# The Isabelle/HOL Algebra Library

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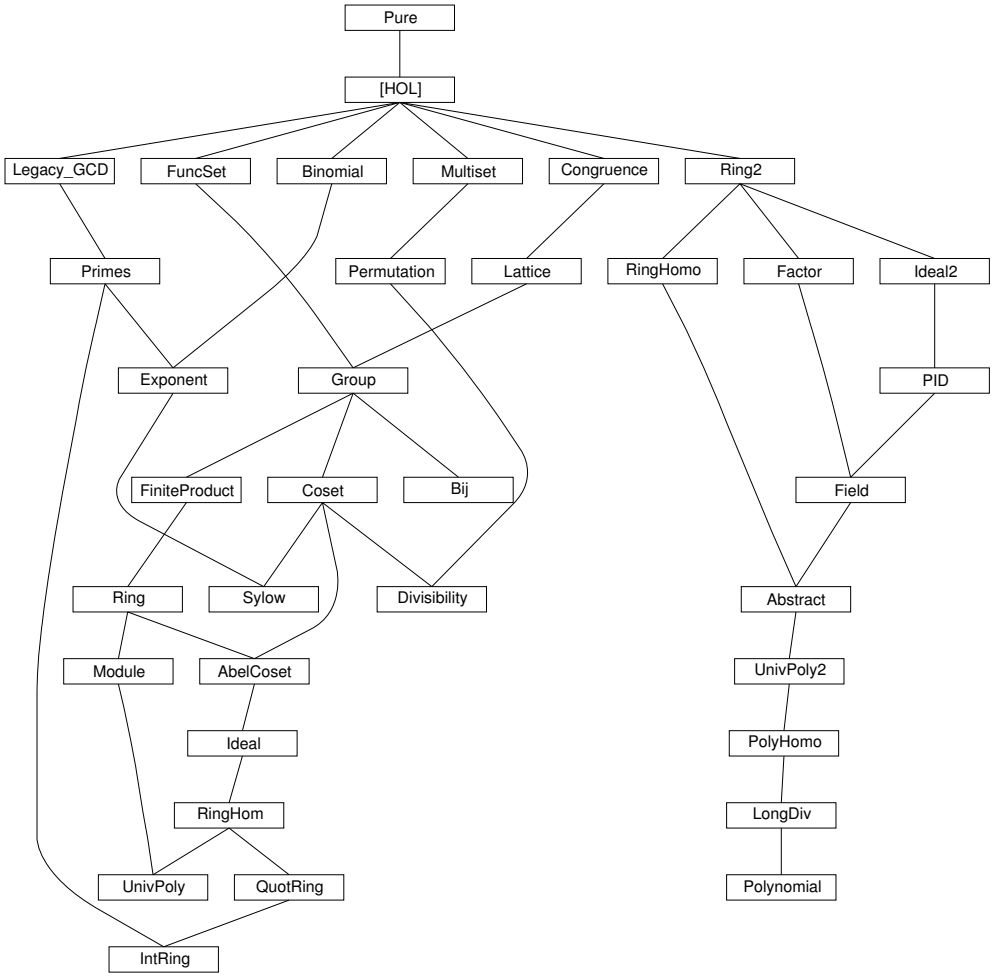

**theory** Congruence
**imports** Main
**begin**

# 1 Objects

## 1.1 Structure with Carrier Set.

**record** 'a partial_object =
  carrier :: "'a set"

## 1.2 Structure with Carrier and Equivalence Relation eq

**record** 'a eq_object = "'a partial_object" +
  eq :: "'a ⇒ 'a ⇒ bool" (**infixl** ".=ı" 50)

**definition**
  elem :: "_ ⇒ 'a ⇒ 'a set ⇒ bool" (**infixl** ".∈ı" 50)
  **where** "x .∈S A ⟷ (∃y ∈ A. x .=S y)"

**definition**
  set_eq :: "_ ⇒ 'a set ⇒ 'a set ⇒ bool" (**infixl** "{.=}ı" 50)
  **where** "A {.=}S B ⟷ ((∀x ∈ A. x .∈S B) ∧ (∀x ∈ B. x .∈S A))"

**definition**
  eq_class_of :: "_ ⇒ 'a ⇒ 'a set" ("class'_ofı _")
  **where** "class_ofS x = {y ∈ carrier S. x .=S y}"

**definition**
  eq_closure_of :: "_ ⇒ 'a set ⇒ 'a set" ("closure'_ofı _")
  **where** "closure_ofS A = {y ∈ carrier S. y .∈S A}"

**definition**
  eq_is_closed :: "_ ⇒ 'a set ⇒ bool" ("is'_closedı _")
  **where** "is_closedS A ⟷ A ⊆ carrier S ∧ closure_ofS A = A"

**abbreviation**
  not_eq :: "_ ⇒ 'a ⇒ 'a ⇒ bool" (**infixl** ".≠ı" 50)
  **where** "x .≠S y == ~(x .=S y)"

**abbreviation**
  not_elem :: "_ ⇒ 'a ⇒ 'a set ⇒ bool" (**infixl** ".∉ı" 50)
  **where** "x .∉S A == ~(x .∈S A)"

**abbreviation**
  set_not_eq :: "_ ⇒ 'a set ⇒ 'a set ⇒ bool" (**infixl** "{.≠}ı" 50)
  **where** "A {.≠}S B == ~(A {.=}S B)"

**locale** equivalence =
  **fixes** S (**structure**)
  **assumes** refl [simp, intro]: "x ∈ carrier S ⟹ x .= x"
    **and** sym [sym]: "⟦ x .= y; x ∈ carrier S; y ∈ carrier S ⟧ ⟹ y .= x"
    **and** trans [trans]: "⟦ x .= y; y .= z; x ∈ carrier S; y ∈ carrier S; z ∈ carrier S ⟧ ⟹ x .= z"

**lemma** elemI:
  **fixes** R (**structure**)
  **assumes** "a' ∈ A" **and** "a .= a'"
  **shows** "a .∈ A"
⟨*proof*⟩

**lemma** (**in** equivalence) elem_exact:
  **assumes** "a ∈ carrier S" **and** "a ∈ A"
  **shows** "a .∈ A"
⟨*proof*⟩

**lemma** elemE:
  **fixes** S (**structure**)
  **assumes** "a .∈ A"
    **and** "⋀a'. ⟦a' ∈ A; a .= a'⟧ ⟹ P"
  **shows** "P"
⟨*proof*⟩

**lemma** (**in** equivalence) elem_cong_l [trans]:
  **assumes** cong: "a' .= a"
    **and** a: "a .∈ A"
    **and** carr: "a ∈ carrier S"  "a' ∈ carrier S"
    **and** Acarr: "A ⊆ carrier S"
  **shows** "a' .∈ A"
⟨*proof*⟩

**lemma** (**in** equivalence) elem_subsetD:
  **assumes** "A ⊆ B"
    **and** aA: "a .∈ A"
  **shows** "a .∈ B"
⟨*proof*⟩

**lemma** (**in** equivalence) mem_imp_elem [simp, intro]:
  "[| x ∈ A; x ∈ carrier S |] ==> x .∈ A"
  ⟨*proof*⟩

**lemma** set_eqI:
  **fixes** R (**structure**)

**assumes** ltr: "⋀a. a ∈ A ⟹ a .∈ B"
**and** rtl: "⋀b. b ∈ B ⟹ b .∈ A"
**shows** "A {.=} B"
⟨*proof*⟩

**lemma** set_eqI2:
**fixes** R (**structure**)
**assumes** ltr: "⋀a b. a ∈ A ⟹ ∃b∈B. a .= b"
**and** rtl: "⋀b. b ∈ B ⟹ ∃a∈A. b .= a"
**shows** "A {.=} B"
⟨*proof*⟩

**lemma** set_eqD1:
**fixes** R (**structure**)
**assumes** AA': "A {.=} A'"
**and** "a ∈ A"
**shows** "∃a'∈A'. a .= a'"
⟨*proof*⟩

**lemma** set_eqD2:
**fixes** R (**structure**)
**assumes** AA': "A {.=} A'"
**and** "a' ∈ A'"
**shows** "∃a∈A. a' .= a"
⟨*proof*⟩

**lemma** set_eqE:
**fixes** R (**structure**)
**assumes** AB: "A {.=} B"
**and** r: "⟦∀a∈A. a .∈ B; ∀b∈B. b .∈ A⟧ ⟹ P"
**shows** "P"
⟨*proof*⟩

**lemma** set_eqE2:
**fixes** R (**structure**)
**assumes** AB: "A {.=} B"
**and** r: "⟦∀a∈A. (∃b∈B. a .= b); ∀b∈B. (∃a∈A. b .= a)⟧ ⟹ P"
**shows** "P"
⟨*proof*⟩

**lemma** set_eqE':
**fixes** R (**structure**)
**assumes** AB: "A {.=} B"
**and** aA: "a ∈ A" **and** bB: "b ∈ B"
**and** r: "⋀a' b'. ⟦a' ∈ A; b .= a'; b' ∈ B; a .= b'⟧ ⟹ P"
**shows** "P"
⟨*proof*⟩

**lemma** (**in** equivalence) eq_elem_cong_r [trans]:

**assumes** a: "a .∈ A"
**and** cong: "A {.=} A'"
**and** carr: "a ∈ carrier S"
**and** Carr: "A ⊆ carrier S" "A' ⊆ carrier S"
**shows** "a .∈ A'"
⟨*proof*⟩

**lemma (in** equivalence) set_eq_sym [sym]:
**assumes** "A {.=} B"
**and** "A ⊆ carrier S" "B ⊆ carrier S"
**shows** "B {.=} A"
⟨*proof*⟩

**lemma (in** equivalence) equal_set_eq_trans [trans]:
**assumes** AB: "A = B" **and** BC: "B {.=} C"
**shows** "A {.=} C"
⟨*proof*⟩

**lemma (in** equivalence) set_eq_equal_trans [trans]:
**assumes** AB: "A {.=} B" **and** BC: "B = C"
**shows** "A {.=} C"
⟨*proof*⟩

**lemma (in** equivalence) set_eq_trans [trans]:
**assumes** AB: "A {.=} B" **and** BC: "B {.=} C"
**and** carr: "A ⊆ carrier S" "B ⊆ carrier S" "C ⊆ carrier S"
**shows** "A {.=} C"
⟨*proof*⟩

**lemma (in** equivalence) set_eq_pairI:
**assumes** xx': "x .= x'"
**and** carr: "x ∈ carrier S" "x' ∈ carrier S" "y ∈ carrier S"
**shows** "{x, y} {.=} {x', y}"
⟨*proof*⟩

**lemma (in** equivalence) is_closedI:
**assumes** closed: "!!x y. [| x .= y; x ∈ A; y ∈ carrier S |] ==> y ∈ A"
**and** S: "A ⊆ carrier S"
**shows** "is_closed A"
⟨*proof*⟩

**lemma (in** equivalence**)** closure_of_eq:
"[| x .= x'; A ⊆ carrier S; x ∈ closure_of A; x ∈ carrier S; x' ∈ carrier S |] ==> x' ∈ closure_of A"
⟨*proof*⟩

**lemma (in** equivalence**)** is_closed_eq [dest]:
"[| x .= x'; x ∈ A; is_closed A; x ∈ carrier S; x' ∈ carrier S |] ==> x' ∈ A"
⟨*proof*⟩

**lemma (in** equivalence**)** is_closed_eq_rev [dest]:
"[| x .= x'; x' ∈ A; is_closed A; x ∈ carrier S; x' ∈ carrier S |] ==> x ∈ A"
⟨*proof*⟩

**lemma** closure_of_closed [simp, intro]:
**fixes** S **(structure)**
**shows** "closure_of A ⊆ carrier S"
⟨*proof*⟩

**lemma** closure_of_memI:
**fixes** S **(structure)**
**assumes** "a .∈ A"
**and** "a ∈ carrier S"
**shows** "a ∈ closure_of A"
⟨*proof*⟩

**lemma** closure_ofI2:
**fixes** S **(structure)**
**assumes** "a .= a'"
**and** "a' ∈ A"
**and** "a ∈ carrier S"
**shows** "a ∈ closure_of A"
⟨*proof*⟩

**lemma** closure_of_memE:
**fixes** S **(structure)**
**assumes** p: "a ∈ closure_of A"
**and** r: "⟦a ∈ carrier S; a .∈ A⟧ ⟹ P"
**shows** "P"
⟨*proof*⟩

**lemma** closure_ofE2:
**fixes** S **(structure)**
**assumes** p: "a ∈ closure_of A"
**and** r: "⋀a'. ⟦a ∈ carrier S; a' ∈ A; a .= a'⟧ ⟹ P"
**shows** "P"
⟨*proof*⟩

**end**

**theory** Lattice
**imports** Congruence
**begin**

# 2 Orders and Lattices

## 2.1 Partial Orders

**record** 'a gorder = "'a eq_object" +
  le :: "['a, 'a] => bool" (**infixl** "⊑\<index>" 50)

**locale** weak_partial_order = equivalence L **for** L (**structure**) +
  **assumes** le_refl [intro, simp]:
      "x ∈ carrier L ==> x ⊑ x"
    **and** weak_le_antisym [intro]:
      "[| x ⊑ y; y ⊑ x; x ∈ carrier L; y ∈ carrier L |] ==> x .= y"
    **and** le_trans [trans]:
      "[| x ⊑ y; y ⊑ z; x ∈ carrier L; y ∈ carrier L; z ∈ carrier L
|] ==> x ⊑ z"
    **and** le_cong:
      "⟦ x .= y; z .= w; x ∈ carrier L; y ∈ carrier L; z ∈ carrier L;
w ∈ carrier L ⟧ ⟹ x ⊑ z ⟷ y ⊑ w"

**definition**
  lless :: "[_, 'a, 'a] => bool" (**infixl** "⊏\<index>" 50)
  **where** "x ⊏L y ⟷ x ⊑L y & x .≠L y"

### 2.1.1 The order relation

**context** weak_partial_order **begin**

**lemma** le_cong_l [intro, trans]:
  "⟦ x .= y; y ⊑ z; x ∈ carrier L; y ∈ carrier L; z ∈ carrier L ⟧ ⟹
x ⊑ z"
  ⟨*proof*⟩

**lemma** le_cong_r [intro, trans]:
  "⟦ x ⊑ y; y .= z; x ∈ carrier L; y ∈ carrier L; z ∈ carrier L ⟧ ⟹
x ⊑ z"
  ⟨*proof*⟩

**lemma** weak_refl [intro, simp]: "⟦ x .= y; x ∈ carrier L; y ∈ carrier L ⟧ ⟹ x ⊑ y"
⟨*proof*⟩

**end**

**lemma** weak_llessI:
**fixes** R (**structure**)
**assumes** "x ⊑ y" **and** "~(x .= y)"
**shows** "x ⊏ y"
⟨*proof*⟩

**lemma** lless_imp_le:
**fixes** R (**structure**)
**assumes** "x ⊏ y"
**shows** "x ⊑ y"
⟨*proof*⟩

**lemma** weak_lless_imp_not_eq:
**fixes** R (**structure**)
**assumes** "x ⊏ y"
**shows** "¬ (x .= y)"
⟨*proof*⟩

**lemma** weak_llessE:
**fixes** R (**structure**)
**assumes** p: "x ⊏ y" **and** e: "⟦x ⊑ y; ¬ (x .= y)⟧ ⟹ P"
**shows** "P"
⟨*proof*⟩

**lemma** (**in** weak_partial_order) lless_cong_l [trans]:
**assumes** xx': "x .= x'"
**and** xy: "x' ⊏ y"
**and** carr: "x ∈ carrier L" "x' ∈ carrier L" "y ∈ carrier L"
**shows** "x ⊏ y"
⟨*proof*⟩

**lemma** (**in** weak_partial_order) lless_cong_r [trans]:
**assumes** xy: "x ⊏ y"
**and** yy': "y .= y'"
**and** carr: "x ∈ carrier L" "y ∈ carrier L" "y' ∈ carrier L"
**shows** "x ⊏ y'"
⟨*proof*⟩

**lemma** (**in** weak_partial_order) lless_antisym:
**assumes** "a ∈ carrier L" "b ∈ carrier L"
**and** "a ⊏ b" "b ⊏ a"
**shows** "P"

⟨*proof*⟩

**lemma (in** weak_partial_order**)** lless_trans [trans]:
**assumes** "a ⊏ b" "b ⊏ c"
**and** carr[simp]: "a ∈ carrier L" "b ∈ carrier L" "c ∈ carrier L"
**shows** "a ⊏ c"
⟨*proof*⟩

### 2.1.2 Upper and lower bounds of a set

**definition**
Upper :: "[_, 'a set] => 'a set"
**where** "Upper L A = {u. (ALL x. x ∈ A ∩ carrier L --> x ⊑$_L$ u)} ∩ carrier L"

**definition**
Lower :: "[_, 'a set] => 'a set"
**where** "Lower L A = {l. (ALL x. x ∈ A ∩ carrier L --> l ⊑$_L$ x)} ∩ carrier L"

**lemma** Upper_closed [intro!, simp]:
"Upper L A ⊆ carrier L"
⟨*proof*⟩

**lemma** Upper_memD [dest]:
**fixes** L **(structure)**
**shows** "[| u ∈ Upper L A; x ∈ A; A ⊆ carrier L |] ==> x ⊑ u ∧ u ∈ carrier L"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** Upper_elemD [dest]:
"[| u .∈ Upper L A; u ∈ carrier L; x ∈ A; A ⊆ carrier L |] ==> x ⊑ u"
⟨*proof*⟩

**lemma** Upper_memI:
**fixes** L **(structure)**
**shows** "[| !! y. y ∈ A ==> y ⊑ x; x ∈ carrier L |] ==> x ∈ Upper L A"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** Upper_elemI:
"[| !! y. y ∈ A ==> y ⊑ x; x ∈ carrier L |] ==> x .∈ Upper L A"
⟨*proof*⟩

**lemma** Upper_antimono:
"A ⊆ B ==> Upper L B ⊆ Upper L A"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** Upper_is_closed [simp]:
"A ⊆ carrier L ==> is_closed (Upper L A)"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** Upper_mem_cong:
**assumes** a'carr: "a' ∈ carrier L" **and** Acarr: "A ⊆ carrier L"
**and** aa': "a .= a'"
**and** aelem: "a ∈ Upper L A"
**shows** "a' ∈ Upper L A"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** Upper_cong:
**assumes** Acarr: "A ⊆ carrier L" **and** A'carr: "A' ⊆ carrier L"
**and** AA': "A {.=} A'"
**shows** "Upper L A = Upper L A'"
⟨*proof*⟩

**lemma** Lower_closed [intro!, simp]:
"Lower L A ⊆ carrier L"
⟨*proof*⟩

**lemma** Lower_memD [dest]:
**fixes** L **(structure)**
**shows** "[| l ∈ Lower L A; x ∈ A; A ⊆ carrier L |] ==> l ⊑ x ∧ l ∈ carrier L"
⟨*proof*⟩

**lemma** Lower_memI:
**fixes** L **(structure)**
**shows** "[| !! y. y ∈ A ==> x ⊑ y; x ∈ carrier L |] ==> x ∈ Lower L A"
⟨*proof*⟩

**lemma** Lower_antimono:
"A ⊆ B ==> Lower L B ⊆ Lower L A"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** Lower_is_closed [simp]:
"A ⊆ carrier L ⟹ is_closed (Lower L A)"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** Lower_mem_cong:
**assumes** a'carr: "a' ∈ carrier L" **and** Acarr: "A ⊆ carrier L"
**and** aa': "a .= a'"
**and** aelem: "a ∈ Lower L A"
**shows** "a' ∈ Lower L A"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** Lower_cong:

**assumes** Acarr: "A ⊆ carrier L" **and** A'carr: "A' ⊆ carrier L"
**and** AA': "A {.=} A'"
**shows** "Lower L A = Lower L A'"
⟨*proof*⟩

### 2.1.3 Least and greatest, as predicate

**definition**
**least** :: "[_, 'a, 'a set] => bool"
**where** "least L l A ⟷ A ⊆ carrier L & l ∈ A & (ALL x : A. l ⊑$_L$ x)"

**definition**
**greatest** :: "[_, 'a, 'a set] => bool"
**where** "greatest L g A ⟷ A ⊆ carrier L & g ∈ A & (ALL x : A. x ⊑$_L$ g)"

Could weaken these to l ∈ carrier L ∧ l .∈ A and g ∈ carrier L ∧ g .∈ A.

**lemma** least_closed [intro, simp]:
"least L l A ==> l ∈ carrier L"
⟨*proof*⟩

**lemma** least_mem:
"least L l A ==> l ∈ A"
⟨*proof*⟩

**lemma** (**in** weak_partial_order) weak_least_unique:
"[| least L x A; least L y A |] ==> x .= y"
⟨*proof*⟩

**lemma** least_le:
**fixes** L (**structure**)
**shows** "[| least L x A; a ∈ A |] ==> x ⊑ a"
⟨*proof*⟩

**lemma** (**in** weak_partial_order) least_cong:
"[| x .= x'; x ∈ carrier L; x' ∈ carrier L; is_closed A |] ==> least L x A = least L x' A"
⟨*proof*⟩

least is not congruent in the second parameter for A {.=} A'

**lemma** (**in** weak_partial_order) least_Upper_cong_l:
**assumes** "x .= x'"
**and** "x ∈ carrier L" "x' ∈ carrier L"
**and** "A ⊆ carrier L"
**shows** "least L x (Upper L A) = least L x' (Upper L A)"
⟨*proof*⟩

**lemma** (**in** weak_partial_order) least_Upper_cong_r:

**assumes** Acarrs: "A ⊆ carrier L" "A' ⊆ carrier L"
**and** AA': "A {.=} A'"
**shows** "least L x (Upper L A) = least L x (Upper L A')"
⟨*proof*⟩

**lemma** least_UpperI:
**fixes** L (**structure**)
**assumes** above: "!! x. x ∈ A ==> x ⊑ s"
**and** below: "!! y. y ∈ Upper L A ==> s ⊑ y"
**and** L: "A ⊆ carrier L" "s ∈ carrier L"
**shows** "least L s (Upper L A)"
⟨*proof*⟩

**lemma** least_Upper_above:
**fixes** L (**structure**)
**shows** "[| least L s (Upper L A); x ∈ A; A ⊆ carrier L |] ==> x ⊑ s"
⟨*proof*⟩

**lemma** greatest_closed [intro, simp]:
"greatest L l A ==> l ∈ carrier L"
⟨*proof*⟩

**lemma** greatest_mem:
"greatest L l A ==> l ∈ A"
⟨*proof*⟩

**lemma** (**in** weak_partial_order) weak_greatest_unique:
"[| greatest L x A; greatest L y A |] ==> x .= y"
⟨*proof*⟩

**lemma** greatest_le:
**fixes** L (**structure**)
**shows** "[| greatest L x A; a ∈ A |] ==> a ⊑ x"
⟨*proof*⟩

**lemma** (**in** weak_partial_order) greatest_cong:
"[| x .= x'; x ∈ carrier L; x' ∈ carrier L; is_closed A |] ==>
greatest L x A = greatest L x' A"
⟨*proof*⟩

greatest is not congruent in the second parameter for A {.=} A'

**lemma** (**in** weak_partial_order) greatest_Lower_cong_l:
**assumes** "x .= x'"
**and** "x ∈ carrier L" "x' ∈ carrier L"
**and** "A ⊆ carrier L"
**shows** "greatest L x (Lower L A) = greatest L x' (Lower L A)"
⟨*proof*⟩

**lemma** (**in** weak_partial_order) greatest_Lower_cong_r:

```
  assumes Acarrs: "A ⊆ carrier L" "A' ⊆ carrier L"
    and AA': "A {.=} A'"
  shows "greatest L x (Lower L A) = greatest L x (Lower L A')"
```
⟨*proof*⟩

**lemma** greatest_LowerI:
```
  fixes L (structure)
  assumes below: "!! x. x ∈ A ==> i ⊑ x"
    and above: "!! y. y ∈ Lower L A ==> y ⊑ i"
    and L: "A ⊆ carrier L"  "i ∈ carrier L"
  shows "greatest L i (Lower L A)"
```
⟨*proof*⟩

**lemma** greatest_Lower_below:
```
  fixes L (structure)
  shows "[| greatest L i (Lower L A); x ∈ A; A ⊆ carrier L |] ==> i ⊑
x"
```
  ⟨*proof*⟩

Supremum and infimum

**definition**
```
  sup :: "[_, 'a set] => 'a" ("⨆⇘_⇙_" [90] 90)
  where "⨆⇘L⇙A = (SOME x. least L x (Upper L A))"
```

**definition**
```
  inf :: "[_, 'a set] => 'a" ("⨅⇘_⇙_" [90] 90)
  where "⨅⇘L⇙A = (SOME x. greatest L x (Lower L A))"
```

**definition**
```
  join :: "[_, 'a, 'a] => 'a" (infixl "⊔⇘_⇙" 65)
  where "x ⊔⇘L⇙ y = ⨆⇘L⇙{x, y}"
```

**definition**
```
  meet :: "[_, 'a, 'a] => 'a" (infixl "⊓⇘_⇙" 70)
  where "x ⊓⇘L⇙ y = ⨅⇘L⇙{x, y}"
```

## 2.2 Lattices

**locale** weak_upper_semilattice = weak_partial_order +
```
  assumes sup_of_two_exists:
    "[| x ∈ carrier L; y ∈ carrier L |] ==> EX s. least L s (Upper L {x,
y})"
```

**locale** weak_lower_semilattice = weak_partial_order +
```
  assumes inf_of_two_exists:
    "[| x ∈ carrier L; y ∈ carrier L |] ==> EX s. greatest L s (Lower
L {x, y})"
```

**locale** weak_lattice = weak_upper_semilattice + weak_lower_semilattice

### 2.2.1 Supremum

**lemma (in** weak_upper_semilattice**)** joinI:
"[| !!l. least L l (Upper L {x, y}) ==> P l; x ∈ carrier L; y ∈ carrier L |]
==> P (x ⊔ y)"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** join_closed [simp]:
"[| x ∈ carrier L; y ∈ carrier L |] ==> x ⊔ y ∈ carrier L"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** join_cong_l:
**assumes** carr: "x ∈ carrier L" "x' ∈ carrier L" "y ∈ carrier L"
**and** xx': "x .= x'"
**shows** "x ⊔ y .= x' ⊔ y"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** join_cong_r:
**assumes** carr: "x ∈ carrier L" "y ∈ carrier L" "y' ∈ carrier L"
**and** yy': "y .= y'"
**shows** "x ⊔ y .= x ⊔ y'"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** sup_of_singletonI:
"x ∈ carrier L ==> least L x (Upper L {x})"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** weak_sup_of_singleton [simp]:
"x ∈ carrier L ==> ⨆{x} .= x"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** sup_of_singleton_closed [simp]:
"x ∈ carrier L ⟹ ⨆{x} ∈ carrier L"
⟨*proof*⟩

Condition on A: supremum exists.

**lemma (in** weak_upper_semilattice**)** sup_insertI:
"[| !!s. least L s (Upper L (insert x A)) ==> P s;
least L a (Upper L A); x ∈ carrier L; A ⊆ carrier L |]
==> P (⨆(insert x A))"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** finite_sup_least:
"[| finite A; A ⊆ carrier L; A ~= {} |] ==> least L (⨆A) (Upper L A)"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** finite_sup_insertI:
**assumes** P: "!!l. least L l (Upper L (insert x A)) ==> P l"
**and** xA: "finite A" "x ∈ carrier L" "A ⊆ carrier L"

**shows** "P (⨆ (insert x A))"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** finite_sup_closed [simp]:
"[| finite A; A ⊆ carrier L; A ~= {} |] ==> ⨆A ∈ carrier L"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** join_left:
"[| x ∈ carrier L; y ∈ carrier L |] ==> x ⊑ x ⊔ y"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** join_right:
"[| x ∈ carrier L; y ∈ carrier L |] ==> y ⊑ x ⊔ y"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** sup_of_two_least:
"[| x ∈ carrier L; y ∈ carrier L |] ==> least L (⨆{x, y}) (Upper L {x, y})"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** join_le:
**assumes** sub: "x ⊑ z" "y ⊑ z"
**and** x: "x ∈ carrier L" **and** y: "y ∈ carrier L" **and** z: "z ∈ carrier L"
**shows** "x ⊔ y ⊑ z"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** weak_join_assoc_lemma:
**assumes** L: "x ∈ carrier L" "y ∈ carrier L" "z ∈ carrier L"
**shows** "x ⊔ (y ⊔ z) .= ⨆{x, y, z}"
⟨*proof*⟩

Commutativity holds for =.

**lemma** join_comm:
**fixes** L (**structure**)
**shows** "x ⊔ y = y ⊔ x"
⟨*proof*⟩

**lemma (in** weak_upper_semilattice**)** weak_join_assoc:
**assumes** L: "x ∈ carrier L" "y ∈ carrier L" "z ∈ carrier L"
**shows** "(x ⊔ y) ⊔ z .= x ⊔ (y ⊔ z)"
⟨*proof*⟩

### 2.2.2 Infimum

**lemma (in** weak_lower_semilattice**)** meetI:
"[| !!i. greatest L i (Lower L {x, y}) ==> P i;
x ∈ carrier L; y ∈ carrier L |]
==> P (x ⊓ y)"

⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** meet_closed [simp]:
"[| x ∈ carrier L; y ∈ carrier L |] ==> x ⊓ y ∈ carrier L"
⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** meet_cong_l:
**assumes** carr: "x ∈ carrier L" "x' ∈ carrier L" "y ∈ carrier L"
**and** xx': "x .= x'"
**shows** "x ⊓ y .= x' ⊓ y"
⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** meet_cong_r:
**assumes** carr: "x ∈ carrier L" "y ∈ carrier L" "y' ∈ carrier L"
**and** yy': "y .= y'"
**shows** "x ⊓ y .= x ⊓ y'"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** inf_of_singletonI:
"x ∈ carrier L ==> greatest L x (Lower L {x})"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** weak_inf_of_singleton [simp]:
"x ∈ carrier L ==> ⨅{x} .= x"
⟨*proof*⟩

**lemma (in** weak_partial_order**)** inf_of_singleton_closed:
"x ∈ carrier L ==> ⨅{x} ∈ carrier L"
⟨*proof*⟩

Condition on A: infimum exists.

**lemma (in** weak_lower_semilattice**)** inf_insertI:
"[| !!i. greatest L i (Lower L (insert x A)) ==> P i;
greatest L a (Lower L A); x ∈ carrier L; A ⊆ carrier L |]
==> P (⨅(insert x A))"
⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** finite_inf_greatest:
"[| finite A; A ⊆ carrier L; A ~= {} |] ==> greatest L (⨅A) (Lower L A)"
⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** finite_inf_insertI:
**assumes** P: "!!i. greatest L i (Lower L (insert x A)) ==> P i"
**and** xA: "finite A" "x ∈ carrier L" "A ⊆ carrier L"
**shows** "P (⨅ (insert x A))"
⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** finite_inf_closed [simp]:

```
  "[| finite A; A ⊆ carrier L; A ~= {} |] ==> ⨅A ∈ carrier L"
```
⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** meet_left:
```
  "[| x ∈ carrier L; y ∈ carrier L |] ==> x ⊓ y ⊑ x"
```
⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** meet_right:
```
  "[| x ∈ carrier L; y ∈ carrier L |] ==> x ⊓ y ⊑ y"
```
⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** inf_of_two_greatest:
```
  "[| x ∈ carrier L; y ∈ carrier L |] ==>
  greatest L (⨅ {x, y}) (Lower L {x, y})"
```
⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** meet_le:
```
  assumes sub: "z ⊑ x"  "z ⊑ y"
    and x: "x ∈ carrier L" and y: "y ∈ carrier L" and z: "z ∈ carrier
L"
  shows "z ⊑ x ⊓ y"
```
⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** weak_meet_assoc_lemma:
```
  assumes L: "x ∈ carrier L"  "y ∈ carrier L"  "z ∈ carrier L"
  shows "x ⊓ (y ⊓ z) .= ⨅{x, y, z}"
```
⟨*proof*⟩

**lemma** meet_comm:
```
  fixes L (structure)
  shows "x ⊓ y = y ⊓ x"
```
⟨*proof*⟩

**lemma (in** weak_lower_semilattice**)** weak_meet_assoc:
```
  assumes L: "x ∈ carrier L"  "y ∈ carrier L"  "z ∈ carrier L"
  shows "(x ⊓ y) ⊓ z .= x ⊓ (y ⊓ z)"
```
⟨*proof*⟩

## 2.3 Total Orders

**locale** weak_total_order = weak_partial_order +
```
  assumes total: "[| x ∈ carrier L; y ∈ carrier L |] ==> x ⊑ y | y ⊑
x"
```

Introduction rule: the usual definition of total order

**lemma (in** weak_partial_order**)** weak_total_orderI:
```
  assumes total: "!!x y. [| x ∈ carrier L; y ∈ carrier L |] ==> x ⊑
y | y ⊑ x"
  shows "weak_total_order L"
```

⟨*proof*⟩

Total orders are lattices.

