
show $(x \in A) = (x \in range\ Rep)$
by (*auto dest*: *Abs-inverse* [*of x*, *symmetric*])
qed

lemma *set-Rep-Abs*: $A = range\ (Rep\ o\ Abs)$
proof (*rule set-ext*)
fix *x*
show $(x \in A) = (x \in range\ (Rep\ o\ Abs))$
by (*auto dest*: *Abs-inverse* [*of x*, *symmetric*])
qed

lemma *Abs-inj-on*: *inj-on Abs A*
unfolding *inj-on-def*
by (*auto dest*: *Abs-inject* [*THEN iffD1*])

lemma *image*: *Abs ' A = UNIV*
by (*auto intro*!: *image-eqI*)

lemmas *td-thm = type-definition-axioms*

lemma *fns1*:
Rep o fa = fr o Rep | fa o Abs = Abs o fr ==> Abs o fr o Rep = fa
by (*auto dest*: *Rep-comp-inverse elim*: *comp-Abs-inverse simp*: *o-assoc*)

lemmas *fns1a = disjI1* [*THEN fns1*]
lemmas *fns1b = disjI2* [*THEN fns1*]

lemma *fns4*:
Rep o fa o Abs = fr ==>
Rep o fa = fr o Rep & fa o Abs = Abs o fr
by (*auto intro*!: *ext*)

end

interpretation *nat-int*: *type-definition int nat Collect* (*op <= 0*)
by (*rule td-nat-int*)

declare
nat-int.Rep-cases [*cases del*]
nat-int.Abs-cases [*cases del*]
nat-int.Rep-induct [*induct del*]
nat-int.Abs-induct [*induct del*]

9.1 Extended form of type definition predicate

lemma *td-conds*:
norm o norm = norm ==> (fr o norm = norm o fr) =
(norm o fr o norm = fr o norm & norm o fr o norm = norm o fr)

apply *safe*
apply (*simp-all add*: *o-assoc* [*symmetric*])
apply (*simp-all add*: *o-assoc*)
done

lemma *fn-comm-power*:
fa o tr = *tr o fr* ==> *fa* ^^ *n o tr* = *tr o fr* ^^ *n*
apply (*rule ext*)
apply (*induct n*)
apply (*auto dest*: *fun-cong*)
done

lemmas *fn-comm-power'* =
ext [*THEN fn-comm-power*, *THEN fun-cong*, *unfolded o-def*, *standard*]

locale *td-ext* = *type-definition* +
fixes *norm*
assumes *eq-norm*: ⋀*x*. *Rep* (*Abs x*) = *norm x*
begin

lemma *Abs-norm* [*simp*]:
Abs (*norm x*) = *Abs x*
using *eq-norm* [*of x*] **by** (*auto elim*: *Rep-inverse'*)

lemma *td-th*:
g o Abs = *f* ==> *f* (*Rep x*) = *g x*
by (*drule comp-Abs-inverse* [*symmetric*]) *simp*

lemma *eq-norm'*: *Rep o Abs* = *norm*
by (*auto simp*: *eq-norm intro*!: *ext*)

lemma *norm-Rep* [*simp*]: *norm* (*Rep x*) = *Rep x*
by (*auto simp*: *eq-norm' intro*: *td-th*)

lemmas *td* = *td-thm*

lemma *set-iff-norm*: *w* : *A* <-> *w* = *norm w*
by (*auto simp*: *set-Rep-Abs eq-norm' eq-norm* [*symmetric*])

lemma *inverse-norm*:
(*Abs n* = *w*) = (*Rep w* = *norm n*)
apply (*rule iffI*)
apply (*clarsimp simp add*: *eq-norm*)
apply (*simp add*: *eq-norm'* [*symmetric*])
done

lemma *norm-eq-iff*:
(*norm x* = *norm y*) = (*Abs x* = *Abs y*)

by (*simp add*: *eq-norm'* [*symmetric*])

lemma *norm-comps*:
Abs o norm = Abs
norm o Rep = Rep
norm o norm = norm
by (*auto simp*: *eq-norm'* [*symmetric*] *o-def*)

lemmas *norm-norm* [*simp*] = *norm-comps*

lemma *fns5*:
Rep o fa o Abs = fr ==>
fr o norm = fr & norm o fr = fr
by (*fold eq-norm'*) (*auto intro*!: *ext*)

lemma *fns2*:
Abs o fr o Rep = fa ==>
(*norm o fr o norm = fr o norm*) = (*Rep o fa = fr o Rep*)
apply (*fold eq-norm'*)
apply *safe*
prefer *2*
apply (*simp add*: *o-assoc*)
apply (*rule ext*)
apply (*drule-tac x=Rep x* **in** *fun-cong*)
apply *auto*
done

lemma *fns3*:
Abs o fr o Rep = fa ==>
(*norm o fr o norm = norm o fr*) = (*fa o Abs = Abs o fr*)
apply (*fold eq-norm'*)
apply *safe*
prefer *2*
apply (*simp add*: *o-assoc* [*symmetric*])
apply (*rule ext*)
apply (*drule fun-cong*)
apply *simp*
done

lemma *fns*:
fr o norm = norm o fr ==>
(*fa o Abs = Abs o fr*) = (*Rep o fa = fr o Rep*)
apply *safe*
apply (*frule fns1b*)
prefer *2*
apply (*frule fns1a*)
apply (*rule fns3* [*THEN iffD1*])
prefer *3*

apply (*rule fns2* [*THEN iffD1*])
apply (*simp-all add*: *o-assoc* [*symmetric*])
apply (*simp-all add*: *o-assoc*)
done

lemma *range-norm*:
range (*Rep o Abs*) = *A*
by (*simp add*: *set-Rep-Abs*)

end

lemmas *td-ext-def'* =
td-ext-def [*unfolded type-definition-def td-ext-axioms-def*]

end

10 WordDefinition: Definition of Word Type

theory *WordDefinition*
imports *Size BinBoolList TdThs*
begin

10.1 Type definition

typedef (**open** *word*) *'a word* = {(*0*::*int*) ..< *2^len-of TYPE*(*'a*::*len0*)}
morphisms *uint Abs-word* **by** *auto*

definition *word-of-int* :: *int* ⇒ *'a*::*len0 word* **where**
— representation of words using unsigned or signed bins, only difference in these is the type class
word-of-int w = *Abs-word* (*bintrunc* (*len-of TYPE* (*'a*)) *w*)

lemma *uint-word-of-int* [*code*]: *uint* (*word-of-int w* :: *'a*::*len0 word*) = *w mod 2* ^ *len-of TYPE*(*'a*)
by (*auto simp add*: *word-of-int-def bintrunc-mod2p intro*: *Abs-word-inverse*)

code-datatype *word-of-int*

notation *fcomp* (**infixl** *o>* *60*)
notation *scomp* (**infixl** *o→* *60*)

instantiation *word* :: ({*len0*, *typerep*}) *random*
begin

definition
random-word i = *Random.range* (*max i* (*2* ^ *len-of TYPE*(*'a*))) *o→* (λ*k*. *Pair* (
let j = *word-of-int* (*Code-Numeral.int-of k*) :: *'a word*

in (*j*, λ-::*unit*. *Code-Evaluation.term-of j*)))

instance ..

end

no-notation *fcomp* (**infixl** *o>* *60*)
no-notation *scomp* (**infixl** *o→* *60*)

10.2 Type conversions and casting

definition *sint* :: *'a* :: *len word* => *int* **where**
— treats the most-significant-bit as a sign bit
sint-uint: *sint w* = *sbintrunc* (*len-of TYPE* (*'a*) − *1*) (*uint w*)

definition *unat* :: *'a* :: *len0 word* => *nat* **where**
unat w = *nat* (*uint w*)

definition *uints* :: *nat* => *int set* **where**
— the sets of integers representing the words
uints n = *range* (*bintrunc n*)

definition *sints* :: *nat* => *int set* **where**
sints n = *range* (*sbintrunc* (*n* − *1*))

definition *unats* :: *nat* => *nat set* **where**
unats n = {*i*. *i* < *2* ^ *n*}

definition *norm-sint* :: *nat* => *int* => *int* **where**
norm-sint n w = (*w* + *2* ^ (*n* − *1*)) *mod* *2* ^ *n* − *2* ^ (*n* − *1*)

definition *scast* :: *'a* :: *len word* => *'b* :: *len word* **where**
— cast a word to a different length
scast w = *word-of-int* (*sint w*)

definition *ucast* :: *'a* :: *len0 word* => *'b* :: *len0 word* **where**
ucast w = *word-of-int* (*uint w*)

instantiation *word* :: (*len0*) *size*
begin

definition
word-size: *size* (*w* :: *'a word*) = *len-of TYPE*(*'a*)

instance ..

end

definition *source-size* :: (*'a* :: *len0 word* => *'b*) => *nat* **where**

— whether a cast (or other) function is to a longer or shorter length
source-size c = (let arb = undefined ; x = c arb in size arb)

definition *target-size* :: (*'a => 'b* :: *len0 word*) => *nat* **where**
target-size c = size (c undefined)

definition *is-up* :: (*'a* :: *len0 word => 'b* :: *len0 word*) => *bool* **where**
is-up c ⟷ source-size c <= target-size c

definition *is-down* :: (*'a* :: *len0 word => 'b* :: *len0 word*) => *bool* **where**
is-down c ⟷ target-size c <= source-size c

definition *of-bl* :: *bool list => 'a* :: *len0 word* **where**
of-bl bl = word-of-int (bl-to-bin bl)

definition *to-bl* :: *'a* :: *len0 word => bool list* **where**
to-bl w = bin-to-bl (len-of TYPE ('a)) (uint w)

definition *word-reverse* :: *'a* :: *len0 word => 'a word* **where**
word-reverse w = of-bl (rev (to-bl w))

definition *word-int-case* :: (*int => 'b*) => (*'a* :: *len0 word*) => *'b* **where**
word-int-case f w = f (uint w)

syntax
of-int :: *int => 'a*
translations
case x of CONST of-int y => b == CONST word-int-case (%y. b) x

10.3 Arithmetic operations

instantiation *word* :: (*len0*) {*number*, *uminus*, *minus*, *plus*, *one*, *zero*, *times*, *Divides.div*, *ord*, *bit*}
begin

definition
word-0-wi: *0 = word-of-int 0*

definition
word-1-wi: *1 = word-of-int 1*

definition
word-add-def: *a + b = word-of-int (uint a + uint b)*

definition
word-sub-wi: *a − b = word-of-int (uint a − uint b)*

definition
word-minus-def: *− a = word-of-int (− uint a)*

definition
word-mult-def: $a * b = word\text{-}of\text{-}int\ (uint\ a * uint\ b)$

definition
word-div-def: $a\ div\ b = word\text{-}of\text{-}int\ (uint\ a\ div\ uint\ b)$

definition
word-mod-def: $a\ mod\ b = word\text{-}of\text{-}int\ (uint\ a\ mod\ uint\ b)$

definition
word-number-of-def: $number\text{-}of\ w = word\text{-}of\text{-}int\ w$

definition
word-le-def: $a \leq b \longleftrightarrow uint\ a \leq uint\ b$

definition
word-less-def: $x < y \longleftrightarrow x \leq y \wedge x \neq (y :: 'a\ word)$

definition
word-and-def:
$(a::'a\ word)\ AND\ b = word\text{-}of\text{-}int\ (uint\ a\ AND\ uint\ b)$

definition
word-or-def:
$(a::'a\ word)\ OR\ b = word\text{-}of\text{-}int\ (uint\ a\ OR\ uint\ b)$

definition
word-xor-def:
$(a::'a\ word)\ XOR\ b = word\text{-}of\text{-}int\ (uint\ a\ XOR\ uint\ b)$

definition
word-not-def:
$NOT\ (a::'a\ word) = word\text{-}of\text{-}int\ (NOT\ (uint\ a))$

instance ..

end

definition
$word\text{-}succ :: 'a :: len0\ word => 'a\ word$
where
$word\text{-}succ\ a = word\text{-}of\text{-}int\ (Int.succ\ (uint\ a))$

definition
$word\text{-}pred :: 'a :: len0\ word => 'a\ word$
where
$word\text{-}pred\ a = word\text{-}of\text{-}int\ (Int.pred\ (uint\ a))$

definition *udvd* :: *'a*::*len word* => *'a*::*len word* => *bool* (**infixl** *udvd 50*) **where**
*a udvd b == EX n>=0. uint b = n * uint a*

definition *word-sle* :: *'a* :: *len word* => *'a word* => *bool* ((-/ <=*s* -) [*50*, *51*] *50*)
where
a <=s b == sint a <= sint b

definition *word-sless* :: *'a* :: *len word* => *'a word* => *bool* ((-/ <*s* -) [*50*, *51*] *50*)
where
(x <s y) == (x <=s y & x ~= y)

10.4 Bit-wise operations

instantiation *word* :: (*len0*) *bits*
begin

definition
word-test-bit-def: *test-bit a = bin-nth (uint a)*

definition
word-set-bit-def: *set-bit a n x =*
word-of-int (bin-sc n (If x bit.B1 bit.B0) (uint a))

definition
word-set-bits-def: *(BITS n. f n) = of-bl (bl-of-nth (len-of TYPE ('a)) f)*

definition
word-lsb-def: *lsb a ⟷ bin-last (uint a) = bit.B1*

definition *shiftl1* :: *'a word* ⇒ *'a word* **where**
shiftl1 w = word-of-int (uint w BIT bit.B0)

definition *shiftr1* :: *'a word* ⇒ *'a word* **where**
— shift right as unsigned or as signed, ie logical or arithmetic
shiftr1 w = word-of-int (bin-rest (uint w))

definition
shiftl-def: *w << n = (shiftl1 ^^ n) w*

definition
shiftr-def: *w >> n = (shiftr1 ^^ n) w*

instance ..

end

instantiation *word* :: (*len*) *bitss*
begin

definition
word-msb-def:
msb a ⟷ *bin-sign* (*sint a*) = *Int.Min*

instance ..

end

definition *setBit* :: *'a* :: *len0 word* => *nat* => *'a word* **where**
setBit w n == *set-bit w n True*

definition *clearBit* :: *'a* :: *len0 word* => *nat* => *'a word* **where**
clearBit w n == *set-bit w n False*

10.5 Shift operations

definition *sshiftr1* :: *'a* :: *len word* => *'a word* **where**
sshiftr1 w == *word-of-int* (*bin-rest* (*sint w*))

definition *bshiftr1* :: *bool* => *'a* :: *len word* => *'a word* **where**
bshiftr1 b w == *of-bl* (*b* # *butlast* (*to-bl w*))

definition *sshiftr* :: *'a* :: *len word* => *nat* => *'a word* (**infixl** >>> *55*) **where**
w >>> *n* == (*sshiftr1* ^^ *n*) *w*

definition *mask* :: *nat* => *'a*::*len word* **where**
mask n == (*1* << *n*) − *1*

definition *revcast* :: *'a* :: *len0 word* => *'b* :: *len0 word* **where**
revcast w == *of-bl* (*takefill False* (*len-of TYPE*('*b*)) (*to-bl w*))

definition *slice1* :: *nat* => *'a* :: *len0 word* => *'b* :: *len0 word* **where**
slice1 n w == *of-bl* (*takefill False n* (*to-bl w*))

definition *slice* :: *nat* => *'a* :: *len0 word* => *'b* :: *len0 word* **where**
slice n w == *slice1* (*size w* − *n*) *w*

10.6 Rotation

definition *rotater1* :: *'a list* => *'a list* **where**
rotater1 ys ==
case ys of [] => [] | *x* # *xs* => *last ys* # *butlast ys*

definition *rotater* :: *nat* => *'a list* => *'a list* **where**
rotater n == *rotater1* ^^ *n*

definition *word-rotr* :: *nat* => *'a* :: *len0 word* => *'a* :: *len0 word* **where**
word-rotr n w == *of-bl* (*rotater n* (*to-bl w*))

definition *word-rotl* :: *nat* => *'a* :: *len0 word* => *'a* :: *len0 word* **where**

word-rotl n w == of-bl (rotate n (to-bl w))

definition *word-roti :: int => 'a :: len0 word => 'a :: len0 word* **where**
word-roti i w == if i >= 0 then word-rotr (nat i) w
else word-rotl (nat (− i)) w

10.7 Split and cat operations

definition *word-cat :: 'a :: len0 word => 'b :: len0 word => 'c :: len0 word* **where**
word-cat a b == word-of-int (bin-cat (uint a) (len-of TYPE ('b)) (uint b))

definition *word-split :: 'a :: len0 word => ('b :: len0 word) * ('c :: len0 word)*
where
word-split a ==
case bin-split (len-of TYPE ('c)) (uint a) of
(u, v) => (word-of-int u, word-of-int v)

definition *word-rcat :: 'a :: len0 word list => 'b :: len0 word* **where**
word-rcat ws ==
word-of-int (bin-rcat (len-of TYPE ('a)) (map uint ws))

definition *word-rsplit :: 'a :: len0 word => 'b :: len word list* **where**
word-rsplit w ==
map word-of-int (bin-rsplit (len-of TYPE ('b)) (len-of TYPE ('a), uint w))

definition *max-word :: 'a::len word* — Largest representable machine integer.
where
max-word ≡ word-of-int (2 ^ len-of TYPE('a) − 1)

primrec *of-bool :: bool ⇒ 'a::len word* **where**
of-bool False = 0
| *of-bool True = 1*

lemmas *of-nth-def = word-set-bits-def*

lemmas *word-size-gt-0 [iff] =*
xtr1 [OF word-size len-gt-0, standard]
lemmas *lens-gt-0 = word-size-gt-0 len-gt-0*
lemmas *lens-not-0 [iff] = lens-gt-0 [THEN gr-implies-not0, standard]*

lemma *uints-num: uints n = {i. 0 ≤ i ∧ i < 2 ^ n}*
by *(simp add: uints-def range-bintrunc)*

lemma *sints-num: sints n = {i. − (2 ^ (n − 1)) ≤ i ∧ i < 2 ^ (n − 1)}*
by *(simp add: sints-def range-sbintrunc)*

lemmas *atLeastLessThan-alt = atLeastLessThan-def [unfolded*
atLeast-def lessThan-def Collect-conj-eq [symmetric]]

lemma *mod-in-reps*: *m > 0 ==> y mod m : {0::int ..< m}*
unfolding *atLeastLessThan-alt* **by** *auto*

lemma
uint-0:*0 <= uint x* **and**
uint-lt: *uint (x::'a::len0 word) < 2 ^ len-of TYPE('a)*
by (*auto simp*: *uint* [*simplified*])

lemma *uint-mod-same*:
uint x mod 2 ^ len-of TYPE('a) = uint (x::'a::len0 word)
by (*simp add*: *int-mod-eq uint-lt uint-0*)

lemma *td-ext-uint*:
td-ext (uint :: 'a word => int) word-of-int (uints (len-of TYPE('a::len0)))
(%w::int. w mod 2 ^ len-of TYPE('a))
apply (*unfold td-ext-def'*)
apply (*simp add*: *uints-num word-of-int-def bintrunc-mod2p*)
apply (*simp add*: *uint-mod-same uint-0 uint-lt*
word.uint-inverse word.Abs-word-inverse int-mod-lem)
done

lemmas *int-word-uint = td-ext-uint* [*THEN td-ext.eq-norm*, *standard*]

interpretation *word-uint*:
td-ext uint::'a::len0 word ⇒ int
word-of-int
uints (len-of TYPE('a::len0))
λw. w mod 2 ^ len-of TYPE('a::len0)
by (*rule td-ext-uint*)

lemmas *td-uint = word-uint.td-thm*

lemmas *td-ext-ubin = td-ext-uint*
[*simplified len-gt-0 no-bintr-alt1* [*symmetric*]]

interpretation *word-ubin*:
td-ext uint::'a::len0 word ⇒ int
word-of-int
uints (len-of TYPE('a::len0))
bintrunc (len-of TYPE('a::len0))
by (*rule td-ext-ubin*)

lemma *sint-sbintrunc'*:
sint (word-of-int bin :: 'a word) =
(sbintrunc (len-of TYPE ('a :: len) − 1) bin)
unfolding *sint-uint*
by (*auto simp*: *word-ubin.eq-norm sbintrunc-bintrunc-lt*)

lemma *uint-sint*:
uint w = bintrunc (len-of TYPE('a)) (sint (w :: 'a :: len word))
unfolding *sint-uint* **by** (*auto simp*: *bintrunc-sbintrunc-le*)

lemma *bintr-uint'*:
n >= size w ==> bintrunc n (uint w) = uint w
apply (*unfold word-size*)
apply (*subst word-ubin.norm-Rep* [*symmetric*])
apply (*simp only*: *bintrunc-bintrunc-min word-size*)
apply (*simp add*: *min-max.inf-absorb2*)
done

lemma *wi-bintr'*:
wb = word-of-int bin ==> n >= size wb ==>
word-of-int (bintrunc n bin) = wb
unfolding *word-size*
by (*clarsimp simp add*: *word-ubin.norm-eq-iff* [*symmetric*] *min-max.inf-absorb1*)

lemmas *bintr-uint = bintr-uint'* [*unfolded word-size*]
lemmas *wi-bintr = wi-bintr'* [*unfolded word-size*]

lemma *td-ext-sbin*:
td-ext (sint :: 'a word => int) word-of-int (sints (len-of TYPE('a::len)))
(sbintrunc (len-of TYPE('a) - 1))
apply (*unfold td-ext-def' sint-uint*)
apply (*simp add : word-ubin.eq-norm*)
apply (*cases len-of TYPE('a)*)
apply (*auto simp add : sints-def*)
apply (*rule sym* [*THEN trans*])
apply (*rule word-ubin.Abs-norm*)
apply (*simp only*: *bintrunc-sbintrunc*)
apply (*drule sym*)
apply *simp*
done

lemmas *td-ext-sint = td-ext-sbin*
[*simplified len-gt-0 no-sbintr-alt2 Suc-pred'* [*symmetric*]]

interpretation *word-sint*:
td-ext sint ::'a::len word => int
word-of-int
sints (len-of TYPE('a::len))
%w. (w + 2^(len-of TYPE('a::len) - 1)) mod 2^len-of TYPE('a::len) -
2 ^ (len-of TYPE('a::len) - 1)
by (*rule td-ext-sint*)

interpretation *word-sbin*:
td-ext sint ::'a::len word => int

word-of-int
sints (*len-of TYPE*('*a*::*len*))
sbintrunc (*len-of TYPE*('*a*::*len*) − *1*)
by (*rule td-ext-sbin*)

lemmas *int-word-sint* = *td-ext-sint* [*THEN td-ext.eq-norm*, *standard*]

lemmas *td-sint* = *word-sint.td*

lemma *word-number-of-alt*: *number-of b* == *word-of-int* (*number-of b*)
unfolding *word-number-of-def* **by** (*simp add*: *number-of-eq*)

lemma *word-no-wi*: *number-of* = *word-of-int*
by (*auto simp*: *word-number-of-def intro*: *ext*)

lemma *to-bl-def'*:
(*to-bl* :: '*a* :: *len0 word* => *bool list*) =
bin-to-bl (*len-of TYPE*('*a*)) *o uint*
by (*auto simp*: *to-bl-def intro*: *ext*)

lemmas *word-reverse-no-def* [*simp*] = *word-reverse-def* [*of number-of w*, *standard*]

lemmas *uints-mod* = *uints-def* [*unfolded no-bintr-alt1*]

lemma *uint-bintrunc*: *uint* (*number-of bin* :: '*a word*) =
number-of (*bintrunc* (*len-of TYPE* ('*a* :: *len0*)) *bin*)
unfolding *word-number-of-def number-of-eq*
by (*auto intro*: *word-ubin.eq-norm*)

lemma *sint-sbintrunc*: *sint* (*number-of bin* :: '*a word*) =
number-of (*sbintrunc* (*len-of TYPE* ('*a* :: *len*) − *1*) *bin*)
unfolding *word-number-of-def number-of-eq*
by (*subst word-sbin.eq-norm*) *simp*

lemma *unat-bintrunc*:
unat (*number-of bin* :: '*a* :: *len0 word*) =
number-of (*bintrunc* (*len-of TYPE*('*a*)) *bin*)
unfolding *unat-def nat-number-of-def*
by (*simp only*: *uint-bintrunc*)

declare
uint-bintrunc [*simp*]
sint-sbintrunc [*simp*]
unat-bintrunc [*simp*]

lemma *size-0-eq*: *size* (*w* :: '*a* :: *len0 word*) = *0* ==> *v* = *w*
apply (*unfold word-size*)
apply (*rule word-uint.Rep-eqD*)

```
  apply (rule box-equals)
    defer
    apply (rule word-ubin.norm-Rep)+
  apply simp
  done

lemmas uint-lem = word-uint.Rep [unfolded uints-num mem-Collect-eq]
lemmas sint-lem = word-sint.Rep [unfolded sints-num mem-Collect-eq]
lemmas uint-ge-0 [iff] = uint-lem [THEN conjunct1, standard]
lemmas uint-lt2p [iff] = uint-lem [THEN conjunct2, standard]
lemmas sint-ge = sint-lem [THEN conjunct1, standard]
lemmas sint-lt = sint-lem [THEN conjunct2, standard]

lemma sign-uint-Pls [simp]:
  bin-sign (uint x) = Int.Pls
  by (simp add: sign-Pls-ge-0 number-of-eq)

lemmas uint-m2p-neg = iffD2 [OF diff-less-0-iff-less uint-lt2p, standard]
lemmas uint-m2p-not-non-neg =
  iffD2 [OF linorder-not-le uint-m2p-neg, standard]

lemma lt2p-lem:
  len-of TYPE('a) <= n ==> uint (w :: 'a :: len0 word) < 2 ^ n
  by (rule xtr8 [OF - uint-lt2p]) simp

lemmas uint-le-0-iff [simp] =
  uint-ge-0 [THEN leD, THEN linorder-antisym-conv1, standard]

lemma uint-nat: uint w == int (unat w)
  unfolding unat-def by auto

lemma uint-number-of:
  uint (number-of b :: 'a :: len0 word) = number-of b mod 2 ^ len-of TYPE('a)
  unfolding word-number-of-alt
  by (simp only: int-word-uint)

lemma unat-number-of:
  bin-sign b = Int.Pls ==>
  unat (number-of b::'a::len0 word) = number-of b mod 2 ^ len-of TYPE ('a)
  apply (unfold unat-def)
  apply (clarsimp simp only: uint-number-of)
  apply (rule nat-mod-distrib [THEN trans])
    apply (erule sign-Pls-ge-0 [THEN iffD1])
   apply (simp-all add: nat-power-eq)
  done

lemma sint-number-of: sint (number-of b :: 'a :: len word) = (number-of b +
    2 ^ (len-of TYPE('a) - 1)) mod 2 ^ len-of TYPE('a) -
    2 ^ (len-of TYPE('a) - 1)
```

unfolding *word-number-of-alt* **by** (*rule int-word-sint*)

lemma *word-of-int-bin* [*simp*] :
(*word-of-int* (*number-of bin*) :: *'a* :: *len0 word*) = (*number-of bin*)
unfolding *word-number-of-alt* **by** *auto*

lemma *word-int-case-wi*:
word-int-case f (*word-of-int i* :: *'b word*) =
f (*i mod 2* ^ *len-of TYPE*(*'b*::*len0*))
unfolding *word-int-case-def* **by** (*simp add*: *word-uint.eq-norm*)

lemma *word-int-split*:
P (*word-int-case f x*) =
(*ALL i. x* = (*word-of-int i* :: *'b* :: *len0 word*) &
0 <= *i* & *i* < *2* ^ *len-of TYPE*(*'b*) −−> *P* (*f i*))
unfolding *word-int-case-def*
by (*auto simp*: *word-uint.eq-norm int-mod-eq'*)

lemma *word-int-split-asm*:
P (*word-int-case f x*) =
(~ (*EX n. x* = (*word-of-int n* :: *'b*::*len0 word*) &
0 <= *n* & *n* < *2* ^ *len-of TYPE*(*'b*::*len0*) & ~ *P* (*f n*)))
unfolding *word-int-case-def*
by (*auto simp*: *word-uint.eq-norm int-mod-eq'*)

lemmas *uint-range'* =
word-uint.Rep [*unfolded uints-num mem-Collect-eq*, *standard*]
lemmas *sint-range'* = *word-sint.Rep* [*unfolded One-nat-def*
sints-num mem-Collect-eq, *standard*]

lemma *uint-range-size*: *0* <= *uint w* & *uint w* < *2* ^ *size w*
unfolding *word-size* **by** (*rule uint-range'*)

lemma *sint-range-size*:
− (*2* ^ (*size w* − *Suc 0*)) <= *sint w* & *sint w* < *2* ^ (*size w* − *Suc 0*)
unfolding *word-size* **by** (*rule sint-range'*)

lemmas *sint-above-size* = *sint-range-size*
[*THEN conjunct2*, *THEN* [*2*] *xtr8*, *folded One-nat-def*, *standard*]

lemmas *sint-below-size* = *sint-range-size*
[*THEN conjunct1*, *THEN* [*2*] *order-trans*, *folded One-nat-def*, *standard*]