**sublocale** weak_total_order < weak: weak_lattice
⟨*proof*⟩

## 2.4 Complete Lattices

**locale** weak_complete_lattice = weak_lattice +
  **assumes** sup_exists:
    "[| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)"
    **and** inf_exists:
    "[| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)"

Introduction rule: the usual definition of complete lattice

**lemma** (**in** weak_partial_order) weak_complete_latticeI:
  **assumes** sup_exists:
    "!!A. [| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)"
    **and** inf_exists:
    "!!A. [| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)"
  **shows** "weak_complete_lattice L"
  ⟨*proof*⟩

**definition**
  top :: "_ => 'a" ("⊤ı")
  **where** "⊤L = sup L (carrier L)"

**definition**
  bottom :: "_ => 'a" ("⊥ı")
  **where** "⊥L = inf L (carrier L)"

**lemma** (**in** weak_complete_lattice) supI:
  "[| !!l. least L l (Upper L A) ==> P l; A ⊆ carrier L |]
  ==> P (⨆A)"
⟨*proof*⟩

**lemma** (**in** weak_complete_lattice) sup_closed [simp]:
  "A ⊆ carrier L ==> ⨆A ∈ carrier L"
  ⟨*proof*⟩

**lemma** (**in** weak_complete_lattice) top_closed [simp, intro]:
  "⊤ ∈ carrier L"
  ⟨*proof*⟩

**lemma** (**in** weak_complete_lattice) infI:
  "[| !!i. greatest L i (Lower L A) ==> P i; A ⊆ carrier L |]
  ==> P (⨅A)"
⟨*proof*⟩

**lemma (in** weak_complete_lattice**)** inf_closed [simp]:
"A ⊆ carrier L ==> ⨅A ∈ carrier L"
⟨*proof*⟩

**lemma (in** weak_complete_lattice**)** bottom_closed [simp, intro]:
"⊥ ∈ carrier L"
⟨*proof*⟩

Jacobson: Theorem 8.1

**lemma** Lower_empty [simp]:
"Lower L {} = carrier L"
⟨*proof*⟩

**lemma** Upper_empty [simp]:
"Upper L {} = carrier L"
⟨*proof*⟩

**theorem (in** weak_partial_order**)** weak_complete_lattice_criterion1:
**assumes** top_exists: "EX g. greatest L g (carrier L)"
**and** inf_exists:
"!!A. [| A ⊆ carrier L; A ~= {} |] ==> EX i. greatest L i (Lower L A)"
**shows** "weak_complete_lattice L"
⟨*proof*⟩

## 2.5 Orders and Lattices where eq is the Equality

**locale** partial_order = weak_partial_order +
**assumes** eq_is_equal: "op .= = op ="
**begin**

**declare** weak_le_antisym [rule del]

**lemma** le_antisym [intro]:
"[| x ⊑ y; y ⊑ x; x ∈ carrier L; y ∈ carrier L |] ==> x = y"
⟨*proof*⟩

**lemma** lless_eq:
"x ⊏ y ⟷ x ⊑ y & x ≠ y"
⟨*proof*⟩

**lemma** lless_asym:
**assumes** "a ∈ carrier L" "b ∈ carrier L"
**and** "a ⊏ b" "b ⊏ a"
**shows** "P"
⟨*proof*⟩

**end**

Least and greatest, as predicate

**lemma (in** partial_order**)** least_unique:
"[| least L x A; least L y A |] ==> x = y"
⟨*proof*⟩

**lemma (in** partial_order**)** greatest_unique:
"[| greatest L x A; greatest L y A |] ==> x = y"
⟨*proof*⟩

Lattices

**locale** upper_semilattice = partial_order +
**assumes** sup_of_two_exists:
"[| x ∈ carrier L; y ∈ carrier L |] ==> EX s. least L s (Upper L {x, y})"

**sublocale** upper_semilattice < weak: weak_upper_semilattice
⟨*proof*⟩

**locale** lower_semilattice = partial_order +
**assumes** inf_of_two_exists:
"[| x ∈ carrier L; y ∈ carrier L |] ==> EX s. greatest L s (Lower L {x, y})"

**sublocale** lower_semilattice < weak: weak_lower_semilattice
⟨*proof*⟩

**locale** lattice = upper_semilattice + lower_semilattice

Supremum

**declare (in** partial_order**)** weak_sup_of_singleton [simp del]

**lemma (in** partial_order**)** sup_of_singleton [simp]:
"x ∈ carrier L ==> ⨆{x} = x"
⟨*proof*⟩

**lemma (in** upper_semilattice**)** join_assoc_lemma:
**assumes** L: "x ∈ carrier L" "y ∈ carrier L" "z ∈ carrier L"
**shows** "x ⊔ (y ⊔ z) = ⨆{x, y, z}"
⟨*proof*⟩

**lemma (in** upper_semilattice**)** join_assoc:
**assumes** L: "x ∈ carrier L" "y ∈ carrier L" "z ∈ carrier L"
**shows** "(x ⊔ y) ⊔ z = x ⊔ (y ⊔ z)"
⟨*proof*⟩

Infimum

**declare (in** partial_order**)** weak_inf_of_singleton [simp del]

**lemma (in** partial_order**)** inf_of_singleton [simp]:
"x ∈ carrier L ==> ⨅{x} = x"
⟨*proof*⟩

Condition on A: infimum exists.

**lemma (in** lower_semilattice**)** meet_assoc_lemma:
**assumes** L: "x ∈ carrier L" "y ∈ carrier L" "z ∈ carrier L"
**shows** "x ⊓ (y ⊓ z) = ⨅{x, y, z}"
⟨*proof*⟩

**lemma (in** lower_semilattice**)** meet_assoc:
**assumes** L: "x ∈ carrier L" "y ∈ carrier L" "z ∈ carrier L"
**shows** "(x ⊓ y) ⊓ z = x ⊓ (y ⊓ z)"
⟨*proof*⟩

Total Orders

**locale** total_order = partial_order +
**assumes** total_order_total: "[| x ∈ carrier L; y ∈ carrier L |] ==>
x ⊑ y | y ⊑ x"

**sublocale** total_order < weak: weak_total_order
⟨*proof*⟩

Introduction rule: the usual definition of total order

**lemma (in** partial_order**)** total_orderI:
**assumes** total: "!!x y. [| x ∈ carrier L; y ∈ carrier L |] ==> x ⊑
y | y ⊑ x"
**shows** "total_order L"
⟨*proof*⟩

Total orders are lattices.

**sublocale** total_order < weak: lattice
⟨*proof*⟩

Complete lattices

**locale** complete_lattice = lattice +
**assumes** sup_exists:
"[| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)"
**and** inf_exists:
"[| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)"

**sublocale** complete_lattice < weak: weak_complete_lattice
⟨*proof*⟩

Introduction rule: the usual definition of complete lattice

**lemma (in** partial_order**)** complete_latticeI:
**assumes** sup_exists:
"!!A. [| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)"

```
    and inf_exists:
    "!!A. [| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)"
  shows "complete_lattice L"
```
⟨*proof*⟩

```
theorem (in partial_order) complete_lattice_criterion1:
  assumes top_exists: "EX g. greatest L g (carrier L)"
    and inf_exists:
      "!!A. [| A ⊆ carrier L; A ~= {} |] ==> EX i. greatest L i (Lower
L A)"
  shows "complete_lattice L"
```
⟨*proof*⟩

### 2.6 Examples

#### 2.6.1 The Powerset of a Set is a Complete Lattice

```
theorem powerset_is_complete_lattice:
  "complete_lattice (| carrier = Pow A, eq = op =, le = op ⊆ |)"
  (is "complete_lattice ?L")
```
⟨*proof*⟩

An other example, that of the lattice of subgroups of a group, can be found in Group theory (Section 3.8).

**end**

```
theory Group
imports Lattice FuncSet
begin
```

# 3 Monoids and Groups

## 3.1 Definitions

Definitions follow [2].

```
record 'a monoid =  "'a partial_object" +
  mult    :: "['a, 'a] ⇒ 'a" (infixl "⊗\<index>" 70)
  one     :: 'a ("1\<index>")
```

**definition**

```
  m_inv :: "('a, 'b) monoid_scheme => 'a => 'a" ("inv\<index> _" [81] 80)
  where "inv_G x = (THE y. y ∈ carrier G & x ⊗_G y = 1_G & y ⊗_G x = 1_G)"
```

**definition**

```
  Units :: "_ => 'a set"
```
— The set of invertible elements

**where** "Units G = {y. y ∈ carrier G & (∃x ∈ carrier G. x ⊗$_G$ y = **1**$_G$ & y ⊗$_G$ x = **1**$_G$)}"

**consts**
pow :: "[('a, 'm) monoid_scheme, 'a, 'b::number] => 'a" (**infixr** "'(^')ı" 75)

**overloading** nat_pow == "pow :: [_, 'a, nat] => 'a"
**begin**
**definition** "nat_pow G a n = nat_rec **1**$_G$ (%u b. b ⊗$_G$ a) n"
**end**

**overloading** int_pow == "pow :: [_, 'a, int] => 'a"
**begin**
**definition** "int_pow G a z =
(let p = nat_rec **1**$_G$ (%u b. b ⊗$_G$ a)
in if neg z then inv$_G$ (p (nat (-z))) else p (nat z))"
**end**

**locale** monoid =
**fixes** G (**structure**)
**assumes** m_closed [intro, simp]:
"⟦x ∈ carrier G; y ∈ carrier G⟧ ⟹ x ⊗ y ∈ carrier G"
**and** m_assoc:
"⟦x ∈ carrier G; y ∈ carrier G; z ∈ carrier G⟧
⟹ (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)"
**and** one_closed [intro, simp]: "**1** ∈ carrier G"
**and** l_one [simp]: "x ∈ carrier G ⟹ **1** ⊗ x = x"
**and** r_one [simp]: "x ∈ carrier G ⟹ x ⊗ **1** = x"

**lemma** monoidI:
**fixes** G (**structure**)
**assumes** m_closed:
"!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y ∈ carrier G"
**and** one_closed: "**1** ∈ carrier G"
**and** m_assoc:
"!!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
(x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)"
**and** l_one: "!!x. x ∈ carrier G ==> **1** ⊗ x = x"
**and** r_one: "!!x. x ∈ carrier G ==> x ⊗ **1** = x"
**shows** "monoid G"
⟨*proof*⟩

**lemma** (**in** monoid) Units_closed [dest]:
"x ∈ Units G ==> x ∈ carrier G"
⟨*proof*⟩

**lemma** (**in** monoid) inv_unique:

**assumes** eq: "y ⊗ x = **1**" "x ⊗ y' = **1**"
**and** G: "x ∈ carrier G" "y ∈ carrier G" "y' ∈ carrier G"
**shows** "y = y'"
⟨*proof*⟩

**lemma** (**in** monoid) Units_m_closed [intro, simp]:
**assumes** x: "x ∈ Units G" **and** y: "y ∈ Units G"
**shows** "x ⊗ y ∈ Units G"
⟨*proof*⟩

**lemma** (**in** monoid) Units_one_closed [intro, simp]:
"**1** ∈ Units G"
⟨*proof*⟩

**lemma** (**in** monoid) Units_inv_closed [intro, simp]:
"x ∈ Units G ==> inv x ∈ carrier G"
⟨*proof*⟩

**lemma** (**in** monoid) Units_l_inv_ex:
"x ∈ Units G ==> ∃y ∈ carrier G. y ⊗ x = **1**"
⟨*proof*⟩

**lemma** (**in** monoid) Units_r_inv_ex:
"x ∈ Units G ==> ∃y ∈ carrier G. x ⊗ y = **1**"
⟨*proof*⟩

**lemma** (**in** monoid) Units_l_inv [simp]:
"x ∈ Units G ==> inv x ⊗ x = **1**"
⟨*proof*⟩

**lemma** (**in** monoid) Units_r_inv [simp]:
"x ∈ Units G ==> x ⊗ inv x = **1**"
⟨*proof*⟩

**lemma** (**in** monoid) Units_inv_Units [intro, simp]:
"x ∈ Units G ==> inv x ∈ Units G"
⟨*proof*⟩

**lemma** (**in** monoid) Units_l_cancel [simp]:
"[| x ∈ Units G; y ∈ carrier G; z ∈ carrier G |] ==>
(x ⊗ y = x ⊗ z) = (y = z)"
⟨*proof*⟩

**lemma** (**in** monoid) Units_inv_inv [simp]:
"x ∈ Units G ==> inv (inv x) = x"
⟨*proof*⟩

**lemma** (**in** monoid) inv_inj_on_Units:
"inj_on (m_inv G) (Units G)"

⟨*proof*⟩

**lemma (in** monoid**)** Units_inv_comm:
  **assumes** inv: "x ⊗ y = **1**"
    **and** G: "x ∈ Units G"  "y ∈ Units G"
  **shows** "y ⊗ x = **1**"
⟨*proof*⟩

Power

**lemma (in** monoid**)** nat_pow_closed [intro, simp]:
  "x ∈ carrier G ==> x (^) (n::nat) ∈ carrier G"
  ⟨*proof*⟩

**lemma (in** monoid**)** nat_pow_0 [simp]:
  "x (^) (0::nat) = **1**"
  ⟨*proof*⟩

**lemma (in** monoid**)** nat_pow_Suc [simp]:
  "x (^) (Suc n) = x (^) n ⊗ x"
  ⟨*proof*⟩

**lemma (in** monoid**)** nat_pow_one [simp]:
  "**1** (^) (n::nat) = **1**"
  ⟨*proof*⟩

**lemma (in** monoid**)** nat_pow_mult:
  "x ∈ carrier G ==> x (^) (n::nat) ⊗ x (^) m = x (^) (n + m)"
  ⟨*proof*⟩

**lemma (in** monoid**)** nat_pow_pow:
  "x ∈ carrier G ==> (x (^) n) (^) m = x (^) (n * m::nat)"
  ⟨*proof*⟩

## 3.2 Groups

A group is a monoid all of whose elements are invertible.

**locale** group = monoid +
  **assumes** Units: "carrier G <= Units G"

**lemma (in** group**)** is_group: "group G" ⟨*proof*⟩

**theorem** groupI:
  **fixes** G **(structure)**
  **assumes** m_closed [simp]:
      "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y ∈ carrier
G"
    **and** one_closed [simp]: "**1** ∈ carrier G"
    **and** m_assoc:
      "!!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>

```
      (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)"
    and l_one [simp]: "!!x. x ∈ carrier G ==> 𝟭 ⊗ x = x"
    and l_inv_ex: "!!x. x ∈ carrier G ==> ∃y ∈ carrier G. y ⊗ x = 𝟭"
  shows "group G"
```
⟨*proof*⟩

**lemma (in** monoid**)** group_l_invI:
```
  assumes l_inv_ex:
    "!!x. x ∈ carrier G ==> ∃y ∈ carrier G. y ⊗ x = 𝟭"
  shows "group G"
```
⟨*proof*⟩

**lemma (in** group**)** Units_eq [simp]:
```
  "Units G = carrier G"
```
⟨*proof*⟩

**lemma (in** group**)** inv_closed [intro, simp]:
```
  "x ∈ carrier G ==> inv x ∈ carrier G"
```
⟨*proof*⟩

**lemma (in** group**)** l_inv_ex [simp]:
```
  "x ∈ carrier G ==> ∃y ∈ carrier G. y ⊗ x = 𝟭"
```
⟨*proof*⟩

**lemma (in** group**)** r_inv_ex [simp]:
```
  "x ∈ carrier G ==> ∃y ∈ carrier G. x ⊗ y = 𝟭"
```
⟨*proof*⟩

**lemma (in** group**)** l_inv [simp]:
```
  "x ∈ carrier G ==> inv x ⊗ x = 𝟭"
```
⟨*proof*⟩

## 3.3 Cancellation Laws and Basic Properties

**lemma (in** group**)** l_cancel [simp]:
```
  "[| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
   (x ⊗ y = x ⊗ z) = (y = z)"
```
⟨*proof*⟩

**lemma (in** group**)** r_inv [simp]:
```
  "x ∈ carrier G ==> x ⊗ inv x = 𝟭"
```
⟨*proof*⟩

**lemma (in** group**)** r_cancel [simp]:
```
  "[| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
   (y ⊗ x = z ⊗ x) = (y = z)"
```
⟨*proof*⟩

**lemma (in** group**)** inv_one [simp]:

```
  "inv 1 = 1"
```
⟨*proof*⟩

**lemma (in** group**)** inv_inv [simp]:
```
  "x ∈ carrier G ==> inv (inv x) = x"
```
⟨*proof*⟩

**lemma (in** group**)** inv_inj:
```
  "inj_on (m_inv G) (carrier G)"
```
⟨*proof*⟩

**lemma (in** group**)** inv_mult_group:
```
  "[| x ∈ carrier G; y ∈ carrier G |] ==> inv (x ⊗ y) = inv y ⊗ inv x"
```
⟨*proof*⟩

**lemma (in** group**)** inv_comm:
```
  "[| x ⊗ y = 1; x ∈ carrier G; y ∈ carrier G |] ==> y ⊗ x = 1"
```
⟨*proof*⟩

**lemma (in** group**)** inv_equality:
```
     "[|y ⊗ x = 1; x ∈ carrier G; y ∈ carrier G|] ==> inv x = y"
```
⟨*proof*⟩

Power

**lemma (in** group**)** int_pow_def2:
```
  "a (^) (z::int) = (if neg z then inv (a (^) (nat (-z))) else a (^) (nat
z))"
```
⟨*proof*⟩

**lemma (in** group**)** int_pow_0 [simp]:
```
  "x (^) (0::int) = 1"
```
⟨*proof*⟩

**lemma (in** group**)** int_pow_one [simp]:
```
  "1 (^) (z::int) = 1"
```
⟨*proof*⟩

## 3.4 Subgroups

**locale** subgroup =
```
  fixes H and G (structure)
  assumes subset: "H ⊆ carrier G"
    and m_closed [intro, simp]: "⟦x ∈ H; y ∈ H⟧ ⟹ x ⊗ y ∈ H"
    and one_closed [simp]: "1 ∈ H"
    and m_inv_closed [intro,simp]: "x ∈ H ⟹ inv x ∈ H"
```

**lemma (in** subgroup**)** is_subgroup:
```
  "subgroup H G"
```
⟨*proof*⟩

**declare (in** subgroup**)** group.intro [intro]

**lemma (in** subgroup**)** mem_carrier [simp]:
"x ∈ H ⟹ x ∈ carrier G"
⟨*proof*⟩

**lemma** subgroup_imp_subset:
"subgroup H G ⟹ H ⊆ carrier G"
⟨*proof*⟩

**lemma (in** subgroup**)** subgroup_is_group [intro]:
**assumes** "group G"
**shows** "group (G⦇carrier := H⦈)"
⟨*proof*⟩

Since H is nonempty, it contains some element x. Since it is closed under inverse, it contains inv x. Since it is closed under product, it contains x ⊗ inv x = **1**.

**lemma (in** group**)** one_in_subset:
"[| H ⊆ carrier G; H ≠ {}; ∀a ∈ H. inv a ∈ H; ∀a∈H. ∀b∈H. a ⊗ b ∈ H |]
==> **1** ∈ H"
⟨*proof*⟩

A characterization of subgroups: closed, non-empty subset.

**lemma (in** group**)** subgroupI:
**assumes** subset: "H ⊆ carrier G" **and** non_empty: "H ≠ {}"
**and** inv: "!!a. a ∈ H ⟹ inv a ∈ H"
**and** mult: "!!a b. ⟦a ∈ H; b ∈ H⟧ ⟹ a ⊗ b ∈ H"
**shows** "subgroup H G"
⟨*proof*⟩

**declare** monoid.one_closed [iff] group.inv_closed [simp]
monoid.l_one [simp] monoid.r_one [simp] group.inv_inv [simp]

**lemma** subgroup_nonempty:
"~ subgroup {} G"
⟨*proof*⟩

**lemma (in** subgroup**)** finite_imp_card_positive:
"finite (carrier G) ==> 0 < card H"
⟨*proof*⟩

## 3.5 Direct Products

**definition**
DirProd :: "_ ⇒ _ ⇒ ('a × 'b) monoid" (**infixr** "××" 80) **where**
"G ×× H =
⦇carrier = carrier G × carrier H,

```
    mult = (λ(g, h) (g', h'). (g ⊗G g', h ⊗H h')),
    one = (1G, 1H)⦈"
```

**lemma** DirProd_monoid:
**assumes** "monoid G" **and** "monoid H"
**shows** "monoid (G ×× H)"
⟨*proof*⟩

Does not use the previous result because it's easier just to use auto.

**lemma** DirProd_group:
**assumes** "group G" **and** "group H"
**shows** "group (G ×× H)"
⟨*proof*⟩

**lemma** carrier_DirProd [simp]:
"carrier (G ×× H) = carrier G × carrier H"
⟨*proof*⟩

**lemma** one_DirProd [simp]:
"1G ×× H = (1G, 1H)"
⟨*proof*⟩

**lemma** mult_DirProd [simp]:
"(g, h) ⊗(G ×× H) (g', h') = (g ⊗G g', h ⊗H h')"
⟨*proof*⟩

**lemma** inv_DirProd [simp]:
**assumes** "group G" **and** "group H"
**assumes** g: "g ∈ carrier G"
**and** h: "h ∈ carrier H"
**shows** "m_inv (G ×× H) (g, h) = (invG g, invH h)"
⟨*proof*⟩

## 3.6 Homomorphisms and Isomorphisms

**definition**
hom :: "_ => _ => ('a => 'b) set" **where**
"hom G H =
{h. h ∈ carrier G -> carrier H &
(∀x ∈ carrier G. ∀y ∈ carrier G. h (x ⊗G y) = h x ⊗H h y)}"

**lemma** (**in** group) hom_compose:
"[|h ∈ hom G H; i ∈ hom H I|] ==> compose (carrier G) i h ∈ hom G I"
⟨*proof*⟩

**definition**
iso :: "_ => _ => ('a => 'b) set" (**infixr** "≅" 60)
**where** "G ≅ H = {h. h ∈ hom G H & bij_betw h (carrier G) (carrier H)}"

**lemma** iso_refl: "(%x. x) ∈ G ≅ G"
⟨*proof*⟩

**lemma** (**in** group) iso_sym:
"h ∈ G ≅ H ⟹ inv_into (carrier G) h ∈ H ≅ G"
⟨*proof*⟩

**lemma** (**in** group) iso_trans:
"[|h ∈ G ≅ H; i ∈ H ≅ I|] ==> (compose (carrier G) i h) ∈ G ≅ I"
⟨*proof*⟩

**lemma** DirProd_commute_iso:
**shows** "(λ(x,y). (y,x)) ∈ (G ×× H) ≅ (H ×× G)"
⟨*proof*⟩

**lemma** DirProd_assoc_iso:
**shows** "(λ(x,y,z). (x,(y,z))) ∈ (G ×× H ×× I) ≅ (G ×× (H ×× I))"
⟨*proof*⟩

Basis for homomorphism proofs: we assume two groups G and H, with a homomorphism h between them

**locale** group_hom = G: group G + H: group H **for** G (**structure**) **and** H (**structure**) +
**fixes** h
**assumes** homh: "h ∈ hom G H"

**lemma** (**in** group_hom) hom_mult [simp]:
"[| x ∈ carrier G; y ∈ carrier G |] ==> h (x ⊗$_G$ y) = h x ⊗$_H$ h y"
⟨*proof*⟩

**lemma** (**in** group_hom) hom_closed [simp]:
"x ∈ carrier G ==> h x ∈ carrier H"
⟨*proof*⟩

**lemma** (**in** group_hom) one_closed [simp]:
"h **1** ∈ carrier H"
⟨*proof*⟩

**lemma** (**in** group_hom) hom_one [simp]:
"h **1** = **1**$_H$"
⟨*proof*⟩

**lemma** (**in** group_hom) inv_closed [simp]:
"x ∈ carrier G ==> h (inv x) ∈ carrier H"
⟨*proof*⟩

**lemma** (**in** group_hom) hom_inv [simp]:
"x ∈ carrier G ==> h (inv x) = inv$_H$ (h x)"
⟨*proof*⟩

## 3.7 Commutative Structures

Naming convention: multiplicative structures that are commutative are called *commutative*, additive structures are called *Abelian*.

**locale** comm_monoid = monoid +
  **assumes** m_comm: "⟦x ∈ carrier G; y ∈ carrier G⟧ ⟹ x ⊗ y = y ⊗ x"

**lemma** (**in** comm_monoid) m_lcomm:
  "⟦x ∈ carrier G; y ∈ carrier G; z ∈ carrier G⟧ ⟹
   x ⊗ (y ⊗ z) = y ⊗ (x ⊗ z)"
⟨*proof*⟩

**lemmas** (**in** comm_monoid) m_ac = m_assoc m_comm m_lcomm

**lemma** comm_monoidI:
  **fixes** G (**structure**)
  **assumes** m_closed:
      "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y ∈ carrier
G"
    **and** one_closed: "**1** ∈ carrier G"
    **and** m_assoc:
      "!!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
      (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)"
    **and** l_one: "!!x. x ∈ carrier G ==> **1** ⊗ x = x"
    **and** m_comm:
      "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y = y ⊗ x"
  **shows** "comm_monoid G"
  ⟨*proof*⟩

**lemma** (**in** monoid) monoid_comm_monoidI:
  **assumes** m_comm:
      "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y = y ⊗ x"
  **shows** "comm_monoid G"
  ⟨*proof*⟩

**lemma** (**in** comm_monoid) nat_pow_distr:
  "[| x ∈ carrier G; y ∈ carrier G |] ==>
  (x ⊗ y) (^) (n::nat) = x (^) n ⊗ y (^) n"
  ⟨*proof*⟩

**locale** comm_group = comm_monoid + group

**lemma** (**in** group) group_comm_groupI:
  **assumes** m_comm: "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==>
      x ⊗ y = y ⊗ x"
  **shows** "comm_group G"
  ⟨*proof*⟩

**lemma** comm_groupI:
**fixes** G (**structure**)
**assumes** m_closed:
"!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y ∈ carrier G"
**and** one_closed: "**1** ∈ carrier G"
**and** m_assoc:
"!!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==> (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)"
**and** m_comm:
"!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y = y ⊗ x"
**and** l_one: "!!x. x ∈ carrier G ==> **1** ⊗ x = x"
**and** l_inv_ex: "!!x. x ∈ carrier G ==> ∃y ∈ carrier G. y ⊗ x = **1**"
**shows** "comm_group G"
⟨*proof*⟩

**lemma** (**in** comm_group) inv_mult:
"[| x ∈ carrier G; y ∈ carrier G |] ==> inv (x ⊗ y) = inv x ⊗ inv y"
⟨*proof*⟩

## 3.8 The Lattice of Subgroups of a Group

**theorem** (**in** group) subgroups_partial_order:
"partial_order (| carrier = {H. subgroup H G}, eq = op =, le = op ⊆ |)"
⟨*proof*⟩

**lemma** (**in** group) subgroup_self:
"subgroup (carrier G) G"
⟨*proof*⟩

**lemma** (**in** group) subgroup_imp_group:
"subgroup H G ==> group (G(| carrier := H |))"
⟨*proof*⟩

**lemma** (**in** group) is_monoid [intro, simp]:
"monoid G"
⟨*proof*⟩

**lemma** (**in** group) subgroup_inv_equality:
"[| subgroup H G; x ∈ H |] ==> m_inv (G (| carrier := H |)) x = inv x"
⟨*proof*⟩

**theorem** (**in** group) subgroups_Inter:
**assumes** subgr: "(!!H. H ∈ A ==> subgroup H G)"
**and** not_empty: "A ~= {}"
**shows** "subgroup (⋂A) G"

⟨*proof*⟩

**theorem** (**in** group) subgroups_complete_lattice:
  "complete_lattice (| carrier = {H. subgroup H G}, eq = op =, le = op ⊆ |)"
    (**is** "complete_lattice ?L")
⟨*proof*⟩

**end**

**theory** FiniteProduct
**imports** Group
**begin**

## 3.9 Product Operator for Commutative Monoids

### 3.9.1 Inductive Definition of a Relation for Products over Sets

Instantiation of locale LC of theory Finite_Set is not possible, because here we have explicit typing rules like x ∈ carrier G. We introduce an explicit argument for the domain D.

**inductive_set**
  foldSetD :: "['a set, 'b => 'a => 'a, 'a] => ('b set * 'a) set"
  **for** D :: "'a set" **and** f :: "'b => 'a => 'a" **and** e :: 'a
  **where**
    emptyI [intro]: "e ∈ D ==> ({}, e) ∈ foldSetD D f e"
  | insertI [intro]: "[| x ~: A; f x y ∈ D; (A, y) ∈ foldSetD D f e |] ==>
                      (insert x A, f x y) ∈ foldSetD D f e"

**inductive_cases** empty_foldSetDE [elim!]: "({}, x) ∈ foldSetD D f e"

**definition**
  foldD :: "['a set, 'b => 'a => 'a, 'a, 'a set] => 'a"
  **where** "foldD D f e A = (THE x. (A, x) ∈ foldSetD D f e)"

**lemma** foldSetD_closed:
  "[| (A, z) ∈ foldSetD D f e ; e ∈ D; !!x y. [| x ∈ A; y ∈ D |] ==> f x y ∈ D
      |] ==> z ∈ D"
  ⟨*proof*⟩

**lemma** Diff1_foldSetD:
  "[| (A - {x}, y) ∈ foldSetD D f e; x ∈ A; f x y ∈ D |] ==>
   (A, f x y) ∈ foldSetD D f e"
  ⟨*proof*⟩

**lemma** `foldSetD_imp_finite [simp]: "(A, x) ∈ foldSetD D f e ==> finite A"`
⟨*proof*⟩

**lemma** `finite_imp_foldSetD:`
`"[| finite A; e ∈ D; !!x y. [| x ∈ A; y ∈ D |] ==> f x y ∈ D |] ==>`
`EX x. (A, x) ∈ foldSetD D f e"`
⟨*proof*⟩

Left-Commutative Operations

**locale** `LCD =`
**fixes** `B :: "'b set"`
**and** `D :: "'a set"`
**and** `f :: "'b => 'a => 'a"` (**infixl** "·" 70)
**assumes** `left_commute:`
`"[| x ∈ B; y ∈ B; z ∈ D |] ==> x · (y · z) = y · (x · z)"`
**and** `f_closed [simp, intro!]: "!!x y. [| x ∈ B; y ∈ D |] ==> f x y ∈ D"`

**lemma (in** `LCD) foldSetD_closed [dest]:`
`"(A, z) ∈ foldSetD D f e ==> z ∈ D"`
⟨*proof*⟩

**lemma (in** `LCD) Diff1_foldSetD:`
`"[| (A - {x}, y) ∈ foldSetD D f e; x ∈ A; A ⊆ B |] ==>`
`(A, f x y) ∈ foldSetD D f e"`
⟨*proof*⟩

**lemma (in** `LCD) foldSetD_imp_finite [simp]:`
`"(A, x) ∈ foldSetD D f e ==> finite A"`
⟨*proof*⟩

**lemma (in** `LCD) finite_imp_foldSetD:`
`"[| finite A; A ⊆ B; e ∈ D |] ==> EX x. (A, x) ∈ foldSetD D f e"`
⟨*proof*⟩

**lemma (in** `LCD) foldSetD_determ_aux:`
`"e ∈ D ==> ∀A x. A ⊆ B & card A < n --> (A, x) ∈ foldSetD D f e -->`
`(∀y. (A, y) ∈ foldSetD D f e --> y = x)"`
⟨*proof*⟩

**lemma (in** `LCD) foldSetD_determ:`
`"[| (A, x) ∈ foldSetD D f e; (A, y) ∈ foldSetD D f e; e ∈ D; A ⊆ B |]`
`==> y = x"`
⟨*proof*⟩

**lemma (in** `LCD) foldD_equality:`
`"[| (A, y) ∈ foldSetD D f e; e ∈ D; A ⊆ B |] ==> foldD D f e A = y"`

⟨*proof*⟩

**lemma** foldD_empty [simp]:
"e ∈ D ==> foldD D f e {} = e"
⟨*proof*⟩

**lemma** (**in** LCD) foldD_insert_aux:
"[| x ~: A; x ∈ B; e ∈ D; A ⊆ B |] ==>
((insert x A, v) ∈ foldSetD D f e) =
(EX y. (A, y) ∈ foldSetD D f e & v = f x y)"
⟨*proof*⟩

**lemma** (**in** LCD) foldD_insert:
"[| finite A; x ~: A; x ∈ B; e ∈ D; A ⊆ B |] ==>
foldD D f e (insert x A) = f x (foldD D f e A)"
⟨*proof*⟩

**lemma** (**in** LCD) foldD_closed [simp]:
"[| finite A; e ∈ D; A ⊆ B |] ==> foldD D f e A ∈ D"
⟨*proof*⟩

**lemma** (**in** LCD) foldD_commute:
"[| finite A; x ∈ B; e ∈ D; A ⊆ B |] ==>
f x (foldD D f e A) = foldD D f (f x e) A"
⟨*proof*⟩

**lemma** Int_mono2:
"[| A ⊆ C; B ⊆ C |] ==> A Int B ⊆ C"
⟨*proof*⟩

**lemma** (**in** LCD) foldD_nest_Un_Int:
"[| finite A; finite C; e ∈ D; A ⊆ B; C ⊆ B |] ==>
foldD D f (foldD D f e C) A = foldD D f (foldD D f e (A Int C)) (A Un C)"
⟨*proof*⟩

**lemma** (**in** LCD) foldD_nest_Un_disjoint:
"[| finite A; finite B; A Int B = {}; e ∈ D; A ⊆ B; C ⊆ B |]
==> foldD D f e (A Un B) = foldD D f (foldD D f e B) A"
⟨*proof*⟩

**declare** foldSetD_imp_finite [simp del]
empty_foldSetDE [rule del]
foldSetD.intros [rule del]
**declare** (**in** LCD)
foldSetD_closed [rule del]

Commutative Monoids

We enter a more restrictive context, with `f :: 'a => 'a => 'a` instead of `'b`

```
=> 'a => 'a.

locale ACeD =
  fixes D :: "'a set"
    and f :: "'a => 'a => 'a"    (infixl "·" 70)
    and e :: 'a
  assumes ident [simp]: "x ∈ D ==> x · e = x"
    and commute: "[| x ∈ D; y ∈ D |] ==> x · y = y · x"
    and assoc: "[| x ∈ D; y ∈ D; z ∈ D |] ==> (x · y) · z = x · (y · z)"
    and e_closed [simp]: "e ∈ D"
    and f_closed [simp]: "[| x ∈ D; y ∈ D |] ==> x · y ∈ D"

lemma (in ACeD) left_commute:
  "[| x ∈ D; y ∈ D; z ∈ D |] ==> x · (y · z) = y · (x · z)"
⟨proof⟩

lemmas (in ACeD) AC = assoc commute left_commute

lemma (in ACeD) left_ident [simp]: "x ∈ D ==> e · x = x"
⟨proof⟩

lemma (in ACeD) foldD_Un_Int:
  "[| finite A; finite B; A ⊆ D; B ⊆ D |] ==>
    foldD D f e A · foldD D f e B =
    foldD D f e (A Un B) · foldD D f e (A Int B)"
  ⟨proof⟩

lemma (in ACeD) foldD_Un_disjoint:
  "[| finite A; finite B; A Int B = {}; A ⊆ D; B ⊆ D |] ==>
    foldD D f e (A Un B) = foldD D f e A · foldD D f e B"
  ⟨proof⟩
```

### 3.9.2 Products over Finite Sets

```
definition
  finprod :: "[('b, 'm) monoid_scheme, 'a => 'b, 'a set] => 'b"
  where "finprod G f A =
   (if finite A
    then foldD (carrier G) (mult G o f) 1_G A
    else undefined)"

syntax
  "_finprod" :: "index => idt => 'a set => 'b => 'b"
      ("(3⨂__:_. _)" [1000, 0, 51, 10] 10)
syntax (xsymbols)
  "_finprod" :: "index => idt => 'a set => 'b => 'b"
      ("(3⨂__∈_. _)" [1000, 0, 51, 10] 10)
syntax (HTML output)
  "_finprod" :: "index => idt => 'a set => 'b => 'b"
      ("(3⨂__∈_. _)" [1000, 0, 51, 10] 10)
```

**translations**
"⨂ i:A. b" == "CONST finprod ◇ (%i. b) A"
— Beware of argument permutation!

**lemma (in** comm_monoid**)** finprod_empty [simp]:
"finprod G f {} = 𝟭"
⟨*proof*⟩

**declare** funcsetI [intro]
funcset_mem [dest]

**context** comm_monoid **begin**

**lemma** finprod_insert [simp]:
"[| finite F; a ∉ F; f ∈ F -> carrier G; f a ∈ carrier G |] ==>
finprod G f (insert a F) = f a ⊗ finprod G f F"
⟨*proof*⟩

**lemma** finprod_one [simp]:
"finite A ==> (⨂ i:A. 𝟭) = 𝟭"
⟨*proof*⟩

**lemma** finprod_closed [simp]:
**fixes** A
**assumes** fin: "finite A" **and** f: "f ∈ A -> carrier G"
**shows** "finprod G f A ∈ carrier G"
⟨*proof*⟩

**lemma** funcset_Int_left [simp, intro]:
"[| f ∈ A -> C; f ∈ B -> C |] ==> f ∈ A Int B -> C"
⟨*proof*⟩

**lemma** funcset_Un_left [iff]:
"(f ∈ A Un B -> C) = (f ∈ A -> C & f ∈ B -> C)"
⟨*proof*⟩

**lemma** finprod_Un_Int:
"[| finite A; finite B; g ∈ A -> carrier G; g ∈ B -> carrier G |] ==>
finprod G g (A Un B) ⊗ finprod G g (A Int B) =
finprod G g A ⊗ finprod G g B"
— The reversed orientation looks more natural, but LOOPS as a simprule!
⟨*proof*⟩

**lemma** finprod_Un_disjoint:
"[| finite A; finite B; A Int B = {};
g ∈ A -> carrier G; g ∈ B -> carrier G |]
==> finprod G g (A Un B) = finprod G g A ⊗ finprod G g B"
⟨*proof*⟩

**lemma** `finprod_multf:`
```
  "[| finite A; f ∈ A -> carrier G; g ∈ A -> carrier G |] ==>
   finprod G (%x. f x ⊗ g x) A = (finprod G f A ⊗ finprod G g A)"
```
⟨*proof*⟩

**lemma** `finprod_cong':`
```
  "[| A = B; g ∈ B -> carrier G;
      !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B"
```
⟨*proof*⟩

**lemma** `finprod_cong:`
```
  "[| A = B; f ∈ B -> carrier G = True;
      !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B"
```

⟨*proof*⟩

Usually, if this rule causes a failed congruence proof error, the reason is that the premise `g ∈ B -> carrier G` cannot be shown. Adding `Pi_def` to the simpset is often useful. For this reason, `comm_monoid.finprod_cong` is not added to the simpset by default.

**end**

**declare** `funcsetI [rule del]`
`  funcset_mem [rule del]`

**context** `comm_monoid` **begin**

**lemma** `finprod_0 [simp]:`
```
  "f ∈ {0::nat} -> carrier G ==> finprod G f {..0} = f 0"
```
⟨*proof*⟩

**lemma** `finprod_Suc [simp]:`
```
  "f ∈ {..Suc n} -> carrier G ==>
   finprod G f {..Suc n} = (f (Suc n) ⊗ finprod G f {..n})"
```
⟨*proof*⟩

**lemma** `finprod_Suc2:`
```
  "f ∈ {..Suc n} -> carrier G ==>
   finprod G f {..Suc n} = (finprod G (%i. f (Suc i)) {..n} ⊗ f 0)"
```
⟨*proof*⟩

**lemma** `finprod_mult [simp]:`
```
  "[| f ∈ {..n} -> carrier G; g ∈ {..n} -> carrier G |] ==>
     finprod G (%i. f i ⊗ g i) {..n::nat} =
     finprod G f {..n} ⊗ finprod G g {..n}"
```
⟨*proof*⟩

**lemma** finprod_reindex:
**assumes** fin: "finite A"
**shows** "f : (h ‘ A) → carrier G ⟹
inj_on h A ==> finprod G f (h ‘ A) = finprod G (%x. f (h x)) A"
⟨*proof*⟩

**lemma** finprod_const:
**assumes** fin [simp]: "finite A"
**and** a [simp]: "a : carrier G"
**shows** "finprod G (%x. a) A = a (^) card A"
⟨*proof*⟩

**lemma** finprod_singleton:
**assumes** i_in_A: "i ∈ A" **and** fin_A: "finite A" **and** f_Pi: "f ∈ A → carrier G"
**shows** "(⨂j∈A. if i = j then f j else 1) = f i"
⟨*proof*⟩

**end**

**end**

**theory** Coset
**imports** Group
**begin**

# 4 Cosets and Quotient Groups

**definition**
r_coset :: "[_, 'a set, 'a] ⇒ 'a set" (**infixl** "#>ı" 60)
**where** "H #>$_G$ a = (⋃h∈H. {h ⊗$_G$ a})"

**definition**
l_coset :: "[_, 'a, 'a set] ⇒ 'a set" (**infixl** "<#ı" 60)
**where** "a <#$_G$ H = (⋃h∈H. {a ⊗$_G$ h})"

**definition**
RCOSETS :: "[_, 'a set] ⇒ ('a set)set" ("rcosetsı _" [81] 80)
**where** "rcosets$_G$ H = (⋃a∈carrier G. {H #>$_G$ a})"

**definition**
set_mult :: "[_, 'a set ,'a set] ⇒ 'a set" (**infixl** "<#>ı" 60)
**where** "H <#>$_G$ K = (⋃h∈H. ⋃k∈K. {h ⊗$_G$ k})"

**definition**

SET_INV :: "[_,'a set] ⇒ 'a set" ("set'_inv‸ _" [81] 80)
**where** "set_inv$_G$ H = (⋃h∈H. {inv$_G$ h})"

**locale** normal = subgroup + group +
**assumes** coset_eq: "(∀x ∈ carrier G. H #> x = x <# H)"

**abbreviation**
normal_rel :: "['a set, ('a, 'b) monoid_scheme] ⇒ bool" (**infixl** "◁" 60) **where**
"H ◁ G ≡ normal H G"

## 4.1 Basic Properties of Cosets

**lemma** (**in** group) coset_mult_assoc:
"[| M ⊆ carrier G; g ∈ carrier G; h ∈ carrier G |]
==> (M #> g) #> h = M #> (g ⊗ h)"
⟨*proof*⟩

**lemma** (**in** group) coset_mult_one [simp]: "M ⊆ carrier G ==> M #> **1** = M"
⟨*proof*⟩

**lemma** (**in** group) coset_mult_inv1:
"[| M #> (x ⊗ (inv y)) = M; x ∈ carrier G ; y ∈ carrier G;
M ⊆ carrier G |] ==> M #> x = M #> y"
⟨*proof*⟩

**lemma** (**in** group) coset_mult_inv2:
"[| M #> x = M #> y; x ∈ carrier G; y ∈ carrier G; M ⊆ carrier G |]
==> M #> (x ⊗ (inv y)) = M "
⟨*proof*⟩

**lemma** (**in** group) coset_join1:
"[| H #> x = H; x ∈ carrier G; subgroup H G |] ==> x ∈ H"
⟨*proof*⟩

**lemma** (**in** group) solve_equation:
"⟦subgroup H G; x ∈ H; y ∈ H⟧ ⟹ ∃h∈H. y = h ⊗ x"
⟨*proof*⟩

**lemma** (**in** group) repr_independence:
"⟦y ∈ H #> x; x ∈ carrier G; subgroup H G⟧ ⟹ H #> x = H #> y"
⟨*proof*⟩

**lemma** (**in** group) coset_join2:
"⟦x ∈ carrier G; subgroup H G; x∈H⟧ ⟹ H #> x = H"
— Alternative proof is to put x = **1** in repr_independence.

⟨*proof*⟩

**lemma (in** monoid**)** r_coset_subset_G:
"[| H ⊆ carrier G; x ∈ carrier G |] ==> H #> x ⊆ carrier G"
⟨*proof*⟩

**lemma (in** group**)** rcosI:
"[| h ∈ H; H ⊆ carrier G; x ∈ carrier G|] ==> h ⊗ x ∈ H #> x"
⟨*proof*⟩

**lemma (in** group**)** rcosetsI:
"⟦H ⊆ carrier G; x ∈ carrier G⟧ ⟹ H #> x ∈ rcosets H"
⟨*proof*⟩

Really needed?

**lemma (in** group**)** transpose_inv:
"[| x ⊗ y = z; x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |]
==> (inv x) ⊗ z = y"
⟨*proof*⟩

**lemma (in** group**)** rcos_self: "[| x ∈ carrier G; subgroup H G |] ==> x ∈ H #> x"
⟨*proof*⟩

Opposite of "repr_independence"

**lemma (in** group**)** repr_independenceD:
**assumes** "subgroup H G"
**assumes** ycarr: "y ∈ carrier G"
**and** repr: "H #> x = H #> y"
**shows** "y ∈ H #> x"
⟨*proof*⟩

Elements of a right coset are in the carrier

**lemma (in** subgroup**)** elemrcos_carrier:
**assumes** "group G"
**assumes** acarr: "a ∈ carrier G"
**and** a': "a' ∈ H #> a"
**shows** "a' ∈ carrier G"
⟨*proof*⟩

**lemma (in** subgroup**)** rcos_const:
**assumes** "group G"
**assumes** hH: "h ∈ H"
**shows** "H #> h = H"
⟨*proof*⟩

Step one for lemma rcos_module

**lemma (in** subgroup**)** rcos_module_imp:

```
  assumes "group G"
  assumes xcarr: "x ∈ carrier G"
      and x'cos: "x' ∈ H #> x"
  shows "(x' ⊗ inv x) ∈ H"
```
⟨*proof*⟩

Step two for lemma `rcos_module`

```
lemma (in subgroup) rcos_module_rev:
  assumes "group G"
  assumes carr: "x ∈ carrier G" "x' ∈ carrier G"
      and xixH: "(x' ⊗ inv x) ∈ H"
  shows "x' ∈ H #> x"
```
⟨*proof*⟩

Module property of right cosets

```
lemma (in subgroup) rcos_module:
  assumes "group G"
  assumes carr: "x ∈ carrier G" "x' ∈ carrier G"
  shows "(x' ∈ H #> x) = (x' ⊗ inv x ∈ H)"
```
⟨*proof*⟩

Right cosets are subsets of the carrier.

```
lemma (in subgroup) rcosets_carrier:
  assumes "group G"
  assumes XH: "X ∈ rcosets H"
  shows "X ⊆ carrier G"
```
⟨*proof*⟩

Multiplication of general subsets

```
lemma (in monoid) set_mult_closed:
  assumes Acarr: "A ⊆ carrier G"
      and Bcarr: "B ⊆ carrier G"
  shows "A <#> B ⊆ carrier G"
```
⟨*proof*⟩

```
lemma (in comm_group) mult_subgroups:
  assumes subH: "subgroup H G"
      and subK: "subgroup K G"
  shows "subgroup (H <#> K) G"
```
⟨*proof*⟩

```
lemma (in subgroup) lcos_module_rev:
  assumes "group G"
  assumes carr: "x ∈ carrier G" "x' ∈ carrier G"
      and xixH: "(inv x ⊗ x') ∈ H"
  shows "x' ∈ x <# H"
```
⟨*proof*⟩

## 4.2 Normal subgroups

**lemma** normal_imp_subgroup: "H ◁ G ⟹ subgroup H G"
⟨*proof*⟩

**lemma** (**in** group) normalI:
"subgroup H G ⟹ (∀x ∈ carrier G. H #> x = x <# H) ⟹ H ◁ G"
⟨*proof*⟩

**lemma** (**in** normal) inv_op_closed1:
"⟦x ∈ carrier G; h ∈ H⟧ ⟹ (inv x) ⊗ h ⊗ x ∈ H"
⟨*proof*⟩

**lemma** (**in** normal) inv_op_closed2:
"⟦x ∈ carrier G; h ∈ H⟧ ⟹ x ⊗ h ⊗ (inv x) ∈ H"
⟨*proof*⟩

Alternative characterization of normal subgroups

**lemma** (**in** group) normal_inv_iff:
"(N ◁ G) =
(subgroup N G & (∀x ∈ carrier G. ∀h ∈ N. x ⊗ h ⊗ (inv x) ∈ N))"
(**is** "_ = ?rhs")
⟨*proof*⟩

## 4.3 More Properties of Cosets

**lemma** (**in** group) lcos_m_assoc:
"[| M ⊆ carrier G; g ∈ carrier G; h ∈ carrier G |]
==> g <# (h <# M) = (g ⊗ h) <# M"
⟨*proof*⟩

**lemma** (**in** group) lcos_mult_one: "M ⊆ carrier G ==> **1** <# M = M"
⟨*proof*⟩

**lemma** (**in** group) l_coset_subset_G:
"[| H ⊆ carrier G; x ∈ carrier G |] ==> x <# H ⊆ carrier G"
⟨*proof*⟩

**lemma** (**in** group) l_coset_swap:
"⟦y ∈ x <# H; x ∈ carrier G; subgroup H G⟧ ⟹ x ∈ y <# H"
⟨*proof*⟩

**lemma** (**in** group) l_coset_carrier:
"[| y ∈ x <# H; x ∈ carrier G; subgroup H G |] ==> y ∈ carrier
G"
⟨*proof*⟩

**lemma** (**in** group) l_repr_imp_subset:
**assumes** y: "y ∈ x <# H" **and** x: "x ∈ carrier G" **and** sb: "subgroup H
G"

**shows** "y <# H ⊆ x <# H"
⟨*proof*⟩

**lemma (in** group) l_repr_independence:
**assumes** y: "y ∈ x <# H" **and** x: "x ∈ carrier G" **and** sb: "subgroup H G"
**shows** "x <# H = y <# H"
⟨*proof*⟩

**lemma (in** group) setmult_subset_G:
"⟦H ⊆ carrier G; K ⊆ carrier G⟧ ⟹ H <#> K ⊆ carrier G"
⟨*proof*⟩

**lemma (in** group) subgroup_mult_id: "subgroup H G ⟹ H <#> H = H"
⟨*proof*⟩

### 4.3.1 Set of Inverses of an r_coset.

**lemma (in** normal) rcos_inv:
**assumes** x: "x ∈ carrier G"
**shows** "set_inv (H #> x) = H #> (inv x)"
⟨*proof*⟩

### 4.3.2 Theorems for <#> with #> or <#.

**lemma (in** group) setmult_rcos_assoc:
"⟦H ⊆ carrier G; K ⊆ carrier G; x ∈ carrier G⟧
⟹ H <#> (K #> x) = (H <#> K) #> x"
⟨*proof*⟩

**lemma (in** group) rcos_assoc_lcos:
"⟦H ⊆ carrier G; K ⊆ carrier G; x ∈ carrier G⟧
⟹ (H #> x) <#> K = H <#> (x <# K)"
⟨*proof*⟩

**lemma (in** normal) rcos_mult_step1:
"⟦x ∈ carrier G; y ∈ carrier G⟧
⟹ (H #> x) <#> (H #> y) = (H <#> (x <# H)) #> y"
⟨*proof*⟩

**lemma (in** normal) rcos_mult_step2:
"⟦x ∈ carrier G; y ∈ carrier G⟧
⟹ (H <#> (x <# H)) #> y = (H <#> (H #> x)) #> y"
⟨*proof*⟩

**lemma (in** normal) rcos_mult_step3:
"⟦x ∈ carrier G; y ∈ carrier G⟧
⟹ (H <#> (H #> x)) #> y = H #> (x ⊗ y)"
⟨*proof*⟩

**lemma (in** normal**)** rcos_sum:
"⟦x ∈ carrier G; y ∈ carrier G⟧
⟹ (H #> x) <#> (H #> y) = H #> (x ⊗ y)"
⟨*proof*⟩

**lemma (in** normal**)** rcosets_mult_eq: "M ∈ rcosets H ⟹ H <#> M = M"
— generalizes subgroup_mult_id
⟨*proof*⟩

### 4.3.3 An Equivalence Relation

**definition**
r_congruent :: "[('a,'b)monoid_scheme, 'a set] ⇒ ('a*'a)set" ("rcong↾
_")
**where** "rcong$_G$ H = {(x,y). x ∈ carrier G & y ∈ carrier G & inv$_G$ x ⊗$_G$
y ∈ H}"

**lemma (in** subgroup**)** equiv_rcong:
**assumes** "group G"
**shows** "equiv (carrier G) (rcong H)"
⟨*proof*⟩

Equivalence classes of rcong correspond to left cosets. Was there a mistake in the definitions? I'd have expected them to correspond to right cosets.

**lemma (in** subgroup**)** l_coset_eq_rcong:
**assumes** "group G"
**assumes** a: "a ∈ carrier G"
**shows** "a <# H = rcong H `` {a}"
⟨*proof*⟩

### 4.3.4 Two Distinct Right Cosets are Disjoint

**lemma (in** group**)** rcos_equation:
**assumes** "subgroup H G"
**assumes** p: "ha ⊗ a = h ⊗ b" "a ∈ carrier G" "b ∈ carrier G" "h ∈ H"
"ha ∈ H" "hb ∈ H"
**shows** "hb ⊗ a ∈ (⋃h∈H. {h ⊗ b})"
⟨*proof*⟩

**lemma (in** group**)** rcos_disjoint:
**assumes** "subgroup H G"
**assumes** p: "a ∈ rcosets H" "b ∈ rcosets H" "a≠b"
**shows** "a ∩ b = {}"
⟨*proof*⟩

## 4.4 Further lemmas for r_congruent

The relation is a congruence

**lemma (in** normal**)** congruent_rcong**:**
**shows** "congruent2 (rcong H) (rcong H) (λa b. a ⊗ b <# H)"
⟨*proof*⟩

## 4.5 Order of a Group and Lagrange's Theorem

**definition**
order :: "('a, 'b) monoid_scheme ⇒ nat"
**where** "order S = card (carrier S)"

**lemma (in** group**)** rcosets_part_G**:**
**assumes** "subgroup H G"
**shows** "⋃(rcosets H) = carrier G"
⟨*proof*⟩

**lemma (in** group**)** cosets_finite**:**
"⟦c ∈ rcosets H; H ⊆ carrier G; finite (carrier G)⟧ ⟹ finite c"
⟨*proof*⟩

The next two lemmas support the proof of card_cosets_equal.

**lemma (in** group**)** inj_on_f**:**
"⟦H ⊆ carrier G; a ∈ carrier G⟧ ⟹ inj_on (λy. y ⊗ inv a) (H #> a)"
⟨*proof*⟩

**lemma (in** group**)** inj_on_g**:**
"⟦H ⊆ carrier G; a ∈ carrier G⟧ ⟹ inj_on (λy. y ⊗ a) H"
⟨*proof*⟩

**lemma (in** group**)** card_cosets_equal**:**
"⟦c ∈ rcosets H; H ⊆ carrier G; finite(carrier G)⟧
⟹ card c = card H"
⟨*proof*⟩

**lemma (in** group**)** rcosets_subset_PowG**:**
"subgroup H G ⟹ rcosets H ⊆ Pow(carrier G)"
⟨*proof*⟩

**theorem (in** group**)** lagrange**:**
"⟦finite(carrier G); subgroup H G⟧
⟹ card(rcosets H) * card(H) = order(G)"
⟨*proof*⟩

## 4.6 Quotient Groups: Factorization of a Group

**definition**
FactGroup :: "[('a,'b) monoid_scheme, 'a set] ⇒ ('a set) monoid" (**infixl** "Mod" 65)

— Actually defined for groups rather than monoids
**where** "FactGroup G H = (|carrier = rcosets$_G$ H, mult = set_mult G, one = H|)"

**lemma (in** normal**)** setmult_closed:
"⟦K1 ∈ rcosets H; K2 ∈ rcosets H⟧ ⟹ K1 <#> K2 ∈ rcosets H"
⟨*proof*⟩

**lemma (in** normal**)** setinv_closed:
"K ∈ rcosets H ⟹ set_inv K ∈ rcosets H"
⟨*proof*⟩

**lemma (in** normal**)** rcosets_assoc:
"⟦M1 ∈ rcosets H; M2 ∈ rcosets H; M3 ∈ rcosets H⟧
⟹ M1 <#> M2 <#> M3 = M1 <#> (M2 <#> M3)"
⟨*proof*⟩

**lemma (in** subgroup**)** subgroup_in_rcosets:
**assumes** "group G"
**shows** "H ∈ rcosets H"
⟨*proof*⟩

**lemma (in** normal**)** rcosets_inv_mult_group_eq:
"M ∈ rcosets H ⟹ set_inv M <#> M = H"
⟨*proof*⟩

**theorem (in** normal**)** factorgroup_is_group:
"group (G Mod H)"
⟨*proof*⟩

**lemma** mult_FactGroup [simp]: "X ⊗$_{(G\ Mod\ H)}$ X' = X <#>$_G$ X'"
⟨*proof*⟩

**lemma (in** normal**)** inv_FactGroup:
"X ∈ carrier (G Mod H) ⟹ inv$_{G\ Mod\ H}$ X = set_inv X"
⟨*proof*⟩

The coset map is a homomorphism from G to the quotient group G Mod H

**lemma (in** normal**)** r_coset_hom_Mod:
"(λa. H #> a) ∈ hom G (G Mod H)"
⟨*proof*⟩

## 4.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

**definition**
kernel :: "('a, 'm) monoid_scheme ⇒ ('b, 'n) monoid_scheme ⇒ ('a ⇒ 'b) ⇒ 'a set"

— the kernel of a homomorphism
**where** "kernel G H h = {x. x ∈ carrier G & h x = $\mathbf{1}_H$}"

**lemma (in** group_hom**)** subgroup_kernel: "subgroup (kernel G H h) G"
⟨*proof*⟩

The kernel of a homomorphism is a normal subgroup

**lemma (in** group_hom**)** normal_kernel: "(kernel G H h) ◁ G"
⟨*proof*⟩

**lemma (in** group_hom**)** FactGroup_nonempty:
**assumes** X: "X ∈ carrier (G Mod kernel G H h)"
**shows** "X ≠ {}"
⟨*proof*⟩

**lemma (in** group_hom**)** FactGroup_contents_mem:
**assumes** X: "X ∈ carrier (G Mod (kernel G H h))"
**shows** "contents (h‘X) ∈ carrier H"
⟨*proof*⟩

**lemma (in** group_hom**)** FactGroup_hom:
"(λX. contents (h‘X)) ∈ hom (G Mod (kernel G H h)) H"
⟨*proof*⟩

Lemma for the following injectivity result

**lemma (in** group_hom**)** FactGroup_subset:
"⟦g ∈ carrier G; g’ ∈ carrier G; h g = h g’⟧
⟹ kernel G H h #> g ⊆ kernel G H h #> g’"
⟨*proof*⟩

**lemma (in** group_hom**)** FactGroup_inj_on:
"inj_on (λX. contents (h ‘ X)) (carrier (G Mod kernel G H h))"
⟨*proof*⟩

If the homomorphism h is onto H, then so is the homomorphism from the quotient group

**lemma (in** group_hom**)** FactGroup_onto:
**assumes** h: "h ‘ carrier G = carrier H"
**shows** "(λX. contents (h ‘ X)) ‘ carrier (G Mod kernel G H h) = carrier H"
⟨*proof*⟩

If h is a homomorphism from G onto H, then the quotient group G Mod kernel G H h is isomorphic to H.

**theorem (in** group_hom**)** FactGroup_iso:
"h ‘ carrier G = carrier H
⟹ (λX. contents (h‘X)) ∈ (G Mod (kernel G H h)) ≅ H"

⟨*proof*⟩

**end**

**theory** Exponent
**imports** Main "~~/src/HOL/Old_Number_Theory/Primes" Binomial
**begin**

# 5 Sylow's Theorem

## 5.1 The Combinatorial Argument Underlying the First Sylow Theorem

**definition**
```
  exponent :: "nat => nat => nat"
  where "exponent p s = (if prime p then (GREATEST r. p^r dvd s) else
0)"
```

Prime Theorems

**lemma** prime_imp_one_less: "prime p ==> Suc 0 < p"
⟨*proof*⟩

**lemma** prime_iff:
  "(prime p) = (Suc 0 < p & (∀a b. p dvd a*b --> (p dvd a) | (p dvd b)))"
⟨*proof*⟩

**lemma** zero_less_prime_power: "prime p ==> 0 < p^a"
⟨*proof*⟩

**lemma** zero_less_card_empty: "[| finite S; S ≠ {} |] ==> 0 < card(S)"
⟨*proof*⟩

**lemma** prime_dvd_cases:
  "[| p*k dvd m*n;  prime p |]
   ==> (∃x. k dvd x*n & m = p*x) | (∃y. k dvd m*y & n = p*y)"
⟨*proof*⟩

**lemma** prime_power_dvd_cases [rule_format (no_asm)]: "prime p
  ==> ∀m n. p^c dvd m*n -->
        (∀a b. a+b = Suc c --> p^a dvd m | p^b dvd n)"
⟨*proof*⟩

**lemma** div_combine:
```
  "[| prime p; ~ (p ^ (Suc r) dvd n);  p^(a+r) dvd n*k |]
   ==> p ^ a dvd k"
```
⟨*proof*⟩

**lemma** Suc_le_power: "Suc 0 < p ==> Suc n <= p^n"
⟨*proof*⟩

**lemma** power_dvd_bound: "[|p^n dvd a;  Suc 0 < p;  a > 0|] ==> n < a"
⟨*proof*⟩

Exponent Theorems

**lemma** exponent_ge [rule_format]:
```
  "[|p^k dvd n;  prime p;  0<n|] ==> k <= exponent p n"
```
⟨*proof*⟩

**lemma** power_exponent_dvd: "s>0 ==> (p ^ exponent p s) dvd s"
⟨*proof*⟩

**lemma** power_Suc_exponent_Not_dvd:
```
  "[|(p * p ^ exponent p s) dvd s;  prime p |] ==> s=0"
```
⟨*proof*⟩

**lemma** exponent_power_eq [simp]: "prime p ==> exponent p (p^a) = a"
⟨*proof*⟩

**lemma** exponent_equalityI:
```
  "!r::nat. (p^r dvd a) = (p^r dvd b) ==> exponent p a = exponent p b"
```
⟨*proof*⟩

**lemma** exponent_eq_0 [simp]: "¬ prime p ==> exponent p s = 0"
⟨*proof*⟩

**lemma** exponent_mult_add1: "[| a > 0; b > 0 |]
```
  ==> (exponent p a) + (exponent p b) <= exponent p (a * b)"
```
⟨*proof*⟩

**lemma** exponent_mult_add2: "[| a > 0; b > 0 |]
```
  ==> exponent p (a * b) <= (exponent p a) + (exponent p b)"
```
⟨*proof*⟩

**lemma** exponent_mult_add: "[| a > 0; b > 0 |]
```
   ==> exponent p (a * b) = (exponent p a) + (exponent p b)"
```
⟨*proof*⟩

**lemma** not_divides_exponent_0: "~ (p dvd n) ==> exponent p n = 0"
⟨*proof*⟩

**lemma** exponent_1_eq_0 [simp]: "exponent p (Suc 0) = 0"
⟨*proof*⟩

Main Combinatorial Argument

**lemma** le_extend_mult: "[| c > 0; a <= b |] ==> a <= b * (c::nat)"
⟨*proof*⟩

**lemma** p_fac_forw_lemma:
"[| (m::nat) > 0; k > 0; k < p^a; (p^r) dvd (p^a)* m - k |] ==> r <= a"
⟨*proof*⟩

**lemma** p_fac_forw: "[| (m::nat) > 0; k>0; k < p^a; (p^r) dvd (p^a)* m - k |]
==> (p^r) dvd (p^a) - k"
⟨*proof*⟩

**lemma** r_le_a_forw:
"[| (k::nat) > 0; k < p^a; p>0; (p^r) dvd (p^a) - k |] ==> r <= a"
⟨*proof*⟩

**lemma** p_fac_backw: "[| m>0; k>0; (p::nat)≠0;  k < p^a;  (p^r) dvd p^a - k |]
==> (p^r) dvd (p^a)*m - k"
⟨*proof*⟩

**lemma** exponent_p_a_m_k_equation: "[| m>0; k>0; (p::nat)≠0;  k < p^a |]
==> exponent p (p^a * m - k) = exponent p (p^a - k)"
⟨*proof*⟩

Suc rules that we have to delete from the simpset

**lemmas** bad_Sucs = binomial_Suc_Suc mult_Suc mult_Suc_right

**lemma** p_not_div_choose_lemma [rule_format]:
"[| ∀i. Suc i < K --> exponent p (Suc i) = exponent p (Suc(j+i))|]
 ==> k<K --> exponent p ((j+k) choose k) = 0"
⟨*proof*⟩

**lemma** p_not_div_choose:
"[| k<K;  k<=n;

```
      ∀j. 0<j & j<K --> exponent p (n - k + (K - j)) = exponent p (K -
j)|]
   ==> exponent p (n choose k) = 0"
```
⟨*proof*⟩

**lemma** const_p_fac_right:
```
  "m>0 ==> exponent p ((p^a * m - Suc 0) choose (p^a - Suc 0)) = 0"
```
⟨*proof*⟩

**lemma** const_p_fac:
```
  "m>0 ==> exponent p (((p^a) * m) choose p^a) = exponent p m"
```
⟨*proof*⟩

**end**

**theory** Sylow
**imports** Coset Exponent
**begin**

See also [3].