lemma *test-bit-eq-iff*: (*test-bit* (*u*::*'a*::*len0 word*) = *test-bit v*) = (*u* = *v*)
unfolding *word-test-bit-def* **by** (*simp add*: *bin-nth-eq-iff*)

lemma *test-bit-size* [*rule-format*] : (*w*::*'a*::*len0 word*) !! *n* −−> *n* < *size w*
apply (*unfold word-test-bit-def*)
apply (*subst word-ubin.norm-Rep* [*symmetric*])

apply (*simp only*: *nth-bintr word-size*)
apply *fast*
done

lemma *word-eqI* [*rule-format*] :
fixes *u* :: *'a*::*len0 word*
shows (*ALL n*. *n* < *size u* −−> *u* !! *n* = *v* !! *n*) ==> *u* = *v*
apply (*rule test-bit-eq-iff* [*THEN iffD1*])
apply (*rule ext*)
apply (*erule allE*)
apply (*erule impCE*)
prefer *2*
apply *assumption*
apply (*auto dest*!: *test-bit-size simp add*: *word-size*)
done

lemmas *word-eqD* = *test-bit-eq-iff* [*THEN iffD2*, *THEN fun-cong*, *standard*]

lemma *test-bit-bin'*: *w* !! *n* = (*n* < *size w* & *bin-nth* (*uint w*) *n*)
unfolding *word-test-bit-def word-size*
by (*simp add*: *nth-bintr* [*symmetric*])

lemmas *test-bit-bin* = *test-bit-bin'* [*unfolded word-size*]

lemma *bin-nth-uint-imp'*: *bin-nth* (*uint w*) *n* −−> *n* < *size w*
apply (*unfold word-size*)
apply (*rule impI*)
apply (*rule nth-bintr* [*THEN iffD1*, *THEN conjunct1*])
apply (*subst word-ubin.norm-Rep*)
apply *assumption*
done

lemma *bin-nth-sint'*:
n >= *size w* −−> *bin-nth* (*sint w*) *n* = *bin-nth* (*sint w*) (*size w* − *1*)
apply (*rule impI*)
apply (*subst word-sbin.norm-Rep* [*symmetric*])
apply (*simp add* : *nth-sbintr word-size*)
apply *auto*
done

lemmas *bin-nth-uint-imp* = *bin-nth-uint-imp'* [*rule-format*, *unfolded word-size*]
lemmas *bin-nth-sint* = *bin-nth-sint'* [*rule-format*, *unfolded word-size*]

lemma *td-bl*:
type-definition (*to-bl* :: *'a*::*len0 word* => *bool list*)
of-bl
{*bl*. *length bl* = *len-of TYPE*(*'a*)}
apply (*unfold type-definition-def of-bl-def to-bl-def*)

apply (*simp add*: *word-ubin.eq-norm*)
apply *safe*
apply (*drule sym*)
apply *simp*
done

interpretation *word-bl*:
type-definition to-bl :: *'a*::*len0 word* => *bool list*
of-bl
{*bl*. *length bl* = *len-of TYPE*(*'a*::*len0*)}
by (*rule td-bl*)

lemma *word-size-bl*: *size w* == *size* (*to-bl w*)
unfolding *word-size* **by** *auto*

lemma *to-bl-use-of-bl*:
(*to-bl w* = *bl*) = (*w* = *of-bl bl* ∧ *length bl* = *length* (*to-bl w*))
by (*fastsimp elim*!: *word-bl.Abs-inverse* [*simplified*])

lemma *to-bl-word-rev*: *to-bl* (*word-reverse w*) = *rev* (*to-bl w*)
unfolding *word-reverse-def* **by** (*simp add*: *word-bl.Abs-inverse*)

lemma *word-rev-rev* [*simp*] : *word-reverse* (*word-reverse w*) = *w*
unfolding *word-reverse-def* **by** (*simp add* : *word-bl.Abs-inverse*)

lemma *word-rev-gal*: *word-reverse w* = *u* ==> *word-reverse u* = *w*
by *auto*

lemmas *word-rev-gal'* = *sym* [*THEN word-rev-gal*, *symmetric*, *standard*]

lemmas *length-bl-gt-0* [*iff*] = *xtr1* [*OF word-bl.Rep'* *len-gt-0*, *standard*]
lemmas *bl-not-Nil* [*iff*] =
length-bl-gt-0 [*THEN length-greater-0-conv* [*THEN iffD1*], *standard*]
lemmas *length-bl-neq-0* [*iff*] = *length-bl-gt-0* [*THEN gr-implies-not0*]

lemma *hd-bl-sign-sint*: *hd* (*to-bl w*) = (*bin-sign* (*sint w*) = *Int.Min*)
apply (*unfold to-bl-def sint-uint*)
apply (*rule trans* [*OF - bl-sbin-sign*])
apply *simp*
done

lemma *of-bl-drop'*:
lend = *length bl* − *len-of TYPE* (*'a* :: *len0*) ==>
of-bl (*drop lend bl*) = (*of-bl bl* :: *'a word*)
apply (*unfold of-bl-def*)
apply (*clarsimp simp add* : *trunc-bl2bin* [*symmetric*])
done

lemmas *of-bl-no* = *of-bl-def* [*folded word-number-of-def*]

lemma *test-bit-of-bl*:
(*of-bl bl*::*'a*::*len0 word*) !! *n* = (*rev bl* ! *n* ∧ *n* < *len-of TYPE*(*'a*) ∧ *n* < *length bl*)
apply (*unfold of-bl-def word-test-bit-def*)
apply (*auto simp add*: *word-size word-ubin.eq-norm nth-bintr bin-nth-of-bl*)
done

lemma *no-of-bl*:
(*number-of bin* ::*'a*::*len0 word*) = *of-bl* (*bin-to-bl* (*len-of TYPE* (*'a*)) *bin*)
unfolding *word-size of-bl-no* **by** (*simp add* : *word-number-of-def*)

lemma *uint-bl*: *to-bl w* == *bin-to-bl* (*size w*) (*uint w*)
unfolding *word-size to-bl-def* **by** *auto*

lemma *to-bl-bin*: *bl-to-bin* (*to-bl w*) = *uint w*
unfolding *uint-bl* **by** (*simp add* : *word-size*)

lemma *to-bl-of-bin*:
to-bl (*word-of-int bin*::*'a*::*len0 word*) = *bin-to-bl* (*len-of TYPE*(*'a*)) *bin*
unfolding *uint-bl* **by** (*clarsimp simp add*: *word-ubin.eq-norm word-size*)

lemmas *to-bl-no-bin* [*simp*] = *to-bl-of-bin* [*folded word-number-of-def*]

lemma *to-bl-to-bin* [*simp*] : *bl-to-bin* (*to-bl w*) = *uint w*
unfolding *uint-bl* **by** (*simp add* : *word-size*)

lemmas *uint-bl-bin* [*simp*] = *trans* [*OF bin-bl-bin word-ubin.norm-Rep*, *standard*]

lemmas *num-AB-u* [*simp*] = *word-uint.Rep-inverse*
[*unfolded o-def word-number-of-def* [*symmetric*], *standard*]
lemmas *num-AB-s* [*simp*] = *word-sint.Rep-inverse*
[*unfolded o-def word-number-of-def* [*symmetric*], *standard*]

lemma *uints-unats*: *uints n* = *int* ‘ *unats n*
apply (*unfold unats-def uints-num*)
apply *safe*
apply (*rule-tac image-eqI*)
apply (*erule-tac nat-0-le* [*symmetric*])
apply *auto*
apply (*erule-tac nat-less-iff* [*THEN iffD2*])
apply (*rule-tac* [*2*] *zless-nat-eq-int-zless* [*THEN iffD1*])
apply (*auto simp add* : *nat-power-eq int-power*)
done

lemma *unats-uints*: *unats n* = *nat* ‘ *uints n*
by (*auto simp add* : *uints-unats image-iff*)

lemmas *bintr-num* = *word-ubin.norm-eq-iff*
[*symmetric*, *folded word-number-of-def*, *standard*]
lemmas *sbintr-num* = *word-sbin.norm-eq-iff*
[*symmetric*, *folded word-number-of-def*, *standard*]

lemmas *num-of-bintr* = *word-ubin.Abs-norm* [*folded word-number-of-def*, *standard*]
lemmas *num-of-sbintr* = *word-sbin.Abs-norm* [*folded word-number-of-def*, *standard*]

lemma *num-of-bintr'*:
bintrunc (*len-of TYPE*('*a* :: *len0*)) *a* = *b* ==>
number-of a = (*number-of b* :: '*a word*)
apply *safe*
apply (*rule-tac num-of-bintr* [*symmetric*])
done

lemma *num-of-sbintr'*:
sbintrunc (*len-of TYPE*('*a* :: *len*) − *1*) *a* = *b* ==>
number-of a = (*number-of b* :: '*a word*)
apply *safe*
apply (*rule-tac num-of-sbintr* [*symmetric*])
done

lemmas *num-abs-bintr* = *sym* [*THEN trans*,
OF num-of-bintr word-number-of-def, *standard*]
lemmas *num-abs-sbintr* = *sym* [*THEN trans*,
OF num-of-sbintr word-number-of-def, *standard*]

lemma *ucast-id*: *ucast w* = *w*
unfolding *ucast-def* **by** *auto*

lemma *scast-id*: *scast w* = *w*
unfolding *scast-def* **by** *auto*

lemma *ucast-bl*: *ucast w* == *of-bl* (*to-bl w*)
unfolding *ucast-def of-bl-def uint-bl*
by (*auto simp add* : *word-size*)

lemma *nth-ucast*:
(*ucast w*::'*a*::*len0 word*) !! *n* = (*w* !! *n* & *n* < *len-of TYPE*('*a*))
apply (*unfold ucast-def test-bit-bin*)
apply (*simp add*: *word-ubin.eq-norm nth-bintr word-size*)
apply (*fast elim*!: *bin-nth-uint-imp*)
done

lemma *ucast-bintr* [*simp*]:
ucast (*number-of w* ::'*a*::*len0 word*) =
number-of (*bintrunc* (*len-of TYPE*('*a*)) *w*)
unfolding *ucast-def* **by** *simp*

lemma *scast-sbintr* [*simp*]:
scast (*number-of w* ::'*a*::*len word*) =
number-of (*sbintrunc* (*len-of TYPE*('*a*) − *Suc 0*) *w*)
unfolding *scast-def* **by** *simp*

lemmas *source-size* = *source-size-def* [*unfolded Let-def word-size*]
lemmas *target-size* = *target-size-def* [*unfolded Let-def word-size*]
lemmas *is-down* = *is-down-def* [*unfolded source-size target-size*]
lemmas *is-up* = *is-up-def* [*unfolded source-size target-size*]

lemmas *is-up-down* = *trans* [*OF is-up is-down* [*symmetric*], *standard*]

lemma *down-cast-same'*: *uc* = *ucast* ==> *is-down uc* ==> *uc* = *scast*
apply (*unfold is-down*)
apply *safe*
apply (*rule ext*)
apply (*unfold ucast-def scast-def uint-sint*)
apply (*rule word-ubin.norm-eq-iff* [*THEN iffD1*])
apply *simp*
done

lemma *word-rev-tf'*:
r = *to-bl* (*of-bl bl*) ==> *r* = *rev* (*takefill False* (*length r*) (*rev bl*))
unfolding *of-bl-def uint-bl*
by (*clarsimp simp add*: *bl-bin-bl-rtf word-ubin.eq-norm word-size*)

lemmas *word-rev-tf* = *refl* [*THEN word-rev-tf'*, *unfolded word-bl.Rep'*, *standard*]

lemmas *word-rep-drop* = *word-rev-tf* [*simplified takefill-alt*,
simplified, *simplified rev-take*, *simplified*]

lemma *to-bl-ucast*:
to-bl (*ucast* (*w*::'*b*::*len0 word*) ::'*a*::*len0 word*) =
replicate (*len-of TYPE*('*a*) − *len-of TYPE*('*b*)) *False* @
drop (*len-of TYPE*('*b*) − *len-of TYPE*('*a*)) (*to-bl w*)
apply (*unfold ucast-bl*)
apply (*rule trans*)
apply (*rule word-rep-drop*)
apply *simp*
done

lemma *ucast-up-app'*:
uc = ucast ==> source-size uc + n = target-size uc ==>
to-bl (uc w) = replicate n False @ (to-bl w)
by (*auto simp add : source-size target-size to-bl-ucast*)

lemma *ucast-down-drop'*:
uc = ucast ==> source-size uc = target-size uc + n ==>
to-bl (uc w) = drop n (to-bl w)
by (*auto simp add : source-size target-size to-bl-ucast*)

lemma *scast-down-drop'*:
sc = scast ==> source-size sc = target-size sc + n ==>
to-bl (sc w) = drop n (to-bl w)
apply (*subgoal-tac sc = ucast*)
apply *safe*
apply *simp*
apply (*erule refl* [*THEN ucast-down-drop'*])
apply (*rule refl* [*THEN down-cast-same', symmetric*])
apply (*simp add : source-size target-size is-down*)
done

lemma *sint-up-scast'*:
sc = scast ==> is-up sc ==> sint (sc w) = sint w
apply (*unfold is-up*)
apply *safe*
apply (*simp add*: *scast-def word-sbin.eq-norm*)
apply (*rule box-equals*)
prefer *3*
apply (*rule word-sbin.norm-Rep*)
apply (*rule sbintrunc-sbintrunc-l*)
defer
apply (*subst word-sbin.norm-Rep*)
apply (*rule refl*)
apply *simp*
done

lemma *uint-up-ucast'*:
uc = ucast ==> is-up uc ==> uint (uc w) = uint w
apply (*unfold is-up*)
apply *safe*
apply (*rule bin-eqI*)
apply (*fold word-test-bit-def*)
apply (*auto simp add*: *nth-ucast*)
apply (*auto simp add*: *test-bit-bin*)
done

lemmas *down-cast-same = refl* [*THEN down-cast-same'*]
lemmas *ucast-up-app = refl* [*THEN ucast-up-app'*]
lemmas *ucast-down-drop = refl* [*THEN ucast-down-drop'*]

lemmas *scast-down-drop = refl [THEN scast-down-drop']*
lemmas *uint-up-ucast = refl [THEN uint-up-ucast']*
lemmas *sint-up-scast = refl [THEN sint-up-scast']*

lemma *ucast-up-ucast'*: *uc = ucast ==> is-up uc ==> ucast (uc w) = ucast w*
apply (*simp (no-asm) add*: *ucast-def*)
apply (*clarsimp simp add*: *uint-up-ucast*)
done

lemma *scast-up-scast'*: *sc = scast ==> is-up sc ==> scast (sc w) = scast w*
apply (*simp (no-asm) add*: *scast-def*)
apply (*clarsimp simp add*: *sint-up-scast*)
done

lemma *ucast-of-bl-up'*:
w = of-bl bl ==> size bl <= size w ==> ucast w = of-bl bl
by (*auto simp add* : *nth-ucast word-size test-bit-of-bl intro*!: *word-eqI*)

lemmas *ucast-up-ucast = refl [THEN ucast-up-ucast']*
lemmas *scast-up-scast = refl [THEN scast-up-scast']*
lemmas *ucast-of-bl-up = refl [THEN ucast-of-bl-up']*

lemmas *ucast-up-ucast-id = trans [OF ucast-up-ucast ucast-id]*
lemmas *scast-up-scast-id = trans [OF scast-up-scast scast-id]*

lemmas *isduu = is-up-down* [**where** *c = ucast*, *THEN iffD2*]
lemmas *isdus = is-up-down* [**where** *c = scast*, *THEN iffD2*]
lemmas *ucast-down-ucast-id = isduu [THEN ucast-up-ucast-id]*
lemmas *scast-down-scast-id = isdus [THEN ucast-up-ucast-id]*

lemma *up-ucast-surj*:
is-up (ucast :: 'b::len0 word => 'a::len0 word) ==>
surj (ucast :: 'a word => 'b word)
by (*rule surjI*, *erule ucast-up-ucast-id*)

lemma *up-scast-surj*:
is-up (scast :: 'b::len word => 'a::len word) ==>
surj (scast :: 'a word => 'b word)
by (*rule surjI*, *erule scast-up-scast-id*)

lemma *down-scast-inj*:
is-down (scast :: 'b::len word => 'a::len word) ==>
inj-on (ucast :: 'a word => 'b word) A
by (*rule inj-on-inverseI*, *erule scast-down-scast-id*)

lemma *down-ucast-inj*:
is-down (ucast :: 'b::len0 word => 'a::len0 word) ==>
inj-on (ucast :: 'a word => 'b word) A
by (*rule inj-on-inverseI*, *erule ucast-down-ucast-id*)

lemma *of-bl-append-same*: *of-bl* (*X* @ *to-bl w*) = *w*
by (*rule word-bl.Rep-eqD*) (*simp add*: *word-rep-drop*)

lemma *ucast-down-no'*:
uc = *ucast* ==> *is-down uc* ==> *uc* (*number-of bin*) = *number-of bin*
apply (*unfold word-number-of-def is-down*)
apply (*clarsimp simp add*: *ucast-def word-ubin.eq-norm*)
apply (*rule word-ubin.norm-eq-iff* [*THEN iffD1*])
apply (*erule bintrunc-bintrunc-ge*)
done

lemmas *ucast-down-no* = *ucast-down-no'* [*OF refl*]

lemma *ucast-down-bl'*: *uc* = *ucast* ==> *is-down uc* ==> *uc* (*of-bl bl*) = *of-bl bl*
unfolding *of-bl-no* **by** *clarify* (*erule ucast-down-no*)

lemmas *ucast-down-bl* = *ucast-down-bl'* [*OF refl*]

lemmas *slice-def'* = *slice-def* [*unfolded word-size*]
lemmas *test-bit-def'* = *word-test-bit-def* [*THEN fun-cong*]

lemmas *word-log-defs* = *word-and-def word-or-def word-xor-def word-not-def*
lemmas *word-log-bin-defs* = *word-log-defs*

Executable equality

instantiation *word* :: ({*len0*}) *eq*
begin

definition *eq-word* :: *'a word* ⇒ *'a word* ⇒ *bool* **where**
eq-word k l ⟷ *HOL.eq* (*uint k*) (*uint l*)

instance proof
qed (*simp add*: *eq eq-word-def*)

end

end

11 WordArith: Word Arithmetic

theory *WordArith*
imports *WordDefinition*
begin

lemma *word-less-alt*: (*a* < *b*) = (*uint a* < *uint b*)
unfolding *word-less-def word-le-def*
by (*auto simp del*: *word-uint.Rep-inject*

simp: *word-uint.Rep-inject* [*symmetric*])

lemma *signed-linorder*: *class.linorder word-sle word-sless*
proof
qed (*unfold word-sle-def word-sless-def*, *auto*)

interpretation *signed*: *linorder word-sle word-sless*
by (*rule signed-linorder*)

lemmas *word-arith-wis* =
word-add-def word-mult-def word-minus-def
word-succ-def word-pred-def word-0-wi word-1-wi

lemma *udvdI*:
$0 \leq n$ ==> *uint b* = n * *uint a* ==> *a udvd b*
by (*auto simp*: *udvd-def*)

lemmas *word-div-no* [*simp*] =
word-div-def [*of number-of a number-of b*, *standard*]

lemmas *word-mod-no* [*simp*] =
word-mod-def [*of number-of a number-of b*, *standard*]

lemmas *word-less-no* [*simp*] =
word-less-def [*of number-of a number-of b*, *standard*]

lemmas *word-le-no* [*simp*] =
word-le-def [*of number-of a number-of b*, *standard*]

lemmas *word-sless-no* [*simp*] =
word-sless-def [*of number-of a number-of b*, *standard*]

lemmas *word-sle-no* [*simp*] =
word-sle-def [*of number-of a number-of b*, *standard*]

lemmas *word-0-wi-Pls* = *word-0-wi* [*folded Pls-def*]
lemmas *word-0-no* = *word-0-wi-Pls* [*folded word-no-wi*]

lemma *int-one-bin*: (*1* :: *int*) == (*Int.Pls BIT bit.B1*)
unfolding *Pls-def Bit-def* **by** *auto*

lemma *word-1-no*:
(*1* :: *'a* :: *len0 word*) == *number-of* (*Int.Pls BIT bit.B1*)
unfolding *word-1-wi word-number-of-def int-one-bin* **by** *auto*

lemma *word-m1-wi*: -1 == *word-of-int* -1
by (*rule word-number-of-alt*)

lemma *word-m1-wi-Min*: $-1 = word\text{-}of\text{-}int\ Int.Min$
by (*simp add*: *word-m1-wi number-of-eq*)

lemma *word-0-bl*: *of-bl* [] = *0*
unfolding *word-0-wi of-bl-def* **by** (*simp add* : *Pls-def*)

lemma *word-1-bl*: *of-bl* [*True*] = *1*
unfolding *word-1-wi of-bl-def*
by (*simp add* : *bl-to-bin-def Bit-def Pls-def*)

lemma *uint-0* [*simp*] : (*uint 0* = *0*)
unfolding *word-0-wi*
by (*simp add*: *word-ubin.eq-norm Pls-def* [*symmetric*])

lemma *of-bl-0* [*simp*] : *of-bl* (*replicate n False*) = *0*
by (*simp add* : *word-0-wi of-bl-def bl-to-bin-rep-False Pls-def*)

lemma *to-bl-0*:
to-bl (*0*::*'a*::*len0 word*) = *replicate* (*len-of TYPE*(*'a*)) *False*
unfolding *uint-bl*
by (*simp add* : *word-size bin-to-bl-Pls Pls-def* [*symmetric*])

lemma *uint-0-iff*: (*uint x* = *0*) = (*x* = *0*)
by (*auto intro*!: *word-uint.Rep-eqD*)

lemma *unat-0-iff*: (*unat x* = *0*) = (*x* = *0*)
unfolding *unat-def* **by** (*auto simp add* : *nat-eq-iff uint-0-iff*)

lemma *unat-0* [*simp*]: *unat 0* = *0*
unfolding *unat-def* **by** *auto*

lemma *size-0-same'*: *size w* = *0* ==> *w* = (*v* :: *'a* :: *len0 word*)
apply (*unfold word-size*)
apply (*rule box-equals*)
defer
apply (*rule word-uint.Rep-inverse*)+
apply (*rule word-ubin.norm-eq-iff* [*THEN iffD1*])
apply *simp*
done

lemmas *size-0-same* = *size-0-same'* [*folded word-size*]

lemmas *unat-eq-0* = *unat-0-iff*
lemmas *unat-eq-zero* = *unat-0-iff*

lemma *unat-gt-0*: (*0* < *unat x*) = (*x* ~= *0*)
by (*auto simp*: *unat-0-iff* [*symmetric*])

lemma *ucast-0* [*simp*] : *ucast 0* = *0*

unfolding *ucast-def*
by *simp* (*simp add*: *word-0-wi*)

lemma *sint-0* [*simp*] : *sint 0 = 0*
unfolding *sint-uint*
by (*simp add*: *Pls-def* [*symmetric*])

lemma *scast-0* [*simp*] : *scast 0 = 0*
apply (*unfold scast-def*)
apply *simp*
apply (*simp add*: *word-0-wi*)
done

lemma *sint-n1* [*simp*] : *sint −1 = −1*
apply (*unfold word-m1-wi-Min*)
apply (*simp add*: *word-sbin.eq-norm*)
apply (*unfold Min-def number-of-eq*)
apply *simp*
done

lemma *scast-n1* [*simp*] : *scast −1 = −1*
apply (*unfold scast-def sint-n1*)
apply (*unfold word-number-of-alt*)
apply (*rule refl*)
done

lemma *uint-1* [*simp*] : *uint* (*1* :: *'a* :: *len word*) = *1*
unfolding *word-1-wi*
by (*simp add*: *word-ubin.eq-norm int-one-bin bintrunc-minus-simps*)

lemma *unat-1* [*simp*] : *unat* (*1* :: *'a* :: *len word*) = *1*
by (*unfold unat-def uint-1*) *auto*

lemma *ucast-1* [*simp*] : *ucast* (*1* :: *'a* :: *len word*) = *1*
unfolding *ucast-def word-1-wi*
by (*simp add*: *word-ubin.eq-norm int-one-bin bintrunc-minus-simps*)

lemmas *arths* =
bintr-ariths [*THEN word-ubin.norm-eq-iff* [*THEN iffD1*],
folded word-ubin.eq-norm, *standard*]

lemma *wi-homs*:
shows
wi-hom-add: *word-of-int a* + *word-of-int b* = *word-of-int* (*a* + *b*) **and**
wi-hom-mult: *word-of-int a* ∗ *word-of-int b* = *word-of-int* (*a* ∗ *b*) **and**
wi-hom-neg: − *word-of-int a* = *word-of-int* (− *a*) **and**
wi-hom-succ: *word-succ* (*word-of-int a*) = *word-of-int* (*Int.succ a*) **and**

wi-hom-pred: *word-pred* (*word-of-int a*) = *word-of-int* (*Int.pred a*)
by (*auto simp*: *word-arith-wis arths*)

lemmas *wi-hom-syms* = *wi-homs* [*symmetric*]

lemma *word-sub-def*: *a* − *b* == *a* + − (*b* :: *'a* :: *len0 word*)
unfolding *word-sub-wi diff-def*
by (*simp only* : *word-uint.Rep-inverse wi-hom-syms*)

lemmas *word-diff-minus* = *word-sub-def* [*THEN meta-eq-to-obj-eq*, *standard*]

lemma *word-of-int-sub-hom*:
(*word-of-int a*) − *word-of-int b* = *word-of-int* (*a* − *b*)
unfolding *word-sub-def diff-def* **by** (*simp only* : *wi-homs*)

lemmas *new-word-of-int-homs* =
word-of-int-sub-hom wi-homs word-0-wi word-1-wi

lemmas *new-word-of-int-hom-syms* = *new-word-of-int-homs* [*symmetric*, *standard*]

lemmas *word-of-int-hom-syms* =
new-word-of-int-hom-syms [*unfolded succ-def pred-def*]

lemmas *word-of-int-homs* =
new-word-of-int-homs [*unfolded succ-def pred-def*]

lemmas *word-of-int-add-hom* = *word-of-int-homs* (*2*)
lemmas *word-of-int-mult-hom* = *word-of-int-homs* (*3*)
lemmas *word-of-int-minus-hom* = *word-of-int-homs* (*4*)
lemmas *word-of-int-succ-hom* = *word-of-int-homs* (*5*)
lemmas *word-of-int-pred-hom* = *word-of-int-homs* (*6*)
lemmas *word-of-int-0-hom* = *word-of-int-homs* (*7*)
lemmas *word-of-int-1-hom* = *word-of-int-homs* (*8*)

lemmas *word-arith-alts* =
word-sub-wi [*unfolded succ-def pred-def*, *standard*]
word-arith-wis [*unfolded succ-def pred-def*, *standard*]

lemmas *word-sub-alt* = *word-arith-alts* (*1*)
lemmas *word-add-alt* = *word-arith-alts* (*2*)
lemmas *word-mult-alt* = *word-arith-alts* (*3*)
lemmas *word-minus-alt* = *word-arith-alts* (*4*)
lemmas *word-succ-alt* = *word-arith-alts* (*5*)
lemmas *word-pred-alt* = *word-arith-alts* (*6*)
lemmas *word-0-alt* = *word-arith-alts* (*7*)
lemmas *word-1-alt* = *word-arith-alts* (*8*)