The combinatorial argument is in theory Exponent

**locale** sylow = group +
```
  fixes p and a and m and calM and RelM
  assumes prime_p:   "prime p"
      and order_G:   "order(G) = (p^a) * m"
      and finite_G [iff]:  "finite (carrier G)"
  defines "calM == {s. s ⊆ carrier(G) & card(s) = p^a}"
      and "RelM == {(N1,N2). N1 ∈ calM & N2 ∈ calM &
                             (∃g ∈ carrier(G). N1 = (N2 #> g) )}"
```

**lemma** (**in** sylow) RelM_refl_on: "refl_on calM RelM"
⟨*proof*⟩

**lemma** (**in** sylow) RelM_sym: "sym RelM"
⟨*proof*⟩

**lemma** (**in** sylow) RelM_trans: "trans RelM"
⟨*proof*⟩

**lemma** (**in** sylow) RelM_equiv: "equiv calM RelM"
⟨*proof*⟩

**lemma** (**in** sylow) M_subset_calM_prep: "M' ∈ calM // RelM  ==> M' ⊆ calM"
⟨*proof*⟩

## 5.2 Main Part of the Proof

**locale** sylow_central = sylow +
  **fixes** H **and** M1 **and** M
  **assumes** M_in_quot: "M ∈ calM // RelM"
      **and** not_dvd_M: "~(p ^ Suc(exponent p m) dvd card(M))"
      **and** M1_in_M: "M1 ∈ M"
  **defines** "H == {g. g∈carrier G & M1 #> g = M1}"

**lemma** (**in** sylow_central) M_subset_calM: "M ⊆ calM"
⟨*proof*⟩

**lemma** (**in** sylow_central) card_M1: "card(M1) = p^a"
⟨*proof*⟩

**lemma** card_nonempty: "0 < card(S) ==> S ≠ {}"
⟨*proof*⟩

**lemma** (**in** sylow_central) exists_x_in_M1: "∃x. x∈M1"
⟨*proof*⟩

**lemma** (**in** sylow_central) M1_subset_G [simp]: "M1 ⊆ carrier G"
⟨*proof*⟩

**lemma** (**in** sylow_central) M1_inj_H: "∃f ∈ H→M1. inj_on f H"
  ⟨*proof*⟩

## 5.3 Discharging the Assumptions of sylow_central

**lemma** (**in** sylow) EmptyNotInEquivSet: "{} ∉ calM // RelM"
⟨*proof*⟩

**lemma** (**in** sylow) existsM1inM: "M ∈ calM // RelM ==> ∃M1. M1 ∈ M"
⟨*proof*⟩

**lemma** (**in** sylow) zero_less_o_G: "0 < order(G)"
⟨*proof*⟩

**lemma** (**in** sylow) zero_less_m: "m > 0"
⟨*proof*⟩

**lemma** (**in** sylow) card_calM: "card(calM) = (p^a) * m choose p^a"
⟨*proof*⟩

**lemma** (**in** sylow) zero_less_card_calM: "card calM > 0"
⟨*proof*⟩

**lemma** (**in** sylow) max_p_div_calM:
     "~ (p ^ Suc(exponent p m) dvd card(calM))"
⟨*proof*⟩

**lemma (in sylow) finite_calM: "finite calM"**
⟨*proof*⟩

**lemma (in sylow) lemma_A1:**
"∃M ∈ calM // RelM. ~ (p ^ Suc(exponent p m) dvd card(M))"
⟨*proof*⟩

### 5.3.1 Introduction and Destruct Rules for H

**lemma (in sylow_central) H_I: "[|g ∈ carrier G; M1 #> g = M1|] ==> g ∈ H"**
⟨*proof*⟩

**lemma (in sylow_central) H_into_carrier_G: "x ∈ H ==> x ∈ carrier G"**
⟨*proof*⟩

**lemma (in sylow_central) in_H_imp_eq: "g : H ==> M1 #> g = M1"**
⟨*proof*⟩

**lemma (in sylow_central) H_m_closed: "[| x∈H; y∈H|] ==> x ⊗ y ∈ H"**
⟨*proof*⟩

**lemma (in sylow_central) H_not_empty: "H ≠ {}"**
⟨*proof*⟩

**lemma (in sylow_central) H_is_subgroup: "subgroup H G"**
⟨*proof*⟩

**lemma (in sylow_central) rcosetGM1g_subset_G:**
"[| g ∈ carrier G; x ∈ M1 #>  g |] ==> x ∈ carrier G"
⟨*proof*⟩

**lemma (in sylow_central) finite_M1: "finite M1"**
⟨*proof*⟩

**lemma (in sylow_central) finite_rcosetGM1g: "g∈carrier G ==> finite (M1 #> g)"**
⟨*proof*⟩

**lemma (in sylow_central) M1_cardeq_rcosetGM1g:**
"g ∈ carrier G ==> card(M1 #> g) = card(M1)"
⟨*proof*⟩

**lemma (in sylow_central) M1_RelM_rcosetGM1g:**
"g ∈ carrier G ==> (M1, M1 #> g) ∈ RelM"
⟨*proof*⟩

## 5.4 Equal Cardinalities of M and the Set of Cosets

Injections between M and $\texttt{rcosets}_{\texttt{G}}$ H show that their cardinalities are equal.

**lemma** ElemClassEquiv:
"[| equiv A r; C ∈ A // r |] ==> ∀x ∈ C. ∀y ∈ C. (x,y)∈r"
⟨*proof*⟩

**lemma (in** sylow_central**)** M_elem_map:
"M2 ∈ M ==> ∃g. g ∈ carrier G & M1 #> g = M2"
⟨*proof*⟩

**lemmas (in** sylow_central**)** M_elem_map_carrier =
M_elem_map [THEN someI_ex, THEN conjunct1]

**lemmas (in** sylow_central**)** M_elem_map_eq =
M_elem_map [THEN someI_ex, THEN conjunct2]

**lemma (in** sylow_central**)** M_funcset_rcosets_H:
"(%x:M. H #> (SOME g. g ∈ carrier G & M1 #> g = x)) ∈ M → rcosets H"
⟨*proof*⟩

**lemma (in** sylow_central**)** inj_M_GmodH: "∃f ∈ M→rcosets H. inj_on f M"
⟨*proof*⟩

### 5.4.1 The Opposite Injection

**lemma (in** sylow_central**)** H_elem_map:
"H1 ∈ rcosets H ==> ∃g. g ∈ carrier G & H #> g = H1"
⟨*proof*⟩

**lemmas (in** sylow_central**)** H_elem_map_carrier =
H_elem_map [THEN someI_ex, THEN conjunct1]

**lemmas (in** sylow_central**)** H_elem_map_eq =
H_elem_map [THEN someI_ex, THEN conjunct2]

**lemma** EquivElemClass:
"[|equiv A r; M ∈ A//r; M1∈M; (M1,M2) ∈ r |] ==> M2 ∈ M"
⟨*proof*⟩

**lemma (in** sylow_central**)** rcosets_H_funcset_M:
"(λC ∈ rcosets H. M1 #> (@g. g ∈ carrier G ∧ H #> g = C)) ∈ rcosets H → M"
⟨*proof*⟩

close to a duplicate of inj_M_GmodH

**lemma (in** sylow_central**)** inj_GmodH_M:
"∃g ∈ rcosets H→M. inj_on g (rcosets H)"
⟨*proof*⟩

**lemma (in** sylow_central**)** calM_subset_PowG: "calM ⊆ Pow(carrier G)"
⟨*proof*⟩

**lemma (in** sylow_central**)** finite_M: "finite M"
⟨*proof*⟩

**lemma (in** sylow_central**)** cardMeqIndexH: "card(M) = card(rcosets H)"
⟨*proof*⟩

**lemma (in** sylow_central**)** index_lem: "card(M) * card(H) = order(G)"
⟨*proof*⟩

**lemma (in** sylow_central**)** lemma_leq1: "p^a ≤ card(H)"
⟨*proof*⟩

**lemma (in** sylow_central**)** lemma_leq2: "card(H) ≤ p^a"
⟨*proof*⟩

**lemma (in** sylow_central**)** card_H_eq: "card(H) = p^a"
⟨*proof*⟩

**lemma (in** sylow**)** sylow_thm: "∃H. subgroup H G & card(H) = p^a"
⟨*proof*⟩

Needed because the locale's automatic definition refers to `semigroup G` and `group_axioms G` rather than simply to `group G`.

**lemma** sylow_eq: "sylow G p a m = (group G & sylow_axioms G p a m)"
⟨*proof*⟩

## 5.5 Sylow's Theorem

**theorem** sylow_thm:
"[| prime p; group(G); order(G) = (p^a) * m; finite (carrier G)|]
==> ∃H. subgroup H G & card(H) = p^a"
⟨*proof*⟩

**end**

**theory** Bij
**imports** Group
**begin**

# 6 Bijections of a Set, Permutation and Automorphism Groups

**definition**
```
  Bij :: "'a set ⇒ ('a ⇒ 'a) set"
```
— Only extensional functions, since otherwise we get too many.
```
  where "Bij S = extensional S ∩ {f. bij_betw f S S}"
```

**definition**
```
  BijGroup :: "'a set ⇒ ('a ⇒ 'a) monoid"
  where "BijGroup S =
    (|carrier = Bij S,
     mult = λg ∈ Bij S. λf ∈ Bij S. compose S g f,
     one = λx ∈ S. x|)"
```

**declare** `Id_compose [simp] compose_Id [simp]`

**lemma** `Bij_imp_extensional: "f ∈ Bij S ⟹ f ∈ extensional S"`
⟨*proof*⟩

**lemma** `Bij_imp_funcset: "f ∈ Bij S ⟹ f ∈ S → S"`
⟨*proof*⟩

## 6.1 Bijections Form a Group

**lemma** `restrict_inv_into_Bij: "f ∈ Bij S ⟹ (λx ∈ S. (inv_into S f) x) ∈ Bij S"`
⟨*proof*⟩

**lemma** `id_Bij: "(λx∈S. x) ∈ Bij S "`
⟨*proof*⟩

**lemma** `compose_Bij: "⟦x ∈ Bij S; y ∈ Bij S⟧ ⟹ compose S x y ∈ Bij S"`
⟨*proof*⟩

**lemma** `Bij_compose_restrict_eq:`
```
     "f ∈ Bij S ⟹ compose S (restrict (inv_into S f) S) f = (λx∈S. x)"
```
⟨*proof*⟩

**theorem** `group_BijGroup: "group (BijGroup S)"`
⟨*proof*⟩

## 6.2 Automorphisms Form a Group

**lemma** `Bij_inv_into_mem: "⟦ f ∈ Bij S;  x ∈ S⟧ ⟹ inv_into S f x ∈ S"`
⟨*proof*⟩

**lemma** Bij_inv_into_lemma:
**assumes** eq: "⋀x y. ⟦x ∈ S; y ∈ S⟧ ⟹ h(g x y) = g (h x) (h y)"
**shows** "⟦h ∈ Bij S; g ∈ S → S → S; x ∈ S; y ∈ S⟧
⟹ inv_into S h (g x y) = g (inv_into S h x) (inv_into S h y)"
⟨*proof*⟩

**definition**
auto :: "('a, 'b) monoid_scheme ⇒ ('a ⇒ 'a) set"
**where** "auto G = hom G G ∩ Bij (carrier G)"

**definition**
AutoGroup :: "('a, 'c) monoid_scheme ⇒ ('a ⇒ 'a) monoid"
**where** "AutoGroup G = BijGroup (carrier G) (|carrier := auto G|)"

**lemma** (**in** group) id_in_auto: "(λx ∈ carrier G. x) ∈ auto G"
⟨*proof*⟩

**lemma** (**in** group) mult_funcset: "mult G ∈ carrier G → carrier G → carrier G"
⟨*proof*⟩

**lemma** (**in** group) restrict_inv_into_hom:
"⟦h ∈ hom G G; h ∈ Bij (carrier G)⟧
⟹ restrict (inv_into (carrier G) h) (carrier G) ∈ hom G G"
⟨*proof*⟩

**lemma** inv_BijGroup:
"f ∈ Bij S ⟹ m_inv (BijGroup S) f = (λx ∈ S. (inv_into S f) x)"
⟨*proof*⟩

**lemma** (**in** group) subgroup_auto:
"subgroup (auto G) (BijGroup (carrier G))"
⟨*proof*⟩

**theorem** (**in** group) AutoGroup: "group (AutoGroup G)"
⟨*proof*⟩

**end**

**theory** Divisibility
**imports** Permutation Coset Group
**begin**

# 7 Factorial Monoids

## 7.1 Monoids with Cancellation Law

**locale** monoid_cancel = monoid +
  **assumes** l_cancel:
        "⟦c ⊗ a = c ⊗ b; a ∈ carrier G; b ∈ carrier G; c ∈ carrier G⟧ ⟹ a = b"
      **and** r_cancel:
        "⟦a ⊗ c = b ⊗ c; a ∈ carrier G; b ∈ carrier G; c ∈ carrier G⟧ ⟹ a = b"

**lemma** (**in** monoid) monoid_cancelI:
  **assumes** l_cancel:
        "⋀a b c. ⟦c ⊗ a = c ⊗ b; a ∈ carrier G; b ∈ carrier G; c ∈ carrier G⟧ ⟹ a = b"
      **and** r_cancel:
        "⋀a b c. ⟦a ⊗ c = b ⊗ c; a ∈ carrier G; b ∈ carrier G; c ∈ carrier G⟧ ⟹ a = b"
  **shows** "monoid_cancel G"
  ⟨*proof*⟩

**lemma** (**in** monoid_cancel) is_monoid_cancel:
  "monoid_cancel G"
  ⟨*proof*⟩

**sublocale** group ⊆ monoid_cancel
  ⟨*proof*⟩

**locale** comm_monoid_cancel = monoid_cancel + comm_monoid

**lemma** comm_monoid_cancelI:
  **fixes** G (**structure**)
  **assumes** "comm_monoid G"
  **assumes** cancel:
        "⋀a b c. ⟦a ⊗ c = b ⊗ c; a ∈ carrier G; b ∈ carrier G; c ∈ carrier G⟧ ⟹ a = b"
  **shows** "comm_monoid_cancel G"
⟨*proof*⟩

**lemma** (**in** comm_monoid_cancel) is_comm_monoid_cancel:
  "comm_monoid_cancel G"
  ⟨*proof*⟩

**sublocale** comm_group ⊆ comm_monoid_cancel
  ⟨*proof*⟩

## 7.2 Products of Units in Monoids

**lemma (in** monoid**)** Units_m_closed[simp, intro]:
  **assumes** h1unit: "h1 ∈ Units G" **and** h2unit: "h2 ∈ Units G"
  **shows** "h1 ⊗ h2 ∈ Units G"
⟨*proof*⟩

**lemma (in** monoid**)** prod_unit_l:
  **assumes** abunit[simp]: "a ⊗ b ∈ Units G" **and** aunit[simp]: "a ∈ Units G"
    **and** carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "b ∈ Units G"
⟨*proof*⟩

**lemma (in** monoid**)** prod_unit_r:
  **assumes** abunit[simp]: "a ⊗ b ∈ Units G" **and** bunit[simp]: "b ∈ Units G"
    **and** carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "a ∈ Units G"
⟨*proof*⟩

**lemma (in** comm_monoid**)** unit_factor:
  **assumes** abunit: "a ⊗ b ∈ Units G"
    **and** [simp]: "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "a ∈ Units G"
⟨*proof*⟩

## 7.3 Divisibility and Association

### 7.3.1 Function definitions

**definition**
  factor :: "[_, 'a, 'a] ⇒ bool" (**infix** "divides⇘_⇙" 65)
  **where** "a divides$_G$ b ⟷ (∃c∈carrier G. b = a ⊗$_G$ c)"

**definition**
  associated :: "[_, 'a, 'a] => bool" (**infix** "∼⇘_⇙" 55)
  **where** "a ∼$_G$ b ⟷ a divides$_G$ b ∧ b divides$_G$ a"

**abbreviation**
  "division_rel G == ⦇carrier = carrier G, eq = op ∼$_G$, le = op divides$_G$⦈"

**definition**
  properfactor :: "[_, 'a, 'a] ⇒ bool"
  **where** "properfactor G a b ⟷ a divides$_G$ b ∧ ¬(b divides$_G$ a)"

**definition**
  irreducible :: "[_, 'a] ⇒ bool"
  **where** "irreducible G a ⟷ a ∉ Units G ∧ (∀b∈carrier G. properfactor G b a ⟶ b ∈ Units G)"

**definition**
```
  prime :: "[_, 'a] ⇒ bool" where
  "prime G p ⟷
    p ∉ Units G ∧
    (∀a∈carrier G. ∀b∈carrier G. p dividesG (a ⊗G b) ⟶ p dividesG
a ∨ p dividesG b)"
```

### 7.3.2 Divisibility

**lemma** `dividesI:`
```
  fixes G (structure)
  assumes carr: "c ∈ carrier G"
    and p: "b = a ⊗ c"
  shows "a divides b"
```
⟨*proof*⟩

**lemma** `dividesI' [intro]:`
```
   fixes G (structure)
  assumes p: "b = a ⊗ c"
    and carr: "c ∈ carrier G"
  shows "a divides b"
```
⟨*proof*⟩

**lemma** `dividesD:`
```
  fixes G (structure)
  assumes "a divides b"
  shows "∃c∈carrier G. b = a ⊗ c"
```
⟨*proof*⟩

**lemma** `dividesE [elim]:`
```
  fixes G (structure)
  assumes d: "a divides b"
    and elim: "⋀c. ⟦b = a ⊗ c; c ∈ carrier G⟧ ⟹ P"
  shows "P"
```
⟨*proof*⟩

**lemma** (**in** `monoid`) `divides_refl[simp, intro!]:`
```
  assumes carr: "a ∈ carrier G"
  shows "a divides a"
```
⟨*proof*⟩

**lemma** (**in** `monoid`) `divides_trans [trans]:`
```
  assumes dvds: "a divides b"  "b divides c"
    and acarr: "a ∈ carrier G"
  shows "a divides c"
```
⟨*proof*⟩

**lemma** (**in** `monoid`) `divides_mult_lI [intro]:`

**assumes** ab: "a divides b"
**and** carr: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**shows** "(c ⊗ a) divides (c ⊗ b)"
⟨*proof*⟩

**lemma** (**in** monoid_cancel) divides_mult_l [simp]:
**assumes** carr: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**shows** "(c ⊗ a) divides (c ⊗ b) = a divides b"
⟨*proof*⟩

**lemma** (**in** comm_monoid) divides_mult_rI [intro]:
**assumes** ab: "a divides b"
**and** carr: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**shows** "(a ⊗ c) divides (b ⊗ c)"
⟨*proof*⟩

**lemma** (**in** comm_monoid_cancel) divides_mult_r [simp]:
**assumes** carr: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**shows** "(a ⊗ c) divides (b ⊗ c) = a divides b"
⟨*proof*⟩

**lemma** (**in** monoid) divides_prod_r:
**assumes** ab: "a divides b"
**and** carr: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**shows** "a divides (b ⊗ c)"
⟨*proof*⟩

**lemma** (**in** comm_monoid) divides_prod_l:
**assumes** carr[intro]: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**and** ab: "a divides b"
**shows** "a divides (c ⊗ b)"
⟨*proof*⟩

**lemma** (**in** monoid) unit_divides:
**assumes** uunit: "u ∈ Units G"
**and** acarr: "a ∈ carrier G"
**shows** "u divides a"
⟨*proof*⟩

**lemma** (**in** comm_monoid) divides_unit:
**assumes** udvd: "a divides u"
**and** carr: "a ∈ carrier G" "u ∈ Units G"
**shows** "a ∈ Units G"
⟨*proof*⟩

**lemma** (**in** comm_monoid) Unit_eq_dividesone:
**assumes** ucarr: "u ∈ carrier G"
**shows** "u ∈ Units G = u divides **1**"

⟨*proof*⟩

### 7.3.3 Association

**lemma** associatedI:
  **fixes** G (**structure**)
  **assumes** "a divides b"  "b divides a"
  **shows** "a ∼ b"
⟨*proof*⟩

**lemma** (**in** monoid) associatedI2:
  **assumes** uunit[simp]: "u ∈ Units G"
    **and** a: "a = b ⊗ u"
    **and** bcarr[simp]: "b ∈ carrier G"
  **shows** "a ∼ b"
⟨*proof*⟩

**lemma** (**in** monoid) associatedI2':
  **assumes** a: "a = b ⊗ u"
    **and** uunit: "u ∈ Units G"
    **and** bcarr: "b ∈ carrier G"
  **shows** "a ∼ b"
⟨*proof*⟩

**lemma** associatedD:
  **fixes** G (**structure**)
  **assumes** "a ∼ b"
  **shows** "a divides b"
⟨*proof*⟩

**lemma** (**in** monoid_cancel) associatedD2:
  **assumes** assoc: "a ∼ b"
    **and** carr: "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "∃u∈Units G. a = b ⊗ u"
⟨*proof*⟩

**lemma** associatedE:
  **fixes** G (**structure**)
  **assumes** assoc: "a ∼ b"
    **and** e: "⟦a divides b; b divides a⟧ ⟹ P"
  **shows** "P"
⟨*proof*⟩

**lemma** (**in** monoid_cancel) associatedE2:
  **assumes** assoc: "a ∼ b"
    **and** e: "⋀u. ⟦a = b ⊗ u; u ∈ Units G⟧ ⟹ P"
    **and** carr: "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "P"
⟨*proof*⟩

**lemma (in** monoid**)** associated_refl [simp, intro!]:
  **assumes** "a ∈ carrier G"
  **shows** "a ∼ a"
⟨*proof*⟩

**lemma (in** monoid**)** associated_sym [sym]:
  **assumes** "a ∼ b"
    **and** "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "b ∼ a"
⟨*proof*⟩

**lemma (in** monoid**)** associated_trans [trans]:
  **assumes** "a ∼ b"  "b ∼ c"
    **and** "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  **shows** "a ∼ c"
⟨*proof*⟩

**lemma (in** monoid**)** division_equiv [intro, simp]:
  "equivalence (division_rel G)"
  ⟨*proof*⟩

### 7.3.4 Division and associativity

**lemma** divides_antisym:
  **fixes** G **(structure)**
  **assumes** "a divides b"  "b divides a"
    **and** "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "a ∼ b"
⟨*proof*⟩

**lemma (in** monoid**)** divides_cong_l [trans]:
  **assumes** xx': "x ∼ x'"
    **and** xdvdy: "x' divides y"
    **and** carr [simp]: "x ∈ carrier G"  "x' ∈ carrier G"  "y ∈ carrier
G"
  **shows** "x divides y"
⟨*proof*⟩

**lemma (in** monoid**)** divides_cong_r [trans]:
  **assumes** xdvdy: "x divides y"
    **and** yy': "y ∼ y'"
    **and** carr[simp]: "x ∈ carrier G"  "y ∈ carrier G"  "y' ∈ carrier G"
  **shows** "x divides y'"
⟨*proof*⟩

**lemma (in** monoid**)** division_weak_partial_order [simp, intro!]:
  "weak_partial_order (division_rel G)"
  ⟨*proof*⟩

### 7.3.5 Multiplication and associativity

**lemma (in** monoid_cancel**)** mult_cong_r:
  **assumes** "b ∼ b'"
    **and** carr: "a ∈ carrier G"  "b ∈ carrier G"  "b' ∈ carrier G"
  **shows** "a ⊗ b ∼ a ⊗ b'"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** mult_cong_l:
  **assumes** "a ∼ a'"
    **and** carr: "a ∈ carrier G"  "a' ∈ carrier G"  "b ∈ carrier G"
  **shows** "a ⊗ b ∼ a' ⊗ b"
⟨*proof*⟩

**lemma (in** monoid_cancel**)** assoc_l_cancel:
  **assumes** carr: "a ∈ carrier G"  "b ∈ carrier G"  "b' ∈ carrier G"
    **and** "a ⊗ b ∼ a ⊗ b'"
  **shows** "b ∼ b'"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** assoc_r_cancel:
  **assumes** "a ⊗ b ∼ a' ⊗ b"
    **and** carr: "a ∈ carrier G"  "a' ∈ carrier G"  "b ∈ carrier G"
  **shows** "a ∼ a'"
⟨*proof*⟩

### 7.3.6 Units

**lemma (in** monoid_cancel**)** assoc_unit_l [trans]:
  **assumes** asc: "a ∼ b" **and** bunit: "b ∈ Units G"
    **and** carr: "a ∈ carrier G"
  **shows** "a ∈ Units G"
⟨*proof*⟩

**lemma (in** monoid_cancel**)** assoc_unit_r [trans]:
  **assumes** aunit: "a ∈ Units G" **and** asc: "a ∼ b"
    **and** bcarr: "b ∈ carrier G"
  **shows** "b ∈ Units G"
⟨*proof*⟩

**lemma (in** comm_monoid**)** Units_cong:
  **assumes** aunit: "a ∈ Units G" **and** asc: "a ∼ b"
    **and** bcarr: "b ∈ carrier G"
  **shows** "b ∈ Units G"
⟨*proof*⟩

**lemma (in** monoid**)** Units_assoc:
  **assumes** units: "a ∈ Units G"  "b ∈ Units G"
  **shows** "a ∼ b"
⟨*proof*⟩

**lemma (in** monoid**)** Units_are_ones:
"Units G {.=}$_{\text{(division\_rel G)}}$ {1}"
⟨*proof*⟩

**lemma (in** comm_monoid**)** Units_Lower:
"Units G = Lower (division_rel G) (carrier G)"
⟨*proof*⟩

### 7.3.7 Proper factors

**lemma** properfactorI:
**fixes** G **(structure)**
**assumes** "a divides b"
**and** "¬(b divides a)"
**shows** "properfactor G a b"
⟨*proof*⟩

**lemma** properfactorI2:
**fixes** G **(structure)**
**assumes** advdb: "a divides b"
**and** neq: "¬(a ∼ b)"
**shows** "properfactor G a b"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** properfactorI3:
**assumes** p: "p = a ⊗ b"
**and** nunit: "b ∉ Units G"
**and** carr: "a ∈ carrier G" "b ∈ carrier G" "p ∈ carrier G"
**shows** "properfactor G a p"
⟨*proof*⟩

**lemma** properfactorE:
**fixes** G **(structure)**
**assumes** pf: "properfactor G a b"
**and** r: "⟦a divides b; ¬(b divides a)⟧ ⟹ P"
**shows** "P"
⟨*proof*⟩

**lemma** properfactorE2:
**fixes** G **(structure)**
**assumes** pf: "properfactor G a b"
**and** elim: "⟦a divides b; ¬(a ∼ b)⟧ ⟹ P"
**shows** "P"
⟨*proof*⟩

**lemma (in** monoid**)** properfactor_unitE:
**assumes** uunit: "u ∈ Units G"
**and** pf: "properfactor G a u"

```
    and acarr: "a ∈ carrier G"
  shows "P"
⟨proof⟩


lemma (in monoid) properfactor_divides:
  assumes pf: "properfactor G a b"
  shows "a divides b"
⟨proof⟩

lemma (in monoid) properfactor_trans1 [trans]:
  assumes dvds: "a divides b"  "properfactor G b c"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "properfactor G a c"
⟨proof⟩

lemma (in monoid) properfactor_trans2 [trans]:
  assumes dvds: "properfactor G a b"  "b divides c"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "properfactor G a c"
⟨proof⟩

lemma properfactor_lless:
  fixes G (structure)
  shows "properfactor G = lless (division_rel G)"
⟨proof⟩

lemma (in monoid) properfactor_cong_l [trans]:
  assumes x'x: "x' ∼ x"
    and pf: "properfactor G x y"
    and carr: "x ∈ carrier G"  "x' ∈ carrier G"  "y ∈ carrier G"
  shows "properfactor G x' y"
⟨proof⟩

lemma (in monoid) properfactor_cong_r [trans]:
  assumes pf: "properfactor G x y"
    and yy': "y ∼ y'"
    and carr: "x ∈ carrier G"  "y ∈ carrier G"  "y' ∈ carrier G"
  shows "properfactor G x y'"
⟨proof⟩

lemma (in monoid_cancel) properfactor_mult_lI [intro]:
  assumes ab: "properfactor G a b"
    and carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
  shows "properfactor G (c ⊗ a) (c ⊗ b)"
⟨proof⟩

lemma (in monoid_cancel) properfactor_mult_l [simp]:
  assumes carr: "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
```

**shows** "properfactor G (c ⊗ a) (c ⊗ b) = properfactor G a b"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** properfactor_mult_rI [intro]:
**assumes** ab: "properfactor G a b"
**and** carr: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**shows** "properfactor G (a ⊗ c) (b ⊗ c)"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** properfactor_mult_r [simp]:
**assumes** carr: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**shows** "properfactor G (a ⊗ c) (b ⊗ c) = properfactor G a b"
⟨*proof*⟩

**lemma (in** monoid**)** properfactor_prod_r:
**assumes** ab: "properfactor G a b"
**and** carr[simp]: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**shows** "properfactor G a (b ⊗ c)"
⟨*proof*⟩

**lemma (in** comm_monoid**)** properfactor_prod_l:
**assumes** ab: "properfactor G a b"
**and** carr[simp]: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**shows** "properfactor G a (c ⊗ b)"
⟨*proof*⟩

## 7.4 Irreducible Elements and Primes

### 7.4.1 Irreducible elements

**lemma** irreducibleI:
**fixes** G **(structure)**
**assumes** "a ∉ Units G"
**and** "⋀b. ⟦b ∈ carrier G; properfactor G b a⟧ ⟹ b ∈ Units G"
**shows** "irreducible G a"
⟨*proof*⟩

**lemma** irreducibleE:
**fixes** G **(structure)**
**assumes** irr: "irreducible G a"
**and** elim: "⟦a ∉ Units G; ∀b. b ∈ carrier G ∧ properfactor G b a
⟶ b ∈ Units G⟧ ⟹ P"
**shows** "P"
⟨*proof*⟩

**lemma** irreducibleD:
**fixes** G **(structure)**
**assumes** irr: "irreducible G a"
**and** pf: "properfactor G b a"
**and** bcarr: "b ∈ carrier G"

**shows** "b ∈ Units G"
⟨*proof*⟩

**lemma (in** monoid_cancel**)** irreducible_cong [trans]:
**assumes** irred: "irreducible G a"
**and** aa': "a ∼ a'"
**and** carr[simp]: "a ∈ carrier G" "a' ∈ carrier G"
**shows** "irreducible G a'"
⟨*proof*⟩

**lemma (in** monoid**)** irreducible_prod_rI:
**assumes** airr: "irreducible G a"
**and** bunit: "b ∈ Units G"
**and** carr[simp]: "a ∈ carrier G" "b ∈ carrier G"
**shows** "irreducible G (a ⊗ b)"
⟨*proof*⟩

**lemma (in** comm_monoid**)** irreducible_prod_lI:
**assumes** birr: "irreducible G b"
**and** aunit: "a ∈ Units G"
**and** carr [simp]: "a ∈ carrier G" "b ∈ carrier G"
**shows** "irreducible G (a ⊗ b)"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** irreducible_prodE [elim]:
**assumes** irr: "irreducible G (a ⊗ b)"
**and** carr[simp]: "a ∈ carrier G" "b ∈ carrier G"
**and** e1: "⟦irreducible G a; b ∈ Units G⟧ ⟹ P"
**and** e2: "⟦a ∈ Units G; irreducible G b⟧ ⟹ P"
**shows** "P"
⟨*proof*⟩

### 7.4.2 Prime elements

**lemma** primeI:
**fixes** G **(structure)**
**assumes** "p ∉ Units G"
**and** "⋀a b. ⟦a ∈ carrier G; b ∈ carrier G; p divides (a ⊗ b)⟧ ⟹
p divides a ∨ p divides b"
**shows** "prime G p"
⟨*proof*⟩

**lemma** primeE:
**fixes** G **(structure)**
**assumes** pprime: "prime G p"
**and** e: "⟦p ∉ Units G; ∀a∈carrier G. ∀b∈carrier G.
p divides a ⊗ b ⟶ p divides a ∨ p divides
b⟧ ⟹ P"
**shows** "P"

⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** prime_divides:
  **assumes** carr: "a ∈ carrier G"  "b ∈ carrier G"
    **and** pprime: "prime G p"
    **and** pdvd: "p divides a ⊗ b"
  **shows** "p divides a ∨ p divides b"
⟨*proof*⟩

**lemma (in** monoid_cancel**)** prime_cong [trans]:
  **assumes** pprime: "prime G p"
    **and** pp': "p ∼ p'"
    **and** carr[simp]: "p ∈ carrier G"  "p' ∈ carrier G"
  **shows** "prime G p'"
⟨*proof*⟩

## 7.5 Factorization and Factorial Monoids

### 7.5.1 Function definitions

**definition**
  factors :: "[_, 'a list, 'a] ⇒ bool"
  **where** "factors G fs a ⟷ (∀x ∈ (set fs). irreducible G x) ∧ foldr (op ⊗$_G$) fs $\mathbf{1}_G$ = a"

**definition**
  wfactors :"[_, 'a list, 'a] ⇒ bool"
  **where** "wfactors G fs a ⟷ (∀x ∈ (set fs). irreducible G x) ∧ foldr (op ⊗$_G$) fs $\mathbf{1}_G$ ∼$_G$ a"

**abbreviation**
  list_assoc :: "('a,_) monoid_scheme ⇒ 'a list ⇒ 'a list ⇒ bool" **(infix** "[∼]$_\wr$" 44**)**
  **where** "list_assoc G == list_all2 (op ∼$_G$)"

**definition**
  essentially_equal :: "[_, 'a list, 'a list] ⇒ bool"
  **where** "essentially_equal G fs1 fs2 ⟷ (∃fs1'. fs1 <~~> fs1' ∧ fs1' [∼]$_G$ fs2)"

**locale** factorial_monoid = comm_monoid_cancel +
  **assumes** factors_exist:
        "⟦a ∈ carrier G; a ∉ Units G⟧ ⟹ ∃fs. set fs ⊆ carrier G ∧ factors G fs a"
      **and** factors_unique:
        "⟦factors G fs a; factors G fs' a; a ∈ carrier G; a ∉ Units G;
          set fs ⊆ carrier G; set fs' ⊆ carrier G⟧ ⟹ essentially_equal G fs fs'"

### 7.5.2 Comparing lists of elements

Association on lists

**lemma (in** monoid**)** listassoc_refl [simp, intro]:
  **assumes** "set as ⊆ carrier G"
  **shows** "as [∼] as"
⟨*proof*⟩

**lemma (in** monoid**)** listassoc_sym [sym]:
  **assumes** "as [∼] bs"
    **and** "set as ⊆ carrier G" **and** "set bs ⊆ carrier G"
  **shows** "bs [∼] as"
⟨*proof*⟩

**lemma (in** monoid**)** listassoc_trans [trans]:
  **assumes** "as [∼] bs" **and** "bs [∼] cs"
    **and** "set as ⊆ carrier G" **and** "set bs ⊆ carrier G" **and** "set cs ⊆ carrier G"
  **shows** "as [∼] cs"
⟨*proof*⟩