11.1 Transferring goals from words to ints

lemma *word-ths*:
shows
word-succ-p1: *word-succ a = a + 1* **and**
word-pred-m1: *word-pred a = a − 1* **and**
word-pred-succ: *word-pred (word-succ a) = a* **and**
word-succ-pred: *word-succ (word-pred a) = a* **and**
word-mult-succ: *word-succ a * b = b + a * b*
by (*rule word-uint.Abs-cases* [*of b*],
rule word-uint.Abs-cases [*of a*],
simp add: *pred-def succ-def add-commute mult-commute*
ring-distribs new-word-of-int-homs)+

lemmas *uint-cong* = *arg-cong* [**where** *f* = *uint*]

lemmas *uint-word-ariths* =
word-arith-alts [*THEN trans* [*OF uint-cong int-word-uint*], *standard*]

lemmas *uint-word-arith-bintrs* = *uint-word-ariths* [*folded bintrunc-mod2p*]

lemmas *sint-word-ariths* = *uint-word-arith-bintrs*
[*THEN uint-sint* [*symmetric*, *THEN trans*],
unfolded uint-sint bintr-arith1s bintr-ariths
len-gt-0 [*THEN bin-sbin-eq-iff'*] *word-sbin.norm-Rep*, *standard*]

lemmas *uint-div-alt* = *word-div-def*
[*THEN trans* [*OF uint-cong int-word-uint*], *standard*]
lemmas *uint-mod-alt* = *word-mod-def*
[*THEN trans* [*OF uint-cong int-word-uint*], *standard*]

lemma *word-pred-0-n1*: *word-pred 0 = word-of-int −1*
unfolding *word-pred-def number-of-eq*
by (*simp add* : *pred-def word-no-wi*)

lemma *word-pred-0-Min*: *word-pred 0 = word-of-int Int.Min*
by (*simp add*: *word-pred-0-n1 number-of-eq*)

lemma *word-m1-Min*: *− 1 = word-of-int Int.Min*
unfolding *Min-def* **by** (*simp only*: *word-of-int-hom-syms*)

lemma *succ-pred-no* [*simp*]:
word-succ (*number-of bin*) = *number-of* (*Int.succ bin*) &
word-pred (*number-of bin*) = *number-of* (*Int.pred bin*)
unfolding *word-number-of-def* **by** (*simp add* : *new-word-of-int-homs*)

lemma *word-sp-01* [*simp*] :
word-succ −1 = 0 & *word-succ 0 = 1* & *word-pred 0 = −1* & *word-pred 1 = 0*
by (*unfold word-0-no word-1-no*) *auto*

lemma *word-of-int-Ex*:
$\exists y.\ x = \textit{word-of-int}\ y$
by (*rule-tac x=uint x* **in** *exI*) *simp*

lemma *word-arith-eqs*:
fixes $a :: 'a::\textit{len0 word}$
fixes $b :: 'a::\textit{len0 word}$
shows
word-add-0: $0 + a = a$ **and**
word-add-0-right: $a + 0 = a$ **and**
word-mult-1: $1 * a = a$ **and**
word-mult-1-right: $a * 1 = a$ **and**
word-add-commute: $a + b = b + a$ **and**
word-add-assoc: $a + b + c = a + (b + c)$ **and**
word-add-left-commute: $a + (b + c) = b + (a + c)$ **and**
word-mult-commute: $a * b = b * a$ **and**
word-mult-assoc: $a * b * c = a * (b * c)$ **and**
word-mult-left-commute: $a * (b * c) = b * (a * c)$ **and**
word-left-distrib: $(a + b) * c = a * c + b * c$ **and**
word-right-distrib: $a * (b + c) = a * b + a * c$ **and**
word-left-minus: $- a + a = 0$ **and**
word-diff-0-right: $a - 0 = a$ **and**
word-diff-self: $a - a = 0$
using *word-of-int-Ex* [*of a*]
word-of-int-Ex [*of b*]
word-of-int-Ex [*of c*]
by (*auto simp*: *word-of-int-hom-syms* [*symmetric*]
zadd-0-right add-commute add-assoc add-left-commute
mult-commute mult-assoc mult-left-commute
left-distrib right-distrib)

lemmas *word-add-ac* = *word-add-commute word-add-assoc word-add-left-commute*
lemmas *word-mult-ac* = *word-mult-commute word-mult-assoc word-mult-left-commute*

lemmas *word-plus-ac0* = *word-add-0 word-add-0-right word-add-ac*
lemmas *word-times-ac1* = *word-mult-1 word-mult-1-right word-mult-ac*

11.2 Order on fixed-length words

lemma *word-order-trans*: $x <= y \Longrightarrow y <= z \Longrightarrow x <= (z :: 'a :: \textit{len0 word})$
unfolding *word-le-def* **by** *auto*

lemma *word-order-refl*: $z <= (z :: 'a :: \textit{len0 word})$
unfolding *word-le-def* **by** *auto*

lemma *word-order-antisym*: $x <= y \Longrightarrow y <= x \Longrightarrow x = (y :: 'a :: \textit{len0 word})$
unfolding *word-le-def* **by** (*auto intro*!: *word-uint.Rep-eqD*)

lemma *word-order-linear*:
y <= x | x <= (y :: 'a :: len0 word)
unfolding *word-le-def* **by** *auto*

lemma *word-zero-le* [*simp*] :
0 <= (y :: 'a :: len0 word)
unfolding *word-le-def* **by** *auto*

instance *word* :: (*len0*) *semigroup-add*
by *intro-classes* (*simp add*: *word-add-assoc*)

instance *word* :: (*len0*) *linorder*
by *intro-classes* (*auto simp*: *word-less-def word-le-def*)

instance *word* :: (*len0*) *ring*
by *intro-classes*
(*auto simp*: *word-arith-eqs word-diff-minus*
word-diff-self [*unfolded word-diff-minus*])

lemma *word-m1-ge* [*simp*] : *word-pred 0 >= y*
unfolding *word-le-def*
by (*simp only* : *word-pred-0-n1 word-uint.eq-norm m1mod2k*) *auto*

lemmas *word-n1-ge* [*simp*] = *word-m1-ge* [*simplified word-sp-01*]

lemmas *word-not-simps* [*simp*] =
word-zero-le [*THEN leD*] *word-m1-ge* [*THEN leD*] *word-n1-ge* [*THEN leD*]

lemma *word-gt-0*: *0 < y = (0 ~= (y :: 'a :: len0 word))*
unfolding *word-less-def* **by** *auto*

lemmas *word-gt-0-no* [*simp*] = *word-gt-0* [*of number-of y, standard*]

lemma *word-sless-alt*: (*a <s b*) == (*sint a < sint b*)
unfolding *word-sle-def word-sless-def*
by (*auto simp add*: *less-le*)

lemma *word-le-nat-alt*: (*a <= b*) = (*unat a <= unat b*)
unfolding *unat-def word-le-def*
by (*rule nat-le-eq-zle* [*symmetric*]) *simp*

lemma *word-less-nat-alt*: (*a < b*) = (*unat a < unat b*)
unfolding *unat-def word-less-alt*
by (*rule nat-less-eq-zless* [*symmetric*]) *simp*

lemma *wi-less*:
(*word-of-int n < (word-of-int m :: 'a :: len0 word*)) =
(*n mod 2 ^ len-of TYPE('a) < m mod 2 ^ len-of TYPE('a)*)

unfolding *word-less-alt* **by** (*simp add*: *word-uint.eq-norm*)

lemma *wi-le*:
(*word-of-int n* <= (*word-of-int m* :: 'a :: *len0 word*)) =
(*n mod 2* ^ *len-of TYPE*('a) <= *m mod 2* ^ *len-of TYPE*('a))
unfolding *word-le-def* **by** (*simp add*: *word-uint.eq-norm*)

lemma *udvd-nat-alt*: *a udvd b* = (*EX n*>=*0*. *unat b* = *n* * *unat a*)
apply (*unfold udvd-def*)
apply *safe*
apply (*simp add*: *unat-def nat-mult-distrib*)
apply (*simp add*: *uint-nat int-mult*)
apply (*rule exI*)
apply *safe*
prefer *2*
apply (*erule notE*)
apply (*rule refl*)
apply *force*
done

lemma *udvd-iff-dvd*: *x udvd y* <-> *unat x dvd unat y*
unfolding *dvd-def udvd-nat-alt* **by** *force*

lemmas *unat-mono* = *word-less-nat-alt* [*THEN iffD1*, *standard*]

lemma *word-zero-neq-one*: *0* < *len-of TYPE* ('a :: *len0*) ==> (*0* :: 'a *word*) ~= *1*
unfolding *word-arith-wis*
by (*auto simp add*: *word-ubin.norm-eq-iff* [*symmetric*] *gr0-conv-Suc*)

lemmas *lenw1-zero-neq-one* = *len-gt-0* [*THEN word-zero-neq-one*]

lemma *no-no* [*simp*] : *number-of* (*number-of b*) = *number-of b*
by (*simp add*: *number-of-eq*)

lemma *unat-minus-one*: *x* ~= *0* ==> *unat* (*x* - *1*) = *unat x* - *1*
apply (*unfold unat-def*)
apply (*simp only*: *int-word-uint word-arith-alts rdmods*)
apply (*subgoal-tac uint x* >= *1*)
prefer *2*
apply (*drule contrapos-nn*)
apply (*erule word-uint.Rep-inverse'* [*symmetric*])
apply (*insert uint-ge-0* [*of x*])[*1*]
apply *arith*
apply (*rule box-equals*)
apply (*rule nat-diff-distrib*)
prefer *2*
apply *assumption*
apply *simp*

```
  apply (subst mod-pos-pos-trivial)
   apply arith
  apply (insert uint-lt2p [of x])[1]
  apply arith
 apply (rule refl)
apply simp
done

lemma measure-unat: p ~= 0 ==> unat (p - 1) < unat p
 by (simp add: unat-minus-one) (simp add: unat-0-iff [symmetric])

lemmas uint-add-ge0 [simp] =
 add-nonneg-nonneg [OF uint-ge-0 uint-ge-0, standard]
lemmas uint-mult-ge0 [simp] =
 mult-nonneg-nonneg [OF uint-ge-0 uint-ge-0, standard]

lemma uint-sub-lt2p [simp]:
 uint (x :: 'a :: len0 word) - uint (y :: 'b :: len0 word) <
  2 ^ len-of TYPE('a)
 using uint-ge-0 [of y] uint-lt2p [of x] by arith
```

11.3 Conditions for the addition (etc) of two words to overflow

```
lemma uint-add-lem:
 (uint x + uint y < 2 ^ len-of TYPE('a)) =
  (uint (x + y :: 'a :: len0 word) = uint x + uint y)
 by (unfold uint-word-ariths) (auto intro!: trans [OF - int-mod-lem])

lemma uint-mult-lem:
 (uint x * uint y < 2 ^ len-of TYPE('a)) =
  (uint (x * y :: 'a :: len0 word) = uint x * uint y)
 by (unfold uint-word-ariths) (auto intro!: trans [OF - int-mod-lem])

lemma uint-sub-lem:
 (uint x >= uint y) = (uint (x - y) = uint x - uint y)
 by (unfold uint-word-ariths) (auto intro!: trans [OF - int-mod-lem])

lemma uint-add-le: uint (x + y) <= uint x + uint y
 unfolding uint-word-ariths by (auto simp: mod-add-if-z)

lemma uint-sub-ge: uint (x - y) >= uint x - uint y
 unfolding uint-word-ariths by (auto simp: mod-sub-if-z)

lemmas uint-sub-if' =
 trans [OF uint-word-ariths(1) mod-sub-if-z, simplified, standard]
lemmas uint-plus-if' =
 trans [OF uint-word-ariths(2) mod-add-if-z, simplified, standard]
```

11.4 Definition of uint_arith

lemma *word-of-int-inverse*:
word-of-int r = a ==> 0 <= r ==> r < 2 ^ len-of TYPE('a) ==>
uint (a::'a::len0 word) = r
apply (*erule word-uint.Abs-inverse' [rotated]*)
apply (*simp add*: *uints-num*)
done

lemma *uint-split*:
fixes *x::'a::len0 word*
shows *P (uint x) =*
(ALL i. word-of-int i = x & 0 <= i & i < 2^len-of TYPE('a) --> P i)
apply (*fold word-int-case-def*)
apply (*auto dest!*: *word-of-int-inverse simp*: *int-word-uint int-mod-eq'*
split: *word-int-split*)
done

lemma *uint-split-asm*:
fixes *x::'a::len0 word*
shows *P (uint x) =*
(~(EX i. word-of-int i = x & 0 <= i & i < 2^len-of TYPE('a) & ~ P i))
by (*auto dest!*: *word-of-int-inverse*
simp: *int-word-uint int-mod-eq'*
split: *uint-split*)

lemmas *uint-splits = uint-split uint-split-asm*

lemmas *uint-arith-simps =*
word-le-def word-less-alt
word-uint.Rep-inject [symmetric]
uint-sub-if' uint-plus-if'

lemma *power-False-cong*: *False ==> a ^ b = c ^ d*
by *auto*

ML ⟪
fun uint-arith-ss-of ss =
ss addsimps @{thms uint-arith-simps}
delsimps @{thms word-uint.Rep-inject}
addsplits @{thms split-if-asm}
addcongs @{thms power-False-cong}

fun uint-arith-tacs ctxt =
let
fun arith-tac' n t =
Arith-Data.verbose-arith-tac ctxt n t
handle Cooper.COOPER - => Seq.empty;

```
    val cs = claset-of ctxt;
    val ss = simpset-of ctxt;
  in
    [ clarify-tac cs 1,
      full-simp-tac (uint-arith-ss-of ss) 1,
      ALLGOALS (full-simp-tac (HOL-ss addsplits @{thms uint-splits}
                                      addcongs @{thms power-False-cong})),
      rewrite-goals-tac @{thms word-size},
      ALLGOALS (fn n => REPEAT (resolve-tac [allI, impI] n) THEN
                        REPEAT (etac conjE n) THEN
                        REPEAT (dtac @{thm word-of-int-inverse} n
                              THEN atac n
                              THEN atac n)),
      TRYALL arith-tac' ]
  end

fun uint-arith-tac ctxt = SELECT-GOAL (EVERY (uint-arith-tacs ctxt))
⟫

method-setup uint-arith =
  ⟪ Scan.succeed (SIMPLE-METHOD' o uint-arith-tac) ⟫
  solving word arithmetic via integers and arith
```

11.5 More on overflows and monotonicity

```
lemma no-plus-overflow-uint-size:
  ((x :: 'a :: len0 word) <= x + y) = (uint x + uint y < 2 ^ size x)
  unfolding word-size by uint-arith

lemmas no-olen-add = no-plus-overflow-uint-size [unfolded word-size]

lemma no-ulen-sub: ((x :: 'a :: len0 word) >= x - y) = (uint y <= uint x)
  by uint-arith

lemma no-olen-add':
  fixes x :: 'a::len0 word
  shows (x ≤ y + x) = (uint y + uint x < 2 ^ len-of TYPE('a))
  by (simp add: word-add-ac add-ac no-olen-add)

lemmas olen-add-eqv = trans [OF no-olen-add no-olen-add' [symmetric], standard]

lemmas uint-plus-simple-iff = trans [OF no-olen-add uint-add-lem, standard]
lemmas uint-plus-simple = uint-plus-simple-iff [THEN iffD1, standard]
lemmas uint-minus-simple-iff = trans [OF no-ulen-sub uint-sub-lem, standard]
lemmas uint-minus-simple-alt = uint-sub-lem [folded word-le-def]
lemmas word-sub-le-iff = no-ulen-sub [folded word-le-def]
lemmas word-sub-le = word-sub-le-iff [THEN iffD2, standard]
```

lemma *word-less-sub1*:
$(x :: 'a :: len\ word) \sim= 0 \Longrightarrow (1 < x) = (0 < x - 1)$
by *uint-arith*

lemma *word-le-sub1*:
$(x :: 'a :: len\ word) \sim= 0 \Longrightarrow (1 <= x) = (0 <= x - 1)$
by *uint-arith*

lemma *sub-wrap-lt*:
$((x :: 'a :: len0\ word) < x - z) = (x < z)$
by *uint-arith*

lemma *sub-wrap*:
$((x :: 'a :: len0\ word) <= x - z) = (z = 0 \mid x < z)$
by *uint-arith*

lemma *plus-minus-not-NULL-ab*:
$(x :: 'a :: len0\ word) <= ab - c \Longrightarrow c <= ab \Longrightarrow c \sim= 0 \Longrightarrow x + c \sim= 0$
by *uint-arith*

lemma *plus-minus-no-overflow-ab*:
$(x :: 'a :: len0\ word) <= ab - c \Longrightarrow c <= ab \Longrightarrow x <= x + c$
by *uint-arith*

lemma *le-minus'*:
$(a :: 'a :: len0\ word) + c <= b \Longrightarrow a <= a + c \Longrightarrow c <= b - a$
by *uint-arith*

lemma *le-plus'*:
$(a :: 'a :: len0\ word) <= b \Longrightarrow c <= b - a \Longrightarrow a + c <= b$
by *uint-arith*

lemmas *le-plus* = *le-plus'* [*rotated*]

lemmas *le-minus* = *leD* [*THEN thin-rl*, *THEN le-minus'*, *standard*]

lemma *word-plus-mono-right*:
$(y :: 'a :: len0\ word) <= z \Longrightarrow x <= x + z \Longrightarrow x + y <= x + z$
by *uint-arith*

lemma *word-less-minus-cancel*:
$y - x < z - x \Longrightarrow x <= z \Longrightarrow (y :: 'a :: len0\ word) < z$
by *uint-arith*

lemma *word-less-minus-mono-left*:
$(y :: 'a :: len0\ word) < z \Longrightarrow x <= y \Longrightarrow y - x < z - x$
by *uint-arith*

lemma *word-less-minus-mono*:

$a < c ==> d < b ==> a - b < a ==> c - d < c$
$==> a - b < c - (d::'a::len\ word)$
by *uint-arith*

lemma *word-le-minus-cancel*:
$y - x <= z - x ==> x <= z ==> (y :: 'a :: len0\ word) <= z$
by *uint-arith*

lemma *word-le-minus-mono-left*:
$(y :: 'a :: len0\ word) <= z ==> x <= y ==> y - x <= z - x$
by *uint-arith*

lemma *word-le-minus-mono*:
$a <= c ==> d <= b ==> a - b <= a ==> c - d <= c$
$==> a - b <= c - (d::'a::len\ word)$
by *uint-arith*

lemma *plus-le-left-cancel-wrap*:
$(x :: 'a :: len0\ word) + y' < x ==> x + y < x ==> (x + y' < x + y) = (y' < y)$
by *uint-arith*

lemma *plus-le-left-cancel-nowrap*:
$(x :: 'a :: len0\ word) <= x + y' ==> x <= x + y ==>$
$(x + y' < x + y) = (y' < y)$
by *uint-arith*

lemma *word-plus-mono-right2*:
$(a :: 'a :: len0\ word) <= a + b ==> c <= b ==> a <= a + c$
by *uint-arith*

lemma *word-less-add-right*:
$(x :: 'a :: len0\ word) < y - z ==> z <= y ==> x + z < y$
by *uint-arith*

lemma *word-less-sub-right*:
$(x :: 'a :: len0\ word) < y + z ==> y <= x ==> x - y < z$
by *uint-arith*

lemma *word-le-plus-either*:
$(x :: 'a :: len0\ word) <= y \mid x <= z ==> y <= y + z ==> x <= y + z$
by *uint-arith*

lemma *word-less-nowrapI*:
$(x :: 'a :: len0\ word) < z - k ==> k <= z ==> 0 < k ==> x < x + k$
by *uint-arith*

lemma *inc-le*: $(i :: 'a :: len\ word) < m ==> i + 1 <= m$
by *uint-arith*

lemma *inc-i*:
(*1* :: *'a* :: *len word*) <= *i* ==> *i* < *m* ==> *1* <= (*i* + *1*) & *i* + *1* <= *m*
by *uint-arith*

lemma *udvd-incr-lem*:
up < *uq* ==> *up* = *ua* + *n* * *uint K* ==>
uq = *ua* + *n'* * *uint K* ==> *up* + *uint K* <= *uq*
apply *clarsimp*
apply (*drule less-le-mult*)
apply *safe*
done

lemma *udvd-incr'*:
p < *q* ==> *uint p* = *ua* + *n* * *uint K* ==>
uint q = *ua* + *n'* * *uint K* ==> *p* + *K* <= *q*
apply (*unfold word-less-alt word-le-def*)
apply (*drule* (*2*) *udvd-incr-lem*)
apply (*erule uint-add-le* [*THEN order-trans*])
done

lemma *udvd-decr'*:
p < *q* ==> *uint p* = *ua* + *n* * *uint K* ==>
uint q = *ua* + *n'* * *uint K* ==> *p* <= *q* - *K*
apply (*unfold word-less-alt word-le-def*)
apply (*drule* (*2*) *udvd-incr-lem*)
apply (*drule le-diff-eq* [*THEN iffD2*])
apply (*erule order-trans*)
apply (*rule uint-sub-ge*)
done

lemmas *udvd-incr-lem0* = *udvd-incr-lem* [**where** *ua=0*, *simplified*]
lemmas *udvd-incr0* = *udvd-incr'* [**where** *ua=0*, *simplified*]
lemmas *udvd-decr0* = *udvd-decr'* [**where** *ua=0*, *simplified*]

lemma *udvd-minus-le'*:
xy < *k* ==> *z udvd xy* ==> *z udvd k* ==> *xy* <= *k* - *z*
apply (*unfold udvd-def*)
apply *clarify*
apply (*erule* (*2*) *udvd-decr0*)
done

ML ⟪ *Delsimprocs Numeral-Simprocs.cancel-factors* ⟫

lemma *udvd-incr2-K*:
p < *a* + *s* ==> *a* <= *a* + *s* ==> *K udvd s* ==> *K udvd p* - *a* ==> *a* <= *p* ==>
0 < *K* ==> *p* <= *p* + *K* & *p* + *K* <= *a* + *s*
apply (*unfold udvd-def*)

apply *clarify*
apply (*simp add*: *uint-arith-simps split*: *split-if-asm*)
prefer *2*
apply (*insert uint-range'* [*of s*])[*1*]
apply *arith*
apply (*drule add-commute* [*THEN xtr1*])
apply (*simp add*: *diff-less-eq* [*symmetric*])
apply (*drule less-le-mult*)
apply *arith*
apply *simp*
done

ML ⟪ *Addsimprocs Numeral-Simprocs.cancel-factors* ⟫

lemma *word-succ-rbl*:
to-bl w = bl ==> to-bl (*word-succ w*) = (*rev* (*rbl-succ* (*rev bl*)))
apply (*unfold word-succ-def*)
apply *clarify*
apply (*simp add*: *to-bl-of-bin*)
apply (*simp add*: *to-bl-def rbl-succ*)
done

lemma *word-pred-rbl*:
to-bl w = bl ==> to-bl (*word-pred w*) = (*rev* (*rbl-pred* (*rev bl*)))
apply (*unfold word-pred-def*)
apply *clarify*
apply (*simp add*: *to-bl-of-bin*)
apply (*simp add*: *to-bl-def rbl-pred*)
done

lemma *word-add-rbl*:
to-bl v = vbl ==> to-bl w = wbl ==>
to-bl (*v + w*) = (*rev* (*rbl-add* (*rev vbl*) (*rev wbl*)))
apply (*unfold word-add-def*)
apply *clarify*
apply (*simp add*: *to-bl-of-bin*)
apply (*simp add*: *to-bl-def rbl-add*)
done

lemma *word-mult-rbl*:
to-bl v = vbl ==> to-bl w = wbl ==>
to-bl (*v * w*) = (*rev* (*rbl-mult* (*rev vbl*) (*rev wbl*)))
apply (*unfold word-mult-def*)
apply *clarify*
apply (*simp add*: *to-bl-of-bin*)
apply (*simp add*: *to-bl-def rbl-mult*)
done

lemma *rtb-rbl-ariths*:
rev (*to-bl w*) = *ys* ⟹ *rev* (*to-bl* (*word-succ w*)) = *rbl-succ ys*

rev (*to-bl w*) = *ys* ⟹ *rev* (*to-bl* (*word-pred w*)) = *rbl-pred ys*

[| *rev* (*to-bl v*) = *ys*; *rev* (*to-bl w*) = *xs* |]
==> *rev* (*to-bl* (*v* ∗ *w*)) = *rbl-mult ys xs*

[| *rev* (*to-bl v*) = *ys*; *rev* (*to-bl w*) = *xs* |]
==> *rev* (*to-bl* (*v* + *w*)) = *rbl-add ys xs*
by (*auto simp*: *rev-swap* [*symmetric*] *word-succ-rbl*
word-pred-rbl word-mult-rbl word-add-rbl)

11.6 Arithmetic type class instantiations

instance *word* :: (*len0*) *comm-monoid-add* **..**

instance *word* :: (*len0*) *comm-monoid-mult*
apply (*intro-classes*)
apply (*simp add*: *word-mult-commute*)
apply (*simp add*: *word-mult-1*)
done

instance *word* :: (*len0*) *comm-semiring*
by (*intro-classes*) (*simp add* : *word-left-distrib*)

instance *word* :: (*len0*) *ab-group-add* **..**

instance *word* :: (*len0*) *comm-ring* **..**

instance *word* :: (*len*) *comm-semiring-1*
by (*intro-classes*) (*simp add*: *lenw1-zero-neq-one*)

instance *word* :: (*len*) *comm-ring-1* **..**

instance *word* :: (*len0*) *comm-semiring-0* **..**

instance *word* :: (*len0*) *order* **..**

lemma *zero-bintrunc*:
iszero (*number-of x* :: *'a* :: *len word*) =
(*bintrunc* (*len-of TYPE*(*'a*)) *x* = *Int.Pls*)
apply (*unfold iszero-def word-0-wi word-no-wi*)
apply (*rule word-ubin.norm-eq-iff* [*symmetric*, *THEN trans*])
apply (*simp add* : *Pls-def* [*symmetric*])
done

lemmas *word-le-0-iff* [*simp*] =

word-zero-le [*THEN leD*, *THEN linorder-antisym-conv1*]

lemma *word-of-nat*: *of-nat n = word-of-int* (*int n*)
by (*induct n*) (*auto simp add* : *word-of-int-hom-syms*)

lemma *word-of-int*: *of-int = word-of-int*
apply (*rule ext*)
apply (*unfold of-int-def*)
apply (*rule contentsI*)
apply *safe*
apply (*simp-all add*: *word-of-nat word-of-int-homs*)
defer
apply (*rule Rep-Integ-ne* [*THEN nonemptyE*])
apply (*rule bexI*)
prefer *2*
apply *assumption*
apply (*auto simp add*: *RI-eq-diff*)
done

lemma *word-of-int-nat*:
0 <= x ==> word-of-int x = of-nat (*nat x*)
by (*simp add*: *of-nat-nat word-of-int*)

lemma *word-number-of-eq*:
number-of w = (*of-int w* :: *'a* :: *len word*)
unfolding *word-number-of-def word-of-int* **by** *auto*

instance *word* :: (*len*) *number-ring*
by (*intro-classes*) (*simp add* : *word-number-of-eq*)

lemma *iszero-word-no* [*simp*] :
iszero (*number-of bin* :: *'a* :: *len word*) =
iszero (*number-of* (*bintrunc* (*len-of TYPE*(*'a*)) *bin*) :: *int*)
apply (*simp add*: *zero-bintrunc number-of-is-id*)
apply (*unfold iszero-def Pls-def*)
apply (*rule refl*)
done

11.7 Word and nat

lemma *td-ext-unat'*:
n = len-of TYPE (*'a* :: *len*) ==>
td-ext (*unat* :: *'a word => nat*) *of-nat*
(*unats n*) (%*i*. *i mod 2* ^ *n*)
apply (*unfold td-ext-def' unat-def word-of-nat unats-uints*)
apply (*auto intro*!: *imageI simp add* : *word-of-int-hom-syms*)
apply (*erule word-uint.Abs-inverse* [*THEN arg-cong*])
apply (*simp add*: *int-word-uint nat-mod-distrib nat-power-eq*)
done

lemmas *td-ext-unat = refl [THEN td-ext-unat']*
lemmas *unat-of-nat = td-ext-unat [THEN td-ext.eq-norm, standard]*

interpretation *word-unat*:
td-ext unat::'a::len word => nat
of-nat
unats (len-of TYPE('a::len))
%i. i mod 2 ^ len-of TYPE('a::len)
by *(rule td-ext-unat)*

lemmas *td-unat = word-unat.td-thm*

lemmas *unat-lt2p [iff] = word-unat.Rep [unfolded unats-def mem-Collect-eq]*

lemma *unat-le*: *y <= unat (z :: 'a :: len word) ==> y : unats (len-of TYPE ('a))*
apply *(unfold unats-def)*
apply *clarsimp*
apply *(rule xtrans, rule unat-lt2p, assumption)*
done

lemma *word-nchotomy*:
ALL w. EX n. (w :: 'a :: len word) = of-nat n & n < 2 ^ len-of TYPE ('a)
apply *(rule allI)*
apply *(rule word-unat.Abs-cases)*
apply *(unfold unats-def)*
apply *auto*
done

lemma *of-nat-eq*:
fixes *w :: 'a::len word*
shows *(of-nat n = w) = (∃ q. n = unat w + q * 2 ^ len-of TYPE('a))*
apply *(rule trans)*
apply *(rule word-unat.inverse-norm)*
apply *(rule iffI)*
apply *(rule mod-eqD)*
apply *simp*
apply *clarsimp*
done

lemma *of-nat-eq-size*:
*(of-nat n = w) = (EX q. n = unat w + q * 2 ^ size w)*
unfolding *word-size* **by** *(rule of-nat-eq)*

lemma *of-nat-0*:
*(of-nat m = (0::'a::len word)) = (∃ q. m = q * 2 ^ len-of TYPE('a))*
by *(simp add: of-nat-eq)*

lemmas *of-nat-2p = mult-1* [*symmetric*, *THEN iffD2* [*OF of-nat-0 exI*]]

lemma *of-nat-gt-0*: *of-nat k ~= 0 ==> 0 < k*
by (*cases k*) *auto*

lemma *of-nat-neq-0*:
0 < k ==> k < 2 ^ len-of TYPE ('a :: len) ==> of-nat k ~= (0 :: 'a word)
by (*clarsimp simp add : of-nat-0*)

lemma *Abs-fnat-hom-add*:
of-nat a + of-nat b = of-nat (a + b)
by *simp*

lemma *Abs-fnat-hom-mult*:
*of-nat a * of-nat b = (of-nat (a * b) :: 'a :: len word)*
by (*simp add*: *word-of-nat word-of-int-mult-hom zmult-int*)

lemma *Abs-fnat-hom-Suc*:
word-succ (of-nat a) = of-nat (Suc a)
by (*simp add*: *word-of-nat word-of-int-succ-hom add-ac*)

lemma *Abs-fnat-hom-0*: *(0::'a::len word) = of-nat 0*
by (*simp add*: *word-of-nat word-0-wi*)

lemma *Abs-fnat-hom-1*: *(1::'a::len word) = of-nat (Suc 0)*
by (*simp add*: *word-of-nat word-1-wi*)

lemmas *Abs-fnat-homs =*
Abs-fnat-hom-add Abs-fnat-hom-mult Abs-fnat-hom-Suc
Abs-fnat-hom-0 Abs-fnat-hom-1

lemma *word-arith-nat-add*:
a + b = of-nat (unat a + unat b)
by *simp*

lemma *word-arith-nat-mult*:
*a * b = of-nat (unat a * unat b)*
by (*simp add*: *Abs-fnat-hom-mult* [*symmetric*])

lemma *word-arith-nat-Suc*:
word-succ a = of-nat (Suc (unat a))
by (*subst Abs-fnat-hom-Suc* [*symmetric*]) *simp*

lemma *word-arith-nat-div*:
a div b = of-nat (unat a div unat b)
by (*simp add*: *word-div-def word-of-nat zdiv-int uint-nat*)

lemma *word-arith-nat-mod*:
a mod b = of-nat (unat a mod unat b)

by (*simp add*: *word-mod-def word-of-nat zmod-int uint-nat*)

lemmas *word-arith-nat-defs* =
word-arith-nat-add word-arith-nat-mult
word-arith-nat-Suc Abs-fnat-hom-0
Abs-fnat-hom-1 word-arith-nat-div
word-arith-nat-mod

lemmas *unat-cong* = *arg-cong* [**where** *f* = *unat*]

lemmas *unat-word-ariths* = *word-arith-nat-defs*
[*THEN trans* [*OF unat-cong unat-of-nat*], *standard*]

lemmas *word-sub-less-iff* = *word-sub-le-iff*
[*simplified linorder-not-less* [*symmetric*], *simplified*]

lemma *unat-add-lem*:
(*unat x* + *unat y* < *2* ^ *len-of TYPE*('*a*)) =
(*unat* (*x* + *y* :: '*a* :: *len word*) = *unat x* + *unat y*)
unfolding *unat-word-ariths*
by (*auto intro*!: *trans* [*OF* - *nat-mod-lem*])

lemma *unat-mult-lem*:
(*unat x* * *unat y* < *2* ^ *len-of TYPE*('*a*)) =
(*unat* (*x* * *y* :: '*a* :: *len word*) = *unat x* * *unat y*)
unfolding *unat-word-ariths*
by (*auto intro*!: *trans* [*OF* - *nat-mod-lem*])

lemmas *unat-plus-if'* =
trans [*OF unat-word-ariths*(*1*) *mod-nat-add*, *simplified*, *standard*]

lemma *le-no-overflow*:
x <= *b* ==> *a* <= *a* + *b* ==> *x* <= *a* + (*b* :: '*a* :: *len0 word*)
apply (*erule order-trans*)
apply (*erule olen-add-eqv* [*THEN iffD1*])
done

lemmas *un-ui-le* = *trans*
[*OF word-le-nat-alt* [*symmetric*]
word-le-def,
standard]

lemma *unat-sub-if-size*:
unat (*x* - *y*) = (*if unat y* <= *unat x*
then unat x - *unat y*
else unat x + *2* ^ *size x* - *unat y*)
apply (*unfold word-size*)
apply (*simp add*: *un-ui-le*)
apply (*auto simp add*: *unat-def uint-sub-if'*)

```
   apply (rule nat-diff-distrib)
    prefer 3
    apply (simp add: algebra-simps)
    apply (rule nat-diff-distrib [THEN trans])
      prefer 3
      apply (subst nat-add-distrib)
       prefer 3
       apply (simp add: nat-power-eq)
      apply auto
  apply uint-arith
  done

lemmas unat-sub-if' = unat-sub-if-size [unfolded word-size]

lemma unat-div: unat ((x :: 'a :: len word) div y) = unat x div unat y
  apply (simp add : unat-word-ariths)
  apply (rule unat-lt2p [THEN xtr7, THEN nat-mod-eq'])
  apply (rule div-le-dividend)
  done

lemma unat-mod: unat ((x :: 'a :: len word) mod y) = unat x mod unat y
  apply (clarsimp simp add : unat-word-ariths)
  apply (cases unat y)
   prefer 2
   apply (rule unat-lt2p [THEN xtr7, THEN nat-mod-eq'])
   apply (rule mod-le-divisor)
   apply auto
  done

lemma uint-div: uint ((x :: 'a :: len word) div y) = uint x div uint y
  unfolding uint-nat by (simp add : unat-div zdiv-int)

lemma uint-mod: uint ((x :: 'a :: len word) mod y) = uint x mod uint y
  unfolding uint-nat by (simp add : unat-mod zmod-int)
```

11.8 Definition of unat_arith tactic

```
lemma unat-split:
  fixes x::'a::len word
  shows P (unat x) =
        (ALL n. of-nat n = x & n < 2^len-of TYPE('a) --> P n)
  by (auto simp: unat-of-nat)

lemma unat-split-asm:
  fixes x::'a::len word
  shows P (unat x) =
        (~(EX n. of-nat n = x & n < 2^len-of TYPE('a) & ~ P n))
  by (auto simp: unat-of-nat)
```

lemmas *of-nat-inverse* =
word-unat.Abs-inverse' [*rotated*, *unfolded unats-def*, *simplified*]

lemmas *unat-splits* = *unat-split unat-split-asm*

lemmas *unat-arith-simps* =
word-le-nat-alt word-less-nat-alt
word-unat.Rep-inject [*symmetric*]
unat-sub-if' unat-plus-if' unat-div unat-mod

ML ⟪
```
fun unat-arith-ss-of ss =
  ss addsimps @{thms unat-arith-simps}
     delsimps @{thms word-unat.Rep-inject}
     addsplits @{thms split-if-asm}
     addcongs @{thms power-False-cong}

fun unat-arith-tacs ctxt =
  let
    fun arith-tac' n t =
      Arith-Data.verbose-arith-tac ctxt n t
        handle Cooper.COOPER - => Seq.empty;
    val cs = claset-of ctxt;
    val ss = simpset-of ctxt;
  in
    [ clarify-tac cs 1,
      full-simp-tac (unat-arith-ss-of ss) 1,
      ALLGOALS (full-simp-tac (HOL-ss addsplits @{thms unat-splits}
                                      addcongs @{thms power-False-cong})),
      rewrite-goals-tac @{thms word-size},
      ALLGOALS (fn n => REPEAT (resolve-tac [allI, impI] n) THEN
                        REPEAT (etac conjE n) THEN
                        REPEAT (dtac @{thm of-nat-inverse} n THEN atac n)),
      TRYALL arith-tac' ]
  end

fun unat-arith-tac ctxt = SELECT-GOAL (EVERY (unat-arith-tacs ctxt))
```
⟫

method-setup *unat-arith* =
⟪ *Scan.succeed* (*SIMPLE-METHOD' o unat-arith-tac*) ⟫
solving word arithmetic via natural numbers and arith

lemma *no-plus-overflow-unat-size*:
((*x* :: *'a* :: *len word*) <= *x* + *y*) = (*unat x* + *unat y* < *2* ^ *size x*)
unfolding *word-size* **by** *unat-arith*

lemma *unat-sub*: *b* <= *a* ==> *unat* (*a* − *b*) = *unat a* − *unat* (*b* :: *'a* :: *len word*)

by *unat-arith*

lemmas *no-olen-add-nat* = *no-plus-overflow-unat-size* [*unfolded word-size*]

lemmas *unat-plus-simple* = *trans* [*OF no-olen-add-nat unat-add-lem*, *standard*]

lemma *word-div-mult*:
(*0* :: *'a* :: *len word*) < *y* ==> *unat x* * *unat y* < *2* ^ *len-of TYPE*(*'a*) ==>
x * *y div y* = *x*
apply *unat-arith*
apply *clarsimp*
apply (*subst unat-mult-lem* [*THEN iffD1*])
apply *auto*
done

lemma *div-lt'*: (*i* :: *'a* :: *len word*) <= *k div x* ==>
unat i * *unat x* < *2* ^ *len-of TYPE*(*'a*)
apply *unat-arith*
apply *clarsimp*
apply (*drule mult-le-mono1*)
apply (*erule order-le-less-trans*)
apply (*rule xtr7* [*OF unat-lt2p div-mult-le*])
done

lemmas *div-lt''* = *order-less-imp-le* [*THEN div-lt'*]

lemma *div-lt-mult*: (*i* :: *'a* :: *len word*) < *k div x* ==> *0* < *x* ==> *i* * *x* < *k*
apply (*frule div-lt''* [*THEN unat-mult-lem* [*THEN iffD1*]])
apply (*simp add*: *unat-arith-simps*)
apply (*drule* (*1*) *mult-less-mono1*)
apply (*erule order-less-le-trans*)
apply (*rule div-mult-le*)
done

lemma *div-le-mult*:
(*i* :: *'a* :: *len word*) <= *k div x* ==> *0* < *x* ==> *i* * *x* <= *k*
apply (*frule div-lt'* [*THEN unat-mult-lem* [*THEN iffD1*]])
apply (*simp add*: *unat-arith-simps*)
apply (*drule mult-le-mono1*)
apply (*erule order-trans*)
apply (*rule div-mult-le*)
done

lemma *div-lt-uint'*:
(*i* :: *'a* :: *len word*) <= *k div x* ==> *uint i* * *uint x* < *2* ^ *len-of TYPE*(*'a*)
apply (*unfold uint-nat*)
apply (*drule div-lt'*)
apply (*simp add*: *zmult-int zless-nat-eq-int-zless* [*symmetric*]
nat-power-eq)

done

lemmas *div-lt-uint''* = *order-less-imp-le* [*THEN div-lt-uint'*]

lemma *word-le-exists'*:
(*x* :: *'a* :: *len0 word*) <= *y* ==>
(*EX z*. *y* = *x* + *z* & *uint x* + *uint z* < 2 ^ *len-of TYPE*(*'a*))
apply (*rule exI*)
apply (*rule conjI*)
apply (*rule zadd-diff-inverse*)
apply *uint-arith*
done

lemmas *plus-minus-not-NULL* = *order-less-imp-le* [*THEN plus-minus-not-NULL-ab*]

lemmas *plus-minus-no-overflow* =
order-less-imp-le [*THEN plus-minus-no-overflow-ab*]

lemmas *mcs* = *word-less-minus-cancel word-less-minus-mono-left*
word-le-minus-cancel word-le-minus-mono-left

lemmas *word-l-diffs* = *mcs* [**where** *y* = *w* + *x*, *unfolded add-diff-cancel*, *standard*]
lemmas *word-diff-ls* = *mcs* [**where** *z* = *w* + *x*, *unfolded add-diff-cancel*, *standard*]
lemmas *word-plus-mcs* = *word-diff-ls*
[**where** *y* = *v* + *x*, *unfolded add-diff-cancel*, *standard*]

lemmas *le-unat-uoi* = *unat-le* [*THEN word-unat.Abs-inverse*]

lemmas *thd* = *refl* [*THEN* [*2*] *split-div-lemma* [*THEN iffD2*], *THEN conjunct1*]

lemma *thd1*:
a div b * *b* ≤ (*a*::*nat*)
using *gt-or-eq-0* [*of b*]
apply (*rule disjE*)
apply (*erule xtr4* [*OF thd mult-commute*])
apply *clarsimp*
done

lemmas *uno-simps* [*THEN le-unat-uoi*, *standard*] =
mod-le-divisor div-le-dividend thd1

lemma *word-mod-div-equality*:
(*n div b*) * *b* + (*n mod b*) = (*n* :: *'a* :: *len word*)
apply (*unfold word-less-nat-alt word-arith-nat-defs*)
apply (*cut-tac y=unat b* **in** *gt-or-eq-0*)
apply (*erule disjE*)
apply (*simp add*: *mod-div-equality uno-simps*)
apply *simp*
done

lemma *word-div-mult-le*: *a div b * b <= (a::'a::len word)*
apply (*unfold word-le-nat-alt word-arith-nat-defs*)
apply (*cut-tac y=unat b* **in** *gt-or-eq-0*)
apply (*erule disjE*)
apply (*simp add*: *div-mult-le uno-simps*)
apply *simp*
done

lemma *word-mod-less-divisor*: *0 < n ==> m mod n < (n :: 'a :: len word)*
apply (*simp only*: *word-less-nat-alt word-arith-nat-defs*)
apply (*clarsimp simp add* : *uno-simps*)
done

lemma *word-of-int-power-hom*:
word-of-int a ^ n = (word-of-int (a ^ n) :: 'a :: len word)
by (*induct n*) (*simp-all add* : *word-of-int-hom-syms power-Suc*)

lemma *word-arith-power-alt*:
a ^ n = (word-of-int (uint a ^ n) :: 'a :: len word)
by (*simp add* : *word-of-int-power-hom* [*symmetric*])

lemma *of-bl-length-less*:
length x = k ==> k < len-of TYPE('a) ==> (of-bl x :: 'a :: len word) < 2 ^ k
apply (*unfold of-bl-no* [*unfolded word-number-of-def*]
word-less-alt word-number-of-alt)
apply *safe*
apply (*simp* (*no-asm*) *add*: *word-of-int-power-hom word-uint.eq-norm*
del: *word-of-int-bin*)
apply (*simp add*: *mod-pos-pos-trivial*)
apply (*subst mod-pos-pos-trivial*)
apply (*rule bl-to-bin-ge0*)
apply (*rule order-less-trans*)
apply (*rule bl-to-bin-lt2p*)
apply *simp*
apply (*rule bl-to-bin-lt2p*)
done

11.9 Cardinality, finiteness of set of words

lemmas *card-lessThan' = card-lessThan* [*unfolded lessThan-def*]

lemmas *card-eq = word-unat.Abs-inj-on* [*THEN card-image*,
unfolded word-unat.image, *unfolded unats-def*, *standard*]

lemmas *card-word = trans* [*OF card-eq card-lessThan'*, *standard*]

lemma *finite-word-UNIV*: *finite* (*UNIV* :: *'a* :: *len word set*)
apply (*rule contrapos-np*)

prefer *2*
apply (*erule card-infinite*)
apply (*simp add*: *card-word*)
done

lemma *card-word-size*:
card (*UNIV* :: *'a* :: *len word set*) = (*2* ^ *size* (*x* :: *'a word*))
unfolding *word-size* **by** (*rule card-word*)

end

12 WordBitwise: Bitwise Operations on Words

theory *WordBitwise*
imports *WordArith*
begin

lemmas *bin-log-bintrs* = *bin-trunc-not bin-trunc-xor bin-trunc-and bin-trunc-or*

lemmas *wils1* = *bin-log-bintrs* [*THEN word-ubin.norm-eq-iff* [*THEN iffD1*],
folded word-ubin.eq-norm, *THEN eq-reflection*, *standard*]

lemmas *word-log-binary-defs* =
word-and-def word-or-def word-xor-def

lemmas *word-no-log-defs* [*simp*] =
word-not-def [**where** *a*=*number-of a*,
unfolded word-no-wi wils1, *folded word-no-wi*, *standard*]
word-log-binary-defs [**where** *a*=*number-of a* **and** *b*=*number-of b*,
unfolded word-no-wi wils1, *folded word-no-wi*, *standard*]

lemmas *word-wi-log-defs* = *word-no-log-defs* [*unfolded word-no-wi*]

lemma *uint-or*: *uint* (*x OR y*) = (*uint x*) *OR* (*uint y*)
by (*simp add*: *word-or-def word-no-wi* [*symmetric*] *number-of-is-id*
bin-trunc-ao(*2*) [*symmetric*])

lemma *uint-and*: *uint* (*x AND y*) = (*uint x*) *AND* (*uint y*)
by (*simp add*: *word-and-def number-of-is-id word-no-wi* [*symmetric*]
bin-trunc-ao(*1*) [*symmetric*])

lemma *word-ops-nth-size*:
n < *size* (*x*::*'a*::*len0 word*) ==>
(*x OR y*) !! *n* = (*x* !! *n* | *y* !! *n*) &

```
  (x AND y) !! n = (x !! n & y !! n) &
  (x XOR y) !! n = (x !! n ~= y !! n) &
  (NOT x) !! n = (~ x !! n)
 unfolding word-size word-no-wi word-test-bit-def word-log-defs
 by (clarsimp simp add : word-ubin.eq-norm nth-bintr bin-nth-ops)

lemma word-ao-nth:
 fixes x :: 'a::len0 word
 shows (x OR y) !! n = (x !! n | y !! n) &
       (x AND y) !! n = (x !! n & y !! n)
 apply (cases n < size x)
  apply (drule-tac y = y in word-ops-nth-size)
  apply simp
 apply (simp add : test-bit-bin word-size)
 done



lemmas bwsimps =
 word-of-int-homs(2)
 word-0-wi-Pls
 word-m1-wi-Min
 word-wi-log-defs

lemma word-bw-assocs:
 fixes x :: 'a::len0 word
 shows
 (x AND y) AND z = x AND y AND z
 (x OR y) OR z = x OR y OR z
 (x XOR y) XOR z = x XOR y XOR z
 using word-of-int-Ex [where x=x]
       word-of-int-Ex [where x=y]
       word-of-int-Ex [where x=z]
 by (auto simp: bwsimps bbw-assocs)

lemma word-bw-comms:
 fixes x :: 'a::len0 word
 shows
 x AND y = y AND x
 x OR y = y OR x
 x XOR y = y XOR x
 using word-of-int-Ex [where x=x]
       word-of-int-Ex [where x=y]
 by (auto simp: bwsimps bin-ops-comm)

lemma word-bw-lcs:
 fixes x :: 'a::len0 word
 shows
 y AND x AND z = x AND y AND z
```

y OR x OR z = x OR y OR z
y XOR x XOR z = x XOR y XOR z
using *word-of-int-Ex* [**where** *x=x*]
word-of-int-Ex [**where** *x=y*]
word-of-int-Ex [**where** *x=z*]
by (*auto simp*: *bwsimps*)

lemma *word-log-esimps* [*simp*]:
fixes *x* :: *'a::len0 word*
shows
x AND 0 = 0
x AND −1 = x
x OR 0 = x
x OR −1 = −1
x XOR 0 = x
x XOR −1 = NOT x
0 AND x = 0
−1 AND x = x
0 OR x = x
−1 OR x = −1
0 XOR x = x
−1 XOR x = NOT x
using *word-of-int-Ex* [**where** *x=x*]
by (*auto simp*: *bwsimps*)

lemma *word-not-dist*:
fixes *x* :: *'a::len0 word*
shows
NOT (x OR y) = NOT x AND NOT y
NOT (x AND y) = NOT x OR NOT y
using *word-of-int-Ex* [**where** *x=x*]
word-of-int-Ex [**where** *x=y*]
by (*auto simp*: *bwsimps bbw-not-dist*)

lemma *word-bw-same*:
fixes *x* :: *'a::len0 word*
shows
x AND x = x
x OR x = x
x XOR x = 0
using *word-of-int-Ex* [**where** *x=x*]
by (*auto simp*: *bwsimps*)

lemma *word-ao-absorbs* [*simp*]:
fixes *x* :: *'a::len0 word*
shows
x AND (y OR x) = x
x OR y AND x = x
x AND (x OR y) = x

y AND x OR x = x
(y OR x) AND x = x
x OR x AND y = x
(x OR y) AND x = x
x AND y OR x = x
using *word-of-int-Ex* [**where** *x=x*]
word-of-int-Ex [**where** *x=y*]
by (*auto simp*: *bwsimps*)

lemma *word-not-not* [*simp*]:
NOT NOT (*x::'a::len0 word*) = *x*
using *word-of-int-Ex* [**where** *x=x*]
by (*auto simp*: *bwsimps*)

lemma *word-ao-dist*:
fixes *x* :: *'a::len0 word*
shows *(x OR y) AND z = x AND z OR y AND z*
using *word-of-int-Ex* [**where** *x=x*]
word-of-int-Ex [**where** *x=y*]
word-of-int-Ex [**where** *x=z*]
by (*auto simp*: *bwsimps bbw-ao-dist simp del*: *bin-ops-comm*)

lemma *word-oa-dist*:
fixes *x* :: *'a::len0 word*
shows *x AND y OR z = (x OR z) AND (y OR z)*
using *word-of-int-Ex* [**where** *x=x*]
word-of-int-Ex [**where** *x=y*]
word-of-int-Ex [**where** *x=z*]
by (*auto simp*: *bwsimps bbw-oa-dist simp del*: *bin-ops-comm*)

lemma *word-add-not* [*simp*]:
fixes *x* :: *'a::len0 word*
shows *x + NOT x = −1*
using *word-of-int-Ex* [**where** *x=x*]
by (*auto simp*: *bwsimps bin-add-not*)

lemma *word-plus-and-or* [*simp*]:
fixes *x* :: *'a::len0 word*
shows *(x AND y) + (x OR y) = x + y*
using *word-of-int-Ex* [**where** *x=x*]
word-of-int-Ex [**where** *x=y*]
by (*auto simp*: *bwsimps plus-and-or*)

lemma *leoa*:
fixes *x* :: *'a::len0 word*
shows *(w = (x OR y)) ==> (y = (w AND y))* **by** *auto*
lemma *leao*:
fixes *x'* :: *'a::len0 word*
shows *(w' = (x' AND y')) ==> (x' = (x' OR w'))* **by** *auto*

lemmas *word-ao-equiv* = *leao* [*COMP leoa* [*COMP iffI*]]

lemma *le-word-or2*: *x* <= *x OR* (*y*::'*a*::*len0 word*)
unfolding *word-le-def uint-or*
by (*auto intro*: *le-int-or*)

lemmas *le-word-or1* = *xtr3* [*OF word-bw-comms* (*2*) *le-word-or2*, *standard*]
lemmas *word-and-le1* =
xtr3 [*OF word-ao-absorbs* (*4*) [*symmetric*] *le-word-or2*, *standard*]
lemmas *word-and-le2* =
xtr3 [*OF word-ao-absorbs* (*8*) [*symmetric*] *le-word-or2*, *standard*]

lemma *bl-word-not*: *to-bl* (*NOT w*) = *map Not* (*to-bl w*)
unfolding *to-bl-def word-log-defs*
by (*simp add*: *bl-not-bin number-of-is-id word-no-wi* [*symmetric*] *bin-to-bl-def*[*symmetric*])

lemma *bl-word-xor*: *to-bl* (*v XOR w*) = *map2 op* ~= (*to-bl v*) (*to-bl w*)
unfolding *to-bl-def word-log-defs bl-xor-bin*
by (*simp add*: *number-of-is-id word-no-wi* [*symmetric*])

lemma *bl-word-or*: *to-bl* (*v OR w*) = *map2 op* | (*to-bl v*) (*to-bl w*)
unfolding *to-bl-def word-log-defs*
by (*simp add*: *bl-or-bin number-of-is-id word-no-wi* [*symmetric*])

lemma *bl-word-and*: *to-bl* (*v AND w*) = *map2 op* & (*to-bl v*) (*to-bl w*)
unfolding *to-bl-def word-log-defs*
by (*simp add*: *bl-and-bin number-of-is-id word-no-wi* [*symmetric*])

lemma *word-lsb-alt*: *lsb* (*w*::'*a*::*len0 word*) = *test-bit w 0*
by (*auto simp*: *word-test-bit-def word-lsb-def*)

lemma *word-lsb-1-0*: *lsb* (*1*::'*a*::*len word*) & ~ *lsb* (*0*::'*b*::*len0 word*)
unfolding *word-lsb-def word-1-no word-0-no* **by** *auto*

lemma *word-lsb-last*: *lsb* (*w*::'*a*::*len word*) = *last* (*to-bl w*)
apply (*unfold word-lsb-def uint-bl bin-to-bl-def*)
apply (*rule-tac bin*=*uint w* **in** *bin-exhaust*)
apply (*cases size w*)
apply *auto*
apply (*auto simp add*: *bin-to-bl-aux-alt*)
done

lemma *word-lsb-int*: *lsb w* = (*uint w mod 2* = *1*)
unfolding *word-lsb-def bin-last-mod* **by** *auto*

lemma *word-msb-sint*: *msb w* = (*sint w* < *0*)
unfolding *word-msb-def*
by (*simp add* : *sign-Min-lt-0 number-of-is-id*)

lemma *word-msb-no'*:
w = number-of bin ==> msb (w::'a::len word) = bin-nth bin (size w − 1)
unfolding *word-msb-def word-number-of-def*
by (*clarsimp simp add*: *word-sbin.eq-norm word-size bin-sign-lem*)

lemmas *word-msb-no = refl* [*THEN word-msb-no'*, *unfolded word-size*]

lemma *word-msb-nth'*: *msb (w::'a::len word) = bin-nth (uint w) (size w − 1)*
apply (*unfold word-size*)
apply (*rule trans* [*OF - word-msb-no*])
apply (*simp add* : *word-number-of-def*)
done

lemmas *word-msb-nth = word-msb-nth'* [*unfolded word-size*]

lemma *word-msb-alt*: *msb (w::'a::len word) = hd (to-bl w)*
apply (*unfold word-msb-nth uint-bl*)
apply (*subst hd-conv-nth*)
apply (*rule length-greater-0-conv* [*THEN iffD1*])
apply *simp*
apply (*simp add* : *nth-bin-to-bl word-size*)
done

lemma *word-set-nth*:
set-bit w n (test-bit w n) = (w::'a::len0 word)
unfolding *word-test-bit-def word-set-bit-def* **by** *auto*

lemma *bin-nth-uint'*:
bin-nth (uint w) n = (rev (bin-to-bl (size w) (uint w)) ! n & n < size w)
apply (*unfold word-size*)
apply (*safe elim*!: *bin-nth-uint-imp*)
apply (*frule bin-nth-uint-imp*)
apply (*fast dest*!: *bin-nth-bl*)+
done

lemmas *bin-nth-uint = bin-nth-uint'* [*unfolded word-size*]

lemma *test-bit-bl*: *w !! n = (rev (to-bl w) ! n & n < size w)*
unfolding *to-bl-def word-test-bit-def word-size*
by (*rule bin-nth-uint*)

lemma *to-bl-nth*: *n < size w ==> to-bl w ! n = w !! (size w − Suc n)*
apply (*unfold test-bit-bl*)
apply *clarsimp*
apply (*rule trans*)
apply (*rule nth-rev-alt*)
apply (*auto simp add*: *word-size*)
done

lemma *test-bit-set*:
fixes *w* :: *'a::len0 word*
shows (*set-bit w n x*) !! *n* = (*n* < *size w* & *x*)
unfolding *word-size word-test-bit-def word-set-bit-def*
by (*clarsimp simp add* : *word-ubin.eq-norm nth-bintr*)

lemma *test-bit-set-gen*:
fixes *w* :: *'a::len0 word*
shows *test-bit* (*set-bit w n x*) *m* =
(*if m* = *n then n* < *size w* & *x else test-bit w m*)
apply (*unfold word-size word-test-bit-def word-set-bit-def*)
apply (*clarsimp simp add*: *word-ubin.eq-norm nth-bintr bin-nth-sc-gen*)
apply (*auto elim*!: *test-bit-size* [*unfolded word-size*]
simp add: *word-test-bit-def* [*symmetric*])
done

lemma *of-bl-rep-False*: *of-bl* (*replicate n False* @ *bs*) = *of-bl bs*
unfolding *of-bl-def bl-to-bin-rep-F* **by** *auto*

lemma *msb-nth'*:
fixes *w* :: *'a::len word*
shows *msb w* = *w* !! (*size w* − *1*)
unfolding *word-msb-nth' word-test-bit-def* **by** *simp*

lemmas *msb-nth* = *msb-nth'* [*unfolded word-size*]

lemmas *msb0* = *len-gt-0* [*THEN diff-Suc-less*, *THEN*
word-ops-nth-size [*unfolded word-size*], *standard*]
lemmas *msb1* = *msb0* [**where** *i* = *0*]
lemmas *word-ops-msb* = *msb1* [*unfolded msb-nth* [*symmetric*, *unfolded One-nat-def*]]

lemmas *lsb0* = *len-gt-0* [*THEN word-ops-nth-size* [*unfolded word-size*], *standard*]
lemmas *word-ops-lsb* = *lsb0* [*unfolded word-lsb-alt*]

lemma *td-ext-nth'*:
n = *size* (*w::'a::len0 word*) ==> *ofn* = *set-bits* ==> [*w*, *ofn g*] = *l* ==>
td-ext test-bit ofn {*f*. *ALL i*. *f i* −−> *i* < *n*} (%*h i*. *h i* & *i* < *n*)
apply (*unfold word-size td-ext-def'*)
apply (*safe del*: *subset-antisym*)
apply (*rule-tac* [*3*] *ext*)
apply (*rule-tac* [*4*] *ext*)
apply (*unfold word-size of-nth-def test-bit-bl*)
apply *safe*
defer
apply (*clarsimp simp*: *word-bl.Abs-inverse*)+
apply (*rule word-bl.Rep-inverse'*)
apply (*rule sym* [*THEN trans*])
apply (*rule bl-of-nth-nth*)

apply *simp*
apply (*rule bl-of-nth-inj*)
apply (*clarsimp simp add* : *test-bit-bl word-size*)
done