**lemma (in** monoid_cancel**)** irrlist_listassoc_cong:
  **assumes** "∀a∈set as. irreducible G a"
    **and** "as [∼] bs"
    **and** "set as ⊆ carrier G" **and** "set bs ⊆ carrier G"
  **shows** "∀a∈set bs. irreducible G a"
⟨*proof*⟩

Permutations

**lemma** perm_map [intro]:
  **assumes** p: "a <~~> b"
  **shows** "map f a <~~> map f b"
⟨*proof*⟩

**lemma** perm_map_switch:
  **assumes** m: "map f a = map f b" **and** p: "b <~~> c"
  **shows** "∃d. a <~~> d ∧ map f d = map f c"
⟨*proof*⟩

**lemma (in** monoid**)** perm_assoc_switch:
   **assumes** a:"as [∼] bs" **and** p: "bs <~~> cs"
   **shows** "∃bs'. as <~~> bs' ∧ bs' [∼] cs"
⟨*proof*⟩

**lemma (in** monoid**)** perm_assoc_switch_r:
   **assumes** p: "as <~~> bs" **and** a:"bs [∼] cs"
   **shows** "∃bs'. as [∼] bs' ∧ bs' <~~> cs"
⟨*proof*⟩

**declare** perm_sym [sym]

**lemma** perm_setP:
  **assumes** perm: "as <~~> bs"
    **and** as: "P (set as)"
  **shows** "P (set bs)"
⟨*proof*⟩

**lemmas** (**in** monoid) perm_closed =
    perm_setP[of _ _ "λas. as ⊆ carrier G"]

**lemmas** (**in** monoid) irrlist_perm_cong =
    perm_setP[of _ _ "λas. ∀a∈as. irreducible G a"]

Essentially equal factorizations

**lemma** (**in** monoid) essentially_equalI:
  **assumes** ex: "fs1 <~~> fs1'"  "fs1' [∼] fs2"
  **shows** "essentially_equal G fs1 fs2"
⟨*proof*⟩

**lemma** (**in** monoid) essentially_equalE:
  **assumes** ee: "essentially_equal G fs1 fs2"
    **and** e: "⋀fs1'. ⟦fs1 <~~> fs1'; fs1' [∼] fs2⟧ ⟹ P"
  **shows** "P"
⟨*proof*⟩

**lemma** (**in** monoid) ee_refl [simp,intro]:
  **assumes** carr: "set as ⊆ carrier G"
  **shows** "essentially_equal G as as"
⟨*proof*⟩

**lemma** (**in** monoid) ee_sym [sym]:
  **assumes** ee: "essentially_equal G as bs"
    **and** carr: "set as ⊆ carrier G"  "set bs ⊆ carrier G"
  **shows** "essentially_equal G bs as"
⟨*proof*⟩

**lemma** (**in** monoid) ee_trans [trans]:
  **assumes** ab: "essentially_equal G as bs" **and** bc: "essentially_equal
G bs cs"
    **and** ascarr: "set as ⊆ carrier G"
    **and** bscarr: "set bs ⊆ carrier G"
    **and** cscarr: "set cs ⊆ carrier G"
  **shows** "essentially_equal G as cs"
⟨*proof*⟩

### 7.5.3 Properties of lists of elements

Multiplication of factors in a list

**lemma (in** monoid**)** multlist_closed [simp, intro]:
**assumes** ascarr: "set fs ⊆ carrier G"
**shows** "foldr (op ⊗) fs **1** ∈ carrier G"
⟨*proof*⟩

**lemma (in** comm_monoid**)** multlist_dividesI :
**assumes** "f ∈ set fs" **and** "f ∈ carrier G" **and** "set fs ⊆ carrier G"
**shows** "f divides (foldr (op ⊗) fs **1**)"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** multlist_listassoc_cong:
**assumes** "fs [∼] fs'"
**and** "set fs ⊆ carrier G" **and** "set fs' ⊆ carrier G"
**shows** "foldr (op ⊗) fs **1** ∼ foldr (op ⊗) fs' **1**"
⟨*proof*⟩

**lemma (in** comm_monoid**)** multlist_perm_cong:
**assumes** prm: "as <~~> bs"
**and** ascarr: "set as ⊆ carrier G"
**shows** "foldr (op ⊗) as **1** = foldr (op ⊗) bs **1**"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** multlist_ee_cong:
**assumes** "essentially_equal G fs fs'"
**and** "set fs ⊆ carrier G" **and** "set fs' ⊆ carrier G"
**shows** "foldr (op ⊗) fs **1** ∼ foldr (op ⊗) fs' **1**"
⟨*proof*⟩

### 7.5.4 Factorization in irreducible elements

**lemma** wfactorsI:
**fixes** G **(structure)**
**assumes** "∀f∈set fs. irreducible G f"
**and** "foldr (op ⊗) fs **1** ∼ a"
**shows** "wfactors G fs a"
⟨*proof*⟩

**lemma** wfactorsE:
**fixes** G **(structure)**
**assumes** wf: "wfactors G fs a"
**and** e: "⟦∀f∈set fs. irreducible G f; foldr (op ⊗) fs **1** ∼ a⟧ ⟹
P"
**shows** "P"
⟨*proof*⟩

**lemma (in** monoid**)** factorsI:
**assumes** "∀f∈set fs. irreducible G f"
**and** "foldr (op ⊗) fs **1** = a"
**shows** "factors G fs a"

⟨*proof*⟩

**lemma** factorsE:
  **fixes** G (**structure**)
  **assumes** f: "factors G fs a"
    **and** e: "⟦∀f∈set fs. irreducible G f; foldr (op ⊗) fs **1** = a⟧ ⟹ P"
  **shows** "P"
⟨*proof*⟩

**lemma** (**in** monoid) factors_wfactors:
  **assumes** "factors G as a" **and** "set as ⊆ carrier G"
  **shows** "wfactors G as a"
⟨*proof*⟩

**lemma** (**in** monoid) wfactors_factors:
  **assumes** "wfactors G as a" **and** "set as ⊆ carrier G"
  **shows** "∃a'. factors G as a' ∧ a' ∼ a"
⟨*proof*⟩

**lemma** (**in** monoid) factors_closed [dest]:
  **assumes** "factors G fs a" **and** "set fs ⊆ carrier G"
  **shows** "a ∈ carrier G"
⟨*proof*⟩

**lemma** (**in** monoid) nunit_factors:
  **assumes** anunit: "a ∉ Units G"
    **and** fs: "factors G as a"
  **shows** "length as > 0"
⟨*proof*⟩

**lemma** (**in** monoid) unit_wfactors [simp]:
  **assumes** aunit: "a ∈ Units G"
  **shows** "wfactors G [] a"
⟨*proof*⟩

**lemma** (**in** comm_monoid_cancel) unit_wfactors_empty:
  **assumes** aunit: "a ∈ Units G"
    **and** wf: "wfactors G fs a"
    **and** carr[simp]: "set fs ⊆ carrier G"
  **shows** "fs = []"
⟨*proof*⟩

Comparing wfactors

**lemma** (**in** comm_monoid_cancel) wfactors_listassoc_cong_l:
  **assumes** fact: "wfactors G fs a"
    **and** asc: "fs [∼] fs'"
    **and** carr: "a ∈ carrier G"  "set fs ⊆ carrier G"  "set fs' ⊆ carrier G"
  **shows** "wfactors G fs' a"

⟨*proof*⟩

**lemma (in** comm_monoid**)** wfactors_perm_cong_l:
  **assumes** "wfactors G fs a"
    **and** "fs <~~> fs'"
    **and** "set fs ⊆ carrier G"
  **shows** "wfactors G fs' a"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** wfactors_ee_cong_l [trans]:
  **assumes** ee: "essentially_equal G as bs"
    **and** bfs: "wfactors G bs b"
    **and** carr: "b ∈ carrier G"  "set as ⊆ carrier G"  "set bs ⊆ carrier G"
  **shows** "wfactors G as b"
⟨*proof*⟩

**lemma (in** monoid**)** wfactors_cong_r [trans]:
  **assumes** fac: "wfactors G fs a" **and** aa': "a ∼ a'"
    **and** carr[simp]: "a ∈ carrier G"  "a' ∈ carrier G"  "set fs ⊆ carrier G"
  **shows** "wfactors G fs a'"
⟨*proof*⟩

### 7.5.5 Essentially equal factorizations

**lemma (in** comm_monoid_cancel**)** unitfactor_ee:
  **assumes** uunit: "u ∈ Units G"
    **and** carr: "set as ⊆ carrier G"
  **shows** "essentially_equal G (as[0 := (as!0 ⊗ u)]) as" (**is** "essentially_equal G ?as' as")
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** factors_cong_unit:
  **assumes** uunit: "u ∈ Units G" **and** anunit: "a ∉ Units G"
    **and** afs: "factors G as a"
    **and** ascarr: "set as ⊆ carrier G"
  **shows** "factors G (as[0 := (as!0 ⊗ u)]) (a ⊗ u)" (**is** "factors G ?as' ?a'")
⟨*proof*⟩

**lemma (in** comm_monoid**)** perm_wfactorsD:
  **assumes** prm: "as <~~> bs"
    **and** afs: "wfactors G as a" **and** bfs: "wfactors G bs b"
    **and** [simp]: "a ∈ carrier G"  "b ∈ carrier G"
    **and** ascarr[simp]: "set as ⊆ carrier G"
  **shows** "a ∼ b"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel) listassoc_wfactorsD:
  **assumes** assoc: "as [∼] bs"
    **and** afs: "wfactors G as a" **and** bfs: "wfactors G bs b"
    **and** [simp]: "a ∈ carrier G"  "b ∈ carrier G"
    **and** [simp]: "set as ⊆ carrier G"  "set bs ⊆ carrier G"
  **shows** "a ∼ b"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel) ee_wfactorsD:
  **assumes** ee: "essentially_equal G as bs"
    **and** afs: "wfactors G as a" **and** bfs: "wfactors G bs b"
    **and** [simp]: "a ∈ carrier G"  "b ∈ carrier G"
    **and** ascarr[simp]: "set as ⊆ carrier G" **and** bscarr[simp]: "set bs
⊆ carrier G"
  **shows** "a ∼ b"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel) ee_factorsD:
  **assumes** ee: "essentially_equal G as bs"
    **and** afs: "factors G as a" **and** bfs:"factors G bs b"
    **and** "set as ⊆ carrier G"  "set bs ⊆ carrier G"
  **shows** "a ∼ b"
⟨*proof*⟩

**lemma (in** factorial_monoid) ee_factorsI:
  **assumes** ab: "a ∼ b"
    **and** afs: "factors G as a" **and** anunit: "a ∉ Units G"
    **and** bfs: "factors G bs b" **and** bnunit: "b ∉ Units G"
    **and** ascarr: "set as ⊆ carrier G" **and** bscarr: "set bs ⊆ carrier G"
  **shows** "essentially_equal G as bs"
⟨*proof*⟩

**lemma (in** factorial_monoid) ee_wfactorsI:
  **assumes** asc: "a ∼ b"
    **and** asf: "wfactors G as a" **and** bsf: "wfactors G bs b"
    **and** acarr[simp]: "a ∈ carrier G" **and** bcarr[simp]: "b ∈ carrier G"
    **and** ascarr[simp]: "set as ⊆ carrier G" **and** bscarr[simp]: "set bs
⊆ carrier G"
  **shows** "essentially_equal G as bs"
⟨*proof*⟩

**lemma (in** factorial_monoid) ee_wfactors:
  **assumes** asf: "wfactors G as a"
    **and** bsf: "wfactors G bs b"
    **and** acarr: "a ∈ carrier G" **and** bcarr: "b ∈ carrier G"
    **and** ascarr: "set as ⊆ carrier G" **and** bscarr: "set bs ⊆ carrier G"
  **shows** asc: "a ∼ b = essentially_equal G as bs"
⟨*proof*⟩

**lemma (in** factorial_monoid**)** wfactors_exist [intro, simp]:
**assumes** acarr[simp]: "a ∈ carrier G"
**shows** "∃fs. set fs ⊆ carrier G ∧ wfactors G fs a"
⟨*proof*⟩

**lemma (in** monoid**)** wfactors_prod_exists [intro, simp]:
**assumes** "∀a ∈ set as. irreducible G a" **and** "set as ⊆ carrier G"
**shows** "∃a. a ∈ carrier G ∧ wfactors G as a"
⟨*proof*⟩

**lemma (in** factorial_monoid**)** wfactors_unique:
**assumes** "wfactors G fs a" **and** "wfactors G fs' a"
**and** "a ∈ carrier G"
**and** "set fs ⊆ carrier G" **and** "set fs' ⊆ carrier G"
**shows** "essentially_equal G fs fs'"
⟨*proof*⟩

**lemma (in** monoid**)** factors_mult_single:
**assumes** "irreducible G a" **and** "factors G fb b" **and** "a ∈ carrier G"
**shows** "factors G (a # fb) (a ⊗ b)"
⟨*proof*⟩

**lemma (in** monoid_cancel**)** wfactors_mult_single:
**assumes** f: "irreducible G a" "wfactors G fb b"
"a ∈ carrier G" "b ∈ carrier G" "set fb ⊆ carrier G"
**shows** "wfactors G (a # fb) (a ⊗ b)"
⟨*proof*⟩

**lemma (in** monoid**)** factors_mult:
**assumes** factors: "factors G fa a" "factors G fb b"
**and** ascarr: "set fa ⊆ carrier G" **and** bscarr:"set fb ⊆ carrier G"
**shows** "factors G (fa @ fb) (a ⊗ b)"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** wfactors_mult [intro]:
**assumes** asf: "wfactors G as a" **and** bsf:"wfactors G bs b"
**and** acarr: "a ∈ carrier G" **and** bcarr: "b ∈ carrier G"
**and** ascarr: "set as ⊆ carrier G" **and** bscarr:"set bs ⊆ carrier G"
**shows** "wfactors G (as @ bs) (a ⊗ b)"
⟨*proof*⟩

**lemma (in** comm_monoid**)** factors_dividesI:
**assumes** "factors G fs a" **and** "f ∈ set fs"
**and** "set fs ⊆ carrier G"
**shows** "f divides a"
⟨*proof*⟩

**lemma (in** comm_monoid**)** wfactors_dividesI:
**assumes** p: "wfactors G fs a"

```
    and fscarr: "set fs ⊆ carrier G" and acarr: "a ∈ carrier G"
    and f: "f ∈ set fs"
  shows "f divides a"
```
⟨*proof*⟩

### 7.5.6 Factorial monoids and wfactors

```
lemma (in comm_monoid_cancel) factorial_monoidI:
  assumes wfactors_exists:
          "⋀a. a ∈ carrier G ⟹ ∃fs. set fs ⊆ carrier G ∧ wfactors
G fs a"
      and wfactors_unique:
          "⋀a fs fs'. ⟦a ∈ carrier G; set fs ⊆ carrier G; set fs' ⊆ carrier
G;
                       wfactors G fs a; wfactors G fs' a⟧ ⟹ essentially_equal
G fs fs'"
  shows "factorial_monoid G"
```
⟨*proof*⟩

## 7.6 Factorizations as Multisets

Gives useful operations like intersection

**abbreviation**
```
  "assocs G x == eq_closure_of (division_rel G) {x}"
```

**definition**
```
  "fmset G as = multiset_of (map (λa. assocs G a) as)"
```

Helper lemmas

```
lemma (in monoid) assocs_repr_independence:
  assumes "y ∈ assocs G x"
    and "x ∈ carrier G"
  shows "assocs G x = assocs G y"
```
⟨*proof*⟩

```
lemma (in monoid) assocs_self:
  assumes "x ∈ carrier G"
  shows "x ∈ assocs G x"
```
⟨*proof*⟩

```
lemma (in monoid) assocs_repr_independenceD:
  assumes repr: "assocs G x = assocs G y"
    and ycarr: "y ∈ carrier G"
  shows "y ∈ assocs G x"
```
⟨*proof*⟩

```
lemma (in comm_monoid) assocs_assoc:
  assumes "a ∈ assocs G b"
    and "b ∈ carrier G"
```

**shows** "a ∼ b"
⟨*proof*⟩

**lemmas** (**in** comm_monoid) assocs_eqD =
assocs_repr_independenceD[THEN assocs_assoc]

### 7.6.1 Comparing multisets

**lemma** (**in** monoid) fmset_perm_cong:
**assumes** prm: "as <~~> bs"
**shows** "fmset G as = fmset G bs"
⟨*proof*⟩

**lemma** (**in** comm_monoid_cancel) eqc_listassoc_cong:
**assumes** "as [∼] bs"
**and** "set as ⊆ carrier G" **and** "set bs ⊆ carrier G"
**shows** "map (assocs G) as = map (assocs G) bs"
⟨*proof*⟩

**lemma** (**in** comm_monoid_cancel) fmset_listassoc_cong:
**assumes** "as [∼] bs"
**and** "set as ⊆ carrier G" **and** "set bs ⊆ carrier G"
**shows** "fmset G as = fmset G bs"
⟨*proof*⟩

**lemma** (**in** comm_monoid_cancel) ee_fmset:
**assumes** ee: "essentially_equal G as bs"
**and** ascarr: "set as ⊆ carrier G" **and** bscarr: "set bs ⊆ carrier G"
**shows** "fmset G as = fmset G bs"
⟨*proof*⟩

**lemma** (**in** monoid_cancel) fmset_ee__hlp_induct:
**assumes** prm: "cas <~~> cbs"
**and** cdef: "cas = map (assocs G) as"  "cbs = map (assocs G) bs"
**shows** "∀as bs. (cas <~~> cbs ∧ cas = map (assocs G) as ∧
cbs = map (assocs G) bs) ⟶ (∃as'. as <~~> as' ∧ map
(assocs G) as' = cbs)"
⟨*proof*⟩

**lemma** (**in** comm_monoid_cancel) fmset_ee:
**assumes** mset: "fmset G as = fmset G bs"
**and** ascarr: "set as ⊆ carrier G" **and** bscarr: "set bs ⊆ carrier G"
**shows** "essentially_equal G as bs"
⟨*proof*⟩

**lemma** (**in** comm_monoid_cancel) ee_is_fmset:
**assumes** "set as ⊆ carrier G" **and** "set bs ⊆ carrier G"
**shows** "essentially_equal G as bs = (fmset G as = fmset G bs)"
⟨*proof*⟩

### 7.6.2 Interpreting multisets as factorizations

**lemma (in** monoid**)** mset_fmsetEx:
**assumes** elems: "⋀X. X ∈ set_of Cs ⟹ ∃x. P x ∧ X = assocs G x"
**shows** "∃cs. (∀c ∈ set cs. P c) ∧ fmset G cs = Cs"
⟨*proof*⟩

**lemma (in** monoid**)** mset_wfactorsEx:
**assumes** elems: "⋀X. X ∈ set_of Cs
⟹ ∃x. (x ∈ carrier G ∧ irreducible G x) ∧ X =
assocs G x"
**shows** "∃c cs. c ∈ carrier G ∧ set cs ⊆ carrier G ∧ wfactors G cs c
∧ fmset G cs = Cs"
⟨*proof*⟩

### 7.6.3 Multiplication on multisets

**lemma (in** factorial_monoid**)** mult_wfactors_fmset:
**assumes** afs: "wfactors G as a" **and** bfs: "wfactors G bs b" **and** cfs:
"wfactors G cs (a ⊗ b)"
**and** carr: "a ∈ carrier G" "b ∈ carrier G"
"set as ⊆ carrier G" "set bs ⊆ carrier G" "set cs ⊆ carrier
G"
**shows** "fmset G cs = fmset G as + fmset G bs"
⟨*proof*⟩

**lemma (in** factorial_monoid**)** mult_factors_fmset:
**assumes** afs: "factors G as a" **and** bfs: "factors G bs b" **and** cfs: "factors
G cs (a ⊗ b)"
**and** "set as ⊆ carrier G" "set bs ⊆ carrier G" "set cs ⊆ carrier
G"
**shows** "fmset G cs = fmset G as + fmset G bs"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** fmset_wfactors_mult:
**assumes** mset: "fmset G cs = fmset G as + fmset G bs"
**and** carr: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
"set as ⊆ carrier G" "set bs ⊆ carrier G" "set cs ⊆ carrier
G"
**and** fs: "wfactors G as a" "wfactors G bs b" "wfactors G cs c"
**shows** "c ∼ a ⊗ b"
⟨*proof*⟩

### 7.6.4 Divisibility on multisets

**lemma (in** factorial_monoid**)** divides_fmsubset:
**assumes** ab: "a divides b"
**and** afs: "wfactors G as a" **and** bfs: "wfactors G bs b"
**and** carr: "a ∈ carrier G" "b ∈ carrier G" "set as ⊆ carrier G"
"set bs ⊆ carrier G"

**shows** "fmset G as ≤ fmset G bs"
⟨*proof*⟩

**lemma (in** comm_monoid_cancel**)** fmsubset_divides:
**assumes** msubset: "fmset G as ≤ fmset G bs"
**and** afs: "wfactors G as a" **and** bfs: "wfactors G bs b"
**and** acarr: "a ∈ carrier G" **and** bcarr: "b ∈ carrier G"
**and** ascarr: "set as ⊆ carrier G" **and** bscarr: "set bs ⊆ carrier G"
**shows** "a divides b"
⟨*proof*⟩

**lemma (in** factorial_monoid**)** divides_as_fmsubset:
**assumes** "wfactors G as a" **and** "wfactors G bs b"
**and** "a ∈ carrier G" **and** "b ∈ carrier G"
**and** "set as ⊆ carrier G" **and** "set bs ⊆ carrier G"
**shows** "a divides b = (fmset G as ≤ fmset G bs)"
⟨*proof*⟩

Proper factors on multisets

**lemma (in** factorial_monoid**)** fmset_properfactor:
**assumes** asubb: "fmset G as ≤ fmset G bs"
**and** anb: "fmset G as ≠ fmset G bs"
**and** "wfactors G as a" **and** "wfactors G bs b"
**and** "a ∈ carrier G" **and** "b ∈ carrier G"
**and** "set as ⊆ carrier G" **and** "set bs ⊆ carrier G"
**shows** "properfactor G a b"
⟨*proof*⟩

**lemma (in** factorial_monoid**)** properfactor_fmset:
**assumes** pf: "properfactor G a b"
**and** "wfactors G as a" **and** "wfactors G bs b"
**and** "a ∈ carrier G" **and** "b ∈ carrier G"
**and** "set as ⊆ carrier G" **and** "set bs ⊆ carrier G"
**shows** "fmset G as ≤ fmset G bs ∧ fmset G as ≠ fmset G bs"
⟨*proof*⟩

## 7.7 Irreducible Elements are Prime

**lemma (in** factorial_monoid**)** irreducible_is_prime:
**assumes** pirr: "irreducible G p"
**and** pcarr: "p ∈ carrier G"
**shows** "prime G p"
⟨*proof*⟩
**lemma (in** factorial_monoid**)** factors_irreducible_is_prime:
**assumes** pirr: "irreducible G p"
**and** pcarr: "p ∈ carrier G"
**shows** "prime G p"
⟨*proof*⟩

## 7.8 Greatest Common Divisors and Lowest Common Multiples

### 7.8.1 Definitions

**definition**
```
  isgcd :: "[('a,_) monoid_scheme, 'a, 'a, 'a] ⇒ bool"  ("(_ gcdof⇘ _
_)" [81,81,81] 80)
  where "x gcdof⇘G⇙ a b ⟷ x divides⇘G⇙ a ∧ x divides⇘G⇙ b ∧
    (∀y∈carrier G. (y divides⇘G⇙ a ∧ y divides⇘G⇙ b ⟶ y divides⇘G⇙ x))"
```

**definition**
```
  islcm :: "[_, 'a, 'a, 'a] ⇒ bool"  ("(_ lcmof⇘ _ _)" [81,81,81] 80)
  where "x lcmof⇘G⇙ a b ⟷ a divides⇘G⇙ x ∧ b divides⇘G⇙ x ∧
    (∀y∈carrier G. (a divides⇘G⇙ y ∧ b divides⇘G⇙ y ⟶ x divides⇘G⇙ y))"
```

**definition**
```
  somegcd :: "('a,_) monoid_scheme ⇒ 'a ⇒ 'a ⇒ 'a"
  where "somegcd G a b = (SOME x. x ∈ carrier G ∧ x gcdof⇘G⇙ a b)"
```

**definition**
```
  somelcm :: "('a,_) monoid_scheme ⇒ 'a ⇒ 'a ⇒ 'a"
  where "somelcm G a b = (SOME x. x ∈ carrier G ∧ x lcmof⇘G⇙ a b)"
```

**definition**
```
  "SomeGcd G A = inf (division_rel G) A"
```

**locale** gcd_condition_monoid = comm_monoid_cancel +
  **assumes** gcdof_exists:
```
        "⟦a ∈ carrier G; b ∈ carrier G⟧ ⟹ ∃c. c ∈ carrier G ∧ c gcdof
a b"
```

**locale** primeness_condition_monoid = comm_monoid_cancel +
  **assumes** irreducible_prime:
```
        "⟦a ∈ carrier G; irreducible G a⟧ ⟹ prime G a"
```

**locale** divisor_chain_condition_monoid = comm_monoid_cancel +
  **assumes** division_wellfounded:
```
        "wf {(x, y). x ∈ carrier G ∧ y ∈ carrier G ∧ properfactor G
x y}"
```

### 7.8.2 Connections to Lattice.thy

**lemma** gcdof_greatestLower:
  **fixes** G (**structure**)
  **assumes** carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "(x ∈ carrier G ∧ x gcdof a b) =
```
       greatest (division_rel G) x (Lower (division_rel G) {a, b})"
```
⟨*proof*⟩

**lemma** lcmof_leastUpper:
  **fixes** G (**structure**)
  **assumes** carr[simp]: "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "(x ∈ carrier G ∧ x lcmof a b) =
        least (division_rel G) x (Upper (division_rel G) {a, b})"
⟨*proof*⟩

**lemma** somegcd_meet:
  **fixes** G (**structure**)
  **assumes** carr: "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "somegcd G a b = meet (division_rel G) a b"
⟨*proof*⟩

**lemma** (**in** monoid) isgcd_divides_l:
  **assumes** "a divides b"
    **and** "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "a gcdof a b"
⟨*proof*⟩

**lemma** (**in** monoid) isgcd_divides_r:
  **assumes** "b divides a"
    **and** "a ∈ carrier G"  "b ∈ carrier G"
  **shows** "b gcdof a b"
⟨*proof*⟩

### 7.8.3 Existence of gcd and lcm

**lemma** (**in** factorial_monoid) gcdof_exists:
  **assumes** acarr: "a ∈ carrier G" **and** bcarr: "b ∈ carrier G"
  **shows** "∃c. c ∈ carrier G ∧ c gcdof a b"
⟨*proof*⟩

**lemma** (**in** factorial_monoid) lcmof_exists:
  **assumes** acarr: "a ∈ carrier G" **and** bcarr: "b ∈ carrier G"
  **shows** "∃c. c ∈ carrier G ∧ c lcmof a b"
⟨*proof*⟩

## 7.9 Conditions for Factoriality

### 7.9.1 Gcd condition

**lemma** (**in** gcd_condition_monoid) division_weak_lower_semilattice [simp]:
  **shows** "weak_lower_semilattice (division_rel G)"
⟨*proof*⟩

**lemma** (**in** gcd_condition_monoid) gcdof_cong_l:
  **assumes** a'a: "a' ∼ a"
    **and** agcd: "a gcdof b c"

**and** a'carr: "a' ∈ carrier G" **and** carr': "a ∈ carrier G"  "b ∈ carrier G"  "c ∈ carrier G"
**shows** "a' gcdof b c"
⟨*proof*⟩

**lemma** (**in** gcd_condition_monoid) gcd_closed [simp]:
**assumes** carr: "a ∈ carrier G"  "b ∈ carrier G"
**shows** "somegcd G a b ∈ carrier G"
⟨*proof*⟩

**lemma** (**in** gcd_condition_monoid) gcd_isgcd:
**assumes** carr: "a ∈ carrier G"  "b ∈ carrier G"
**shows** "(somegcd G a b) gcdof a b"
⟨*proof*⟩

**lemma** (**in** gcd_condition_monoid) gcd_exists:
**assumes** carr: "a ∈ carrier G"  "b ∈ carrier G"
**shows** "∃x∈carrier G. x = somegcd G a b"
⟨*proof*⟩

**lemma** (**in** gcd_condition_monoid) gcd_divides_l:
**assumes** carr: "a ∈ carrier G"  "b ∈ carrier G"
**shows** "(somegcd G a b) divides a"
⟨*proof*⟩

**lemma** (**in** gcd_condition_monoid) gcd_divides_r:
**assumes** carr: "a ∈ carrier G"  "b ∈ carrier G"
**shows** "(somegcd G a b) divides b"
⟨*proof*⟩

**lemma** (**in** gcd_condition_monoid) gcd_divides:
**assumes** sub: "z divides x"  "z divides y"
**and** L: "x ∈ carrier G"  "y ∈ carrier G"  "z ∈ carrier G"
**shows** "z divides (somegcd G x y)"
⟨*proof*⟩

**lemma** (**in** gcd_condition_monoid) gcd_cong_l:
**assumes** xx': "x ∼ x'"
**and** carr: "x ∈ carrier G"  "x' ∈ carrier G"  "y ∈ carrier G"
**shows** "somegcd G x y ∼ somegcd G x' y"
⟨*proof*⟩

**lemma** (**in** gcd_condition_monoid) gcd_cong_r:
**assumes** carr: "x ∈ carrier G"  "y ∈ carrier G"  "y' ∈ carrier G"
**and** yy': "y ∼ y'"
**shows** "somegcd G x y ∼ somegcd G x y'"
⟨*proof*⟩

**lemma (in** gcd_condition_monoid**)** gcdI:
**assumes** dvd: "a divides b" "a divides c"
**and** others: "∀y∈carrier G. y divides b ∧ y divides c ⟶ y divides a"
**and** acarr: "a ∈ carrier G" **and** bcarr: "b ∈ carrier G" **and** ccarr: "c ∈ carrier G"
**shows** "a ∼ somegcd G b c"
⟨*proof*⟩

**lemma (in** gcd_condition_monoid**)** gcdI2:
**assumes** "a gcdof b c"
**and** "a ∈ carrier G" **and** bcarr: "b ∈ carrier G" **and** ccarr: "c ∈ carrier G"
**shows** "a ∼ somegcd G b c"
⟨*proof*⟩

**lemma (in** gcd_condition_monoid**)** SomeGcd_ex:
**assumes** "finite A" "A ⊆ carrier G" "A ≠ {}"
**shows** "∃x∈ carrier G. x = SomeGcd G A"
⟨*proof*⟩

**lemma (in** gcd_condition_monoid**)** gcd_assoc:
**assumes** carr: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**shows** "somegcd G (somegcd G a b) c ∼ somegcd G a (somegcd G b c)"
⟨*proof*⟩

**lemma (in** gcd_condition_monoid**)** gcd_mult:
**assumes** acarr: "a ∈ carrier G" **and** bcarr: "b ∈ carrier G" **and** ccarr: "c ∈ carrier G"
**shows** "c ⊗ somegcd G a b ∼ somegcd G (c ⊗ a) (c ⊗ b)"
⟨*proof*⟩

**lemma (in** monoid**)** assoc_subst:
**assumes** ab: "a ∼ b"
**and** cP: "ALL a b. a : carrier G & b : carrier G & a ∼ b
--> f a : carrier G & f b : carrier G & f a ∼ f b"
**and** carr: "a ∈ carrier G" "b ∈ carrier G"
**shows** "f a ∼ f b"
⟨*proof*⟩

**lemma (in** gcd_condition_monoid**)** relprime_mult:
**assumes** abrelprime: "somegcd G a b ∼ 1" **and** acrelprime: "somegcd G a c ∼ 1"
**and** carr[simp]: "a ∈ carrier G" "b ∈ carrier G" "c ∈ carrier G"
**shows** "somegcd G a (b ⊗ c) ∼ 1"
⟨*proof*⟩

**lemma (in** gcd_condition_monoid**)** primeness_condition:

```
  "primeness_condition_monoid G"
```
⟨*proof*⟩

**sublocale** gcd_condition_monoid ⊆ primeness_condition_monoid
  ⟨*proof*⟩

### 7.9.2 Divisor chain condition

**lemma (in** divisor_chain_condition_monoid) wfactors_exist:
  **assumes** acarr: "a ∈ carrier G"
  **shows** "∃as. set as ⊆ carrier G ∧ wfactors G as a"
⟨*proof*⟩

### 7.9.3 Primeness condition

**lemma (in** comm_monoid_cancel) multlist_prime_pos:
  **assumes** carr: "a ∈ carrier G"  "set as ⊆ carrier G"
    **and** aprime: "prime G a"
    **and** "a divides (foldr (op ⊗) as **1**)"
  **shows** "∃i<length as. a divides (as!i)"
⟨*proof*⟩

**lemma (in** primeness_condition_monoid) wfactors_unique__hlp_induct:
  "∀a as'. a ∈ carrier G ∧ set as ⊆ carrier G ∧ set as' ⊆ carrier G
∧
           wfactors G as a ∧ wfactors G as' a ⟶ essentially_equal G
as as'"
⟨*proof*⟩

**lemma (in** primeness_condition_monoid) wfactors_unique:
  **assumes** "wfactors G as a"  "wfactors G as' a"
    **and** "a ∈ carrier G"  "set as ⊆ carrier G"  "set as' ⊆ carrier G"
  **shows** "essentially_equal G as as'"
⟨*proof*⟩

### 7.9.4 Application to factorial monoids

Number of factors for wellfoundedness

**definition**
  factorcount :: "_ ⇒ 'a ⇒ nat" **where**
  "factorcount G a =
    (THE c. (ALL as. set as ⊆ carrier G ∧ wfactors G as a ⟶ c = length
as))"

**lemma (in** monoid) ee_length:
  **assumes** ee: "essentially_equal G as bs"
  **shows** "length as = length bs"
⟨*proof*⟩

**lemma (in** factorial_monoid**)** factorcount_exists:
**assumes** carr[simp]: "a ∈ carrier G"
**shows** "EX c. ALL as. set as ⊆ carrier G ∧ wfactors G as a ⟶ c = length as"
⟨*proof*⟩

**lemma (in** factorial_monoid**)** factorcount_unique:
**assumes** afs: "wfactors G as a"
**and** acarr[simp]: "a ∈ carrier G" **and** ascarr[simp]: "set as ⊆ carrier G"
**shows** "factorcount G a = length as"
⟨*proof*⟩

**lemma (in** factorial_monoid**)** divides_fcount:
**assumes** dvd: "a divides b"
**and** acarr: "a ∈ carrier G" **and** bcarr:"b ∈ carrier G"
**shows** "factorcount G a <= factorcount G b"
⟨*proof*⟩

**lemma (in** factorial_monoid**)** associated_fcount:
**assumes** acarr: "a ∈ carrier G" **and** bcarr:"b ∈ carrier G"
**and** asc: "a ∼ b"
**shows** "factorcount G a = factorcount G b"
⟨*proof*⟩

**lemma (in** factorial_monoid**)** properfactor_fcount:
**assumes** acarr: "a ∈ carrier G" **and** bcarr:"b ∈ carrier G"
**and** pf: "properfactor G a b"
**shows** "factorcount G a < factorcount G b"
⟨*proof*⟩

**sublocale** factorial_monoid ⊆ divisor_chain_condition_monoid
⟨*proof*⟩

**sublocale** factorial_monoid ⊆ primeness_condition_monoid
⟨*proof*⟩

**lemma (in** factorial_monoid**)** primeness_condition:
**shows** "primeness_condition_monoid G"
⟨*proof*⟩

**lemma (in** factorial_monoid**)** gcd_condition [simp]:
**shows** "gcd_condition_monoid G"
⟨*proof*⟩

**sublocale** factorial_monoid ⊆ gcd_condition_monoid
⟨*proof*⟩

**lemma (in** factorial_monoid**) division_weak_lattice [simp]:**
**shows** "weak_lattice (division_rel G)"
⟨*proof*⟩

### 7.10 Factoriality Theorems

**theorem** factorial_condition_one:
**shows** "(divisor_chain_condition_monoid G ∧ primeness_condition_monoid G) =
factorial_monoid G"
⟨*proof*⟩

**theorem** factorial_condition_two:
**shows** "(divisor_chain_condition_monoid G ∧ gcd_condition_monoid G) = factorial_monoid G"
⟨*proof*⟩

**end**

**theory** Ring
**imports** FiniteProduct
**uses** ("ringsimp.ML")
**begin**

## 8 The Algebraic Hierarchy of Rings

### 8.1 Abelian Groups

**record** 'a ring = "'a monoid" +
zero :: 'a ("**0**$_2$")
add :: "['a, 'a] => 'a" (**infixl** "⊕$_2$" 65)

Derived operations.