lemmas *td-ext-nth* = *td-ext-nth'* [*OF refl refl refl*, *unfolded word-size*]

interpretation *test-bit*:
td-ext op !! :: *'a::len0 word => nat => bool*
set-bits
{*f*. ∀ *i*. *f i* ⟶ *i* < *len-of TYPE('a::len0)*}
(λ*h i*. *h i* ∧ *i* < *len-of TYPE('a::len0)*)
by (*rule td-ext-nth*)

declare *test-bit.Rep'* [*simp del*]
declare *test-bit.Rep'* [*rule del*]

lemmas *td-nth* = *test-bit.td-thm*

lemma *word-set-set-same*:
fixes *w* :: *'a::len0 word*
shows *set-bit* (*set-bit w n x*) *n y* = *set-bit w n y*
by (*rule word-eqI*) (*simp add* : *test-bit-set-gen word-size*)

lemma *word-set-set-diff*:
fixes *w* :: *'a::len0 word*
assumes *m* ~= *n*
shows *set-bit* (*set-bit w m x*) *n y* = *set-bit* (*set-bit w n y*) *m x*
by (*rule word-eqI*) (*clarsimp simp add* : *test-bit-set-gen word-size prems*)

lemma *test-bit-no'*:
fixes *w* :: *'a::len0 word*
shows *w* = *number-of bin* ==> *test-bit w n* = (*n* < *size w* & *bin-nth bin n*)
unfolding *word-test-bit-def word-number-of-def word-size*
by (*simp add* : *nth-bintr* [*symmetric*] *word-ubin.eq-norm*)

lemmas *test-bit-no* =
refl [*THEN test-bit-no'*, *unfolded word-size*, *THEN eq-reflection*, *standard*]

lemma *nth-0*: ~ (*0::'a::len0 word*) !! *n*
unfolding *test-bit-no word-0-no* **by** *auto*

lemma *nth-sint*:
fixes *w* :: *'a::len word*
defines *l* ≡ *len-of TYPE* (*'a*)
shows *bin-nth* (*sint w*) *n* = (*if n* < *l* − *1 then w* !! *n else w* !! (*l* − *1*))
unfolding *sint-uint l-def*
by (*clarsimp simp add*: *nth-sbintr word-test-bit-def* [*symmetric*])

lemma *word-lsb-no*:
lsb (*number-of bin* :: *'a* :: *len word*) = (*bin-last bin* = *bit.B1*)
unfolding *word-lsb-alt test-bit-no* **by** *auto*

lemma *word-set-no*:
set-bit (*number-of bin*::*'a*::*len0 word*) *n b* =
number-of (*bin-sc n* (*if b then bit.B1 else bit.B0*) *bin*)
apply (*unfold word-set-bit-def word-number-of-def* [*symmetric*])
apply (*rule word-eqI*)
apply (*clarsimp simp*: *word-size bin-nth-sc-gen number-of-is-id*
test-bit-no nth-bintr)
done

lemmas *setBit-no* = *setBit-def* [*THEN trans* [*OF meta-eq-to-obj-eq word-set-no*],
simplified if-simps, *THEN eq-reflection*, *standard*]
lemmas *clearBit-no* = *clearBit-def* [*THEN trans* [*OF meta-eq-to-obj-eq word-set-no*],
simplified if-simps, *THEN eq-reflection*, *standard*]

lemma *to-bl-n1*:
to-bl (−*1*::*'a*::*len0 word*) = *replicate* (*len-of TYPE* (*'a*)) *True*
apply (*rule word-bl.Abs-inverse'*)
apply *simp*
apply (*rule word-eqI*)
apply (*clarsimp simp add*: *word-size test-bit-no*)
apply (*auto simp add*: *word-bl.Abs-inverse test-bit-bl word-size*)
done

lemma *word-msb-n1*: *msb* (−*1*::*'a*::*len word*)
unfolding *word-msb-alt word-msb-alt to-bl-n1* **by** *simp*

declare *word-set-set-same* [*simp*] *word-set-nth* [*simp*]
test-bit-no [*simp*] *word-set-no* [*simp*] *nth-0* [*simp*]
setBit-no [*simp*] *clearBit-no* [*simp*]
word-lsb-no [*simp*] *word-msb-no* [*simp*] *word-msb-n1* [*simp*] *word-lsb-1-0* [*simp*]

lemma *word-set-nth-iff*:
(*set-bit w n b* = *w*) = (*w* !! *n* = *b* | *n* >= *size* (*w*::*'a*::*len0 word*))
apply (*rule iffI*)
apply (*rule disjCI*)
apply (*drule word-eqD*)
apply (*erule sym* [*THEN trans*])
apply (*simp add*: *test-bit-set*)
apply (*erule disjE*)
apply *clarsimp*
apply (*rule word-eqI*)
apply (*clarsimp simp add* : *test-bit-set-gen*)
apply (*drule test-bit-size*)
apply *force*
done

lemma *test-bit-2p'*:
w = word-of-int (2 ^ n) ==>
w !! m = (m = n & m < size (w :: 'a :: len word))
unfolding *word-test-bit-def word-size*
by (*auto simp add*: *word-ubin.eq-norm nth-bintr nth-2p-bin*)

lemmas *test-bit-2p = refl* [*THEN test-bit-2p'*, *unfolded word-size*]

lemma *nth-w2p*:
((2::'a::len word) ^ n) !! m ⟷ m = n ∧ m < len-of TYPE('a::len)
unfolding *test-bit-2p* [*symmetric*] *word-of-int* [*symmetric*]
by (*simp add*: *of-int-power*)

lemma *uint-2p*:
(0::'a::len word) < 2 ^ n ==> uint (2 ^ n::'a::len word) = 2 ^ n
apply (*unfold word-arith-power-alt*)
apply (*case-tac len-of TYPE ('a)*)
apply *clarsimp*
apply (*case-tac nat*)
apply *clarsimp*
apply (*case-tac n*)
apply (*clarsimp simp add : word-1-wi* [*symmetric*])
apply (*clarsimp simp add : word-0-wi* [*symmetric*])
apply (*drule word-gt-0* [*THEN iffD1*])
apply (*safe intro!: word-eqI bin-nth-lem ext*)
apply (*auto simp add*: *test-bit-2p nth-2p-bin word-test-bit-def* [*symmetric*])
done

lemma *word-of-int-2p*: *(word-of-int (2 ^ n) :: 'a :: len word) = 2 ^ n*
apply (*unfold word-arith-power-alt*)
apply (*case-tac len-of TYPE ('a)*)
apply *clarsimp*
apply (*case-tac nat*)
apply (*rule word-ubin.norm-eq-iff* [*THEN iffD1*])
apply (*rule box-equals*)
apply (*rule-tac* [*2*] *bintr-ariths* (*1*))+
apply (*clarsimp simp add : number-of-is-id*)
apply *simp*
done

lemma *bang-is-le*: *x !! m ==> 2 ^ m <= (x :: 'a :: len word)*
apply (*rule xtr3*)
apply (*rule-tac* [*2*] *y = x* **in** *le-word-or2*)
apply (*rule word-eqI*)
apply (*auto simp add*: *word-ao-nth nth-w2p word-size*)
done

lemma *word-clr-le*:

fixes *w* :: *'a*::*len0 word*
shows *w >= set-bit w n False*
apply (*unfold word-set-bit-def word-le-def word-ubin.eq-norm*)
apply *simp*
apply (*rule order-trans*)
apply (*rule bintr-bin-clr-le*)
apply *simp*
done

lemma *word-set-ge*:
fixes *w* :: *'a*::*len word*
shows *w <= set-bit w n True*
apply (*unfold word-set-bit-def word-le-def word-ubin.eq-norm*)
apply *simp*
apply (*rule order-trans* [*OF - bintr-bin-set-ge*])
apply *simp*
done

end

13 WordShift: Shifting, Rotating, and Splitting Words

theory *WordShift*
imports *WordBitwise*
begin

13.1 Bit shifting

lemma *shiftl1-number* [*simp*] :
shiftl1 (*number-of w*) = *number-of* (*w BIT bit.B0*)
apply (*unfold shiftl1-def word-number-of-def*)
apply (*simp add*: *word-ubin.norm-eq-iff* [*symmetric*] *word-ubin.eq-norm*
del: *BIT-simps*)
apply (*subst refl* [*THEN bintrunc-BIT-I*, *symmetric*])
apply (*subst bintrunc-bintrunc-min*)
apply *simp*
done

lemma *shiftl1-0* [*simp*] : *shiftl1 0 = 0*
unfolding *word-0-no shiftl1-number* **by** *auto*

lemmas *shiftl1-def-u = shiftl1-def* [*folded word-number-of-def*]

lemma *shiftl1-def-s*: *shiftl1 w = number-of* (*sint w BIT bit.B0*)
by (*rule trans* [*OF - shiftl1-number*]) *simp*

lemma *shiftr1-0* [*simp*] : *shiftr1 0 = 0*
unfolding *shiftr1-def*
by *simp* (*simp add*: *word-0-wi*)

lemma *sshiftr1-0* [*simp*] : *sshiftr1 0 = 0*
apply (*unfold sshiftr1-def*)
apply *simp*
apply (*simp add* : *word-0-wi*)
done

lemma *sshiftr1-n1* [*simp*] : *sshiftr1 −1 = −1*
unfolding *sshiftr1-def* **by** *auto*

lemma *shiftl-0* [*simp*] : (*0*::*'a*::*len0 word*) << *n* = *0*
unfolding *shiftl-def* **by** (*induct n*) *auto*

lemma *shiftr-0* [*simp*] : (*0*::*'a*::*len0 word*) >> *n* = *0*
unfolding *shiftr-def* **by** (*induct n*) *auto*

lemma *sshiftr-0* [*simp*] : *0* >>> *n* = *0*
unfolding *sshiftr-def* **by** (*induct n*) *auto*

lemma *sshiftr-n1* [*simp*] : *−1* >>> *n* = *−1*
unfolding *sshiftr-def* **by** (*induct n*) *auto*

lemma *nth-shiftl1*: *shiftl1 w* !! *n* = (*n* < *size w* & *n* > *0* & *w* !! (*n* − *1*))
apply (*unfold shiftl1-def word-test-bit-def*)
apply (*simp add*: *nth-bintr word-ubin.eq-norm word-size*)
apply (*cases n*)
apply *auto*
done

lemma *nth-shiftl'* [*rule-format*]:
ALL n. ((*w*::*'a*::*len0 word*) << *m*) !! *n* = (*n* < *size w* & *n* >= *m* & *w* !! (*n* − *m*))
apply (*unfold shiftl-def*)
apply (*induct m*)
apply (*force elim*!: *test-bit-size*)
apply (*clarsimp simp add* : *nth-shiftl1 word-size*)
apply *arith*
done

lemmas *nth-shiftl* = *nth-shiftl'* [*unfolded word-size*]

lemma *nth-shiftr1*: *shiftr1 w* !! *n* = *w* !! *Suc n*
apply (*unfold shiftr1-def word-test-bit-def*)
apply (*simp add*: *nth-bintr word-ubin.eq-norm*)
apply *safe*

apply (*drule bin-nth.Suc* [*THEN iffD2*, *THEN bin-nth-uint-imp*])
apply *simp*
done

lemma *nth-shiftr*:
⋀*n*. ((*w*::'*a*::*len0 word*) >> *m*) !! *n* = *w* !! (*n* + *m*)
apply (*unfold shiftr-def*)
apply (*induct m*)
apply (*auto simp add* : *nth-shiftr1*)
done

lemma *uint-shiftr1*: *uint* (*shiftr1 w*) = *bin-rest* (*uint w*)
apply (*unfold shiftr1-def word-ubin.eq-norm bin-rest-trunc-i*)
apply (*subst bintr-uint* [*symmetric*, *OF order-refl*])
apply (*simp only* : *bintrunc-bintrunc-l*)
apply *simp*
done

lemma *nth-sshiftr1*:
sshiftr1 w !! *n* = (*if n* = *size w* − *1 then w* !! *n else w* !! *Suc n*)
apply (*unfold sshiftr1-def word-test-bit-def*)
apply (*simp add*: *nth-bintr word-ubin.eq-norm*
bin-nth.Suc [*symmetric*] *word-size*
del: *bin-nth.simps*)
apply (*simp add*: *nth-bintr uint-sint del* : *bin-nth.simps*)
apply (*auto simp add*: *bin-nth-sint*)
done

lemma *nth-sshiftr* [*rule-format*] :
ALL n. *sshiftr w m* !! *n* = (*n* < *size w* &
(*if n* + *m* >= *size w then w* !! (*size w* − *1*) *else w* !! (*n* + *m*)))
apply (*unfold sshiftr-def*)
apply (*induct-tac m*)
apply (*simp add*: *test-bit-bl*)
apply (*clarsimp simp add*: *nth-sshiftr1 word-size*)
apply *safe*
apply *arith*
apply *arith*
apply (*erule thin-rl*)
apply (*case-tac n*)
apply *safe*
apply *simp*
apply *simp*
apply (*erule thin-rl*)
apply (*case-tac n*)
apply *safe*
apply *simp*

```
   apply simp
  apply arith+
 done

lemma shiftr1-div-2: uint (shiftr1 w) = uint w div 2
 apply (unfold shiftr1-def bin-rest-div)
 apply (rule word-uint.Abs-inverse)
 apply (simp add: uints-num pos-imp-zdiv-nonneg-iff)
 apply (rule xtr7)
  prefer 2
  apply (rule zdiv-le-dividend)
   apply auto
 done

lemma sshiftr1-div-2: sint (sshiftr1 w) = sint w div 2
 apply (unfold sshiftr1-def bin-rest-div [symmetric])
 apply (simp add: word-sbin.eq-norm)
 apply (rule trans)
  defer
  apply (subst word-sbin.norm-Rep [symmetric])
  apply (rule refl)
 apply (subst word-sbin.norm-Rep [symmetric])
 apply (unfold One-nat-def)
 apply (rule sbintrunc-rest)
 done

lemma shiftr-div-2n: uint (shiftr w n) = uint w div 2 ^ n
 apply (unfold shiftr-def)
 apply (induct n)
  apply simp
 apply (simp add: shiftr1-div-2 mult-commute
                  zdiv-zmult2-eq [symmetric])
 done

lemma sshiftr-div-2n: sint (sshiftr w n) = sint w div 2 ^ n
 apply (unfold sshiftr-def)
 apply (induct n)
  apply simp
 apply (simp add: sshiftr1-div-2 mult-commute
                  zdiv-zmult2-eq [symmetric])
 done
```

13.1.1 shift functions in terms of lists of bools

```
lemmas bshiftr1-no-bin [simp] =
  bshiftr1-def [where w=number-of w, unfolded to-bl-no-bin, standard]

lemma bshiftr1-bl: to-bl (bshiftr1 b w) = b # butlast (to-bl w)
 unfolding bshiftr1-def by (rule word-bl.Abs-inverse) simp
```

lemma *shiftl1-of-bl*: *shiftl1* (*of-bl bl*) = *of-bl* (*bl* @ [*False*])
unfolding *uint-bl of-bl-no*
by (*simp add*: *bl-to-bin-aux-append bl-to-bin-def*)

lemma *shiftl1-bl*: *shiftl1* (*w*::'*a*::*len0 word*) = *of-bl* (*to-bl w* @ [*False*])
proof −
have *shiftl1 w* = *shiftl1* (*of-bl* (*to-bl w*)) **by** *simp*
also have ... = *of-bl* (*to-bl w* @ [*False*]) **by** (*rule shiftl1-of-bl*)
finally show *?thesis* **.**
qed

lemma *bl-shiftl1*:
to-bl (*shiftl1* (*w* :: '*a* :: *len word*)) = *tl* (*to-bl w*) @ [*False*]
apply (*simp add*: *shiftl1-bl word-rep-drop drop-Suc drop-Cons'*)
apply (*fast intro*!: *Suc-leI*)
done

lemma *shiftr1-bl*: *shiftr1 w* = *of-bl* (*butlast* (*to-bl w*))
apply (*unfold shiftr1-def uint-bl of-bl-def*)
apply (*simp add*: *butlast-rest-bin word-size*)
apply (*simp add*: *bin-rest-trunc* [*symmetric*, *unfolded One-nat-def*])
done

lemma *bl-shiftr1*:
to-bl (*shiftr1* (*w* :: '*a* :: *len word*)) = *False* # *butlast* (*to-bl w*)
unfolding *shiftr1-bl*
by (*simp add* : *word-rep-drop len-gt-0* [*THEN Suc-leI*])

lemma *shiftl1-rev*:
shiftl1 (*w* :: '*a* :: *len word*) = *word-reverse* (*shiftr1* (*word-reverse w*))
apply (*unfold word-reverse-def*)
apply (*rule word-bl.Rep-inverse'* [*symmetric*])
apply (*simp add*: *bl-shiftl1 bl-shiftr1 word-bl.Abs-inverse*)
apply (*cases to-bl w*)
apply *auto*
done

lemma *shiftl-rev*:
shiftl (*w* :: '*a* :: *len word*) *n* = *word-reverse* (*shiftr* (*word-reverse w*) *n*)
apply (*unfold shiftl-def shiftr-def*)
apply (*induct n*)
apply (*auto simp add* : *shiftl1-rev*)
done

lemmas *rev-shiftl* =
shiftl-rev [**where** *w* = *word-reverse w*, *simplified*, *standard*]

lemmas *shiftr-rev* = *rev-shiftl* [*THEN word-rev-gal'*, *standard*]
lemmas *rev-shiftr* = *shiftl-rev* [*THEN word-rev-gal'*, *standard*]

lemma *bl-sshiftr1*:
  to-bl (*sshiftr1* (*w* :: *'a* :: *len word*)) = *hd* (*to-bl w*) # *butlast* (*to-bl w*)
  apply (*unfold sshiftr1-def uint-bl word-size*)
  apply (*simp add*: *butlast-rest-bin word-ubin.eq-norm*)
  apply (*simp add*: *sint-uint*)
  apply (*rule nth-equalityI*)
   apply *clarsimp*
  apply *clarsimp*
  apply (*case-tac i*)
   apply (*simp-all add*: *hd-conv-nth length-0-conv* [*symmetric*]
                        nth-bin-to-bl bin-nth.Suc [*symmetric*]
                        nth-sbintr
                  del: *bin-nth.Suc*)
   apply *force*
  apply (*rule impI*)
  apply (*rule-tac f* = *bin-nth* (*uint w*) **in** *arg-cong*)
  apply *simp*
  done

lemma *drop-shiftr*:
  drop n (*to-bl* ((*w* :: *'a* :: *len word*) >> *n*)) = *take* (*size w* − *n*) (*to-bl w*)
  apply (*unfold shiftr-def*)
  apply (*induct n*)
   prefer *2*
   apply (*simp add*: *drop-Suc bl-shiftr1 butlast-drop* [*symmetric*])
   apply (*rule butlast-take* [*THEN trans*])
  apply (*auto simp*: *word-size*)
  done

lemma *drop-sshiftr*:
  drop n (*to-bl* ((*w* :: *'a* :: *len word*) >>> *n*)) = *take* (*size w* − *n*) (*to-bl w*)
  apply (*unfold sshiftr-def*)
  apply (*induct n*)
   prefer *2*
   apply (*simp add*: *drop-Suc bl-sshiftr1 butlast-drop* [*symmetric*])
   apply (*rule butlast-take* [*THEN trans*])
  apply (*auto simp*: *word-size*)
  done

lemma *take-shiftr* [*rule-format*] :
  n <= *size* (*w* :: *'a* :: *len word*) −−> *take n* (*to-bl* (*w* >> *n*)) =
    replicate n False
  apply (*unfold shiftr-def*)
  apply (*induct n*)
   prefer *2*

```
  apply (simp add: bl-shiftr1)
  apply (rule impI)
  apply (rule take-butlast [THEN trans])
 apply (auto simp: word-size)
 done

lemma take-sshiftr' [rule-format] :
 n <= size (w :: 'a :: len word) --> hd (to-bl (w >>> n)) = hd (to-bl w) &
   take n (to-bl (w >>> n)) = replicate n (hd (to-bl w))
 apply (unfold sshiftr-def)
 apply (induct n)
  prefer 2
  apply (simp add: bl-sshiftr1)
  apply (rule impI)
  apply (rule take-butlast [THEN trans])
 apply (auto simp: word-size)
 done

lemmas hd-sshiftr = take-sshiftr' [THEN conjunct1, standard]
lemmas take-sshiftr = take-sshiftr' [THEN conjunct2, standard]

lemma atd-lem: take n xs = t ==> drop n xs = d ==> xs = t @ d
 by (auto intro: append-take-drop-id [symmetric])

lemmas bl-shiftr = atd-lem [OF take-shiftr drop-shiftr]
lemmas bl-sshiftr = atd-lem [OF take-sshiftr drop-sshiftr]

lemma shiftl-of-bl: of-bl bl << n = of-bl (bl @ replicate n False)
 unfolding shiftl-def
 by (induct n) (auto simp: shiftl1-of-bl replicate-app-Cons-same)

lemma shiftl-bl:
 (w::'a::len0 word) << (n::nat) = of-bl (to-bl w @ replicate n False)
proof -
 have w << n = of-bl (to-bl w) << n by simp
 also have ... = of-bl (to-bl w @ replicate n False) by (rule shiftl-of-bl)
 finally show ?thesis .
qed

lemmas shiftl-number [simp] = shiftl-def [where w=number-of w, standard]

lemma bl-shiftl:
 to-bl (w << n) = drop n (to-bl w) @ replicate (min (size w) n) False
 by (simp add: shiftl-bl word-rep-drop word-size)

lemma shiftl-zero-size:
 fixes x :: 'a::len0 word
 shows size x <= n ==> x << n = 0
 apply (unfold word-size)
```

apply (*rule word-eqI*)
apply (*clarsimp simp add*: *shiftl-bl word-size test-bit-of-bl nth-append*)
done

lemma *shiftl1-2t*: *shiftl1* (*w* :: *'a* :: *len word*) = *2* ∗ *w*
apply (*simp add*: *shiftl1-def-u*)
apply (*simp only*: *double-number-of-Bit0* [*symmetric*])
apply *simp*
done

lemma *shiftl1-p*: *shiftl1* (*w* :: *'a* :: *len word*) = *w* + *w*
apply (*simp add*: *shiftl1-def-u*)
apply (*simp only*: *double-number-of-Bit0* [*symmetric*])
apply *simp*
done

lemma *shiftl-t2n*: *shiftl* (*w* :: *'a* :: *len word*) *n* = *2* ^ *n* ∗ *w*
unfolding *shiftl-def*
by (*induct n*) (*auto simp*: *shiftl1-2t power-Suc*)

lemma *shiftr1-bintr* [*simp*]:
(*shiftr1* (*number-of w*) :: *'a* :: *len0 word*) =
number-of (*bin-rest* (*bintrunc* (*len-of TYPE* (*'a*)) *w*))
unfolding *shiftr1-def word-number-of-def*
by (*simp add* : *word-ubin.eq-norm*)

lemma *sshiftr1-sbintr* [*simp*] :
(*sshiftr1* (*number-of w*) :: *'a* :: *len word*) =
number-of (*bin-rest* (*sbintrunc* (*len-of TYPE* (*'a*) − *1*) *w*))
unfolding *sshiftr1-def word-number-of-def*
by (*simp add* : *word-sbin.eq-norm*)

lemma *shiftr-no'*:
w = *number-of bin* ==>
(*w*::*'a*::*len0 word*) >> *n* = *number-of* ((*bin-rest* ^^ *n*) (*bintrunc* (*size w*) *bin*))
apply *clarsimp*
apply (*rule word-eqI*)
apply (*auto simp*: *nth-shiftr nth-rest-power-bin nth-bintr word-size*)
done

lemma *sshiftr-no'*:
w = *number-of bin* ==> *w* >>> *n* = *number-of* ((*bin-rest* ^^ *n*)
(*sbintrunc* (*size w* − *1*) *bin*))
apply *clarsimp*
apply (*rule word-eqI*)
apply (*auto simp*: *nth-sshiftr nth-rest-power-bin nth-sbintr word-size*)
apply (*subgoal-tac na* + *n* = *len-of TYPE*(*'a*) − *Suc 0*, *simp*, *simp*)+

done

lemmas *sshiftr-no* [*simp*] =
sshiftr-no' [**where** *w* = *number-of w*, *OF refl*, *unfolded word-size*, *standard*]

lemmas *shiftr-no* [*simp*] =
shiftr-no' [**where** *w* = *number-of w*, *OF refl*, *unfolded word-size*, *standard*]

lemma *shiftr1-bl-of'*:
us = *shiftr1* (*of-bl bl*) ==> *length bl* <= *size us* ==>
us = *of-bl* (*butlast bl*)
by (*clarsimp simp*: *shiftr1-def of-bl-def word-size butlast-rest-bl2bin*
word-ubin.eq-norm trunc-bl2bin)

lemmas *shiftr1-bl-of* = *refl* [*THEN shiftr1-bl-of'*, *unfolded word-size*]

lemma *shiftr-bl-of'* [*rule-format*]:
us = *of-bl bl* >> *n* ==> *length bl* <= *size us* -->
us = *of-bl* (*take* (*length bl* − *n*) *bl*)
apply (*unfold shiftr-def*)
apply *hypsubst*
apply (*unfold word-size*)
apply (*induct n*)
apply *clarsimp*
apply *clarsimp*
apply (*subst shiftr1-bl-of*)
apply *simp*
apply (*simp add*: *butlast-take*)
done

lemmas *shiftr-bl-of* = *refl* [*THEN shiftr-bl-of'*, *unfolded word-size*]

lemmas *shiftr-bl* = *word-bl.Rep'* [*THEN eq-imp-le*, *THEN shiftr-bl-of*,
simplified word-size, *simplified*, *THEN eq-reflection*, *standard*]

lemma *msb-shift'*: *msb* (*w*::'*a*::*len word*) <-> (*w* >> (*size w* − *1*)) ~= *0*
apply (*unfold shiftr-bl word-msb-alt*)
apply (*simp add*: *word-size Suc-le-eq take-Suc*)
apply (*cases hd* (*to-bl w*))
apply (*auto simp*: *word-1-bl word-0-bl*
of-bl-rep-False [**where** *n=1* **and** *bs=*[], *simplified*])
done

lemmas *msb-shift* = *msb-shift'* [*unfolded word-size*]

lemma *align-lem-or* [*rule-format*] :
ALL x m. *length x* = *n* + *m* --> *length y* = *n* + *m* -->
drop m x = *replicate n False* --> *take m y* = *replicate m False* -->
map2 op | *x y* = *take m x* @ *drop m y*

apply (*induct-tac y*)
apply *force*
apply *clarsimp*
apply (*case-tac x*, *force*)
apply (*case-tac m*, *auto*)
apply (*drule sym*)
apply *auto*
apply (*induct-tac list*, *auto*)
done

lemma *align-lem-and* [*rule-format*] :
ALL x m. length x = n + m −−> length y = n + m −−>
drop m x = replicate n False −−> take m y = replicate m False −−>
map2 op & x y = replicate (n + m) False
apply (*induct-tac y*)
apply *force*
apply *clarsimp*
apply (*case-tac x*, *force*)
apply (*case-tac m*, *auto*)
apply (*drule sym*)
apply *auto*
apply (*induct-tac list*, *auto*)
done

lemma *aligned-bl-add-size'*:
size x − n = m ==> n <= size x ==> drop m (to-bl x) = replicate n False ==>
take m (to-bl y) = replicate m False ==>
to-bl (x + y) = take m (to-bl x) @ drop m (to-bl y)
apply (*subgoal-tac x AND y = 0*)
prefer *2*
apply (*rule word-bl.Rep-eqD*)
apply (*simp add*: *bl-word-and to-bl-0*)
apply (*rule align-lem-and* [*THEN trans*])
apply (*simp-all add*: *word-size*)[*5*]
apply *simp*
apply (*subst word-plus-and-or* [*symmetric*])
apply (*simp add* : *bl-word-or*)
apply (*rule align-lem-or*)
apply (*simp-all add*: *word-size*)
done

lemmas *aligned-bl-add-size = refl* [*THEN aligned-bl-add-size'*]

13.1.2 Mask

lemma *nth-mask'*: *m = mask n ==> test-bit m i = (i < n & i < size m)*
apply (*unfold mask-def test-bit-bl*)
apply (*simp only*: *word-1-bl* [*symmetric*] *shiftl-of-bl*)

apply (*clarsimp simp add*: *word-size*)
apply (*simp only*: *of-bl-no mask-lem number-of-succ add-diff-cancel2*)
apply (*fold of-bl-no*)
apply (*simp add*: *word-1-bl*)
apply (*rule test-bit-of-bl* [*THEN trans*, *unfolded test-bit-bl word-size*])
apply *auto*
done

lemmas *nth-mask* [*simp*] = *refl* [*THEN nth-mask'*]

lemma *mask-bl*: *mask n* = *of-bl* (*replicate n True*)
by (*auto simp add* : *test-bit-of-bl word-size intro*: *word-eqI*)

lemma *mask-bin*: *mask n* = *number-of* (*bintrunc n Int.Min*)
by (*auto simp add*: *nth-bintr word-size intro*: *word-eqI*)

lemma *and-mask-bintr*: *w AND mask n* = *number-of* (*bintrunc n* (*uint w*))
apply (*rule word-eqI*)
apply (*simp add*: *nth-bintr word-size word-ops-nth-size*)
apply (*auto simp add*: *test-bit-bin*)
done

lemma *and-mask-no*: *number-of i AND mask n* = *number-of* (*bintrunc n i*)
by (*auto simp add* : *nth-bintr word-size word-ops-nth-size intro*: *word-eqI*)

lemmas *and-mask-wi* = *and-mask-no* [*unfolded word-number-of-def*]

lemma *bl-and-mask*:
to-bl (*w AND mask n* :: *'a* :: *len word*) =
replicate (*len-of TYPE*('*a*) − *n*) *False* @
drop (*len-of TYPE*('*a*) − *n*) (*to-bl w*)
apply (*rule nth-equalityI*)
apply *simp*
apply (*clarsimp simp add*: *to-bl-nth word-size*)
apply (*simp add*: *word-size word-ops-nth-size*)
apply (*auto simp add*: *word-size test-bit-bl nth-append nth-rev*)
done

lemmas *and-mask-mod-2p* =
and-mask-bintr [*unfolded word-number-of-alt no-bintr-alt*]

lemma *and-mask-lt-2p*: *uint* (*w AND mask n*) < *2* ^ *n*
apply (*simp add* : *and-mask-bintr no-bintr-alt*)
apply (*rule xtr8*)
prefer *2*
apply (*rule pos-mod-bound*)
apply *auto*
done

lemmas *eq-mod-iff* = *trans* [*symmetric*, *OF int-mod-lem eq-sym-conv*]

lemma *mask-eq-iff*: (*w AND mask n*) = *w* <−> *uint w* < *2* ^ *n*
apply (*simp add*: *and-mask-bintr word-number-of-def*)
apply (*simp add*: *word-ubin.inverse-norm*)
apply (*simp add*: *eq-mod-iff bintrunc-mod2p min-def*)
apply (*fast intro*!: *lt2p-lem*)
done

lemma *and-mask-dvd*: *2* ^ *n dvd uint w* = (*w AND mask n* = *0*)
apply (*simp add*: *dvd-eq-mod-eq-0 and-mask-mod-2p*)
apply (*simp add*: *word-uint.norm-eq-iff* [*symmetric*] *word-of-int-homs*)
apply (*subst word-uint.norm-Rep* [*symmetric*])
apply (*simp only*: *bintrunc-bintrunc-min bintrunc-mod2p* [*symmetric*] *min-def*)
apply *auto*
done

lemma *and-mask-dvd-nat*: *2* ^ *n dvd unat w* = (*w AND mask n* = *0*)
apply (*unfold unat-def*)
apply (*rule trans* [*OF* - *and-mask-dvd*])
apply (*unfold dvd-def*)
apply *auto*
apply (*drule uint-ge-0* [*THEN nat-int.Abs-inverse'* [*simplified*], *symmetric*])
apply (*simp add* : *int-mult int-power*)
apply (*simp add* : *nat-mult-distrib nat-power-eq*)
done

lemma *word-2p-lem*:
n < *size w* ==> *w* < *2* ^ *n* = (*uint* (*w* :: *'a* :: *len word*) < *2* ^ *n*)
apply (*unfold word-size word-less-alt word-number-of-alt*)
apply (*clarsimp simp add*: *word-of-int-power-hom word-uint.eq-norm*
int-mod-eq'
simp del: *word-of-int-bin*)
done

lemma *less-mask-eq*: *x* < *2* ^ *n* ==> *x AND mask n* = (*x* :: *'a* :: *len word*)
apply (*unfold word-less-alt word-number-of-alt*)
apply (*clarsimp simp add*: *and-mask-mod-2p word-of-int-power-hom*
word-uint.eq-norm
simp del: *word-of-int-bin*)
apply (*drule xtr8* [*rotated*])
apply (*rule int-mod-le*)
apply (*auto simp add* : *mod-pos-pos-trivial*)
done

lemmas *mask-eq-iff-w2p* =
trans [*OF mask-eq-iff word-2p-lem* [*symmetric*], *standard*]

lemmas *and-mask-less'* =

iffD2 [*OF word-2p-lem and-mask-lt-2p*, *simplified word-size*, *standard*]

lemma *and-mask-less-size*: $n < size\ x \Longrightarrow x\ AND\ mask\ n < 2\hat{}n$
unfolding *word-size* **by** (*erule and-mask-less'*)

lemma *word-mod-2p-is-mask'*:
$c = 2\ \hat{}\ n \Longrightarrow c > 0 \Longrightarrow x\ mod\ c = (x :: 'a :: len\ word)\ AND\ mask\ n$
by (*clarsimp simp add*: *word-mod-def uint-2p and-mask-mod-2p*)

lemmas *word-mod-2p-is-mask* = *refl* [*THEN word-mod-2p-is-mask'*]

lemma *mask-eqs*:
$(a\ AND\ mask\ n) + b\ AND\ mask\ n = a + b\ AND\ mask\ n$
$a + (b\ AND\ mask\ n)\ AND\ mask\ n = a + b\ AND\ mask\ n$
$(a\ AND\ mask\ n) - b\ AND\ mask\ n = a - b\ AND\ mask\ n$
$a - (b\ AND\ mask\ n)\ AND\ mask\ n = a - b\ AND\ mask\ n$
$a * (b\ AND\ mask\ n)\ AND\ mask\ n = a * b\ AND\ mask\ n$
$(b\ AND\ mask\ n) * a\ AND\ mask\ n = b * a\ AND\ mask\ n$
$(a\ AND\ mask\ n) + (b\ AND\ mask\ n)\ AND\ mask\ n = a + b\ AND\ mask\ n$
$(a\ AND\ mask\ n) - (b\ AND\ mask\ n)\ AND\ mask\ n = a - b\ AND\ mask\ n$
$(a\ AND\ mask\ n) * (b\ AND\ mask\ n)\ AND\ mask\ n = a * b\ AND\ mask\ n$
$- (a\ AND\ mask\ n)\ AND\ mask\ n = - a\ AND\ mask\ n$
$word\text{-}succ\ (a\ AND\ mask\ n)\ AND\ mask\ n = word\text{-}succ\ a\ AND\ mask\ n$
$word\text{-}pred\ (a\ AND\ mask\ n)\ AND\ mask\ n = word\text{-}pred\ a\ AND\ mask\ n$
using *word-of-int-Ex* [**where** x=a] *word-of-int-Ex* [**where** x=b]
by (*auto simp*: *and-mask-wi bintr-ariths bintr-arith1s new-word-of-int-homs*)

lemma *mask-power-eq*:
$(x\ AND\ mask\ n)\ \hat{}\ k\ AND\ mask\ n = x\ \hat{}\ k\ AND\ mask\ n$
using *word-of-int-Ex* [**where** x=x]
by (*clarsimp simp*: *and-mask-wi word-of-int-power-hom bintr-ariths*)

13.1.3 Revcast

lemmas *revcast-def'* = *revcast-def* [*simplified*]
lemmas *revcast-def''* = *revcast-def'* [*simplified word-size*]
lemmas *revcast-no-def* [*simp*] =
revcast-def' [**where** w=*number-of w*, *unfolded word-size*, *standard*]

lemma *to-bl-revcast*:
$to\text{-}bl\ (revcast\ w :: 'a :: len0\ word) =$
takefill False (*len-of TYPE* ('a')) (*to-bl w*)
apply (*unfold revcast-def' word-size*)
apply (*rule word-bl.Abs-inverse*)
apply *simp*
done

lemma *revcast-rev-ucast'*:
$cs = [rc, uc] \Longrightarrow rc = revcast\ (word\text{-}reverse\ w) \Longrightarrow uc = ucast\ w \Longrightarrow$

```
    rc = word-reverse uc
  apply (unfold ucast-def revcast-def' Let-def word-reverse-def)
  apply (clarsimp simp add : to-bl-of-bin takefill-bintrunc)
  apply (simp add : word-bl.Abs-inverse word-size)
  done

lemmas revcast-rev-ucast = revcast-rev-ucast' [OF refl refl refl]

lemmas revcast-ucast = revcast-rev-ucast
  [where w = word-reverse w, simplified word-rev-rev, standard]

lemmas ucast-revcast = revcast-rev-ucast [THEN word-rev-gal', standard]
lemmas ucast-rev-revcast = revcast-ucast [THEN word-rev-gal', standard]


— linking revcast and cast via shift

lemmas wsst-TYs = source-size target-size word-size

lemma revcast-down-uu':
  rc = revcast ==> source-size rc = target-size rc + n ==>
    rc (w :: 'a :: len word) = ucast (w >> n)
  apply (simp add: revcast-def')
  apply (rule word-bl.Rep-inverse')
  apply (rule trans, rule ucast-down-drop)
   prefer 2
   apply (rule trans, rule drop-shiftr)
   apply (auto simp: takefill-alt wsst-TYs)
  done

lemma revcast-down-us':
  rc = revcast ==> source-size rc = target-size rc + n ==>
    rc (w :: 'a :: len word) = ucast (w >>> n)
  apply (simp add: revcast-def')
  apply (rule word-bl.Rep-inverse')
  apply (rule trans, rule ucast-down-drop)
   prefer 2
   apply (rule trans, rule drop-sshiftr)
   apply (auto simp: takefill-alt wsst-TYs)
  done

lemma revcast-down-su':
  rc = revcast ==> source-size rc = target-size rc + n ==>
    rc (w :: 'a :: len word) = scast (w >> n)
  apply (simp add: revcast-def')
  apply (rule word-bl.Rep-inverse')
  apply (rule trans, rule scast-down-drop)
   prefer 2
   apply (rule trans, rule drop-shiftr)
```

```
 apply (auto simp: takefill-alt wsst-TYs)
done

lemma revcast-down-ss':
  rc = revcast ==> source-size rc = target-size rc + n ==>
    rc (w :: 'a :: len word) = scast (w >>> n)
  apply (simp add: revcast-def')
  apply (rule word-bl.Rep-inverse')
  apply (rule trans, rule scast-down-drop)
   prefer 2
   apply (rule trans, rule drop-sshiftr)
   apply (auto simp: takefill-alt wsst-TYs)
  done

lemmas revcast-down-uu = refl [THEN revcast-down-uu']
lemmas revcast-down-us = refl [THEN revcast-down-us']
lemmas revcast-down-su = refl [THEN revcast-down-su']
lemmas revcast-down-ss = refl [THEN revcast-down-ss']

lemma cast-down-rev:
  uc = ucast ==> source-size uc = target-size uc + n ==>
    uc w = revcast ((w :: 'a :: len word) << n)
  apply (unfold shiftl-rev)
  apply clarify
  apply (simp add: revcast-rev-ucast)
  apply (rule word-rev-gal')
  apply (rule trans [OF - revcast-rev-ucast])
  apply (rule revcast-down-uu [symmetric])
  apply (auto simp add: wsst-TYs)
  done

lemma revcast-up':
  rc = revcast ==> source-size rc + n = target-size rc ==>
    rc w = (ucast w :: 'a :: len word) << n
  apply (simp add: revcast-def')
  apply (rule word-bl.Rep-inverse')
  apply (simp add: takefill-alt)
  apply (rule bl-shiftl [THEN trans])
  apply (subst ucast-up-app)
  apply (auto simp add: wsst-TYs)
  done

lemmas revcast-up = refl [THEN revcast-up']

lemmas rc1 = revcast-up [THEN
  revcast-rev-ucast [symmetric, THEN trans, THEN word-rev-gal, symmetric]]
lemmas rc2 = revcast-down-uu [THEN
  revcast-rev-ucast [symmetric, THEN trans, THEN word-rev-gal, symmetric]]
```

lemmas *ucast-up* =
rc1 [*simplified rev-shiftr* [*symmetric*] *revcast-ucast* [*symmetric*]]
lemmas *ucast-down* =
rc2 [*simplified rev-shiftr revcast-ucast* [*symmetric*]]

13.1.4 Slices

lemmas *slice1-no-bin* [*simp*] =
slice1-def [**where** *w*=*number-of w*, *unfolded to-bl-no-bin*, *standard*]

lemmas *slice-no-bin* [*simp*] =
trans [*OF slice-def* [*THEN meta-eq-to-obj-eq*]
slice1-no-bin [*THEN meta-eq-to-obj-eq*],
unfolded word-size, *standard*]

lemma *slice1-0* [*simp*] : *slice1 n 0 = 0*
unfolding *slice1-def* **by** (*simp add* : *to-bl-0*)

lemma *slice-0* [*simp*] : *slice n 0 = 0*
unfolding *slice-def* **by** *auto*

lemma *slice-take'*: *slice n w = of-bl* (*take* (*size w − n*) (*to-bl w*))
unfolding *slice-def' slice1-def*
by (*simp add* : *takefill-alt word-size*)

lemmas *slice-take* = *slice-take'* [*unfolded word-size*]

— shiftr to a word of the same size is just slice, slice is just shiftr then ucast
lemmas *shiftr-slice* = *trans*
[*OF shiftr-bl* [*THEN meta-eq-to-obj-eq*] *slice-take* [*symmetric*], *standard*]

lemma *slice-shiftr*: *slice n w = ucast* (*w >> n*)
apply (*unfold slice-take shiftr-bl*)
apply (*rule ucast-of-bl-up* [*symmetric*])
apply (*simp add*: *word-size*)
done

lemma *nth-slice*:
(*slice n w* :: *'a* :: *len0 word*) !! *m* =
(*w* !! (*m* + *n*) & *m* < *len-of TYPE* (*'a*))
unfolding *slice-shiftr*
by (*simp add* : *nth-ucast nth-shiftr*)

lemma *slice1-down-alt'*:
sl = *slice1 n w* ==> *fs* = *size sl* ==> *fs* + *k* = *n* ==>
to-bl sl = *takefill False fs* (*drop k* (*to-bl w*))
unfolding *slice1-def word-size of-bl-def uint-bl*
by (*clarsimp simp*: *word-ubin.eq-norm bl-bin-bl-rep-drop drop-takefill*)

lemma *slice1-up-alt'*:
sl = slice1 n w ==> fs = size sl ==> fs = n + k ==>
to-bl sl = takefill False fs (replicate k False @ (to-bl w))
apply (*unfold slice1-def word-size of-bl-def uint-bl*)
apply (*clarsimp simp*: *word-ubin.eq-norm bl-bin-bl-rep-drop*
takefill-append [*symmetric*])
apply (*rule-tac f = %k. takefill False* (*len-of TYPE('a)*)
(*replicate k False @ bin-to-bl* (*len-of TYPE('b)*) (*uint w*)) **in** *arg-cong*)
apply *arith*
done

lemmas *sd1 = slice1-down-alt'* [*OF refl refl*, *unfolded word-size*]
lemmas *su1 = slice1-up-alt'* [*OF refl refl*, *unfolded word-size*]
lemmas *slice1-down-alt = le-add-diff-inverse* [*THEN sd1*]
lemmas *slice1-up-alts =*
le-add-diff-inverse [*symmetric*, *THEN su1*]
le-add-diff-inverse2 [*symmetric*, *THEN su1*]

lemma *ucast-slice1*: *ucast w = slice1* (*size w*) *w*
unfolding *slice1-def ucast-bl*
by (*simp add* : *takefill-same' word-size*)

lemma *ucast-slice*: *ucast w = slice 0 w*
unfolding *slice-def* **by** (*simp add* : *ucast-slice1*)

lemmas *slice-id = trans* [*OF ucast-slice* [*symmetric*] *ucast-id*]

lemma *revcast-slice1'*:
rc = revcast w ==> slice1 (*size rc*) *w = rc*
unfolding *slice1-def revcast-def'* **by** (*simp add* : *word-size*)

lemmas *revcast-slice1 = refl* [*THEN revcast-slice1'*]

lemma *slice1-tf-tf'*:
to-bl (*slice1 n w* :: *'a* :: *len0 word*) =
rev (*takefill False* (*len-of TYPE('a)*) (*rev* (*takefill False n* (*to-bl w*))))
unfolding *slice1-def* **by** (*rule word-rev-tf*)

lemmas *slice1-tf-tf = slice1-tf-tf'*
[*THEN word-bl.Rep-inverse'*, *symmetric*, *standard*]

lemma *rev-slice1*:
n + k = len-of TYPE('a) + len-of TYPE('b) ⟹
slice1 n (*word-reverse w* :: *'b* :: *len0 word*) =
word-reverse (*slice1 k w* :: *'a* :: *len0 word*)
apply (*unfold word-reverse-def slice1-tf-tf*)
apply (*rule word-bl.Rep-inverse'*)
apply (*rule rev-swap* [*THEN iffD1*])
apply (*rule trans* [*symmetric*])

```
  apply (rule tf-rev)
   apply (simp add: word-bl.Abs-inverse)
  apply (simp add: word-bl.Abs-inverse)
  done

lemma rev-slice':
  res = slice n (word-reverse w) ==> n + k + size res = size w ==>
    res = word-reverse (slice k w)
  apply (unfold slice-def word-size)
  apply clarify
  apply (rule rev-slice1)
  apply arith
  done

lemmas rev-slice = refl [THEN rev-slice', unfolded word-size]

lemmas sym-notr =
  not-iff [THEN iffD2, THEN not-sym, THEN not-iff [THEN iffD1]]
```

— problem posed by TPHOLs referee: criterion for overflow of addition of signed integers

```
lemma sofl-test:
  (sint (x :: 'a :: len word) + sint y = sint (x + y)) =
     ((((x+y) XOR x) AND ((x+y) XOR y)) >> (size x - 1) = 0)
  apply (unfold word-size)
  apply (cases len-of TYPE('a), simp)
  apply (subst msb-shift [THEN sym-notr])
  apply (simp add: word-ops-msb)
  apply (simp add: word-msb-sint)
  apply safe
       apply simp-all
     apply (unfold sint-word-ariths)
     apply (unfold word-sbin.set-iff-norm [symmetric] sints-num)
     apply safe
        apply (insert sint-range' [where x=x])
        apply (insert sint-range' [where x=y])
        defer
        apply (simp (no-asm), arith)
       apply (simp (no-asm), arith)
      defer
      defer
      apply (simp (no-asm), arith)
     apply (simp (no-asm), arith)
    apply (rule notI [THEN notnotD],
           drule leI not-leE,
           drule sbintrunc-inc sbintrunc-dec,
           simp)+
  done
```

13.2 Split and cat

lemmas *word-split-bin′ = word-split-def* [*THEN meta-eq-to-obj-eq*, *standard*]
lemmas *word-cat-bin′ = word-cat-def* [*THEN meta-eq-to-obj-eq*, *standard*]

lemma *word-rsplit-no*:
(*word-rsplit* (*number-of bin* :: *'b* :: *len0 word*) :: *'a word list*) =
map number-of (*bin-rsplit* (*len-of TYPE*(*'a* :: *len*))
(*len-of TYPE*(*'b*), *bintrunc* (*len-of TYPE*(*'b*)) *bin*))
apply (*unfold word-rsplit-def word-no-wi*)
apply (*simp add*: *word-ubin.eq-norm*)
done

lemmas *word-rsplit-no-cl* [*simp*] = *word-rsplit-no*
[*unfolded bin-rsplitl-def bin-rsplit-l* [*symmetric*]]

lemma *test-bit-cat*:
wc = *word-cat a b* ==> *wc* !! *n* = (*n* < *size wc* &
(*if n* < *size b then b* !! *n else a* !! (*n* − *size b*)))
apply (*unfold word-cat-bin′ test-bit-bin*)
apply (*auto simp add* : *word-ubin.eq-norm nth-bintr bin-nth-cat word-size*)
apply (*erule bin-nth-uint-imp*)
done

lemma *word-cat-bl*: *word-cat a b* = *of-bl* (*to-bl a* @ *to-bl b*)
apply (*unfold of-bl-def to-bl-def word-cat-bin′*)
apply (*simp add*: *bl-to-bin-app-cat*)
done

lemma *of-bl-append*:
(*of-bl* (*xs* @ *ys*) :: *'a* :: *len word*) = *of-bl xs* ∗ *2^*(*length ys*) + *of-bl ys*
apply (*unfold of-bl-def*)
apply (*simp add*: *bl-to-bin-app-cat bin-cat-num*)
apply (*simp add*: *word-of-int-power-hom* [*symmetric*] *new-word-of-int-hom-syms*)
done

lemma *of-bl-False* [*simp*]:
of-bl (*False*#*xs*) = *of-bl xs*
by (*rule word-eqI*)
(*auto simp add*: *test-bit-of-bl nth-append*)

lemma *of-bl-True*:
(*of-bl* (*True*#*xs*)::*'a*::*len word*) = *2^length xs* + *of-bl xs*
by (*subst of-bl-append* [**where** *xs*=[*True*], *simplified*])
(*simp add*: *word-1-bl*)

lemma *of-bl-Cons*:
of-bl (*x*#*xs*) = *of-bool x* ∗ *2^length xs* + *of-bl xs*
by (*cases x*) (*simp-all add*: *of-bl-True*)

lemma *split-uint-lem*: *bin-split n (uint (w :: 'a :: len0 word)) = (a, b) ==>*
a = bintrunc (len-of TYPE('a) − n) a & b = bintrunc (len-of TYPE('a)) b
apply (*frule word-ubin.norm-Rep* [*THEN ssubst*])
apply (*drule bin-split-trunc1*)
apply (*drule sym* [*THEN trans*])
apply *assumption*
apply *safe*
done

lemma *word-split-bl'*:
std = size c − size b ==> (word-split c = (a, b)) ==>
(a = of-bl (take std (to-bl c)) & b = of-bl (drop std (to-bl c)))
apply (*unfold word-split-bin'*)
apply *safe*
defer
apply (*clarsimp split*: *prod.splits*)
apply (*drule word-ubin.norm-Rep* [*THEN ssubst*])
apply (*drule split-bintrunc*)
apply (*simp add* : *of-bl-def bl2bin-drop word-size*
word-ubin.norm-eq-iff [*symmetric*] *min-def del* : *word-ubin.norm-Rep*)
apply (*clarsimp split*: *prod.splits*)
apply (*frule split-uint-lem* [*THEN conjunct1*])
apply (*unfold word-size*)
apply (*cases len-of TYPE('a) >= len-of TYPE('b)*)
defer
apply (*simp add*: *word-0-bl word-0-wi-Pls*)
apply (*simp add* : *of-bl-def to-bl-def*)
apply (*subst bin-split-take1* [*symmetric*])
prefer *2*
apply *assumption*
apply *simp*
apply (*erule thin-rl*)
apply (*erule arg-cong* [*THEN trans*])
apply (*simp add* : *word-ubin.norm-eq-iff* [*symmetric*])
done

lemma *word-split-bl*: *std = size c − size b ==>*
(a = of-bl (take std (to-bl c)) & b = of-bl (drop std (to-bl c))) <->
word-split c = (a, b)
apply (*rule iffI*)
defer
apply (*erule* (*1*) *word-split-bl'*)
apply (*case-tac word-split c*)
apply (*auto simp add* : *word-size*)
apply (*frule word-split-bl'* [*rotated*])
apply (*auto simp add* : *word-size*)
done

lemma *word-split-bl-eq*:

```
  (word-split (c::'a::len word) :: ('c :: len0 word * 'd :: len0 word)) =
     (of-bl (take (len-of TYPE('a::len) - len-of TYPE('d::len0)) (to-bl c)),
      of-bl (drop (len-of TYPE('a) - len-of TYPE('d)) (to-bl c)))
 apply (rule word-split-bl [THEN iffD1])
 apply (unfold word-size)
 apply (rule refl conjI)+
 done

— keep quantifiers for use in simplification
lemma test-bit-split':
  word-split c = (a, b) --> (ALL n m. b !! n = (n < size b & c !! n) &
    a !! m = (m < size a & c !! (m + size b)))
 apply (unfold word-split-bin' test-bit-bin)
 apply (clarify)
 apply (clarsimp simp: word-ubin.eq-norm nth-bintr word-size split: prod.splits)
 apply (drule bin-nth-split)
 apply safe
     apply (simp-all add: add-commute)
  apply (erule bin-nth-uint-imp)+
 done

lemmas test-bit-split =
  test-bit-split' [THEN mp, simplified all-simps, standard]

lemma test-bit-split-eq: word-split c = (a, b) <->
  ((ALL n::nat. b !! n = (n < size b & c !! n)) &
    (ALL m::nat. a !! m = (m < size a & c !! (m + size b))))
 apply (rule-tac iffI)
  apply (rule-tac conjI)
   apply (erule test-bit-split [THEN conjunct1])
  apply (erule test-bit-split [THEN conjunct2])
 apply (case-tac word-split c)
 apply (frule test-bit-split)
 apply (erule trans)
 apply (fastsimp intro ! : word-eqI simp add : word-size)
 done

— this odd result is analogous to ucast-id, result to the length given by the result
type

lemma word-cat-id: word-cat a b = b
 unfolding word-cat-bin' by (simp add: word-ubin.inverse-norm)

— limited hom result
lemma word-cat-hom:
  len-of TYPE('a::len0) <= len-of TYPE('b::len0) + len-of TYPE ('c::len0)
  ==>
  (word-cat (word-of-int w :: 'b word) (b :: 'c word) :: 'a word) =
  word-of-int (bin-cat w (size b) (uint b))
```

apply (*unfold word-cat-def word-size*)
apply (*clarsimp simp add*: *word-ubin.norm-eq-iff* [*symmetric*]
word-ubin.eq-norm bintr-cat min-max.inf-absorb1)
done

lemma *word-cat-split-alt*:
size w <= size u + size v ==> word-split w = (u, v) ==> word-cat u v = w
apply (*rule word-eqI*)
apply (*drule test-bit-split*)
apply (*clarsimp simp add* : *test-bit-cat word-size*)
apply *safe*
apply *arith*
done

lemmas *word-cat-split-size* =
sym [*THEN* [*2*] *word-cat-split-alt* [*symmetric*], *standard*]

13.2.1 Split and slice

lemma *split-slices*:
word-split w = (u, v) ==> u = slice (size v) w & v = slice 0 w
apply (*drule test-bit-split*)
apply (*rule conjI*)
apply (*rule word-eqI*, *clarsimp simp*: *nth-slice word-size*)+
done

lemma *slice-cat1'*:
wc = word-cat a b ==> size wc >= size a + size b ==> slice (size b) wc = a
apply *safe*
apply (*rule word-eqI*)
apply (*simp add*: *nth-slice test-bit-cat word-size*)
done

lemmas *slice-cat1* = *refl* [*THEN slice-cat1'*]
lemmas *slice-cat2* = *trans* [*OF slice-id word-cat-id*]

lemma *cat-slices*:
a = slice n c ==> b = slice 0 c ==> n = size b ==>
size a + size b >= size c ==> word-cat a b = c
apply *safe*
apply (*rule word-eqI*)
apply (*simp add*: *nth-slice test-bit-cat word-size*)
apply *safe*
apply *arith*
done

lemma *word-split-cat-alt*:
w = word-cat u v ==> size u + size v <= size w ==> word-split w = (u, v)
apply (*case-tac word-split ?w*)

apply (*rule trans*, *assumption*)
apply (*drule test-bit-split*)
apply *safe*
apply (*rule word-eqI*, *clarsimp simp*: *test-bit-cat word-size*)+
done

lemmas *word-cat-bl-no-bin* [*simp*] =
word-cat-bl [**where** *a*=*number-of a*
and *b*=*number-of b*,
unfolded to-bl-no-bin, *standard*]

lemmas *word-split-bl-no-bin* [*simp*] =
word-split-bl-eq [**where** *c*=*number-of c*, *unfolded to-bl-no-bin*, *standard*]