**definition**
a_inv :: "[('a, 'm) ring_scheme, 'a ] => 'a" ("⊖$_2$ _" [81] 80)
**where** "a_inv R = m_inv (| carrier = carrier R, mult = add R, one = zero R |)"

**definition**
a_minus :: "[('a, 'm) ring_scheme, 'a, 'a] => 'a" (**infixl** "⊖$_2$" 65)
**where** "[| x ∈ carrier R; y ∈ carrier R |] ==> x ⊖$_R$ y = x ⊕$_R$ (⊖$_R$ y)"

**locale** abelian_monoid =
**fixes** G (**structure**)
**assumes** a_comm_monoid:
"comm_monoid (| carrier = carrier G, mult = add G, one = zero G |)"

The following definition is redundant but simple to use.

**locale** abelian_group = abelian_monoid +
  **assumes** a_comm_group:
    "comm_group (| carrier = carrier G, mult = add G, one = zero G |)"

## 8.2 Basic Properties

**lemma** abelian_monoidI:
  **fixes** R (**structure**)
  **assumes** a_closed:
      "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==> x ⊕ y ∈ carrier R"
    **and** zero_closed: "**0** ∈ carrier R"
    **and** a_assoc:
      "!!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |] ==>
      (x ⊕ y) ⊕ z = x ⊕ (y ⊕ z)"
    **and** l_zero: "!!x. x ∈ carrier R ==> **0** ⊕ x = x"
    **and** a_comm:
      "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==> x ⊕ y = y ⊕ x"
  **shows** "abelian_monoid R"
  ⟨*proof*⟩

**lemma** abelian_groupI:
  **fixes** R (**structure**)
  **assumes** a_closed:
      "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==> x ⊕ y ∈ carrier R"
    **and** zero_closed: "zero R ∈ carrier R"
    **and** a_assoc:
      "!!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |] ==>
      (x ⊕ y) ⊕ z = x ⊕ (y ⊕ z)"
    **and** a_comm:
      "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==> x ⊕ y = y ⊕ x"
    **and** l_zero: "!!x. x ∈ carrier R ==> **0** ⊕ x = x"
    **and** l_inv_ex: "!!x. x ∈ carrier R ==> EX y : carrier R. y ⊕ x = **0**"
  **shows** "abelian_group R"
  ⟨*proof*⟩

**lemma** (**in** abelian_monoid) a_monoid:
  "monoid (| carrier = carrier G, mult = add G, one = zero G |)"
⟨*proof*⟩

**lemma** (**in** abelian_group) a_group:
  "group (| carrier = carrier G, mult = add G, one = zero G |)"
  ⟨*proof*⟩

**lemmas** monoid_record_simps = partial_object.simps monoid.simps

**lemma** (**in** abelian_monoid) a_closed [intro, simp]:
  "⟦ x ∈ carrier G; y ∈ carrier G ⟧ ⟹ x ⊕ y ∈ carrier G"

⟨*proof*⟩

**lemma** (**in** abelian_monoid) zero_closed [intro, simp]:
"**0** ∈ carrier G"
⟨*proof*⟩

**lemma** (**in** abelian_group) a_inv_closed [intro, simp]:
"x ∈ carrier G ==> ⊖ x ∈ carrier G"
⟨*proof*⟩

**lemma** (**in** abelian_group) minus_closed [intro, simp]:
"[| x ∈ carrier G; y ∈ carrier G |] ==> x ⊖ y ∈ carrier G"
⟨*proof*⟩

**lemma** (**in** abelian_group) a_l_cancel [simp]:
"[| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
(x ⊕ y = x ⊕ z) = (y = z)"
⟨*proof*⟩

**lemma** (**in** abelian_group) a_r_cancel [simp]:
"[| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
(y ⊕ x = z ⊕ x) = (y = z)"
⟨*proof*⟩

**lemma** (**in** abelian_monoid) a_assoc:
"⟦x ∈ carrier G; y ∈ carrier G; z ∈ carrier G⟧ ⟹
(x ⊕ y) ⊕ z = x ⊕ (y ⊕ z)"
⟨*proof*⟩

**lemma** (**in** abelian_monoid) l_zero [simp]:
"x ∈ carrier G ==> **0** ⊕ x = x"
⟨*proof*⟩

**lemma** (**in** abelian_group) l_neg:
"x ∈ carrier G ==> ⊖ x ⊕ x = **0**"
⟨*proof*⟩

**lemma** (**in** abelian_monoid) a_comm:
"⟦x ∈ carrier G; y ∈ carrier G⟧ ⟹ x ⊕ y = y ⊕ x"
⟨*proof*⟩

**lemma** (**in** abelian_monoid) a_lcomm:
"⟦x ∈ carrier G; y ∈ carrier G; z ∈ carrier G⟧ ⟹
x ⊕ (y ⊕ z) = y ⊕ (x ⊕ z)"
⟨*proof*⟩

**lemma** (**in** abelian_monoid) r_zero [simp]:
"x ∈ carrier G ==> x ⊕ **0** = x"
⟨*proof*⟩

**lemma (in** abelian_group**)** r_neg:
"x ∈ carrier G ==> x ⊕ (⊖ x) = **0**"
⟨*proof*⟩

**lemma (in** abelian_group**)** minus_zero [simp]:
"⊖ **0** = **0**"
⟨*proof*⟩

**lemma (in** abelian_group**)** minus_minus [simp]:
"x ∈ carrier G ==> ⊖ (⊖ x) = x"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_inv_inj:
"inj_on (a_inv G) (carrier G)"
⟨*proof*⟩

**lemma (in** abelian_group**)** minus_add:
"[| x ∈ carrier G; y ∈ carrier G |] ==> ⊖ (x ⊕ y) = ⊖ x ⊕ ⊖ y"
⟨*proof*⟩

**lemma (in** abelian_group**)** minus_equality:
"[| x ∈ carrier G; y ∈ carrier G; y ⊕ x = **0** |] ==> ⊖ x = y"
⟨*proof*⟩

**lemma (in** abelian_monoid**)** minus_unique:
"[| x ∈ carrier G; y ∈ carrier G; y' ∈ carrier G;
y ⊕ x = **0**; x ⊕ y' = **0** |] ==> y = y'"
⟨*proof*⟩

**lemmas (in** abelian_monoid**)** a_ac = a_assoc a_comm a_lcomm

Derive an abelian_group from a comm_group

**lemma** comm_group_abelian_groupI:
**fixes** G (**structure**)
**assumes** cg: "comm_group (|carrier = carrier G, mult = add G, one = zero G|)"
**shows** "abelian_group G"
⟨*proof*⟩

## 8.3 Sums over Finite Sets

This definition makes it easy to lift lemmas from finprod.

**definition**
finsum :: "[('b, 'm) ring_scheme, 'a => 'b, 'a set] => 'b" **where**
"finsum G f A = finprod (| carrier = carrier G, mult = add G, one = zero G |) f A"

**syntax**

```
  "_finsum" :: "index => idt => 'a set => 'b => 'b"
      ("(3⨁ __:_. _)" [1000, 0, 51, 10] 10)
```
**syntax** (xsymbols)
```
  "_finsum" :: "index => idt => 'a set => 'b => 'b"
      ("(3⨁ __∈_. _)" [1000, 0, 51, 10] 10)
```
**syntax** (HTML **output**)
```
  "_finsum" :: "index => idt => 'a set => 'b => 'b"
      ("(3⨁ __∈_. _)" [1000, 0, 51, 10] 10)
```
**translations**
```
  "⨁ i:A. b" == "CONST finsum ◇ (%i. b) A"
```
— Beware of argument permutation!

**context** abelian_monoid **begin**

**lemma** finsum_empty [simp]:
```
  "finsum G f {} = 𝟬"
```
⟨*proof*⟩

**lemma** finsum_insert [simp]:
```
  "[| finite F; a ∉ F; f ∈ F -> carrier G; f a ∈ carrier G |]
  ==> finsum G f (insert a F) = f a ⊕ finsum G f F"
```
⟨*proof*⟩

**lemma** finsum_zero [simp]:
```
  "finite A ==> (⨁ i∈A. 𝟬) = 𝟬"
```
⟨*proof*⟩

**lemma** finsum_closed [simp]:
**fixes** A
**assumes** fin: "finite A" **and** f: "f ∈ A -> carrier G"
**shows** "finsum G f A ∈ carrier G"
⟨*proof*⟩

**lemma** finsum_Un_Int:
```
  "[| finite A; finite B; g ∈ A -> carrier G; g ∈ B -> carrier G |] ==>
     finsum G g (A Un B) ⊕ finsum G g (A Int B) =
     finsum G g A ⊕ finsum G g B"
```
⟨*proof*⟩

**lemma** finsum_Un_disjoint:
```
  "[| finite A; finite B; A Int B = {};
      g ∈ A -> carrier G; g ∈ B -> carrier G |]
   ==> finsum G g (A Un B) = finsum G g A ⊕ finsum G g B"
```
⟨*proof*⟩

**lemma** finsum_addf:
```
  "[| finite A; f ∈ A -> carrier G; g ∈ A -> carrier G |] ==>
```

```
   finsum G (%x. f x ⊕ g x) A = (finsum G f A ⊕ finsum G g A)"
```
⟨*proof*⟩

**lemma** `finsum_cong':`
```
  "[| A = B; g : B -> carrier G;
      !!i. i : B ==> f i = g i |] ==> finsum G f A = finsum G g B"
```
⟨*proof*⟩

**lemma** `finsum_0 [simp]:`
```
  "f : {0::nat} -> carrier G ==> finsum G f {..0} = f 0"
```
⟨*proof*⟩

**lemma** `finsum_Suc [simp]:`
```
  "f : {..Suc n} -> carrier G ==>
   finsum G f {..Suc n} = (f (Suc n) ⊕ finsum G f {..n})"
```
⟨*proof*⟩

**lemma** `finsum_Suc2:`
```
  "f : {..Suc n} -> carrier G ==>
   finsum G f {..Suc n} = (finsum G (%i. f (Suc i)) {..n} ⊕ f 0)"
```
⟨*proof*⟩

**lemma** `finsum_add [simp]:`
```
  "[| f : {..n} -> carrier G; g : {..n} -> carrier G |] ==>
     finsum G (%i. f i ⊕ g i) {..n::nat} =
     finsum G f {..n} ⊕ finsum G g {..n}"
```
⟨*proof*⟩

**lemma** `finsum_cong:`
```
  "[| A = B; f : B -> carrier G;
      !!i. i : B =simp=> f i = g i |] ==> finsum G f A = finsum G g B"
```
⟨*proof*⟩

Usually, if this rule causes a failed congruence proof error, the reason is that the premise `g ∈ B -> carrier G` cannot be shown. Adding `Pi_def` to the simpset is often useful.

**lemma** `finsum_reindex:`
```
  assumes fin: "finite A"
    shows "f : (h ‘ A) → carrier G ⟹
        inj_on h A ==> finsum G f (h ‘ A) = finsum G (%x. f (h x)) A"
```
⟨*proof*⟩

**lemma** `finsum_singleton:`
```
  assumes i_in_A: "i ∈ A" and fin_A: "finite A" and f_Pi: "f ∈ A →
carrier G"
```

**shows** "(⨁j∈A. if i = j then f j else **0**) = f i"
⟨*proof*⟩

**end**

## 8.4 Rings: Basic Definitions

**locale** ring = abelian_group R + monoid R **for** R (**structure**) +
**assumes** l_distr: "[| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |]
==> (x ⊕ y) ⊗ z = x ⊗ z ⊕ y ⊗ z"
**and** r_distr: "[| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |]
==> z ⊗ (x ⊕ y) = z ⊗ x ⊕ z ⊗ y"

**locale** cring = ring + comm_monoid R

**locale** "domain" = cring +
**assumes** one_not_zero [simp]: "**1** ~= **0**"
**and** integral: "[| a ⊗ b = **0**; a ∈ carrier R; b ∈ carrier R |] ==>
a = **0** | b = **0**"

**locale** field = "domain" +
**assumes** field_Units: "Units R = carrier R - {**0**}"

## 8.5 Rings

**lemma** ringI:
**fixes** R (**structure**)
**assumes** abelian_group: "abelian_group R"
**and** monoid: "monoid R"
**and** l_distr: "!!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier
R |]
==> (x ⊕ y) ⊗ z = x ⊗ z ⊕ y ⊗ z"
**and** r_distr: "!!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier
R |]
==> z ⊗ (x ⊕ y) = z ⊗ x ⊕ z ⊗ y"
**shows** "ring R"
⟨*proof*⟩

**lemma** (**in** ring) is_abelian_group:
"abelian_group R"
⟨*proof*⟩

**lemma** (**in** ring) is_monoid:
"monoid R"
⟨*proof*⟩

**lemma** (**in** ring) is_ring:
"ring R"
⟨*proof*⟩

**lemmas** ring_record_simps = monoid_record_simps ring.simps

**lemma** cringI:
  **fixes** R (**structure**)
  **assumes** abelian_group: "abelian_group R"
    **and** comm_monoid: "comm_monoid R"
    **and** l_distr: "!!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |]
      ==> (x ⊕ y) ⊗ z = x ⊗ z ⊕ y ⊗ z"
  **shows** "cring R"
⟨*proof*⟩

**lemma** (**in** cring) is_cring:
  "cring R" ⟨*proof*⟩

### 8.5.1 Normaliser for Rings

**lemma** (**in** abelian_group) r_neg2:
  "[| x ∈ carrier G; y ∈ carrier G |] ==> x ⊕ (⊖ x ⊕ y) = y"
⟨*proof*⟩

**lemma** (**in** abelian_group) r_neg1:
  "[| x ∈ carrier G; y ∈ carrier G |] ==> ⊖ x ⊕ (x ⊕ y) = y"
⟨*proof*⟩

The following proofs are from Jacobson, Basic Algebra I, pp. 88–89

**lemma** (**in** ring) l_null [simp]:
  "x ∈ carrier R ==> **0** ⊗ x = **0**"
⟨*proof*⟩

**lemma** (**in** ring) r_null [simp]:
  "x ∈ carrier R ==> x ⊗ **0** = **0**"
⟨*proof*⟩

**lemma** (**in** ring) l_minus:
  "[| x ∈ carrier R; y ∈ carrier R |] ==> ⊖ x ⊗ y = ⊖ (x ⊗ y)"
⟨*proof*⟩

**lemma** (**in** ring) r_minus:
  "[| x ∈ carrier R; y ∈ carrier R |] ==> x ⊗ ⊖ y = ⊖ (x ⊗ y)"
⟨*proof*⟩

**lemma** (**in** abelian_group) minus_eq:
  "[| x ∈ carrier G; y ∈ carrier G |] ==> x ⊖ y = x ⊕ ⊖ y"
  ⟨*proof*⟩

Setup algebra method: compute distributive normal form in locale contexts

⟨*ML*⟩

**lemmas (in** ring**)** ring_simprules
  [algebra ring "zero R" "add R" "a_inv R" "a_minus R" "one R" "mult R"]
=
  a_closed zero_closed a_inv_closed minus_closed m_closed one_closed
  a_assoc l_zero l_neg a_comm m_assoc l_one l_distr minus_eq
  r_zero r_neg r_neg2 r_neg1 minus_add minus_minus minus_zero
  a_lcomm r_distr l_null r_null l_minus r_minus

**lemmas (in** cring**)**
  [algebra del: ring "zero R" "add R" "a_inv R" "a_minus R" "one R" "mult R"] =
  -

**lemmas (in** cring**)** cring_simprules
  [algebra add: cring "zero R" "add R" "a_inv R" "a_minus R" "one R" "mult R"] =
  a_closed zero_closed a_inv_closed minus_closed m_closed one_closed
  a_assoc l_zero l_neg a_comm m_assoc l_one l_distr m_comm minus_eq
  r_zero r_neg r_neg2 r_neg1 minus_add minus_minus minus_zero
  a_lcomm m_lcomm r_distr l_null r_null l_minus r_minus

**lemma (in** cring**)** nat_pow_zero:
  "(n::nat) ~= 0 ==> **0** (^) n = **0**"
  ⟨*proof*⟩

**lemma (in** ring**)** one_zeroD:
  **assumes** onezero: "**1** = **0**"
  **shows** "carrier R = {**0**}"
⟨*proof*⟩

**lemma (in** ring**)** one_zeroI:
  **assumes** carrzero: "carrier R = {**0**}"
  **shows** "**1** = **0**"
⟨*proof*⟩

**lemma (in** ring**)** carrier_one_zero:
  **shows** "(carrier R = {**0**}) = (**1** = **0**)"
  ⟨*proof*⟩

**lemma (in** ring**)** carrier_one_not_zero:
  **shows** "(carrier R ≠ {**0**}) = (**1** ≠ **0**)"
  ⟨*proof*⟩

Two examples for use of method algebra

**lemma**
  **fixes** R **(structure) and** S **(structure)**

**assumes** "ring R" "cring S"
**assumes** RS: "a ∈ carrier R" "b ∈ carrier R" "c ∈ carrier S" "d ∈ carrier S"
**shows** "a ⊕ ⊖ (a ⊕ ⊖ b) = b & c ⊗$_S$ d = d ⊗$_S$ c"
⟨*proof*⟩
⟨*ML*⟩
⟨*proof*⟩

**lemma**
**fixes** R **(structure)**
**assumes** "ring R"
**assumes** R: "a ∈ carrier R" "b ∈ carrier R"
**shows** "a ⊖ (a ⊖ b) = b"
⟨*proof*⟩

### 8.5.2 Sums over Finite Sets

**lemma (in** ring**)** finsum_ldistr:
"[| finite A; a ∈ carrier R; f ∈ A -> carrier R |] ==>
finsum R f A ⊗ a = finsum R (%i. f i ⊗ a) A"
⟨*proof*⟩

**lemma (in** ring**)** finsum_rdistr:
"[| finite A; a ∈ carrier R; f ∈ A -> carrier R |] ==>
a ⊗ finsum R f A = finsum R (%i. a ⊗ f i) A"
⟨*proof*⟩

## 8.6 Integral Domains

**lemma (in** "domain"**)** zero_not_one [simp]:
"**0** ~= **1**"
⟨*proof*⟩

**lemma (in** "domain"**)** integral_iff:
"[| a ∈ carrier R; b ∈ carrier R |] ==> (a ⊗ b = **0**) = (a = **0** | b = **0**)"
⟨*proof*⟩

**lemma (in** "domain"**)** m_lcancel:
**assumes** prem: "a ~= **0**"
**and** R: "a ∈ carrier R" "b ∈ carrier R" "c ∈ carrier R"
**shows** "(a ⊗ b = a ⊗ c) = (b = c)"
⟨*proof*⟩

**lemma (in** "domain"**)** m_rcancel:
**assumes** prem: "a ~= **0**"
**and** R: "a ∈ carrier R" "b ∈ carrier R" "c ∈ carrier R"
**shows** conc: "(b ⊗ a = c ⊗ a) = (b = c)"
⟨*proof*⟩

## 8.7 Fields

Field would not need to be derived from domain, the properties for domain follow from the assumptions of field

**lemma (in** cring**)** cring_fieldI:
  **assumes** field_Units: "Units R = carrier R - {**0**}"
  **shows** "field R"
⟨*proof*⟩

Another variant to show that something is a field

**lemma (in** cring**)** cring_fieldI2:
  **assumes** notzero: "**0** ≠ **1**"
  **and** invex: "⋀a. ⟦a ∈ carrier R; a ≠ **0**⟧ ⟹ ∃b∈carrier R. a ⊗ b = **1**"
  **shows** "field R"
  ⟨*proof*⟩

## 8.8 Morphisms

**definition**
  ring_hom :: "[('a, 'm) ring_scheme, ('b, 'n) ring_scheme] => ('a => 'b) set"
  **where** "ring_hom R S =
    {h. h ∈ carrier R -> carrier S &
      (ALL x y. x ∈ carrier R & y ∈ carrier R -->
        h (x ⊗$_R$ y) = h x ⊗$_S$ h y & h (x ⊕$_R$ y) = h x ⊕$_S$ h y) &
      h **1**$_R$ = **1**$_S$}"

**lemma** ring_hom_memI:
  **fixes** R **(structure) and** S **(structure)**
  **assumes** hom_closed: "!!x. x ∈ carrier R ==> h x ∈ carrier S"
    **and** hom_mult: "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==>
      h (x ⊗ y) = h x ⊗$_S$ h y"
    **and** hom_add: "!!x y. [| x ∈ carrier R; y ∈ carrier R |] ==>
      h (x ⊕ y) = h x ⊕$_S$ h y"
    **and** hom_one: "h **1** = **1**$_S$"
  **shows** "h ∈ ring_hom R S"
  ⟨*proof*⟩

**lemma** ring_hom_closed:
  "[| h ∈ ring_hom R S; x ∈ carrier R |] ==> h x ∈ carrier S"
  ⟨*proof*⟩

**lemma** ring_hom_mult:
  **fixes** R **(structure) and** S **(structure)**
  **shows**
    "[| h ∈ ring_hom R S; x ∈ carrier R; y ∈ carrier R |] ==>
    h (x ⊗ y) = h x ⊗$_S$ h y"
    ⟨*proof*⟩

**lemma** ring_hom_add:
  **fixes** R **(structure) and** S **(structure)**
  **shows**
    "[| h ∈ ring_hom R S; x ∈ carrier R; y ∈ carrier R |] ==>
    h (x ⊕ y) = h x ⊕$_S$ h y"
    ⟨*proof*⟩

**lemma** ring_hom_one:
  **fixes** R **(structure) and** S **(structure)**
  **shows** "h ∈ ring_hom R S ==> h **1** = **1**$_S$"
  ⟨*proof*⟩

**locale** ring_hom_cring = R: cring R + S: cring S
    **for** R **(structure) and** S **(structure)** +
  **fixes** h
  **assumes** homh [simp, intro]: "h ∈ ring_hom R S"
  **notes** hom_closed [simp, intro] = ring_hom_closed [OF homh]
    **and** hom_mult [simp] = ring_hom_mult [OF homh]
    **and** hom_add [simp] = ring_hom_add [OF homh]
    **and** hom_one [simp] = ring_hom_one [OF homh]

**lemma (in** ring_hom_cring**)** hom_zero [simp]:
  "h **0** = **0**$_S$"
⟨*proof*⟩

**lemma (in** ring_hom_cring**)** hom_a_inv [simp]:
  "x ∈ carrier R ==> h (⊖ x) = ⊖$_S$ h x"
⟨*proof*⟩

**lemma (in** ring_hom_cring**)** hom_finsum [simp]:
  "[| finite A; f ∈ A -> carrier R |] ==>
  h (finsum R f A) = finsum S (h o f) A"
⟨*proof*⟩

**lemma (in** ring_hom_cring**)** hom_finprod:
  "[| finite A; f ∈ A -> carrier R |] ==>
  h (finprod R f A) = finprod S (h o f) A"
⟨*proof*⟩

**declare** ring_hom_cring.hom_finprod [simp]

**lemma** id_ring_hom [simp]:
  "id ∈ ring_hom R R"
  ⟨*proof*⟩

**end**

**theory** AbelCoset
**imports** Coset Ring
**begin**

## 8.9 More Lifting from Groups to Abelian Groups

### 8.9.1 Definitions

Hiding <+> from Sum_Type until I come up with better syntax here

**no_notation** Plus (**infixr** "<+>" 65)

**definition**
  a_r_coset    :: "[_, 'a set, 'a] ⇒ 'a set"    (**infixl** "+>\<index>" 60)
  **where** "a_r_coset G = r_coset ⦇carrier = carrier G, mult = add G, one = zero G⦈"

**definition**
  a_l_coset    :: "[_, 'a, 'a set] ⇒ 'a set"    (**infixl** "<+\<index>" 60)
  **where** "a_l_coset G = l_coset ⦇carrier = carrier G, mult = add G, one = zero G⦈"

**definition**
  A_RCOSETS  :: "[_, 'a set] ⇒ ('a set)set"   ("a'_rcosets\<index> _" [81] 80)
  **where** "A_RCOSETS G H = RCOSETS ⦇carrier = carrier G, mult = add G, one = zero G⦈ H"

**definition**
  set_add  :: "[_, 'a set ,'a set] ⇒ 'a set" (**infixl** "<+>\<index>" 60)
  **where** "set_add G = set_mult ⦇carrier = carrier G, mult = add G, one = zero G⦈"

**definition**
  A_SET_INV :: "[_,'a set] ⇒ 'a set"  ("a'_set'_inv\<index> _" [81] 80)
  **where** "A_SET_INV G H = SET_INV ⦇carrier = carrier G, mult = add G, one = zero G⦈ H"

**definition**
  a_r_congruent :: "[('a,'b)ring_scheme, 'a set] ⇒ ('a*'a)set"  ("racong\<index> _")
  **where** "a_r_congruent G = r_congruent ⦇carrier = carrier G, mult = add G, one = zero G⦈"

**definition**
  A_FactGroup :: "[('a,'b) ring_scheme, 'a set] ⇒ ('a set) monoid" (**infixl** "A'_Mod" 65)
    — Actually defined for groups rather than monoids
  **where** "A_FactGroup G H = FactGroup ⦇carrier = carrier G, mult = add G, one = zero G⦈ H"

**definition**
  a_kernel :: "('a, 'm) ring_scheme ⇒ ('b, 'n) ring_scheme ⇒ ('a ⇒ 'b) ⇒ 'a set"
    — the kernel of a homomorphism (additive)
  **where** "a_kernel G H h =
    kernel ⦇carrier = carrier G, mult = add G, one = zero G⦈
      ⦇carrier = carrier H, mult = add H, one = zero H⦈ h"

**locale** abelian_group_hom = G: abelian_group G + H: abelian_group H
    **for** G (**structure**) **and** H (**structure**) +
  **fixes** h
  **assumes** a_group_hom: "group_hom (| carrier = carrier G, mult = add G, one = zero G |)
                                    (| carrier = carrier H, mult = add H, one = zero H |) h"

**lemmas** a_r_coset_defs =
  a_r_coset_def r_coset_def

**lemma** a_r_coset_def':
  **fixes** G (**structure**)
  **shows** "H +> a ≡ ⋃h∈H. {h ⊕ a}"
⟨*proof*⟩

**lemmas** a_l_coset_defs =
  a_l_coset_def l_coset_def

**lemma** a_l_coset_def':
  **fixes** G (**structure**)
  **shows** "a <+ H ≡ ⋃h∈H. {a ⊕ h}"
⟨*proof*⟩

**lemmas** A_RCOSETS_defs =
  A_RCOSETS_def RCOSETS_def

**lemma** A_RCOSETS_def':
  **fixes** G (**structure**)
  **shows** "a_rcosets H ≡ ⋃a∈carrier G. {H +> a}"
⟨*proof*⟩

**lemmas** set_add_defs =
  set_add_def set_mult_def

**lemma** set_add_def':
  **fixes** G (**structure**)
  **shows** "H <+> K ≡ ⋃h∈H. ⋃k∈K. {h ⊕ k}"
⟨*proof*⟩

**lemmas** A_SET_INV_defs =
  A_SET_INV_def SET_INV_def

**lemma** A_SET_INV_def':
  **fixes** G (**structure**)
  **shows** "a_set_inv H ≡ ⋃h∈H. {⊖ h}"
⟨*proof*⟩

### 8.9.2 Cosets

**lemma** (**in** abelian_group) a_coset_add_assoc:
     "[| M ⊆ carrier G; g ∈ carrier G; h ∈ carrier G |]
      ==> (M +> g) +> h = M +> (g ⊕ h)"
⟨*proof*⟩

**lemma** (**in** abelian_group) a_coset_add_zero [simp]:
  "M ⊆ carrier G ==> M +> **0** = M"
⟨*proof*⟩

**lemma** (**in** abelian_group) a_coset_add_inv1:
     "[| M +> (x ⊕ (⊖ y)) = M;  x ∈ carrier G ; y ∈ carrier G;
         M ⊆ carrier G |] ==> M +> x = M +> y"
⟨*proof*⟩

**lemma** (**in** abelian_group) a_coset_add_inv2:
     "[| M +> x = M +> y;  x ∈ carrier G;  y ∈ carrier G;  M ⊆ carrier
G |]
      ==> M +> (x ⊕ (⊖ y)) = M"
⟨*proof*⟩

**lemma** (**in** abelian_group) a_coset_join1:
     "[| H +> x = H;  x ∈ carrier G;  subgroup H (|carrier = carrier G,
mult = add G, one = zero G|) |] ==> x ∈ H"
⟨*proof*⟩

**lemma** (**in** abelian_group) a_solve_equation:
    "⟦subgroup H (|carrier = carrier G, mult = add G, one = zero G|);
x ∈ H; y ∈ H⟧ ⟹ ∃h∈H. y = h ⊕ x"
⟨*proof*⟩

**lemma** (**in** abelian_group) a_repr_independence:
     "⟦y ∈ H +> x;  x ∈ carrier G; subgroup H (|carrier = carrier G, mult
= add G, one = zero G|) ⟧ ⟹ H +> x = H +> y"
⟨*proof*⟩

**lemma** (**in** abelian_group) a_coset_join2:
     "⟦x ∈ carrier G;  subgroup H (|carrier = carrier G, mult = add G,
one = zero G|); x∈H⟧ ⟹ H +> x = H"
⟨*proof*⟩

**lemma (in** abelian_monoid**)** a_r_coset_subset_G:
"[| H ⊆ carrier G; x ∈ carrier G |] ==> H +> x ⊆ carrier G"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_rcosI:
"[| h ∈ H; H ⊆ carrier G; x ∈ carrier G|] ==> h ⊕ x ∈ H +> x"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_rcosetsI:
"⟦H ⊆ carrier G; x ∈ carrier G⟧ ⟹ H +> x ∈ a_rcosets H"
⟨*proof*⟩

Really needed?