— this odd result arises from the fact that the statement of the result implies that the decoded words are of the same type, and therefore of the same length, as the original word

lemma *word-rsplit-same*: *word-rsplit w* = [*w*]
unfolding *word-rsplit-def* **by** (*simp add* : *bin-rsplit-all*)

lemma *word-rsplit-empty-iff-size*:
(*word-rsplit w* = []) = (*size w* = *0*)
unfolding *word-rsplit-def bin-rsplit-def word-size*
by (*simp add*: *bin-rsplit-aux-simp-alt Let-def split*: *Product-Type.split-split*)

lemma *test-bit-rsplit*:
sw = *word-rsplit w* ==> *m* < *size* (*hd sw* :: *'a* :: *len word*) ==>
k < *length sw* ==> (*rev sw* ! *k*) !! *m* = (*w* !! (*k* ∗ *size* (*hd sw*) + *m*))
apply (*unfold word-rsplit-def word-test-bit-def*)
apply (*rule trans*)
apply (*rule-tac f* = %*x*. *bin-nth x m* **in** *arg-cong*)
apply (*rule nth-map* [*symmetric*])
apply *simp*
apply (*rule bin-nth-rsplit*)
apply *simp-all*
apply (*simp add* : *word-size rev-map*)
apply (*rule trans*)
defer
apply (*rule map-ident* [*THEN fun-cong*])
apply (*rule refl* [*THEN map-cong*])
apply (*simp add* : *word-ubin.eq-norm*)
apply (*erule bin-rsplit-size-sign* [*OF len-gt-0 refl*])
done

lemma *word-rcat-bl*: *word-rcat wl* == *of-bl* (*concat* (*map to-bl wl*))
unfolding *word-rcat-def to-bl-def' of-bl-def*
by (*clarsimp simp add* : *bin-rcat-bl*)

lemma *size-rcat-lem'*:
size (*concat* (*map to-bl wl*)) = *length wl* ∗ *size* (*hd wl*)
unfolding *word-size* **by** (*induct wl*) *auto*

lemmas *size-rcat-lem* = *size-rcat-lem'* [*unfolded word-size*]

lemmas *td-gal-lt-len* = *len-gt-0* [*THEN td-gal-lt*, *standard*]

lemma *nth-rcat-lem'* [*rule-format*] :
sw = *size* (*hd wl* :: 'a :: *len word*) ==> (*ALL n*. *n* < *size wl* ∗ *sw* −−>
rev (*concat* (*map to-bl wl*)) ! *n* =
rev (*to-bl* (*rev wl* ! (*n div sw*))) ! (*n mod sw*))
apply (*unfold word-size*)
apply (*induct wl*)
apply *clarsimp*
apply (*clarsimp simp add* : *nth-append size-rcat-lem*)
apply (*simp* (*no-asm-use*) *only*: *mult-Suc* [*symmetric*]
td-gal-lt-len less-Suc-eq-le mod-div-equality')
apply *clarsimp*
done

lemmas *nth-rcat-lem* = *refl* [*THEN nth-rcat-lem'*, *unfolded word-size*]

lemma *test-bit-rcat*:
sw = *size* (*hd wl* :: 'a :: *len word*) ==> *rc* = *word-rcat wl* ==> *rc* !! *n* =
(*n* < *size rc* & *n div sw* < *size wl* & (*rev wl*) ! (*n div sw*) !! (*n mod sw*))
apply (*unfold word-rcat-bl word-size*)
apply (*clarsimp simp add* :
test-bit-of-bl size-rcat-lem word-size td-gal-lt-len)
apply *safe*
apply (*auto simp add* :
test-bit-bl word-size td-gal-lt-len [*THEN iffD2*, *THEN nth-rcat-lem*])
done

lemma *foldl-eq-foldr* [*rule-format*] :
ALL x. *foldl op* + *x xs* = *foldr op* + (*x* # *xs*) (*0* :: 'a :: *comm-monoid-add*)
by (*induct xs*) (*auto simp add* : *add-assoc*)

lemmas *test-bit-cong* = *arg-cong* [**where** *f* = *test-bit*, *THEN fun-cong*]

lemmas *test-bit-rsplit-alt* =
trans [*OF nth-rev-alt* [*THEN test-bit-cong*]
test-bit-rsplit [*OF refl asm-rl diff-Suc-less*]]

— lazy way of expressing that u and v, and su and sv, have same types
lemma *word-rsplit-len-indep'*:
[*u*,*v*] = *p* ==> [*su*,*sv*] = *q* ==> *word-rsplit u* = *su* ==>
word-rsplit v = *sv* ==> *length su* = *length sv*
apply (*unfold word-rsplit-def*)

apply (*auto simp add* : *bin-rsplit-len-indep*)
done

lemmas *word-rsplit-len-indep* = *word-rsplit-len-indep'* [*OF refl refl refl refl*]

lemma *length-word-rsplit-size*:
n = *len-of TYPE* ('*a* :: *len*) ==>
(*length* (*word-rsplit w* :: '*a word list*) <= *m*) = (*size w* <= *m* * *n*)
apply (*unfold word-rsplit-def word-size*)
apply (*clarsimp simp add* : *bin-rsplit-len-le*)
done

lemmas *length-word-rsplit-lt-size* =
length-word-rsplit-size [*unfolded Not-eq-iff linorder-not-less* [*symmetric*]]

lemma *length-word-rsplit-exp-size*:
n = *len-of TYPE* ('*a* :: *len*) ==>
length (*word-rsplit w* :: '*a word list*) = (*size w* + *n* − *1*) *div n*
unfolding *word-rsplit-def* **by** (*clarsimp simp add* : *word-size bin-rsplit-len*)

lemma *length-word-rsplit-even-size*:
n = *len-of TYPE* ('*a* :: *len*) ==> *size w* = *m* * *n* ==>
length (*word-rsplit w* :: '*a word list*) = *m*
by (*clarsimp simp add* : *length-word-rsplit-exp-size given-quot-alt*)

lemmas *length-word-rsplit-exp-size'* = *refl* [*THEN length-word-rsplit-exp-size*]

lemmas *tdle* = *iffD2* [*OF split-div-lemma refl*, *THEN conjunct1*]
lemmas *dtle* = *xtr4* [*OF tdle mult-commute*]

lemma *word-rcat-rsplit*: *word-rcat* (*word-rsplit w*) = *w*
apply (*rule word-eqI*)
apply (*clarsimp simp add* : *test-bit-rcat word-size*)
apply (*subst refl* [*THEN test-bit-rsplit*])
apply (*simp-all add*: *word-size*
refl [*THEN length-word-rsplit-size* [*simplified not-less* [*symmetric*], *simplified*]])
apply *safe*
apply (*erule xtr7*, *rule len-gt-0* [*THEN dtle*])+
done

lemma *size-word-rsplit-rcat-size'*:
word-rcat (*ws* :: '*a* :: *len word list*) = *frcw* ==>
size frcw = *length ws* * *len-of TYPE* ('*a*) ==>
size (*hd* [*word-rsplit frcw*, *ws*]) = *size ws*
apply (*clarsimp simp add* : *word-size length-word-rsplit-exp-size'*)
apply (*fast intro*: *given-quot-alt*)
done

lemmas *size-word-rsplit-rcat-size* =
size-word-rsplit-rcat-size' [*simplified*]

lemma *msrevs*:
fixes *n*::*nat*
shows *0* < *n* ⟹ (*k* * *n* + *m*) *div n* = *m div n* + *k*
and (*k* * *n* + *m*) *mod n* = *m mod n*
by (*auto simp*: *add-commute*)

lemma *word-rsplit-rcat-size'*:
word-rcat (*ws* :: *'a* :: *len word list*) = *frcw* ==>
size frcw = *length ws* * *len-of TYPE* (*'a*) ==> *word-rsplit frcw* = *ws*
apply (*frule size-word-rsplit-rcat-size*, *assumption*)
apply (*clarsimp simp add* : *word-size*)
apply (*rule nth-equalityI*, *assumption*)
apply *clarsimp*
apply (*rule word-eqI*)
apply (*rule trans*)
apply (*rule test-bit-rsplit-alt*)
apply (*clarsimp simp*: *word-size*)+
apply (*rule trans*)
apply (*rule test-bit-rcat* [*OF refl refl*])
apply (*simp add* : *word-size msrevs*)
apply (*subst nth-rev*)
apply *arith*
apply (*simp add* : *le0* [*THEN* [*2*] *xtr7*, *THEN diff-Suc-less*])
apply *safe*
apply (*simp add* : *diff-mult-distrib*)
apply (*rule mpl-lem*)
apply (*cases size ws*)
apply *simp-all*
done

lemmas *word-rsplit-rcat-size* = *refl* [*THEN word-rsplit-rcat-size'*]

13.3 Rotation

lemmas *rotater-0'* [*simp*] = *rotater-def* [**where** *n* = *0*, *simplified*]

lemmas *word-rot-defs* = *word-roti-def word-rotr-def word-rotl-def*

lemma *rotate-eq-mod*:
m mod length xs = *n mod length xs* ==> *rotate m xs* = *rotate n xs*
apply (*rule box-equals*)
defer
apply (*rule rotate-conv-mod* [*symmetric*])+
apply *simp*
done

lemmas *rotate-eqs* [*standard*] =
trans [*OF rotate0* [*THEN fun-cong*] *id-apply*]
rotate-rotate [*symmetric*]
rotate-id
rotate-conv-mod
rotate-eq-mod

13.3.1 Rotation of list to right

lemma *rotate1-rl'*: *rotater1* (*l* @ [*a*]) = *a* # *l*
unfolding *rotater1-def* **by** (*cases l*) *auto*

lemma *rotate1-rl* [*simp*] : *rotater1* (*rotate1 l*) = *l*
apply (*unfold rotater1-def*)
apply (*cases l*)
apply (*case-tac* [*2*] *list*)
apply *auto*
done

lemma *rotate1-lr* [*simp*] : *rotate1* (*rotater1 l*) = *l*
unfolding *rotater1-def* **by** (*cases l*) *auto*

lemma *rotater1-rev'*: *rotater1* (*rev xs*) = *rev* (*rotate1 xs*)
apply (*cases xs*)
apply (*simp add* : *rotater1-def*)
apply (*simp add* : *rotate1-rl'*)
done

lemma *rotater-rev'*: *rotater n* (*rev xs*) = *rev* (*rotate n xs*)
unfolding *rotater-def* **by** (*induct n*) (*auto intro*: *rotater1-rev'*)

lemmas *rotater-rev* = *rotater-rev'* [**where** *xs* = *rev ys*, *simplified*, *standard*]

lemma *rotater-drop-take*:
rotater n xs =
drop (*length xs* − *n mod length xs*) *xs* @
take (*length xs* − *n mod length xs*) *xs*
by (*clarsimp simp add* : *rotater-rev rotate-drop-take rev-take rev-drop*)

lemma *rotater-Suc* [*simp*] :
rotater (*Suc n*) *xs* = *rotater1* (*rotater n xs*)
unfolding *rotater-def* **by** *auto*

lemma *rotate-inv-plus* [*rule-format*] :
ALL k. *k* = *m* + *n* −−> *rotater k* (*rotate n xs*) = *rotater m xs* &
rotate k (*rotater n xs*) = *rotate m xs* &
rotater n (*rotate k xs*) = *rotate m xs* &
rotate n (*rotater k xs*) = *rotater m xs*
unfolding *rotater-def rotate-def*

by (*induct n*) (*auto intro*: *funpow-swap1* [*THEN trans*])

lemmas *rotate-inv-rel* = *le-add-diff-inverse2* [*symmetric*, *THEN rotate-inv-plus*]

lemmas *rotate-inv-eq* = *order-refl* [*THEN rotate-inv-rel*, *simplified*]

lemmas *rotate-lr* [*simp*] = *rotate-inv-eq* [*THEN conjunct1*, *standard*]
lemmas *rotate-rl* [*simp*] =
rotate-inv-eq [*THEN conjunct2*, *THEN conjunct1*, *standard*]

lemma *rotate-gal*: (*rotater n xs* = *ys*) = (*rotate n ys* = *xs*)
by *auto*

lemma *rotate-gal'*: (*ys* = *rotater n xs*) = (*xs* = *rotate n ys*)
by *auto*

lemma *length-rotater* [*simp*]:
length (*rotater n xs*) = *length xs*
by (*simp add* : *rotater-rev*)

lemmas *rrs0* = *rotate-eqs* [*THEN restrict-to-left*,
simplified rotate-gal [*symmetric*] *rotate-gal'* [*symmetric*], *standard*]
lemmas *rrs1* = *rrs0* [*THEN refl* [*THEN rev-iffD1*]]
lemmas *rotater-eqs* = *rrs1* [*simplified length-rotater*, *standard*]
lemmas *rotater-0* = *rotater-eqs* (*1*)
lemmas *rotater-add* = *rotater-eqs* (*2*)

13.3.2 map, map2, commuting with rotate(r)

lemma *last-map*: *xs* ~= [] ==> *last* (*map f xs*) = *f* (*last xs*)
by (*induct xs*) *auto*

lemma *butlast-map*:
xs ~= [] ==> *butlast* (*map f xs*) = *map f* (*butlast xs*)
by (*induct xs*) *auto*

lemma *rotater1-map*: *rotater1* (*map f xs*) = *map f* (*rotater1 xs*)
unfolding *rotater1-def*
by (*cases xs*) (*auto simp add*: *last-map butlast-map*)

lemma *rotater-map*:
rotater n (*map f xs*) = *map f* (*rotater n xs*)
unfolding *rotater-def*
by (*induct n*) (*auto simp add* : *rotater1-map*)

lemma *but-last-zip* [*rule-format*] :
ALL ys. length xs = *length ys* --> *xs* ~= [] -->
last (*zip xs ys*) = (*last xs*, *last ys*) &
butlast (*zip xs ys*) = *zip* (*butlast xs*) (*butlast ys*)

```
  apply (induct xs)
  apply auto
    apply ((case-tac ys, auto simp: neq-Nil-conv)[1])+
  done

lemma but-last-map2 [rule-format] :
  ALL ys. length xs = length ys --> xs ~= [] -->
    last (map2 f xs ys) = f (last xs) (last ys) &
    butlast (map2 f xs ys) = map2 f (butlast xs) (butlast ys)
  apply (induct xs)
  apply auto
    apply (unfold map2-def)
    apply ((case-tac ys, auto simp: neq-Nil-conv)[1])+
  done

lemma rotater1-zip:
  length xs = length ys ==>
    rotater1 (zip xs ys) = zip (rotater1 xs) (rotater1 ys)
  apply (unfold rotater1-def)
  apply (cases xs)
   apply auto
   apply ((case-tac ys, auto simp: neq-Nil-conv but-last-zip)[1])+
  done

lemma rotater1-map2:
  length xs = length ys ==>
    rotater1 (map2 f xs ys) = map2 f (rotater1 xs) (rotater1 ys)
  unfolding map2-def by (simp add: rotater1-map rotater1-zip)

lemmas lrth =
  box-equals [OF asm-rl length-rotater [symmetric]
                 length-rotater [symmetric],
              THEN rotater1-map2]

lemma rotater-map2:
  length xs = length ys ==>
    rotater n (map2 f xs ys) = map2 f (rotater n xs) (rotater n ys)
  by (induct n) (auto intro!: lrth)

lemma rotate1-map2:
  length xs = length ys ==>
    rotate1 (map2 f xs ys) = map2 f (rotate1 xs) (rotate1 ys)
  apply (unfold map2-def)
  apply (cases xs)
   apply (cases ys, auto simp add : rotate1-def)+
  done

lemmas lth = box-equals [OF asm-rl length-rotate [symmetric]
  length-rotate [symmetric], THEN rotate1-map2]
```

lemma *rotate-map2*:
length xs = length ys ==>
rotate n (map2 f xs ys) = map2 f (rotate n xs) (rotate n ys)
by (*induct n*) (*auto intro*!: *lth*)

— corresponding equalities for word rotation

lemma *to-bl-rotl*:
to-bl (word-rotl n w) = rotate n (to-bl w)
by (*simp add*: *word-bl.Abs-inverse' word-rotl-def*)

lemmas *blrs0 = rotate-eqs [THEN to-bl-rotl [THEN trans]]*

lemmas *word-rotl-eqs =*
blrs0 [simplified word-bl.Rep' word-bl.Rep-inject to-bl-rotl [symmetric]]

lemma *to-bl-rotr*:
to-bl (word-rotr n w) = rotater n (to-bl w)
by (*simp add*: *word-bl.Abs-inverse' word-rotr-def*)

lemmas *brrs0 = rotater-eqs [THEN to-bl-rotr [THEN trans]]*

lemmas *word-rotr-eqs =*
brrs0 [simplified word-bl.Rep' word-bl.Rep-inject to-bl-rotr [symmetric]]

declare *word-rotr-eqs (1) [simp]*
declare *word-rotl-eqs (1) [simp]*

lemma
word-rot-rl [simp]:
word-rotl k (word-rotr k v) = v **and**
word-rot-lr [simp]:
word-rotr k (word-rotl k v) = v
by (*auto simp add*: *to-bl-rotr to-bl-rotl word-bl.Rep-inject [symmetric]*)

lemma
word-rot-gal:
(word-rotr n v = w) = (word-rotl n w = v) **and**
word-rot-gal':
(w = word-rotr n v) = (v = word-rotl n w)
by (*auto simp*: *to-bl-rotr to-bl-rotl word-bl.Rep-inject [symmetric]*
dest: *sym*)

lemma *word-rotr-rev*:
word-rotr n w = word-reverse (word-rotl n (word-reverse w))
by (*simp add*: *word-bl.Rep-inject [symmetric] to-bl-word-rev*
to-bl-rotr to-bl-rotl rotater-rev)

```
lemma word-roti-0 [simp]: word-roti 0 w = w
  by (unfold word-rot-defs) auto

lemmas abl-cong = arg-cong [where f = of-bl]

lemma word-roti-add:
  word-roti (m + n) w = word-roti m (word-roti n w)
proof −
  have rotater-eq-lem:
    ⋀m n xs. m = n ==> rotater m xs = rotater n xs
    by auto

  have rotate-eq-lem:
    ⋀m n xs. m = n ==> rotate m xs = rotate n xs
    by auto

  note rpts [symmetric, standard] =
    rotate-inv-plus [THEN conjunct1]
    rotate-inv-plus [THEN conjunct2, THEN conjunct1]
    rotate-inv-plus [THEN conjunct2, THEN conjunct2, THEN conjunct1]
    rotate-inv-plus [THEN conjunct2, THEN conjunct2, THEN conjunct2]

  note rrp = trans [symmetric, OF rotate-rotate rotate-eq-lem]
  note rrrp = trans [symmetric, OF rotater-add [symmetric] rotater-eq-lem]

  show ?thesis
  apply (unfold word-rot-defs)
  apply (simp only: split: split-if)
  apply (safe intro!: abl-cong)
  apply (simp-all only: to-bl-rotl [THEN word-bl.Rep-inverse']
                        to-bl-rotl
                        to-bl-rotr [THEN word-bl.Rep-inverse']
                        to-bl-rotr)
  apply (rule rrp rrrp rpts,
         simp add: nat-add-distrib [symmetric]
                   nat-diff-distrib [symmetric])+
  done
qed

lemma word-roti-conv-mod': word-roti n w = word-roti (n mod int (size w)) w
  apply (unfold word-rot-defs)
  apply (cut-tac y=size w in gt-or-eq-0)
  apply (erule disjE)
   apply simp-all
  apply (safe intro!: abl-cong)
   apply (rule rotater-eqs)
   apply (simp add: word-size nat-mod-distrib)
  apply (simp add: rotater-add [symmetric] rotate-gal [symmetric])
```

apply (*rule rotater-eqs*)
apply (*simp add*: *word-size nat-mod-distrib*)
apply (*rule int-eq-0-conv* [*THEN iffD1*])
apply (*simp only*: *zmod-int zadd-int* [*symmetric*])
apply (*simp add*: *rdmods*)
done

lemmas *word-roti-conv-mod* = *word-roti-conv-mod'* [*unfolded word-size*]

13.3.3 Word rotation commutes with bit-wise operations

locale *word-rotate*

context *word-rotate*
begin

lemmas *word-rot-defs'* = *to-bl-rotl to-bl-rotr*

lemmas *blwl-syms* [*symmetric*] = *bl-word-not bl-word-and bl-word-or bl-word-xor*

lemmas *lbl-lbl* = *trans* [*OF word-bl.Rep' word-bl.Rep'* [*symmetric*]]

lemmas *ths-map2* [*OF lbl-lbl*] = *rotate-map2 rotater-map2*

lemmas *ths-map* [**where** *xs* = *to-bl v*, *standard*] = *rotate-map rotater-map*

lemmas *th1s* [*simplified word-rot-defs'* [*symmetric*]] = *ths-map2 ths-map*

lemma *word-rot-logs*:
word-rotl n (*NOT v*) = *NOT word-rotl n v*
word-rotr n (*NOT v*) = *NOT word-rotr n v*
word-rotl n (*x AND y*) = *word-rotl n x AND word-rotl n y*
word-rotr n (*x AND y*) = *word-rotr n x AND word-rotr n y*
word-rotl n (*x OR y*) = *word-rotl n x OR word-rotl n y*
word-rotr n (*x OR y*) = *word-rotr n x OR word-rotr n y*
word-rotl n (*x XOR y*) = *word-rotl n x XOR word-rotl n y*
word-rotr n (*x XOR y*) = *word-rotr n x XOR word-rotr n y*
by (*rule word-bl.Rep-eqD*,
rule word-rot-defs' [*THEN trans*],
simp only: *blwl-syms* [*symmetric*],
rule th1s [*THEN trans*],
rule refl)+
end

lemmas *word-rot-logs* = *word-rotate.word-rot-logs*

lemmas *bl-word-rotl-dt* = *trans* [*OF to-bl-rotl rotate-drop-take*,
simplified word-bl.Rep', *standard*]

lemmas *bl-word-rotr-dt = trans* [*OF to-bl-rotr rotater-drop-take*,
simplified word-bl.Rep', *standard*]

lemma *bl-word-roti-dt'*:
n = nat ((− i) mod int (size (w :: 'a :: len word))) ==>
to-bl (word-roti i w) = drop n (to-bl w) @ take n (to-bl w)
apply (*unfold word-roti-def*)
apply (*simp add*: *bl-word-rotl-dt bl-word-rotr-dt word-size*)
apply *safe*
apply (*simp add*: *zmod-zminus1-eq-if*)
apply *safe*
apply (*simp add*: *nat-mult-distrib*)
apply (*simp add*: *nat-diff-distrib* [*OF pos-mod-sign pos-mod-conj*
[*THEN conjunct2*, *THEN order-less-imp-le*]]
nat-mod-distrib)
apply (*simp add*: *nat-mod-distrib*)
done

lemmas *bl-word-roti-dt = bl-word-roti-dt'* [*unfolded word-size*]

lemmas *word-rotl-dt = bl-word-rotl-dt*
[*THEN word-bl.Rep-inverse'* [*symmetric*], *standard*]
lemmas *word-rotr-dt = bl-word-rotr-dt*
[*THEN word-bl.Rep-inverse'* [*symmetric*], *standard*]
lemmas *word-roti-dt = bl-word-roti-dt*
[*THEN word-bl.Rep-inverse'* [*symmetric*], *standard*]

lemma *word-rotx-0* [*simp*] : *word-rotr i 0 = 0 & word-rotl i 0 = 0*
by (*simp add* : *word-rotr-dt word-rotl-dt to-bl-0 replicate-add* [*symmetric*])

lemma *word-roti-0'* [*simp*] : *word-roti n 0 = 0*
unfolding *word-roti-def* **by** *auto*

lemmas *word-rotr-dt-no-bin'* [*simp*] =
word-rotr-dt [**where** *w=number-of w*, *unfolded to-bl-no-bin*, *standard*]

lemmas *word-rotl-dt-no-bin'* [*simp*] =
word-rotl-dt [**where** *w=number-of w*, *unfolded to-bl-no-bin*, *standard*]

declare *word-roti-def* [*simp*]

end

14 Boolean-Algebra: Boolean Algebras

theory *Boolean-Algebra*
imports *Main*

begin

locale *boolean* =
  fixes *conj* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $\sqcap$ *70*)
  fixes *disj* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $\sqcup$ *65*)
  fixes *compl* :: $'a \Rightarrow 'a$ ($\sim$ - [*81*] *80*)
  fixes *zero* :: $'a$ (**0**)
  fixes *one* :: $'a$ (**1**)
  assumes *conj-assoc*: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
  assumes *disj-assoc*: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
  assumes *conj-commute*: $x \sqcap y = y \sqcap x$
  assumes *disj-commute*: $x \sqcup y = y \sqcup x$
  assumes *conj-disj-distrib*: $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
  assumes *disj-conj-distrib*: $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
  assumes *conj-one-right* [*simp*]: $x \sqcap \mathbf{1} = x$
  assumes *disj-zero-right* [*simp*]: $x \sqcup \mathbf{0} = x$
  assumes *conj-cancel-right* [*simp*]: $x \sqcap \sim x = \mathbf{0}$
  assumes *disj-cancel-right* [*simp*]: $x \sqcup \sim x = \mathbf{1}$

sublocale *boolean* $<$ *conj*!: *abel-semigroup conj* **proof**
qed (*fact conj-assoc conj-commute*)+

sublocale *boolean* $<$ *disj*!: *abel-semigroup disj* **proof**
qed (*fact disj-assoc disj-commute*)+

context *boolean*
begin

lemmas *conj-left-commute* = *conj.left-commute*

lemmas *disj-left-commute* = *disj.left-commute*

lemmas *conj-ac* = *conj.assoc conj.commute conj.left-commute*
lemmas *disj-ac* = *disj.assoc disj.commute disj.left-commute*

lemma *dual*: *boolean disj conj compl one zero*
apply (*rule boolean.intro*)
apply (*rule disj-assoc*)
apply (*rule conj-assoc*)
apply (*rule disj-commute*)
apply (*rule conj-commute*)
apply (*rule disj-conj-distrib*)
apply (*rule conj-disj-distrib*)
apply (*rule disj-zero-right*)
apply (*rule conj-one-right*)
apply (*rule disj-cancel-right*)
apply (*rule conj-cancel-right*)
done

14.1 Complement

lemma *complement-unique*:
assumes *1*: $a \sqcap x = \mathbf{0}$
assumes *2*: $a \sqcup x = \mathbf{1}$
assumes *3*: $a \sqcap y = \mathbf{0}$
assumes *4*: $a \sqcup y = \mathbf{1}$
shows $x = y$
proof −
have $(a \sqcap x) \sqcup (x \sqcap y) = (a \sqcap y) \sqcup (x \sqcap y)$ **using** *1 3* **by** *simp*
hence $(x \sqcap a) \sqcup (x \sqcap y) = (y \sqcap a) \sqcup (y \sqcap x)$ **using** *conj-commute* **by** *simp*
hence $x \sqcap (a \sqcup y) = y \sqcap (a \sqcup x)$ **using** *conj-disj-distrib* **by** *simp*
hence $x \sqcap \mathbf{1} = y \sqcap \mathbf{1}$ **using** *2 4* **by** *simp*
thus $x = y$ **using** *conj-one-right* **by** *simp*
qed

lemma *compl-unique*: $[\![x \sqcap y = \mathbf{0};\ x \sqcup y = \mathbf{1}]\!] \Longrightarrow \sim x = y$
by (*rule complement-unique* [*OF conj-cancel-right disj-cancel-right*])

lemma *double-compl* [*simp*]: $\sim (\sim x) = x$
proof (*rule compl-unique*)
from *conj-cancel-right* **show** $\sim x \sqcap x = \mathbf{0}$ **by** (*simp only*: *conj-commute*)
from *disj-cancel-right* **show** $\sim x \sqcup x = \mathbf{1}$ **by** (*simp only*: *disj-commute*)
qed

lemma *compl-eq-compl-iff* [*simp*]: $(\sim x = \sim y) = (x = y)$
by (*rule inj-eq* [*OF inj-on-inverseI*], *rule double-compl*)

14.2 Conjunction

lemma *conj-absorb* [*simp*]: $x \sqcap x = x$
proof −
have $x \sqcap x = (x \sqcap x) \sqcup \mathbf{0}$ **using** *disj-zero-right* **by** *simp*
also have ... $= (x \sqcap x) \sqcup (x \sqcap \sim x)$ **using** *conj-cancel-right* **by** *simp*
also have ... $= x \sqcap (x \sqcup \sim x)$ **using** *conj-disj-distrib* **by** (*simp only*:)
also have ... $= x \sqcap \mathbf{1}$ **using** *disj-cancel-right* **by** *simp*
also have ... $= x$ **using** *conj-one-right* **by** *simp*
finally show *?thesis* **.**
qed

lemma *conj-zero-right* [*simp*]: $x \sqcap \mathbf{0} = \mathbf{0}$
proof −
have $x \sqcap \mathbf{0} = x \sqcap (x \sqcap \sim x)$ **using** *conj-cancel-right* **by** *simp*
also have ... $= (x \sqcap x) \sqcap \sim x$ **using** *conj-assoc* **by** (*simp only*:)
also have ... $= x \sqcap \sim x$ **using** *conj-absorb* **by** *simp*
also have ... $= \mathbf{0}$ **using** *conj-cancel-right* **by** *simp*
finally show *?thesis* **.**
qed

lemma *compl-one* [*simp*]: $\sim \mathbf{1} = \mathbf{0}$

by (*rule compl-unique* [*OF conj-zero-right disj-zero-right*])

lemma *conj-zero-left* [*simp*]: $\mathbf{0} \sqcap x = \mathbf{0}$
by (*subst conj-commute*) (*rule conj-zero-right*)

lemma *conj-one-left* [*simp*]: $\mathbf{1} \sqcap x = x$
by (*subst conj-commute*) (*rule conj-one-right*)

lemma *conj-cancel-left* [*simp*]: $\sim x \sqcap x = \mathbf{0}$
by (*subst conj-commute*) (*rule conj-cancel-right*)

lemma *conj-left-absorb* [*simp*]: $x \sqcap (x \sqcap y) = x \sqcap y$
by (*simp only*: *conj-assoc* [*symmetric*] *conj-absorb*)

lemma *conj-disj-distrib2*:
$(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
by (*simp only*: *conj-commute conj-disj-distrib*)

lemmas *conj-disj-distribs* =
conj-disj-distrib conj-disj-distrib2

14.3 Disjunction

lemma *disj-absorb* [*simp*]: $x \sqcup x = x$
by (*rule boolean.conj-absorb* [*OF dual*])

lemma *disj-one-right* [*simp*]: $x \sqcup \mathbf{1} = \mathbf{1}$
by (*rule boolean.conj-zero-right* [*OF dual*])

lemma *compl-zero* [*simp*]: $\sim \mathbf{0} = \mathbf{1}$
by (*rule boolean.compl-one* [*OF dual*])

lemma *disj-zero-left* [*simp*]: $\mathbf{0} \sqcup x = x$
by (*rule boolean.conj-one-left* [*OF dual*])

lemma *disj-one-left* [*simp*]: $\mathbf{1} \sqcup x = \mathbf{1}$
by (*rule boolean.conj-zero-left* [*OF dual*])

lemma *disj-cancel-left* [*simp*]: $\sim x \sqcup x = \mathbf{1}$
by (*rule boolean.conj-cancel-left* [*OF dual*])

lemma *disj-left-absorb* [*simp*]: $x \sqcup (x \sqcup y) = x \sqcup y$
by (*rule boolean.conj-left-absorb* [*OF dual*])

lemma *disj-conj-distrib2*:
$(y \sqcap z) \sqcup x = (y \sqcup x) \sqcap (z \sqcup x)$
by (*rule boolean.conj-disj-distrib2* [*OF dual*])

lemmas *disj-conj-distribs* =

disj-conj-distrib disj-conj-distrib2

14.4 De Morgan's Laws

lemma *de-Morgan-conj* [*simp*]: $\sim (x \sqcap y) = \sim x \sqcup \sim y$
proof (*rule compl-unique*)
  have $(x \sqcap y) \sqcap (\sim x \sqcup \sim y) = ((x \sqcap y) \sqcap \sim x) \sqcup ((x \sqcap y) \sqcap \sim y)$
    by (*rule conj-disj-distrib*)
  also have ... $= (y \sqcap (x \sqcap \sim x)) \sqcup (x \sqcap (y \sqcap \sim y))$
    by (*simp only*: *conj-ac*)
  finally show $(x \sqcap y) \sqcap (\sim x \sqcup \sim y) = \mathbf{0}$
    by (*simp only*: *conj-cancel-right conj-zero-right disj-zero-right*)
next
  have $(x \sqcap y) \sqcup (\sim x \sqcup \sim y) = (x \sqcup (\sim x \sqcup \sim y)) \sqcap (y \sqcup (\sim x \sqcup \sim y))$
    by (*rule disj-conj-distrib2*)
  also have ... $= (\sim y \sqcup (x \sqcup \sim x)) \sqcap (\sim x \sqcup (y \sqcup \sim y))$
    by (*simp only*: *disj-ac*)
  finally show $(x \sqcap y) \sqcup (\sim x \sqcup \sim y) = \mathbf{1}$
    by (*simp only*: *disj-cancel-right disj-one-right conj-one-right*)
qed

lemma *de-Morgan-disj* [*simp*]: $\sim (x \sqcup y) = \sim x \sqcap \sim y$
by (*rule boolean.de-Morgan-conj* [*OF dual*])

end

14.5 Symmetric Difference

locale *boolean-xor* = *boolean* +
  fixes *xor* :: $'a \Rightarrow 'a \Rightarrow 'a$ (**infixr** $\oplus$ *65*)
  assumes *xor-def*: $x \oplus y = (x \sqcap \sim y) \sqcup (\sim x \sqcap y)$

sublocale *boolean-xor* < *xor*!: *abel-semigroup xor* **proof**
  fix $x\ y\ z$ :: $'a$
  let *?t* $= (x \sqcap y \sqcap z) \sqcup (x \sqcap \sim y \sqcap \sim z) \sqcup$
        $(\sim x \sqcap y \sqcap \sim z) \sqcup (\sim x \sqcap \sim y \sqcap z)$
  have *?t* $\sqcup (z \sqcap x \sqcap \sim x) \sqcup (z \sqcap y \sqcap \sim y) =$
      ?t $\sqcup (x \sqcap y \sqcap \sim y) \sqcup (x \sqcap z \sqcap \sim z)$
    by (*simp only*: *conj-cancel-right conj-zero-right*)
  thus $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
    apply (*simp only*: *xor-def de-Morgan-disj de-Morgan-conj double-compl*)
    apply (*simp only*: *conj-disj-distribs conj-ac disj-ac*)
    done
  show $x \oplus y = y \oplus x$
    by (*simp only*: *xor-def conj-commute disj-commute*)
qed

context *boolean-xor*
begin

lemmas *xor-assoc* = *xor.assoc*
lemmas *xor-commute* = *xor.commute*
lemmas *xor-left-commute* = *xor.left-commute*

lemmas *xor-ac* = *xor.assoc xor.commute xor.left-commute*

lemma *xor-def2*:
$x \oplus y = (x \sqcup y) \sqcap (\sim x \sqcup \sim y)$
by (*simp only*: *xor-def conj-disj-distribs*
disj-ac conj-ac conj-cancel-right disj-zero-left)

lemma *xor-zero-right* [*simp*]: $x \oplus \mathbf{0} = x$
by (*simp only*: *xor-def compl-zero conj-one-right conj-zero-right disj-zero-right*)

lemma *xor-zero-left* [*simp*]: $\mathbf{0} \oplus x = x$
by (*subst xor-commute*) (*rule xor-zero-right*)

lemma *xor-one-right* [*simp*]: $x \oplus \mathbf{1} = \sim x$
by (*simp only*: *xor-def compl-one conj-zero-right conj-one-right disj-zero-left*)

lemma *xor-one-left* [*simp*]: $\mathbf{1} \oplus x = \sim x$
by (*subst xor-commute*) (*rule xor-one-right*)

lemma *xor-self* [*simp*]: $x \oplus x = \mathbf{0}$
by (*simp only*: *xor-def conj-cancel-right conj-cancel-left disj-zero-right*)

lemma *xor-left-self* [*simp*]: $x \oplus (x \oplus y) = y$
by (*simp only*: *xor-assoc* [*symmetric*] *xor-self xor-zero-left*)

lemma *xor-compl-left* [*simp*]: $\sim x \oplus y = \sim (x \oplus y)$
apply (*simp only*: *xor-def de-Morgan-disj de-Morgan-conj double-compl*)
apply (*simp only*: *conj-disj-distribs*)
apply (*simp only*: *conj-cancel-right conj-cancel-left*)
apply (*simp only*: *disj-zero-left disj-zero-right*)
apply (*simp only*: *disj-ac conj-ac*)
done

lemma *xor-compl-right* [*simp*]: $x \oplus \sim y = \sim (x \oplus y)$
apply (*simp only*: *xor-def de-Morgan-disj de-Morgan-conj double-compl*)
apply (*simp only*: *conj-disj-distribs*)
apply (*simp only*: *conj-cancel-right conj-cancel-left*)
apply (*simp only*: *disj-zero-left disj-zero-right*)
apply (*simp only*: *disj-ac conj-ac*)
done

lemma *xor-cancel-right*: $x \oplus \sim x = \mathbf{1}$
by (*simp only*: *xor-compl-right xor-self compl-zero*)

lemma *xor-cancel-left*: $\sim x \oplus x = \mathbf{1}$

by (*simp only*: *xor-compl-left xor-self compl-zero*)

lemma *conj-xor-distrib*: $x \sqcap (y \oplus z) = (x \sqcap y) \oplus (x \sqcap z)$
proof −
  have $(x \sqcap y \sqcap \sim z) \sqcup (x \sqcap \sim y \sqcap z) =$
      $(y \sqcap x \sqcap \sim x) \sqcup (z \sqcap x \sqcap \sim x) \sqcup (x \sqcap y \sqcap \sim z) \sqcup (x \sqcap \sim y \sqcap z)$
    by (*simp only*: *conj-cancel-right conj-zero-right disj-zero-left*)
  thus $x \sqcap (y \oplus z) = (x \sqcap y) \oplus (x \sqcap z)$
    by (*simp* (*no-asm-use*) *only*:
      xor-def de-Morgan-disj de-Morgan-conj double-compl
      conj-disj-distribs conj-ac disj-ac)
qed

lemma *conj-xor-distrib2*:
  $(y \oplus z) \sqcap x = (y \sqcap x) \oplus (z \sqcap x)$
proof −
  have $x \sqcap (y \oplus z) = (x \sqcap y) \oplus (x \sqcap z)$
    by (*rule conj-xor-distrib*)
  thus $(y \oplus z) \sqcap x = (y \sqcap x) \oplus (z \sqcap x)$
    by (*simp only*: *conj-commute*)
qed

lemmas *conj-xor-distribs* =
   conj-xor-distrib conj-xor-distrib2

end

end

15 WordGenLib: Miscellaneous Library for Words

theory *WordGenLib*
imports *WordShift Boolean-Algebra*
begin

declare *of-nat-2p* [*simp*]

lemma *word-int-cases*:
  $[\![\bigwedge n.\ [\![(x :: 'a{::}len0\ word) = word\text{-}of\text{-}int\ n;\ 0 \le n;\ n < 2\hat{}len\text{-}of\ TYPE('a)]\!] \Longrightarrow P]\!]$
   $\Longrightarrow P$
  by (*cases x rule*: *word-uint.Abs-cases*) (*simp add*: *uints-num*)

lemma *word-nat-cases* [*cases type*: *word*]:
  $[\![\bigwedge n.\ [\![(x :: 'a{::}len\ word) = of\text{-}nat\ n;\ n < 2\hat{}len\text{-}of\ TYPE('a)]\!] \Longrightarrow P]\!]$
   $\Longrightarrow P$
  by (*cases x rule*: *word-unat.Abs-cases*) (*simp add*: *unats-def*)

lemma *max-word-eq*:
(*max-word*::*'a*::*len word*) = *2*^*len-of TYPE*(*'a*) − *1*
by (*simp add*: *max-word-def word-of-int-hom-syms word-of-int-2p*)

lemma *max-word-max* [*simp*,*intro*!]:
n ≤ *max-word*
by (*cases n rule*: *word-int-cases*)
(*simp add*: *max-word-def word-le-def int-word-uint int-mod-eq'*)

lemma *word-of-int-2p-len*:
word-of-int (*2* ^ *len-of TYPE*(*'a*)) = (*0*::*'a*::*len0 word*)
by (*subst word-uint.Abs-norm* [*symmetric*])
(*simp add*: *word-of-int-hom-syms*)

lemma *word-pow-0*:
(*2*::*'a*::*len word*) ^ *len-of TYPE*(*'a*) = *0*
proof −
have *word-of-int* (*2* ^ *len-of TYPE*(*'a*)) = (*0*::*'a word*)
by (*rule word-of-int-2p-len*)
thus *?thesis* **by** (*simp add*: *word-of-int-2p*)
qed

lemma *max-word-wrap*: *x* + *1* = *0* ⟹ *x* = *max-word*
apply (*simp add*: *max-word-eq*)
apply *uint-arith*
apply *auto*
apply (*simp add*: *word-pow-0*)
done

lemma *max-word-minus*:
max-word = (−*1*::*'a*::*len word*)
proof −
have −*1* + *1* = (*0*::*'a word*) **by** *simp*
thus *?thesis* **by** (*rule max-word-wrap* [*symmetric*])
qed

lemma *max-word-bl* [*simp*]:
to-bl (*max-word*::*'a*::*len word*) = *replicate* (*len-of TYPE*(*'a*)) *True*
by (*subst max-word-minus to-bl-n1*)+ *simp*

lemma *max-test-bit* [*simp*]:
(*max-word*::*'a*::*len word*) !! *n* = (*n* < *len-of TYPE*(*'a*))
by (*auto simp add*: *test-bit-bl word-size*)

lemma *word-and-max* [*simp*]:
x AND max-word = *x*
by (*rule word-eqI*) (*simp add*: *word-ops-nth-size word-size*)

lemma *word-or-max* [*simp*]:

x OR max-word = max-word
by (*rule word-eqI*) (*simp add*: *word-ops-nth-size word-size*)

lemma *word-ao-dist2*:
x AND (y OR z) = x AND y OR x AND (z::'a::len0 word)
by (*rule word-eqI*) (*auto simp add*: *word-ops-nth-size word-size*)

lemma *word-oa-dist2*:
x OR y AND z = (x OR y) AND (x OR (z::'a::len0 word))
by (*rule word-eqI*) (*auto simp add*: *word-ops-nth-size word-size*)

lemma *word-and-not* [*simp*]:
x AND NOT x = (0::'a::len0 word)
by (*rule word-eqI*) (*auto simp add*: *word-ops-nth-size word-size*)

lemma *word-or-not* [*simp*]:
x OR NOT x = max-word
by (*rule word-eqI*) (*auto simp add*: *word-ops-nth-size word-size*)

lemma *word-boolean*:
boolean (op AND) (op OR) bitNOT 0 max-word
apply (*rule boolean.intro*)
apply (*rule word-bw-assocs*)
apply (*rule word-bw-assocs*)
apply (*rule word-bw-comms*)
apply (*rule word-bw-comms*)
apply (*rule word-ao-dist2*)
apply (*rule word-oa-dist2*)
apply (*rule word-and-max*)
apply (*rule word-log-esimps*)
apply (*rule word-and-not*)
apply (*rule word-or-not*)
done

interpretation *word-bool-alg*:
boolean op AND op OR bitNOT 0 max-word
by (*rule word-boolean*)

lemma *word-xor-and-or*:
x XOR y = x AND NOT y OR NOT x AND (y::'a::len0 word)
by (*rule word-eqI*) (*auto simp add*: *word-ops-nth-size word-size*)

interpretation *word-bool-alg*:
boolean-xor op AND op OR bitNOT 0 max-word op XOR
apply (*rule boolean-xor.intro*)
apply (*rule word-boolean*)
apply (*rule boolean-xor-axioms.intro*)
apply (*rule word-xor-and-or*)
done

lemma *shiftr-0* [*iff*]:
(*x*::'*a*::*len0 word*) >> *0* = *x*
by (*simp add*: *shiftr-bl*)

lemma *shiftl-0* [*simp*]:
(*x* :: '*a* :: *len word*) << *0* = *x*
by (*simp add*: *shiftl-t2n*)

lemma *shiftl-1* [*simp*]:
(*1*::'*a*::*len word*) << *n* = *2*^*n*
by (*simp add*: *shiftl-t2n*)

lemma *uint-lt-0* [*simp*]:
uint x < *0* = *False*
by (*simp add*: *linorder-not-less*)

lemma *shiftr1-1* [*simp*]:
shiftr1 (*1*::'*a*::*len word*) = *0*
by (*simp add*: *shiftr1-def word-0-alt*)

lemma *shiftr-1*[*simp*]:
(*1*::'*a*::*len word*) >> *n* = (*if n* = *0 then 1 else 0*)
by (*induct n*) (*auto simp*: *shiftr-def*)

lemma *word-less-1* [*simp*]:
((*x*::'*a*::*len word*) < *1*) = (*x* = *0*)
by (*simp add*: *word-less-nat-alt unat-0-iff*)

lemma *to-bl-mask*:
to-bl (*mask n* :: '*a*::*len word*) =
replicate (*len-of TYPE*('*a*) − *n*) *False* @
replicate (*min* (*len-of TYPE*('*a*)) *n*) *True*
by (*simp add*: *mask-bl word-rep-drop min-def*)

lemma *map-replicate-True*:
n = *length xs* ==>
map (λ(*x*,*y*). *x* & *y*) (*zip xs* (*replicate n True*)) = *xs*
by (*induct xs arbitrary*: *n*) *auto*

lemma *map-replicate-False*:
n = *length xs* ==> *map* (λ(*x*,*y*). *x* & *y*)
(*zip xs* (*replicate n False*)) = *replicate n False*
by (*induct xs arbitrary*: *n*) *auto*

lemma *bl-and-mask*:
fixes *w* :: '*a*::*len word*
fixes *n*
defines *n'* ≡ *len-of TYPE*('*a*) − *n*

```
  shows to-bl (w AND mask n) =  replicate n' False @ drop n' (to-bl w)
proof −
  note [simp] = map-replicate-True map-replicate-False
  have to-bl (w AND mask n) =
        map2 op & (to-bl w) (to-bl (mask n::'a::len word))
    by (simp add: bl-word-and)
  also
  have to-bl w = take n' (to-bl w) @ drop n' (to-bl w) by simp
  also
  have map2 op & ... (to-bl (mask n::'a::len word)) =
        replicate n' False @ drop n' (to-bl w)
    unfolding to-bl-mask n'-def map2-def
    by (subst zip-append) auto
  finally
  show ?thesis .
qed

lemma drop-rev-takefill:
  length xs ≤ n ==>
    drop (n − length xs) (rev (takefill False n (rev xs))) = xs
  by (simp add: takefill-alt rev-take)

lemma map-nth-0 [simp]:
  map (op !! (0::'a::len0 word)) xs = replicate (length xs) False
  by (induct xs) auto

lemma uint-plus-if-size:
  uint (x + y) =
  (if uint x + uint y < 2^size x then
      uint x + uint y
   else
      uint x + uint y − 2^size x)
  by (simp add: word-arith-alts int-word-uint mod-add-if-z
                word-size)

lemma unat-plus-if-size:
  unat (x + (y::'a::len word)) =
  (if unat x + unat y < 2^size x then
      unat x + unat y
   else
      unat x + unat y − 2^size x)
  apply (subst word-arith-nat-defs)
  apply (subst unat-of-nat)
  apply (simp add:  mod-nat-add word-size)
  done

lemma word-neq-0-conv [simp]:
  fixes w :: 'a :: len word
  shows (w ≠ 0) = (0 < w)
```

proof −
have *0* ≤ *w* **by** (*rule word-zero-le*)
thus *?thesis* **by** (*auto simp add*: *word-less-def*)
qed

lemma *max-lt*:
unat (*max a b div c*) = *unat* (*max a b*) *div unat* (*c*:: *'a* :: *len word*)
apply (*subst word-arith-nat-defs*)
apply (*subst word-unat.eq-norm*)
apply (*subst mod-if*)
apply *clarsimp*
apply (*erule notE*)
apply (*insert div-le-dividend* [*of unat* (*max a b*) *unat c*])
apply (*erule order-le-less-trans*)
apply (*insert unat-lt2p* [*of max a b*])
apply *simp*
done

lemma *uint-sub-if-size*:
uint (*x* − *y*) =
(*if uint y* ≤ *uint x then*
uint x − *uint y*
else
uint x − *uint y* + *2^size x*)
by (*simp add*: *word-arith-alts int-word-uint mod-sub-if-z*
word-size)

lemma *unat-sub-simple*:
x ≤ *y* ==> *unat* (*y* − *x*) = *unat y* − *unat x*
by (*simp add*: *unat-def uint-sub-if-size word-le-def nat-diff-distrib*)

lemmas *unat-sub* = *unat-sub-simple*

lemma *word-less-sub1*:
fixes *x* :: *'a* :: *len word*
shows *x* ≠ *0* ==> *1* < *x* = (*0* < *x* − *1*)
by (*simp add*: *unat-sub-if-size word-less-nat-alt*)

lemma *word-le-sub1*:
fixes *x* :: *'a* :: *len word*
shows *x* ≠ *0* ==> *1* ≤ *x* = (*0* ≤ *x* − *1*)
by (*simp add*: *unat-sub-if-size order-le-less word-less-nat-alt*)

lemmas *word-less-sub1-numberof* [*simp*] =
word-less-sub1 [*of number-of w*, *standard*]
lemmas *word-le-sub1-numberof* [*simp*] =
word-le-sub1 [*of number-of w*, *standard*]

lemma *word-of-int-minus*:

word-of-int (*2^len-of TYPE*('*a*) − *i*) = (*word-of-int* (−*i*)::'*a*::*len word*)
proof −
have *x*: *2^len-of TYPE*('*a*) − *i* = −*i* + *2^len-of TYPE*('*a*) **by** *simp*
show *?thesis*
apply (*subst x*)
apply (*subst word-uint.Abs-norm* [*symmetric*], *subst mod-add-self2*)
apply *simp*
done
qed

lemmas *word-of-int-inj* =
word-uint.Abs-inject [*unfolded uints-num*, *simplified*]

lemma *word-le-less-eq*:
(*x* ::'*z*::*len word*) ≤ *y* = (*x* = *y* ∨ *x* < *y*)
by (*auto simp add*: *word-less-def*)

lemma *mod-plus-cong*:
assumes *1*: (*b*::*int*) = *b'*
and *2*: *x mod b'* = *x' mod b'*
and *3*: *y mod b'* = *y' mod b'*
and *4*: *x'* + *y'* = *z'*
shows (*x* + *y*) *mod b* = *z' mod b'*
proof −
from *1 2*[*symmetric*] *3*[*symmetric*] **have** (*x* + *y*) *mod b* = (*x' mod b'* + *y' mod b'*) *mod b'*
by (*simp add*: *mod-add-eq*[*symmetric*])
also have ... = (*x'* + *y'*) *mod b'*
by (*simp add*: *mod-add-eq*[*symmetric*])
finally show *?thesis* **by** (*simp add*: *4*)
qed

lemma *mod-minus-cong*:
assumes *1*: (*b*::*int*) = *b'*
and *2*: *x mod b'* = *x' mod b'*
and *3*: *y mod b'* = *y' mod b'*
and *4*: *x'* − *y'* = *z'*
shows (*x* − *y*) *mod b* = *z' mod b'*
using *assms*
apply (*subst zmod-zsub-left-eq*)
apply (*subst zmod-zsub-right-eq*)
apply (*simp add*: *zmod-zsub-left-eq* [*symmetric*] *zmod-zsub-right-eq* [*symmetric*])
done

lemma *word-induct-less*:
⟦*P* (*0*::'*a*::*len word*); ⋀*n*. ⟦*n* < *m*; *P n*⟧ ⟹ *P* (*1* + *n*)⟧ ⟹ *P m*
apply (*cases m*)
apply *atomize*
apply (*erule rev-mp*)+

apply (*rule-tac x=m* **in** *spec*)
apply (*induct-tac n*)
apply *simp*
apply *clarsimp*
apply (*erule impE*)
apply *clarsimp*
apply (*erule-tac x=n* **in** *allE*)
apply (*erule impE*)
apply (*simp add*: *unat-arith-simps*)
apply (*clarsimp simp*: *unat-of-nat*)
apply *simp*
apply (*erule-tac x=of-nat na* **in** *allE*)
apply (*erule impE*)
apply (*simp add*: *unat-arith-simps*)
apply (*clarsimp simp*: *unat-of-nat*)
apply *simp*
done

lemma *word-induct*:
⟦*P* (*0*::*'a*::*len word*); ⋀*n*. *P n* ⟹ *P* (*1* + *n*)⟧ ⟹ *P m*
by (*erule word-induct-less*, *simp*)

lemma *word-induct2* [*induct type*]:
⟦*P 0*; ⋀*n*. ⟦*1* + *n* ≠ *0*; *P n*⟧ ⟹ *P* (*1* + *n*)⟧ ⟹ *P* (*n*::*'b*::*len word*)
apply (*rule word-induct*, *simp*)
apply (*case-tac 1+n = 0*, *auto*)
done

definition *word-rec* :: *'a* ⇒ (*'b*::*len word* ⇒ *'a* ⇒ *'a*) ⇒ *'b word* ⇒ *'a* **where**
word-rec forZero forSuc n ≡ *nat-rec forZero* (*forSuc* ∘ *of-nat*) (*unat n*)

lemma *word-rec-0*: *word-rec z s 0* = *z*
by (*simp add*: *word-rec-def*)

lemma *word-rec-Suc*:
1 + *n* ≠ (*0*::*'a*::*len word*) ⟹ *word-rec z s* (*1* + *n*) = *s n* (*word-rec z s n*)
apply (*simp add*: *word-rec-def unat-word-ariths*)
apply (*subst nat-mod-eq'*)
apply (*cut-tac x=n* **in** *unat-lt2p*)
apply (*drule Suc-mono*)
apply (*simp add*: *less-Suc-eq-le*)
apply (*simp only*: *order-less-le*, *simp*)
apply (*erule contrapos-pn*, *simp*)
apply (*drule arg-cong*[**where** *f=of-nat*])
apply *simp*
apply (*subst* (*asm*) *word-unat.Rep-inverse*[*of n*])
apply *simp*
apply *simp*
done

lemma *word-rec-Pred*:
n ≠ *0* ⟹ *word-rec z s n* = *s* (*n* − *1*) (*word-rec z s* (*n* − *1*))
apply (*rule subst*[**where** *t*=*n* **and** *s*=*1* + (*n* − *1*)])
apply *simp*
apply (*subst word-rec-Suc*)
apply *simp*
apply *simp*
done

lemma *word-rec-in*:
f (*word-rec z* (λ-. *f*) *n*) = *word-rec* (*f z*) (λ-. *f*) *n*
by (*induct n*) (*simp-all add*: *word-rec-0 word-rec-Suc*)

lemma *word-rec-in2*:
f n (*word-rec z f n*) = *word-rec* (*f 0 z*) (*f* ∘ *op* + *1*) *n*
by (*induct n*) (*simp-all add*: *word-rec-0 word-rec-Suc*)

lemma *word-rec-twice*:
m ≤ *n* ⟹ *word-rec z f n* = *word-rec* (*word-rec z f* (*n* − *m*)) (*f* ∘ *op* + (*n* − *m*)) *m*
apply (*erule rev-mp*)
apply (*rule-tac x*=*z* **in** *spec*)
apply (*rule-tac x*=*f* **in** *spec*)
apply (*induct n*)
apply (*simp add*: *word-rec-0*)
apply *clarsimp*
apply (*rule-tac t*=*1* + *n* − *m* **and** *s*=*1* + (*n* − *m*) **in** *subst*)
apply *simp*
apply (*case-tac 1* + (*n* − *m*) = *0*)
apply (*simp add*: *word-rec-0*)
apply (*rule-tac f* = *word-rec ?a ?b* **in** *arg-cong*)
apply (*rule-tac t*=*m* **and** *s*=*m* + (*1* + (*n* − *m*)) **in** *subst*)
apply *simp*
apply (*simp* (*no-asm-use*))
apply (*simp add*: *word-rec-Suc word-rec-in2*)
apply (*erule impE*)
apply *uint-arith*
apply (*drule-tac x*=*x* ∘ *op* + *1* **in** *spec*)
apply (*drule-tac x*=*x 0 xa* **in** *spec*)
apply *simp*
apply (*rule-tac t*=λ*a. x* (*1* + (*n* − *m* + *a*)) **and** *s*=λ*a. x* (*1* + (*n* − *m*) + *a*)
in *subst*)
apply (*clarsimp simp add*: *expand-fun-eq*)
apply (*rule-tac t*=(*1* + (*n* − *m* + *xb*)) **and** *s*=*1* + (*n* − *m*) + *xb* **in** *subst*)
apply *simp*
apply (*rule refl*)
apply (*rule refl*)
done

lemma *word-rec-id*: *word-rec z* (λ-. *id*) *n* = *z*
by (*induct n*) (*auto simp add*: *word-rec-0 word-rec-Suc*)

lemma *word-rec-id-eq*: ∀ *m* < *n*. *f m* = *id* ⟹ *word-rec z f n* = *z*
apply (*erule rev-mp*)
apply (*induct n*)
apply (*auto simp add*: *word-rec-0 word-rec-Suc*)
apply (*drule spec*, *erule mp*)
apply *uint-arith*
apply (*drule-tac x*=*n* **in** *spec*, *erule impE*)
apply *uint-arith*
apply *simp*
done

lemma *word-rec-max*:
∀ *m*≥*n*. *m* ≠ −*1* ⟶ *f m* = *id* ⟹ *word-rec z f* −*1* = *word-rec z f n*
apply (*subst word-rec-twice*[**where** *n*=−*1* **and** *m*=−*1* − *n*])
apply *simp*
apply *simp*
apply (*rule word-rec-id-eq*)
apply *clarsimp*
apply (*drule spec*, *rule mp*, *erule mp*)
apply (*rule word-plus-mono-right2*[*OF - order-less-imp-le*])
prefer *2*
apply *assumption*
apply *simp*
apply (*erule contrapos-pn*)
apply *simp*
apply (*drule arg-cong*[**where** *f*=λ*x*. *x* − *n*])
apply *simp*
done

lemma *unatSuc*:
1 + *n* ≠ (*0*::'*a*::*len word*) ⟹ *unat* (*1* + *n*) = *Suc* (*unat n*)
by *unat-arith*

lemmas *word-no-1* [*simp*] = *word-1-no* [*symmetric*, *unfolded BIT-simps*]
lemmas *word-no-0* [*simp*] = *word-0-no* [*symmetric*]

declare *word-0-bl* [*simp*]
declare *bin-to-bl-def* [*simp*]
declare *to-bl-0* [*simp*]
declare *of-bl-True* [*simp*]

end

16 Word: Word Library interface

theory *Word*
imports *WordGenLib*
uses ~~/*src*/*HOL*/*Tools*/*SMT*/*smt-word.ML*
begin

setup ⟪ *SMT-Word.setup* ⟫

see *Examples*/*WordExamples.thy* for examples

end

References

[1] Jeremy Dawson. Isabelle theories for machine words. In Michael Goldsmith and Bill Roscoe, editors, *Seventh International Workshop on Automated Verification of Critical Systems (AVOCS'07)*, Electronic Notes in Theoretical Computer Science, page 15, Oxford, September 2007. Elsevier. to appear.