**lemma (in** abelian_group**)** a_transpose_inv:
"[| x ⊕ y = z; x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |]
==> (⊖ x) ⊕ z = y"
⟨*proof*⟩

### 8.9.3 Subgroups

**locale** additive_subgroup =
**fixes** H **and** G **(structure)**
**assumes** a_subgroup: "subgroup H (|carrier = carrier G, mult = add G, one = zero G|)"

**lemma (in** additive_subgroup**)** is_additive_subgroup:
**shows** "additive_subgroup H G"
⟨*proof*⟩

**lemma** additive_subgroupI:
**fixes** G **(structure)**
**assumes** a_subgroup: "subgroup H (|carrier = carrier G, mult = add G, one = zero G|)"
**shows** "additive_subgroup H G"
⟨*proof*⟩

**lemma (in** additive_subgroup**)** a_subset:
"H ⊆ carrier G"
⟨*proof*⟩

**lemma (in** additive_subgroup**)** a_closed [intro, simp]:
"⟦x ∈ H; y ∈ H⟧ ⟹ x ⊕ y ∈ H"
⟨*proof*⟩

**lemma (in** additive_subgroup**)** zero_closed [simp]:
"**0** ∈ H"
⟨*proof*⟩

**lemma (in** additive_subgroup**)** a_inv_closed [intro,simp]:
"x ∈ H ⟹ ⊖ x ∈ H"
⟨*proof*⟩

### 8.9.4 Additive subgroups are normal

Every subgroup of an abelian_group is normal

**locale** abelian_subgroup = additive_subgroup + abelian_group G +
**assumes** a_normal: "normal H (|carrier = carrier G, mult = add G, one = zero G|)"

**lemma (in** abelian_subgroup**)** is_abelian_subgroup:
**shows** "abelian_subgroup H G"
⟨*proof*⟩

**lemma** abelian_subgroupI:
**assumes** a_normal: "normal H (|carrier = carrier G, mult = add G, one = zero G|)"
**and** a_comm: "!!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊕$_G$ y = y ⊕$_G$ x"
**shows** "abelian_subgroup H G"
⟨*proof*⟩

**lemma** abelian_subgroupI2:
**fixes** G **(structure)**
**assumes** a_comm_group: "comm_group (|carrier = carrier G, mult = add G, one = zero G|)"
**and** a_subgroup: "subgroup H (|carrier = carrier G, mult = add G, one = zero G|)"
**shows** "abelian_subgroup H G"
⟨*proof*⟩

**lemma** abelian_subgroupI3:
**fixes** G **(structure)**
**assumes** asg: "additive_subgroup H G"
**and** ag: "abelian_group G"
**shows** "abelian_subgroup H G"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_coset_eq:
"(∀x ∈ carrier G. H +> x = x <+ H)"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_inv_op_closed1:
**shows** "⟦x ∈ carrier G; h ∈ H⟧ ⟹ (⊖ x) ⊕ h ⊕ x ∈ H"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_inv_op_closed2:
**shows** "⟦x ∈ carrier G; h ∈ H⟧ ⟹ x ⊕ h ⊕ (⊖ x) ∈ H"

⟨*proof*⟩

Alternative characterization of normal subgroups

**lemma (in** abelian_group**)** a_normal_inv_iff:
"(N ◁ (|carrier = carrier G, mult = add G, one = zero G|)) =
(subgroup N (|carrier = carrier G, mult = add G, one = zero G|) &
(∀x ∈ carrier G. ∀h ∈ N. x ⊕ h ⊕ (⊖ x) ∈ N))"
**(is** "_ = ?rhs")
⟨*proof*⟩

**lemma (in** abelian_group**)** a_lcos_m_assoc:
"[| M ⊆ carrier G; g ∈ carrier G; h ∈ carrier G |]
==> g <+ (h <+ M) = (g ⊕ h) <+ M"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_lcos_mult_one:
"M ⊆ carrier G ==> **0** <+ M = M"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_l_coset_subset_G:
"[| H ⊆ carrier G; x ∈ carrier G |] ==> x <+ H ⊆ carrier G"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_l_coset_swap:
"⟦y ∈ x <+ H; x ∈ carrier G; subgroup H (|carrier = carrier G, mult
= add G, one = zero G|)⟧ ⟹ x ∈ y <+ H"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_l_coset_carrier:
"[| y ∈ x <+ H; x ∈ carrier G; subgroup H (|carrier = carrier G,
mult = add G, one = zero G|) |] ==> y ∈ carrier G"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_l_repr_imp_subset:
**assumes** y: "y ∈ x <+ H" **and** x: "x ∈ carrier G" **and** sb: "subgroup H
(|carrier = carrier G, mult = add G, one = zero G|)"
**shows** "y <+ H ⊆ x <+ H"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_l_repr_independence:
**assumes** y: "y ∈ x <+ H" **and** x: "x ∈ carrier G" **and** sb: "subgroup H
(|carrier = carrier G, mult = add G, one = zero G|)"
**shows** "x <+ H = y <+ H"
⟨*proof*⟩

**lemma (in** abelian_group**)** setadd_subset_G:
"⟦H ⊆ carrier G; K ⊆ carrier G⟧ ⟹ H <+> K ⊆ carrier G"

⟨*proof*⟩

**lemma (in** abelian_group**)** subgroup_add_id: "subgroup H (|carrier = carrier G, mult = add G, one = zero G|) ⟹ H <+> H = H"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcos_inv:
**assumes** x: "x ∈ carrier G"
**shows** "a_set_inv (H +> x) = H +> (⊖ x)"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_setmult_rcos_assoc:
"⟦H ⊆ carrier G; K ⊆ carrier G; x ∈ carrier G⟧
⟹ H <+> (K +> x) = (H <+> K) +> x"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_rcos_assoc_lcos:
"⟦H ⊆ carrier G; K ⊆ carrier G; x ∈ carrier G⟧
⟹ (H +> x) <+> K = H <+> (x <+ K)"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcos_sum:
"⟦x ∈ carrier G; y ∈ carrier G⟧
⟹ (H +> x) <+> (H +> y) = H +> (x ⊕ y)"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** rcosets_add_eq:
"M ∈ a_rcosets H ⟹ H <+> M = M"
— generalizes subgroup_mult_id
⟨*proof*⟩

### 8.9.5 Congruence Relation

**lemma (in** abelian_subgroup**)** a_equiv_rcong:
**shows** "equiv (carrier G) (racong H)"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_l_coset_eq_rcong:
**assumes** a: "a ∈ carrier G"
**shows** "a <+ H = racong H `` {a}"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcos_equation:
**shows**
"⟦ha ⊕ a = h ⊕ b; a ∈ carrier G; b ∈ carrier G;
h ∈ H; ha ∈ H; hb ∈ H⟧
⟹ hb ⊕ a ∈ (⋃h∈H. {h ⊕ b})"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcos_disjoint:
**shows** "⟦a ∈ a_rcosets H; b ∈ a_rcosets H; a≠b⟧ ⟹ a ∩ b = {}"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcos_self:
**shows** "x ∈ carrier G ⟹ x ∈ H +> x"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcosets_part_G:
**shows** "⋃(a_rcosets H) = carrier G"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_cosets_finite:
"⟦c ∈ a_rcosets H; H ⊆ carrier G; finite (carrier G)⟧ ⟹ finite c"
⟨*proof*⟩

**lemma (in** abelian_group**)** a_card_cosets_equal:
"⟦c ∈ a_rcosets H; H ⊆ carrier G; finite(carrier G)⟧
⟹ card c = card H"
⟨*proof*⟩

**lemma (in** abelian_group**)** rcosets_subset_PowG:
"additive_subgroup H G ⟹ a_rcosets H ⊆ Pow(carrier G)"
⟨*proof*⟩

**theorem (in** abelian_group**)** a_lagrange:
"⟦finite(carrier G); additive_subgroup H G⟧
⟹ card(a_rcosets H) * card(H) = order(G)"
⟨*proof*⟩

### 8.9.6 Factorization

**lemmas** A_FactGroup_defs = A_FactGroup_def FactGroup_def

**lemma** A_FactGroup_def':
**fixes** G **(structure)**
**shows** "G A_Mod H ≡ (|carrier = a_rcosets$_G$ H, mult = set_add G, one = H|)"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_setmult_closed:
"⟦K1 ∈ a_rcosets H; K2 ∈ a_rcosets H⟧ ⟹ K1 <+> K2 ∈ a_rcosets H"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_setinv_closed:
"K ∈ a_rcosets H ⟹ a_set_inv K ∈ a_rcosets H"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcosets_assoc:
"⟦M1 ∈ a_rcosets H; M2 ∈ a_rcosets H; M3 ∈ a_rcosets H⟧
⟹ M1 <+> M2 <+> M3 = M1 <+> (M2 <+> M3)"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_subgroup_in_rcosets:
"H ∈ a_rcosets H"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcosets_inv_mult_group_eq:
"M ∈ a_rcosets H ⟹ a_set_inv M <+> M = H"
⟨*proof*⟩

**theorem (in** abelian_subgroup**)** a_factorgroup_is_group:
"group (G A_Mod H)"
⟨*proof*⟩

Since the Factorization is based on an *abelian* subgroup, is results in a commutative group

**theorem (in** abelian_subgroup**)** a_factorgroup_is_comm_group:
"comm_group (G A_Mod H)"
⟨*proof*⟩

**lemma** add_A_FactGroup [simp]: "X ⊗$_{(G\ A\_Mod\ H)}$ X' = X <+>$_G$ X'"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_inv_FactGroup:
"X ∈ carrier (G A_Mod H) ⟹ inv$_{G\ A_Mod\ H}$ X = a_set_inv X"
⟨*proof*⟩

The coset map is a homomorphism from G to the quotient group G Mod H

**lemma (in** abelian_subgroup**)** a_r_coset_hom_A_Mod:
"(λa. H +> a) ∈ hom ⦇carrier = carrier G, mult = add G, one = zero G⦈
(G A_Mod H)"
⟨*proof*⟩

The isomorphism theorems have been omitted from lifting, at least for now

### 8.9.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

**lemmas** a_kernel_defs =
a_kernel_def kernel_def

**lemma** a_kernel_def':
"a_kernel R S h = {x ∈ carrier R. h x = $\mathbf{0}_S$}"
⟨*proof*⟩

### 8.9.8 Homomorphisms

**lemma** abelian_group_homI:
  **assumes** "abelian_group G"
  **assumes** "abelian_group H"
  **assumes** a_group_hom: "group_hom (| carrier = carrier G, mult = add G, one = zero G |)
                                    (| carrier = carrier H, mult = add H, one = zero H |) h"
  **shows** "abelian_group_hom G H h"
⟨*proof*⟩

**lemma** (**in** abelian_group_hom) is_abelian_group_hom:
  "abelian_group_hom G H h"
  ⟨*proof*⟩

**lemma** (**in** abelian_group_hom) hom_add [simp]:
  "[| x : carrier G; y : carrier G |]
       ==> h (x ⊕$_G$ y) = h x ⊕$_H$ h y"
⟨*proof*⟩

**lemma** (**in** abelian_group_hom) hom_closed [simp]:
  "x ∈ carrier G ⟹ h x ∈ carrier H"
⟨*proof*⟩

**lemma** (**in** abelian_group_hom) zero_closed [simp]:
  "h **0** ∈ carrier H"
⟨*proof*⟩

**lemma** (**in** abelian_group_hom) hom_zero [simp]:
  "h **0** = **0**$_H$"
⟨*proof*⟩

**lemma** (**in** abelian_group_hom) a_inv_closed [simp]:
  "x ∈ carrier G ==> h (⊖x) ∈ carrier H"
⟨*proof*⟩

**lemma** (**in** abelian_group_hom) hom_a_inv [simp]:
  "x ∈ carrier G ==> h (⊖x) = ⊖$_H$ (h x)"
⟨*proof*⟩

**lemma** (**in** abelian_group_hom) additive_subgroup_a_kernel:
  "additive_subgroup (a_kernel G H h) G"
⟨*proof*⟩

The kernel of a homomorphism is an abelian subgroup

**lemma** (**in** abelian_group_hom) abelian_subgroup_a_kernel:
  "abelian_subgroup (a_kernel G H h) G"
⟨*proof*⟩

**lemma (in** abelian_group_hom**)** A_FactGroup_nonempty:
**assumes** X: "X ∈ carrier (G A_Mod a_kernel G H h)"
**shows** "X ≠ {}"
⟨*proof*⟩

**lemma (in** abelian_group_hom**)** FactGroup_contents_mem:
**assumes** X: "X ∈ carrier (G A_Mod (a_kernel G H h))"
**shows** "contents (h‘X) ∈ carrier H"
⟨*proof*⟩

**lemma (in** abelian_group_hom**)** A_FactGroup_hom:
"(λX. contents (h‘X)) ∈ hom (G A_Mod (a_kernel G H h))
(⦇carrier = carrier H, mult = add H, one = zero H⦈)"
⟨*proof*⟩

**lemma (in** abelian_group_hom**)** A_FactGroup_inj_on:
"inj_on (λX. contents (h ‘ X)) (carrier (G A_Mod a_kernel G H h))"
⟨*proof*⟩

If the homomorphism h is onto H, then so is the homomorphism from the quotient group

**lemma (in** abelian_group_hom**)** A_FactGroup_onto:
**assumes** h: "h ‘ carrier G = carrier H"
**shows** "(λX. contents (h ‘ X)) ‘ carrier (G A_Mod a_kernel G H h) = carrier H"
⟨*proof*⟩

If h is a homomorphism from G onto H, then the quotient group G Mod kernel G H h is isomorphic to H.

**theorem (in** abelian_group_hom**)** A_FactGroup_iso:
"h ‘ carrier G = carrier H
⟹ (λX. contents (h‘X)) ∈ (G A_Mod (a_kernel G H h)) ≅
(| carrier = carrier H, mult = add H, one = zero H |)"
⟨*proof*⟩

### 8.9.9 Cosets

Not eveything from CosetExt.thy is lifted here.

**lemma (in** additive_subgroup**)** a_Hcarr [simp]:
**assumes** hH: "h ∈ H"
**shows** "h ∈ carrier G"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_elemrcos_carrier:
**assumes** acarr: "a ∈ carrier G"
**and** a': "a' ∈ H +> a"
**shows** "a' ∈ carrier G"

⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcos_const:
  **assumes** hH: "h ∈ H"
  **shows** "H +> h = H"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcos_module_imp:
  **assumes** xcarr: "x ∈ carrier G"
      **and** x'cos: "x' ∈ H +> x"
  **shows** "(x' ⊕ ⊖x) ∈ H"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcos_module_rev:
  **assumes** "x ∈ carrier G" "x' ∈ carrier G"
      **and** "(x' ⊕ ⊖x) ∈ H"
  **shows** "x' ∈ H +> x"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcos_module:
  **assumes** "x ∈ carrier G" "x' ∈ carrier G"
  **shows** "(x' ∈ H +> x) = (x' ⊕ ⊖x ∈ H)"
⟨*proof*⟩
**lemma (in** abelian_subgroup**)** a_rcos_module_minus:
  **assumes** "ring G"
  **assumes** carr: "x ∈ carrier G" "x' ∈ carrier G"
  **shows** "(x' ∈ H +> x) = (x' ⊖ x ∈ H)"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_repr_independence':
  **assumes** y: "y ∈ H +> x"
      **and** xcarr: "x ∈ carrier G"
  **shows** "H +> x = H +> y"
  ⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_repr_independenceD:
  **assumes** ycarr: "y ∈ carrier G"
      **and** repr:  "H +> x = H +> y"
  **shows** "y ∈ H +> x"
⟨*proof*⟩

**lemma (in** abelian_subgroup**)** a_rcosets_carrier:
  "X ∈ a_rcosets H ⟹ X ⊆ carrier G"
⟨*proof*⟩

### 8.9.10 Addition of Subgroups

**lemma (in** abelian_monoid**)** set_add_closed:

**assumes** Acarr: "A ⊆ carrier G"
**and** Bcarr: "B ⊆ carrier G"
**shows** "A <+> B ⊆ carrier G"
⟨*proof*⟩

**lemma** (**in** abelian_group) add_additive_subgroups:
**assumes** subH: "additive_subgroup H G"
**and** subK: "additive_subgroup K G"
**shows** "additive_subgroup (H <+> K) G"
⟨*proof*⟩

**end**

**theory** Ideal
**imports** Ring AbelCoset
**begin**

# 9 Ideals

## 9.1 Definitions

### 9.1.1 General definition

**locale** ideal = additive_subgroup I R + ring R **for** I **and** R (**structure**) +
**assumes** I_l_closed: "⟦a ∈ I; x ∈ carrier R⟧ ⟹ x ⊗ a ∈ I"
**and** I_r_closed: "⟦a ∈ I; x ∈ carrier R⟧ ⟹ a ⊗ x ∈ I"

**sublocale** ideal ⊆ abelian_subgroup I R
⟨*proof*⟩

**lemma** (**in** ideal) is_ideal:
"ideal I R"
⟨*proof*⟩

**lemma** idealI:
**fixes** R (**structure**)
**assumes** "ring R"
**assumes** a_subgroup: "subgroup I (|carrier = carrier R, mult = add R,
one = zero R|)"
**and** I_l_closed: "⋀a x. ⟦a ∈ I; x ∈ carrier R⟧ ⟹ x ⊗ a ∈ I"
**and** I_r_closed: "⋀a x. ⟦a ∈ I; x ∈ carrier R⟧ ⟹ a ⊗ x ∈ I"
**shows** "ideal I R"
⟨*proof*⟩

### 9.1.2 Ideals Generated by a Subset of carrier R

**definition**

```
  genideal :: "('a, 'b) ring_scheme ⇒ 'a set ⇒ 'a set"  ("Idl⇘ _" [80]
79)
  where "genideal R S = Inter {I. ideal I R ∧ S ⊆ I}"
```

### 9.1.3 Principal Ideals

```
locale principalideal = ideal +
  assumes generate: "∃i ∈ carrier R. I = Idl {i}"

lemma (in principalideal) is_principalideal:
  shows "principalideal I R"
```
⟨*proof*⟩

```
lemma principalidealI:
  fixes R (structure)
  assumes "ideal I R"
  assumes generate: "∃i ∈ carrier R. I = Idl {i}"
  shows "principalideal I R"
```
⟨*proof*⟩

### 9.1.4 Maximal Ideals

```
locale maximalideal = ideal +
  assumes I_notcarr: "carrier R ≠ I"
      and I_maximal: "⟦ideal J R; I ⊆ J; J ⊆ carrier R⟧ ⟹ J = I ∨ J
= carrier R"

lemma (in maximalideal) is_maximalideal:
 shows "maximalideal I R"
```
⟨*proof*⟩

```
lemma maximalidealI:
  fixes R
  assumes "ideal I R"
  assumes I_notcarr: "carrier R ≠ I"
     and I_maximal: "⋀J. ⟦ideal J R; I ⊆ J; J ⊆ carrier R⟧ ⟹ J = I
∨ J = carrier R"
  shows "maximalideal I R"
```
⟨*proof*⟩

### 9.1.5 Prime Ideals

```
locale primeideal = ideal + cring +
  assumes I_notcarr: "carrier R ≠ I"
      and I_prime: "⟦a ∈ carrier R; b ∈ carrier R; a ⊗ b ∈ I⟧ ⟹ a ∈
I ∨ b ∈ I"

lemma (in primeideal) is_primeideal:
 shows "primeideal I R"
```
⟨*proof*⟩

**lemma** primeidealI:
  **fixes** R **(structure)**
  **assumes** "ideal I R"
  **assumes** "cring R"
  **assumes** I_notcarr: "carrier R ≠ I"
      **and** I_prime: "⋀a b. ⟦a ∈ carrier R; b ∈ carrier R; a ⊗ b ∈ I⟧
⟹ a ∈ I ∨ b ∈ I"
  **shows** "primeideal I R"
⟨*proof*⟩

**lemma** primeidealI2:
  **fixes** R **(structure)**
  **assumes** "additive_subgroup I R"
  **assumes** "cring R"
  **assumes** I_l_closed: "⋀a x. ⟦a ∈ I; x ∈ carrier R⟧ ⟹ x ⊗ a ∈ I"
      **and** I_r_closed: "⋀a x. ⟦a ∈ I; x ∈ carrier R⟧ ⟹ a ⊗ x ∈ I"
      **and** I_notcarr: "carrier R ≠ I"
      **and** I_prime: "⋀a b. ⟦a ∈ carrier R; b ∈ carrier R; a ⊗ b ∈ I⟧
⟹ a ∈ I ∨ b ∈ I"
  **shows** "primeideal I R"
⟨*proof*⟩

## 9.2 Special Ideals

**lemma (in** ring**)** zeroideal:
  **shows** "ideal {**0**} R"
⟨*proof*⟩

**lemma (in** ring**)** oneideal:
  **shows** "ideal (carrier R) R"
⟨*proof*⟩

**lemma (in** "domain"**)** zeroprimeideal:
 **shows** "primeideal {**0**} R"
⟨*proof*⟩

## 9.3 General Ideal Properies

**lemma (in** ideal**)** one_imp_carrier:
  **assumes** I_one_closed: "**1** ∈ I"
  **shows** "I = carrier R"
⟨*proof*⟩

**lemma (in** ideal**)** Icarr:
  **assumes** iI: "i ∈ I"
  **shows** "i ∈ carrier R"
⟨*proof*⟩

## 9.4 Intersection of Ideals

**Intersection of two ideals** The intersection of any two ideals is again an ideal in `R`

**lemma (in** `ring`**)** `i_intersect`**:**
**assumes** "ideal I R"
**assumes** "ideal J R"
**shows** "ideal (I ∩ J) R"
⟨*proof*⟩

The intersection of any Number of Ideals is again an Ideal in `R`

**lemma (in** `ring`**)** `i_Intersect`**:**
**assumes** `Sideals`: "⋀I. I ∈ S ⟹ ideal I R"
**and** `notempty`: "S ≠ {}"
**shows** "ideal (Inter S) R"
⟨*proof*⟩

## 9.5 Addition of Ideals

**lemma (in** `ring`**)** `add_ideals`**:**
**assumes** `idealI`: "ideal I R"
**and** `idealJ`: "ideal J R"
**shows** "ideal (I <+> J) R"
⟨*proof*⟩

## 9.6 Ideals generated by a subset of `carrier R`

`genideal` generates an ideal

**lemma (in** `ring`**)** `genideal_ideal`**:**
**assumes** `Scarr`: "S ⊆ carrier R"
**shows** "ideal (Idl S) R"
⟨*proof*⟩

**lemma (in** `ring`**)** `genideal_self`**:**
**assumes** "S ⊆ carrier R"
**shows** "S ⊆ Idl S"
⟨*proof*⟩

**lemma (in** `ring`**)** `genideal_self'`**:**
**assumes** `carr`: "i ∈ carrier R"
**shows** "i ∈ Idl {i}"
⟨*proof*⟩

`genideal` generates the minimal ideal

**lemma (in** `ring`**)** `genideal_minimal`**:**
**assumes** `a`: "ideal I R"
**and** `b`: "S ⊆ I"
**shows** "Idl S ⊆ I"

⟨*proof*⟩

Generated ideals and subsets

**lemma (in** ring**)** Idl_subset_ideal:
  **assumes** Iideal: "ideal I R"
      **and** Hcarr: "H ⊆ carrier R"
  **shows** "(Idl H ⊆ I) = (H ⊆ I)"
⟨*proof*⟩

**lemma (in** ring**)** subset_Idl_subset:
  **assumes** Icarr: "I ⊆ carrier R"
      **and** HI: "H ⊆ I"
  **shows** "Idl H ⊆ Idl I"
⟨*proof*⟩

**lemma (in** ring**)** Idl_subset_ideal':
  **assumes** acarr: "a ∈ carrier R" **and** bcarr: "b ∈ carrier R"
  **shows** "(Idl {a} ⊆ Idl {b}) = (a ∈ Idl {b})"
⟨*proof*⟩

**lemma (in** ring**)** genideal_zero:
  "Idl {**0**} = {**0**}"
⟨*proof*⟩

**lemma (in** ring**)** genideal_one:
  "Idl {**1**} = carrier R"
⟨*proof*⟩

Generation of Principal Ideals in Commutative Rings

**definition**
  cgenideal :: "('a, 'b) monoid_scheme ⇒ 'a ⇒ 'a set"  ("PIdl↾ _" [80] 79)
  **where** "cgenideal R a = {x ⊗$_R$ a | x. x ∈ carrier R}"

genhideal (?) really generates an ideal

**lemma (in** cring**)** cgenideal_ideal:
  **assumes** acarr: "a ∈ carrier R"
  **shows** "ideal (PIdl a) R"
⟨*proof*⟩

**lemma (in** ring**)** cgenideal_self:
  **assumes** icarr: "i ∈ carrier R"
  **shows** "i ∈ PIdl i"
⟨*proof*⟩

cgenideal is minimal

**lemma (in** ring**)** cgenideal_minimal:
  **assumes** "ideal J R"

**assumes** aJ: "a ∈ J"
**shows** "PIdl a ⊆ J"
⟨*proof*⟩

**lemma (in** cring**)** cgenideal_eq_genideal:
**assumes** icarr: "i ∈ carrier R"
**shows** "PIdl i = Idl {i}"
⟨*proof*⟩

**lemma (in** cring**)** cgenideal_eq_rcos:
"PIdl i = carrier R #> i"
⟨*proof*⟩

**lemma (in** cring**)** cgenideal_is_principalideal:
**assumes** icarr: "i ∈ carrier R"
**shows** "principalideal (PIdl i) R"
⟨*proof*⟩

## 9.7 Union of Ideals

**lemma (in** ring**)** union_genideal:
**assumes** idealI: "ideal I R"
**and** idealJ: "ideal J R"
**shows** "Idl (I ∪ J) = I <+> J"
⟨*proof*⟩

## 9.8 Properties of Principal Ideals

**0** generates the zero ideal

**lemma (in** ring**)** zero_genideal:
**shows** "Idl {**0**} = {**0**}"
⟨*proof*⟩

**1** generates the unit ideal

**lemma (in** ring**)** one_genideal:
**shows** "Idl {**1**} = carrier R"
⟨*proof*⟩

The zero ideal is a principal ideal

**corollary (in** ring**)** zeropideal:
**shows** "principalideal {**0**} R"
⟨*proof*⟩

The unit ideal is a principal ideal

**corollary (in** ring**)** onepideal:
**shows** "principalideal (carrier R) R"
⟨*proof*⟩

Every principal ideal is a right coset of the carrier

**lemma (in** principalideal**)** rcos_generate**:**
**assumes** "cring R"
**shows** "∃x∈I. I = carrier R #> x"
⟨*proof*⟩

## 9.9 Prime Ideals

**lemma (in** ideal**)** primeidealCD**:**
**assumes** "cring R"
**assumes** notprime**:** "¬ primeideal I R"
**shows** "carrier R = I ∨ (∃a b. a ∈ carrier R ∧ b ∈ carrier R ∧ a ⊗ b ∈ I ∧ a ∉ I ∧ b ∉ I)"
⟨*proof*⟩

**lemma (in** ideal**)** primeidealCE**:**
**assumes** "cring R"
**assumes** notprime**:** "¬ primeideal I R"
**obtains** "carrier R = I"
| "∃a b. a ∈ carrier R ∧ b ∈ carrier R ∧ a ⊗ b ∈ I ∧ a ∉ I ∧ b ∉ I"
⟨*proof*⟩

If {**0**} is a prime ideal of a commutative ring, the ring is a domain

**lemma (in** cring**)** zeroprimeideal_domainI**:**
**assumes** pi**:** "primeideal {**0**} R"
**shows** "domain R"
⟨*proof*⟩

**corollary (in** cring**)** domain_eq_zeroprimeideal**:**
**shows** "domain R = primeideal {**0**} R"
⟨*proof*⟩

## 9.10 Maximal Ideals

**lemma (in** ideal**)** helper_I_closed**:**
**assumes** carr**:** "a ∈ carrier R" "x ∈ carrier R" "y ∈ carrier R"
**and** axI**:** "a ⊗ x ∈ I"
**shows** "a ⊗ (x ⊗ y) ∈ I"
⟨*proof*⟩

**lemma (in** ideal**)** helper_max_prime**:**
**assumes** "cring R"
**assumes** acarr**:** "a ∈ carrier R"
**shows** "ideal {x∈carrier R. a ⊗ x ∈ I} R"
⟨*proof*⟩

In a cring every maximal ideal is prime

**lemma (in** cring**)** maximalideal_is_prime**:**
**assumes** "maximalideal I R"

**shows** "primeideal I R"
⟨*proof*⟩

### 9.11 Derived Theorems

— A non-zero cring that has only the two trivial ideals is a field
**lemma (in** cring**)** trivialideals_fieldI:
**assumes** carrnzero: "carrier R ≠ {0}"
**and** haveideals: "{I. ideal I R} = {{0}, carrier R}"
**shows** "field R"
⟨*proof*⟩

**lemma (in** field**)** all_ideals:
**shows** "{I. ideal I R} = {{0}, carrier R}"
⟨*proof*⟩
**lemma (in** cring**)** trivialideals_eq_field:
**assumes** carrnzero: "carrier R ≠ {0}"
**shows** "({I. ideal I R} = {{0}, carrier R}) = field R"
⟨*proof*⟩

Like zeroprimeideal for domains

**lemma (in** field**)** zeromaximalideal:
"maximalideal {0} R"
⟨*proof*⟩

**lemma (in** cring**)** zeromaximalideal_fieldI:
**assumes** zeromax: "maximalideal {0} R"
**shows** "field R"
⟨*proof*⟩

**lemma (in** cring**)** zeromaximalideal_eq_field:
"maximalideal {0} R = field R"
⟨*proof*⟩

**end**

**theory** RingHom
**imports** Ideal
**begin**

## 10 Homomorphisms of Non-Commutative Rings

Lifting existing lemmas in a ring_hom_ring locale

**locale** ring_hom_ring = R: ring R + S: ring S
**for** R **(structure) and** S **(structure)** +
**fixes** h

**assumes** homh: "h ∈ ring_hom R S"
**notes** hom_mult [simp] = ring_hom_mult [OF homh]
**and** hom_one [simp] = ring_hom_one [OF homh]

**sublocale** ring_hom_cring ⊆ ring: ring_hom_ring
⟨*proof*⟩

**sublocale** ring_hom_ring ⊆ abelian_group: abelian_group_hom R S
⟨*proof*⟩

**lemma (in** ring_hom_ring**)** is_ring_hom_ring:
"ring_hom_ring R S h"
⟨*proof*⟩

**lemma** ring_hom_ringI:
**fixes** R **(structure) and** S **(structure)**
**assumes** "ring R" "ring S"
**assumes**
hom_closed: "!!x. x ∈ carrier R ==> h x ∈ carrier S"
**and** compatible_mult: "!!x y. [| x : carrier R; y : carrier R |]
==> h (x ⊗ y) = h x ⊗$_S$ h y"
**and** compatible_add: "!!x y. [| x : carrier R; y : carrier R |] ==>
h (x ⊕ y) = h x ⊕$_S$ h y"
**and** compatible_one: "h **1** = **1**$_S$"
**shows** "ring_hom_ring R S h"
⟨*proof*⟩

**lemma** ring_hom_ringI2:
**assumes** "ring R" "ring S"
**assumes** h: "h ∈ ring_hom R S"
**shows** "ring_hom_ring R S h"
⟨*proof*⟩

**lemma** ring_hom_ringI3:
**fixes** R **(structure) and** S **(structure)**
**assumes** "abelian_group_hom R S h" "ring R" "ring S"
**assumes** compatible_mult: "!!x y. [| x : carrier R; y : carrier R |]
==> h (x ⊗ y) = h x ⊗$_S$ h y"
**and** compatible_one: "h **1** = **1**$_S$"
**shows** "ring_hom_ring R S h"
⟨*proof*⟩

**lemma** ring_hom_cringI:
**assumes** "ring_hom_ring R S h" "cring R" "cring S"
**shows** "ring_hom_cring R S h"
⟨*proof*⟩

### 10.1 The Kernel of a Ring Homomorphism

— the kernel of a ring homomorphism is an ideal
**lemma (in** ring_hom_ring**)** kernel_is_ideal:
  **shows** "ideal (a_kernel R S h) R"
⟨*proof*⟩

Elements of the kernel are mapped to zero

**lemma (in** abelian_group_hom**)** kernel_zero [simp]:
  "i ∈ a_kernel R S h ⟹ h i = $\mathbf{0}_S$"
⟨*proof*⟩

### 10.2 Cosets

Cosets of the kernel correspond to the elements of the image of the homomorphism

**lemma (in** ring_hom_ring**)** rcos_imp_homeq:
  **assumes** acarr: "a ∈ carrier R"
      **and** xrcos: "x ∈ a_kernel R S h +> a"
  **shows** "h x = h a"
⟨*proof*⟩

**lemma (in** ring_hom_ring**)** homeq_imp_rcos:
  **assumes** acarr: "a ∈ carrier R"
      **and** xcarr: "x ∈ carrier R"
      **and** hx: "h x = h a"
  **shows** "x ∈ a_kernel R S h +> a"
⟨*proof*⟩

**corollary (in** ring_hom_ring**)** rcos_eq_homeq:
  **assumes** acarr: "a ∈ carrier R"
  **shows** "(a_kernel R S h) +> a = {x ∈ carrier R. h x = h a}"
⟨*proof*⟩

**end**

**theory** QuotRing
**imports** RingHom
**begin**

# 11 Quotient Rings

### 11.1 Multiplication on Cosets

**definition**
  rcoset_mult :: "[('a, _) ring_scheme, 'a set, 'a set, 'a set] ⇒ 'a set"

("[mod _:] _ ⨂ ≀ _" [81,81,81] 80)
**where** "rcoset_mult R I A B = (⋃a∈A. ⋃b∈B. I +>$_R$ (a ⊗$_R$ b))"

rcoset_mult fulfils the properties required by congruences

**lemma (in** ideal**)** rcoset_mult_add:
"⟦x ∈ carrier R; y ∈ carrier R⟧ ⟹ [mod I:] (I +> x) ⨂ (I +> y) = I +> (x ⊗ y)"
⟨*proof*⟩

## 11.2 Quotient Ring Definition

**definition**
FactRing :: "[('a,'b) ring_scheme, 'a set] ⇒ ('a set) ring" **(infixl** "Quot" 65**)**
**where** "FactRing R I =
(|carrier = a_rcosets$_R$ I, mult = rcoset_mult R I, one = (I +>$_R$ 1$_R$), zero = I, add = set_add R|)"

## 11.3 Factorization over General Ideals

The quotient is a ring

**lemma (in** ideal**)** quotient_is_ring:
**shows** "ring (R Quot I)"
⟨*proof*⟩

This is a ring homomorphism

**lemma (in** ideal**)** rcos_ring_hom:
"(op +> I) ∈ ring_hom R (R Quot I)"
⟨*proof*⟩

**lemma (in** ideal**)** rcos_ring_hom_ring:
"ring_hom_ring R (R Quot I) (op +> I)"
⟨*proof*⟩

The quotient of a cring is also commutative

**lemma (in** ideal**)** quotient_is_cring:
**assumes** "cring R"
**shows** "cring (R Quot I)"
⟨*proof*⟩

Cosets as a ring homomorphism on crings

**lemma (in** ideal**)** rcos_ring_hom_cring:
**assumes** "cring R"
**shows** "ring_hom_cring R (R Quot I) (op +> I)"
⟨*proof*⟩

### 11.4 Factorization over Prime Ideals

The quotient ring generated by a prime ideal is a domain

**lemma (in** primeideal**)** quotient_is_domain:
  **shows** "domain (R Quot I)"
⟨*proof*⟩

Generating right cosets of a prime ideal is a homomorphism on commutative rings

**lemma (in** primeideal**)** rcos_ring_hom_cring:
  **shows** "ring_hom_cring R (R Quot I) (op +> I)"
⟨*proof*⟩

### 11.5 Factorization over Maximal Ideals

In a commutative ring, the quotient ring over a maximal ideal is a field. The proof follows "W. Adkins, S. Weintraub: Algebra – An Approach via Module Theory"

**lemma (in** maximalideal**)** quotient_is_field:
  **assumes** "cring R"
  **shows** "field (R Quot I)"
⟨*proof*⟩

**end**

**theory** IntRing
**imports** QuotRing Lattice Int "~~/src/HOL/Old_Number_Theory/Primes"
**begin**

## 12 The Ring of Integers

### 12.1 Some properties of int

**lemma** dvds_eq_abseq:
  "(l dvd k ∧ k dvd l) = (abs l = abs (k::int))"
⟨*proof*⟩

### 12.2 $\mathcal{Z}$: The Set of Integers as Algebraic Structure

**definition**
  int_ring :: "int ring" ("$\mathcal{Z}$") **where**
  "int_ring = (|carrier = UNIV, mult = op *, one = 1, zero = 0, add = op +|)"

**lemma** int_Zcarr [intro!, simp]:
  "k ∈ carrier $\mathcal{Z}$"

⟨*proof*⟩

**lemma** int_is_cring:
"cring ℤ"
⟨*proof*⟩

## 12.3 Interpretations

Since definitions of derived operations are global, their interpretation needs to be done as early as possible — that is, with as few assumptions as possible.

**interpretation** int: monoid ℤ
**where** "carrier ℤ = UNIV"
**and** "mult ℤ x y = x * y"
**and** "one ℤ = 1"
**and** "pow ℤ x n = x^n"
⟨*proof*⟩

**interpretation** int: comm_monoid ℤ
**where** "finprod ℤ f A = (if finite A then setprod f A else undefined)"
⟨*proof*⟩

**interpretation** int: abelian_monoid ℤ
**where** "zero ℤ = 0"
**and** "add ℤ x y = x + y"
**and** "finsum ℤ f A = (if finite A then setsum f A else undefined)"
⟨*proof*⟩

**interpretation** int: abelian_group ℤ
**where** "a_inv ℤ x = - x"
**and** "a_minus ℤ x y = x - y"
⟨*proof*⟩

**interpretation** int: "domain" ℤ
⟨*proof*⟩

Removal of occurrences of UNIV in interpretation result — experimental.

**lemma** UNIV:
"x ∈ UNIV = True"
"A ⊆ UNIV = True"
"(ALL x : UNIV. P x) = (ALL x. P x)"
"(EX x : UNIV. P x) = (EX x. P x)"
"(True --> Q) = Q"
"(True ==> PROP R) == PROP R"
⟨*proof*⟩

**interpretation** int : partial_order "(| carrier = UNIV::int set, eq = op =, le = op ≤ |)"
**where** "carrier (| carrier = UNIV::int set, eq = op =, le = op ≤ |) = UNIV"

**and** "le (| carrier = UNIV::int set, eq = op =, le = op ≤ |) x y = (x ≤ y)"
**and** "lless (| carrier = UNIV::int set, eq = op =, le = op ≤ |) x y = (x < y)"
⟨*proof*⟩

**interpretation** int : lattice "(| carrier = UNIV::int set, eq = op =, le = op ≤ |)"
**where** "join (| carrier = UNIV::int set, eq = op =, le = op ≤ |) x y = max x y"
**and** "meet (| carrier = UNIV::int set, eq = op =, le = op ≤ |) x y = min x y"
⟨*proof*⟩

**interpretation** int : total_order "(| carrier = UNIV::int set, eq = op =, le = op ≤ |)"
⟨*proof*⟩

## 12.4 Generated Ideals of $\mathcal{Z}$

**lemma** int_Idl:
"Idl$_{\mathcal{Z}}$ {a} = {x * a | x. True}"
⟨*proof*⟩

**lemma** multiples_principalideal:
"principalideal {x * a | x. True } $\mathcal{Z}$"
⟨*proof*⟩

**lemma** prime_primeideal:
**assumes** prime: "prime (nat p)"
**shows** "primeideal (Idl$_{\mathcal{Z}}$ {p}) $\mathcal{Z}$"
⟨*proof*⟩

## 12.5 Ideals and Divisibility

**lemma** int_Idl_subset_ideal:
"Idl$_{\mathcal{Z}}$ {k} ⊆ Idl$_{\mathcal{Z}}$ {l} = (k ∈ Idl$_{\mathcal{Z}}$ {l})"
⟨*proof*⟩

**lemma** Idl_subset_eq_dvd:
"(Idl$_{\mathcal{Z}}$ {k} ⊆ Idl$_{\mathcal{Z}}$ {l}) = (l dvd k)"
⟨*proof*⟩

**lemma** dvds_eq_Idl:
"(l dvd k ∧ k dvd l) = (Idl$_{\mathcal{Z}}$ {k} = Idl$_{\mathcal{Z}}$ {l})"
⟨*proof*⟩

**lemma** Idl_eq_abs:
"(Idl$_{\mathcal{Z}}$ {k} = Idl$_{\mathcal{Z}}$ {l}) = (abs l = abs k)"

⟨*proof*⟩

## 12.6 Ideals and the Modulus

**definition**
```
  ZMod :: "int => int => int set"
  where "ZMod k r = (Idl_Z {k}) +>_Z r"
```

**lemmas** `ZMod_defs =`
```
  ZMod_def genideal_def
```

**lemma** `rcos_zfact:`
```
  assumes kIl: "k ∈ ZMod l r"
  shows "EX x. k = x * l + r"
```
⟨*proof*⟩

**lemma** `ZMod_imp_zmod:`
```
  assumes zmods: "ZMod m a = ZMod m b"
  shows "a mod m = b mod m"
```
⟨*proof*⟩

**lemma** `ZMod_mod:`
```
  shows "ZMod m a = ZMod m (a mod m)"
```
⟨*proof*⟩

**lemma** `zmod_imp_ZMod:`
```
  assumes modeq: "a mod m = b mod m"
  shows "ZMod m a = ZMod m b"
```
⟨*proof*⟩

**corollary** `ZMod_eq_mod:`
```
  shows "(ZMod m a = ZMod m b) = (a mod m = b mod m)"
```
⟨*proof*⟩

## 12.7 Factorization

**definition**
```
  ZFact :: "int ⇒ int set ring"
  where "ZFact k = Z Quot (Idl_Z {k})"
```

**lemmas** `ZFact_defs = ZFact_def FactRing_def`

**lemma** `ZFact_is_cring:`
```
  shows "cring (ZFact k)"
```
⟨*proof*⟩

**lemma** `ZFact_zero:`
```
  "carrier (ZFact 0) = (⋃a. {{a}})"
```
⟨*proof*⟩

**lemma** ZFact_one:
  "carrier (ZFact 1) = {UNIV}"
⟨*proof*⟩

**lemma** ZFact_prime_is_domain:
  **assumes** pprime: "prime (nat p)"
  **shows** "domain (ZFact p)"
⟨*proof*⟩

**end**

**theory** Module
**imports** Ring
**begin**

# 13 Modules over an Abelian Group

## 13.1 Definitions

**record** ('a, 'b) module = "'b ring" +
  smult :: "['a, 'b] => 'b" (**infixl** "⊙₂" 70)

**locale** module = R: cring + M: abelian_group M **for** M (**structure**) +
  **assumes** smult_closed [simp, intro]:
      "[| a ∈ carrier R; x ∈ carrier M |] ==> a ⊙$_M$ x ∈ carrier M"
    **and** smult_l_distr:
      "[| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
      (a ⊕ b) ⊙$_M$ x = a ⊙$_M$ x ⊕$_M$ b ⊙$_M$ x"
    **and** smult_r_distr:
      "[| a ∈ carrier R; x ∈ carrier M; y ∈ carrier M |] ==>
      a ⊙$_M$ (x ⊕$_M$ y) = a ⊙$_M$ x ⊕$_M$ a ⊙$_M$ y"
    **and** smult_assoc1:
      "[| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
      (a ⊗ b) ⊙$_M$ x = a ⊙$_M$ (b ⊙$_M$ x)"
    **and** smult_one [simp]:
      "x ∈ carrier M ==> **1** ⊙$_M$ x = x"

**locale** algebra = module + cring M +
  **assumes** smult_assoc2:
      "[| a ∈ carrier R; x ∈ carrier M; y ∈ carrier M |] ==>
      (a ⊙$_M$ x) ⊗$_M$ y = a ⊙$_M$ (x ⊗$_M$ y)"

**lemma** moduleI:
  **fixes** R (**structure**) **and** M (**structure**)
  **assumes** cring: "cring R"
    **and** abelian_group: "abelian_group M"
    **and** smult_closed:

```
      "!!a x. [| a ∈ carrier R; x ∈ carrier M |] ==> a ⊙M x ∈ carrier
M"
    and smult_l_distr:
      "!!a b x. [| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
      (a ⊕ b) ⊙M x = (a ⊙M x) ⊕M (b ⊙M x)"
    and smult_r_distr:
      "!!a x y. [| a ∈ carrier R; x ∈ carrier M; y ∈ carrier M |] ==>
      a ⊙M (x ⊕M y) = (a ⊙M x) ⊕M (a ⊙M y)"
    and smult_assoc1:
      "!!a b x. [| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
      (a ⊗ b) ⊙M x = a ⊙M (b ⊙M x)"
    and smult_one:
      "!!x. x ∈ carrier M ==> 1 ⊙M x = x"
  shows "module R M"
  ⟨proof⟩

lemma algebraI:
  fixes R (structure) and M (structure)
  assumes R_cring: "cring R"
    and M_cring: "cring M"
    and smult_closed:
      "!!a x. [| a ∈ carrier R; x ∈ carrier M |] ==> a ⊙M x ∈ carrier
M"
    and smult_l_distr:
      "!!a b x. [| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
      (a ⊕ b) ⊙M x = (a ⊙M x) ⊕M (b ⊙M x)"
    and smult_r_distr:
      "!!a x y. [| a ∈ carrier R; x ∈ carrier M; y ∈ carrier M |] ==>
      a ⊙M (x ⊕M y) = (a ⊙M x) ⊕M (a ⊙M y)"
    and smult_assoc1:
      "!!a b x. [| a ∈ carrier R; b ∈ carrier R; x ∈ carrier M |] ==>
      (a ⊗ b) ⊙M x = a ⊙M (b ⊙M x)"
    and smult_one:
      "!!x. x ∈ carrier M ==> (one R) ⊙M x = x"
    and smult_assoc2:
      "!!a x y. [| a ∈ carrier R; x ∈ carrier M; y ∈ carrier M |] ==>
      (a ⊙M x) ⊗M y = a ⊙M (x ⊗M y)"
  shows "algebra R M"
⟨proof⟩

lemma (in algebra) R_cring:
  "cring R"
  ⟨proof⟩

lemma (in algebra) M_cring:
  "cring M"
  ⟨proof⟩

lemma (in algebra) module:
```

```
  "module R M"
```
⟨*proof*⟩

### 13.2 Basic Properties of Algebras

**lemma (in** algebra**)** smult_l_null [simp]:
```
  "x ∈ carrier M ==> 0 ⊙M x = 0M"
```
⟨*proof*⟩

**lemma (in** algebra**)** smult_r_null [simp]:
```
  "a ∈ carrier R ==> a ⊙M 0M = 0M"
```
⟨*proof*⟩

**lemma (in** algebra**)** smult_l_minus:
```
  "[| a ∈ carrier R; x ∈ carrier M |] ==> (⊖a) ⊙M x = ⊖M (a ⊙M x)"
```
⟨*proof*⟩

**lemma (in** algebra**)** smult_r_minus:
```
  "[| a ∈ carrier R; x ∈ carrier M |] ==> a ⊙M (⊖Mx) = ⊖M (a ⊙M x)"
```
⟨*proof*⟩

**end**

**theory** UnivPoly
**imports** Module RingHom
**begin**

## 14 Univariate Polynomials

Polynomials are formalised as modules with additional operations for extracting coefficients from polynomials and for obtaining monomials from coefficients and exponents (record `up_ring`). The carrier set is a set of bounded functions from Nat to the coefficient domain. Bounded means that these functions return zero above a certain bound (the degree). There is a chapter on the formalisation of polynomials in the PhD thesis [1], which was implemented with axiomatic type classes. This was later ported to Locales.

### 14.1 The Constructor for Univariate Polynomials

Functions with finite support.

**locale** bound =
```
  fixes z :: 'a
    and n :: nat
    and f :: "nat => 'a"
  assumes bound: "!!m. n < m ⟹ f m = z"
```

**declare** bound.intro [intro!]
  **and** bound.bound [dest]

**lemma** bound_below:
  **assumes** bound: "bound z m f" **and** nonzero: "f n ≠ z" **shows** "n ≤ m"
⟨*proof*⟩

**record** ('a, 'p) up_ring = "('a, 'p) module" +
  monom :: "['a, nat] => 'p"
  coeff :: "['p, nat] => 'a"

**definition**
  up :: "('a, 'm) ring_scheme => (nat => 'a) set"
  **where** "up R = {f. f ∈ UNIV -> carrier R & (EX n. bound $\mathbf{0}_R$ n f)}"

**definition** UP :: "('a, 'm) ring_scheme => ('a, nat => 'a) up_ring"
  **where** "UP R = (|
   carrier = up R,
   mult = (%p:up R. %q:up R. %n. $\bigoplus_R$i ∈ {..n}. p i $\otimes_R$ q (n-i)),
   one = (%i. if i=0 then $\mathbf{1}_R$ else $\mathbf{0}_R$),
   zero = (%i. $\mathbf{0}_R$),
   add = (%p:up R. %q:up R. %i. p i $\oplus_R$ q i),
   smult = (%a:carrier R. %p:up R. %i. a $\otimes_R$ p i),
   monom = (%a:carrier R. %n i. if i=n then a else $\mathbf{0}_R$),
   coeff = (%p:up R. %n. p n) |)"

Properties of the set of polynomials up.

**lemma** mem_upI [intro]:
  "[| !!n. f n ∈ carrier R; EX n. bound (zero R) n f |] ==> f ∈ up R"
  ⟨*proof*⟩

**lemma** mem_upD [dest]:
  "f ∈ up R ==> f n ∈ carrier R"
  ⟨*proof*⟩

**context** ring
**begin**

**lemma** bound_upD [dest]: "f ∈ up R ==> EX n. bound $\mathbf{0}$ n f" ⟨*proof*⟩

**lemma** up_one_closed: "(%n. if n = 0 then $\mathbf{1}$ else $\mathbf{0}$) ∈ up R" ⟨*proof*⟩

**lemma** up_smult_closed: "[| a ∈ carrier R; p ∈ up R |] ==> (%i. a ⊗ p i) ∈ up R" ⟨*proof*⟩

**lemma** up_add_closed:
  "[| p ∈ up R; q ∈ up R |] ==> (%i. p i ⊕ q i) ∈ up R"
⟨*proof*⟩

**lemma** up_a_inv_closed:
"p ∈ up R ==> (%i. ⊖ (p i)) ∈ up R"
⟨*proof*⟩

**lemma** up_minus_closed:
"[| p ∈ up R; q ∈ up R |] ==> (%i. p i ⊖ q i) ∈ up R"
⟨*proof*⟩

**lemma** up_mult_closed:
"[| p ∈ up R; q ∈ up R |] ==>
(%n. ⨁i ∈ {..n}. p i ⊗ q (n-i)) ∈ up R"
⟨*proof*⟩

**end**

## 14.2 Effect of Operations on Coefficients

**locale** UP =
**fixes** R (**structure**) **and** P (**structure**)
**defines** P_def: "P == UP R"

**locale** UP_ring = UP + R: ring R

**locale** UP_cring = UP + R: cring R

**sublocale** UP_cring < UP_ring
⟨*proof*⟩

**locale** UP_domain = UP + R: "domain" R

**sublocale** UP_domain < UP_cring
⟨*proof*⟩

**context** UP
**begin**

Temporarily declare P ≡ UP R as simp rule.

**declare** P_def [simp]

**lemma** up_eqI:
**assumes** prem: "!!n. coeff P p n = coeff P q n" **and** R: "p ∈ carrier P" "q ∈ carrier P"
**shows** "p = q"
⟨*proof*⟩

**lemma** coeff_closed [simp]:
"p ∈ carrier P ==> coeff P p n ∈ carrier R" ⟨*proof*⟩

**end**

**context** UP_ring
**begin**

**lemma** coeff_monom [simp]:
"a ∈ carrier R ==> coeff P (monom P a m) n = (if m=n then a else **0**)"
⟨*proof*⟩

**lemma** coeff_zero [simp]: "coeff P **0**$_P$ n = **0**" ⟨*proof*⟩

**lemma** coeff_one [simp]: "coeff P **1**$_P$ n = (if n=0 then **1** else **0**)"
⟨*proof*⟩

**lemma** coeff_smult [simp]:
"[| a ∈ carrier R; p ∈ carrier P |] ==> coeff P (a ⊙$_P$ p) n = a ⊗ coeff P p n"
⟨*proof*⟩

**lemma** coeff_add [simp]:
"[| p ∈ carrier P; q ∈ carrier P |] ==> coeff P (p ⊕$_P$ q) n = coeff P p n ⊕ coeff P q n"
⟨*proof*⟩

**lemma** coeff_mult [simp]:
"[| p ∈ carrier P; q ∈ carrier P |] ==> coeff P (p ⊗$_P$ q) n = (⨁i ∈ {..n}. coeff P p i ⊗ coeff P q (n-i))"
⟨*proof*⟩

**end**

## 14.3 Polynomials Form a Ring.

**context** UP_ring
**begin**

Operations are closed over P.

**lemma** UP_mult_closed [simp]:
"[| p ∈ carrier P; q ∈ carrier P |] ==> p ⊗$_P$ q ∈ carrier P" ⟨*proof*⟩

**lemma** UP_one_closed [simp]:
"**1**$_P$ ∈ carrier P" ⟨*proof*⟩

**lemma** UP_zero_closed [intro, simp]:
"**0**$_P$ ∈ carrier P" ⟨*proof*⟩

**lemma** UP_a_closed [intro, simp]:

"[| p ∈ carrier P; q ∈ carrier P |] ==> p ⊕P q ∈ carrier P" ⟨*proof*⟩

**lemma** monom_closed [simp]:
"a ∈ carrier R ==> monom P a n ∈ carrier P" ⟨*proof*⟩

**lemma** UP_smult_closed [simp]:
"[| a ∈ carrier R; p ∈ carrier P |] ==> a ⊙P p ∈ carrier P" ⟨*proof*⟩

**end**

**declare (in** UP) P_def [simp del]

Algebraic ring properties

**context** UP_ring
**begin**

**lemma** UP_a_assoc:
**assumes** R: "p ∈ carrier P" "q ∈ carrier P" "r ∈ carrier P"
**shows** "(p ⊕P q) ⊕P r = p ⊕P (q ⊕P r)" ⟨*proof*⟩

**lemma** UP_l_zero [simp]:
**assumes** R: "p ∈ carrier P"
**shows** "**0**P ⊕P p = p" ⟨*proof*⟩

**lemma** UP_l_neg_ex:
**assumes** R: "p ∈ carrier P"
**shows** "EX q : carrier P. q ⊕P p = **0**P"
⟨*proof*⟩

**lemma** UP_a_comm:
**assumes** R: "p ∈ carrier P" "q ∈ carrier P"
**shows** "p ⊕P q = q ⊕P p" ⟨*proof*⟩

**lemma** UP_m_assoc:
**assumes** R: "p ∈ carrier P" "q ∈ carrier P" "r ∈ carrier P"
**shows** "(p ⊗P q) ⊗P r = p ⊗P (q ⊗P r)"
⟨*proof*⟩

**lemma** UP_r_one [simp]:
**assumes** R: "p ∈ carrier P" **shows** "p ⊗P **1**P = p"
⟨*proof*⟩

**lemma** UP_l_one [simp]:
**assumes** R: "p ∈ carrier P"
**shows** "**1**P ⊗P p = p"
⟨*proof*⟩

**lemma** UP_l_distr:
**assumes** R: "p ∈ carrier P" "q ∈ carrier P" "r ∈ carrier P"

**shows** "(p $\oplus_P$ q) $\otimes_P$ r = (p $\otimes_P$ r) $\oplus_P$ (q $\otimes_P$ r)"
⟨*proof*⟩

**lemma** UP_r_distr:
**assumes** R: "p ∈ carrier P" "q ∈ carrier P" "r ∈ carrier P"
**shows** "r $\otimes_P$ (p $\oplus_P$ q) = (r $\otimes_P$ p) $\oplus_P$ (r $\otimes_P$ q)"
⟨*proof*⟩

**theorem** UP_ring: "ring P"
⟨*proof*⟩

**end**

## 14.4 Polynomials Form a Commutative Ring.

**context** UP_cring
**begin**

**lemma** UP_m_comm:
**assumes** R1: "p ∈ carrier P" **and** R2: "q ∈ carrier P" **shows** "p $\otimes_P$ q = q $\otimes_P$ p"
⟨*proof*⟩

## 14.5 Polynomials over a commutative ring for a commutative ring

**theorem** UP_cring:
"cring P" ⟨*proof*⟩

**end**

**context** UP_ring
**begin**

**lemma** UP_a_inv_closed [intro, simp]:
"p ∈ carrier P ==> $\ominus_P$ p ∈ carrier P"
⟨*proof*⟩

**lemma** coeff_a_inv [simp]:
**assumes** R: "p ∈ carrier P"
**shows** "coeff P ($\ominus_P$ p) n = $\ominus$ (coeff P p n)"
⟨*proof*⟩

**end**

**sublocale** UP_ring < P: ring P ⟨*proof*⟩
**sublocale** UP_cring < P: cring P ⟨*proof*⟩

## 14.6 Polynomials Form an Algebra

**context** UP_ring
**begin**

**lemma** UP_smult_l_distr:
"[| a ∈ carrier R; b ∈ carrier R; p ∈ carrier P |] ==>
(a ⊕ b) ⊙P p = a ⊙P p ⊕P b ⊙P p"
⟨*proof*⟩

**lemma** UP_smult_r_distr:
"[| a ∈ carrier R; p ∈ carrier P; q ∈ carrier P |] ==>
a ⊙P (p ⊕P q) = a ⊙P p ⊕P a ⊙P q"
⟨*proof*⟩

**lemma** UP_smult_assoc1:
"[| a ∈ carrier R; b ∈ carrier R; p ∈ carrier P |] ==>
(a ⊗ b) ⊙P p = a ⊙P (b ⊙P p)"
⟨*proof*⟩

**lemma** UP_smult_zero [simp]:
"p ∈ carrier P ==> **0** ⊙P p = **0**P"
⟨*proof*⟩

**lemma** UP_smult_one [simp]:
"p ∈ carrier P ==> **1** ⊙P p = p"
⟨*proof*⟩

**lemma** UP_smult_assoc2:
"[| a ∈ carrier R; p ∈ carrier P; q ∈ carrier P |] ==>
(a ⊙P p) ⊗P q = a ⊙P (p ⊗P q)"
⟨*proof*⟩

**end**

Interpretation of lemmas from algebra.

**lemma** (**in** cring) cring:
"cring R" ⟨*proof*⟩

**lemma** (**in** UP_cring) UP_algebra:
"algebra R P" ⟨*proof*⟩

**sublocale** UP_cring < algebra R P ⟨*proof*⟩

## 14.7 Further Lemmas Involving Monomials

**context** UP_ring
**begin**

**lemma** monom_zero [simp]:

"monom P **0** n = **0**$_P$" ⟨*proof*⟩

**lemma** monom_mult_is_smult:
**assumes** R: "a ∈ carrier R" "p ∈ carrier P"
**shows** "monom P a 0 ⊗$_P$ p = a ⊙$_P$ p"
⟨*proof*⟩

**lemma** monom_one [simp]:
"monom P **1** 0 = **1**$_P$"
⟨*proof*⟩

**lemma** monom_add [simp]:
"[| a ∈ carrier R; b ∈ carrier R |] ==>
monom P (a ⊕ b) n = monom P a n ⊕$_P$ monom P b n"
⟨*proof*⟩

**lemma** monom_one_Suc:
"monom P **1** (Suc n) = monom P **1** n ⊗$_P$ monom P **1** 1"
⟨*proof*⟩

**lemma** monom_one_Suc2:
"monom P **1** (Suc n) = monom P **1** 1 ⊗$_P$ monom P **1** n"
⟨*proof*⟩

The following corollary follows from lemmas monom P **1** (Suc ?n) = monom P **1** ?n ⊗$_P$ monom P **1** 1 and monom P **1** (Suc ?n) = monom P **1** 1 ⊗$_P$ monom P **1** ?n, and is trivial in UP_cring

**corollary** monom_one_comm: **shows** "monom P **1** k ⊗$_P$ monom P **1** 1 = monom P **1** 1 ⊗$_P$ monom P **1** k"
⟨*proof*⟩

**lemma** monom_mult_smult:
"[| a ∈ carrier R; b ∈ carrier R |] ==> monom P (a ⊗ b) n = a ⊙$_P$ monom P b n"
⟨*proof*⟩

**lemma** monom_one_mult:
"monom P **1** (n + m) = monom P **1** n ⊗$_P$ monom P **1** m"
⟨*proof*⟩

**lemma** monom_one_mult_comm: "monom P **1** n ⊗$_P$ monom P **1** m = monom P **1** m ⊗$_P$ monom P **1** n"
⟨*proof*⟩

**lemma** monom_mult [simp]:
**assumes** a_in_R: "a ∈ carrier R" **and** b_in_R: "b ∈ carrier R"
**shows** "monom P (a ⊗ b) (n + m) = monom P a n ⊗$_P$ monom P b m"
⟨*proof*⟩

**lemma** monom_a_inv [simp]:
"a ∈ carrier R ==> monom P (⊖ a) n = ⊖$_P$ monom P a n"
⟨*proof*⟩

**lemma** monom_inj:
"inj_on (%a. monom P a n) (carrier R)"
⟨*proof*⟩

**end**

## 14.8 The Degree Function

**definition**
deg :: "[('a, 'm) ring_scheme, nat => 'a] => nat"
**where** "deg R p = (LEAST n. bound $\mathbf{0}_R$ n (coeff (UP R) p))"

**context** UP_ring
**begin**

**lemma** deg_aboveI:
"[| (!!m. n < m ==> coeff P p m = **0**); p ∈ carrier P |] ==> deg R p <= n"
⟨*proof*⟩

**lemma** deg_aboveD:
**assumes** "deg R p < m" **and** "p ∈ carrier P"
**shows** "coeff P p m = **0**"
⟨*proof*⟩

**lemma** deg_belowI:
**assumes** non_zero: "n ~= 0 ==> coeff P p n ~= **0**"
**and** R: "p ∈ carrier P"
**shows** "n <= deg R p"
— Logically, this is a slightly stronger version of deg_aboveD
⟨*proof*⟩

**lemma** lcoeff_nonzero_deg:
**assumes** deg: "deg R p ~= 0" **and** R: "p ∈ carrier P"
**shows** "coeff P p (deg R p) ~= **0**"
⟨*proof*⟩

**lemma** lcoeff_nonzero_nonzero:
**assumes** deg: "deg R p = 0" **and** nonzero: "p ~= $\mathbf{0}_P$" **and** R: "p ∈ carrier P"
**shows** "coeff P p 0 ~= **0**"
⟨*proof*⟩

**lemma** lcoeff_nonzero:
  **assumes** neq: "p ~= $\mathbf{0}_P$" **and** R: "p ∈ carrier P"
  **shows** "coeff P p (deg R p) ~= **0**"
⟨*proof*⟩

**lemma** deg_eqI:
  "[| !!m. n < m ==> coeff P p m = **0**;
      !!n. n ~= 0 ==> coeff P p n ~= **0**; p ∈ carrier P |] ==> deg R p = n"
⟨*proof*⟩

Degree and polynomial operations

**lemma** deg_add [simp]:
  "p ∈ carrier P ⟹ q ∈ carrier P ⟹
  deg R (p $\oplus_P$ q) <= max (deg R p) (deg R q)"
⟨*proof*⟩

**lemma** deg_monom_le:
  "a ∈ carrier R ==> deg R (monom P a n) <= n"
  ⟨*proof*⟩

**lemma** deg_monom [simp]:
  "[| a ~= **0**; a ∈ carrier R |] ==> deg R (monom P a n) = n"
  ⟨*proof*⟩

**lemma** deg_const [simp]:
  **assumes** R: "a ∈ carrier R" **shows** "deg R (monom P a 0) = 0"
⟨*proof*⟩

**lemma** deg_zero [simp]:
  "deg R $\mathbf{0}_P$ = 0"
⟨*proof*⟩

**lemma** deg_one [simp]:
  "deg R $\mathbf{1}_P$ = 0"
⟨*proof*⟩

**lemma** deg_uminus [simp]:
  **assumes** R: "p ∈ carrier P" **shows** "deg R ($\ominus_P$ p) = deg R p"
⟨*proof*⟩

The following lemma is later *overwritten* by the most specific one for domains, deg_smult.

**lemma** deg_smult_ring [simp]:
  "[| a ∈ carrier R; p ∈ carrier P |] ==>
  deg R (a $\odot_P$ p) <= (if a = **0** then 0 else deg R p)"
  ⟨*proof*⟩

**end**

**context** UP_domain
**begin**

**lemma** deg_smult [simp]:
  **assumes** R: "a ∈ carrier R" "p ∈ carrier P"
  **shows** "deg R (a ⊙P p) = (if a = **0** then 0 else deg R p)"
⟨*proof*⟩

**end**

**context** UP_ring
**begin**

**lemma** deg_mult_ring:
  **assumes** R: "p ∈ carrier P" "q ∈ carrier P"
  **shows** "deg R (p ⊗P q) <= deg R p + deg R q"
⟨*proof*⟩

**end**

**context** UP_domain
**begin**

**lemma** deg_mult [simp]:
  "[| p ~= **0**P; q ~= **0**P; p ∈ carrier P; q ∈ carrier P |] ==>
  deg R (p ⊗P q) = deg R p + deg R q"
⟨*proof*⟩

**end**

The following lemmas also can be lifted to UP_ring.

**context** UP_ring
**begin**

**lemma** coeff_finsum:
  **assumes** fin: "finite A"
  **shows** "p ∈ A -> carrier P ==>
    coeff P (finsum P p A) k = (⨁i ∈ A. coeff P (p i) k)"
  ⟨*proof*⟩

**lemma** up_repr:
  **assumes** R: "p ∈ carrier P"
  **shows** "(⨁P i ∈ {..deg R p}. monom P (coeff P p i) i) = p"
⟨*proof*⟩

**lemma** up_repr_le:
  "[| deg R p <= n; p ∈ carrier P |] ==>
  (⨁P i ∈ {..n}. monom P (coeff P p i) i) = p"

⟨*proof*⟩

**end**

## 14.9 Polynomials over Integral Domains

**lemma** domainI:
  **assumes** cring: "cring R"
    **and** one_not_zero: "one R ~= zero R"
    **and** integral: "!!a b. [| mult R a b = zero R; a ∈ carrier R;
      b ∈ carrier R |] ==> a = zero R | b = zero R"
  **shows** "domain R"
  ⟨*proof*⟩

**context** UP_domain
**begin**

**lemma** UP_one_not_zero:
  "$\mathbf{1}_P$ ~= $\mathbf{0}_P$"
⟨*proof*⟩

**lemma** UP_integral:
  "[| p $\otimes_P$ q = $\mathbf{0}_P$; p ∈ carrier P; q ∈ carrier P |] ==> p = $\mathbf{0}_P$ | q = $\mathbf{0}_P$"
⟨*proof*⟩

**theorem** UP_domain:
  "domain P"
  ⟨*proof*⟩

**end**

Interpretation of theorems from domain.

**sublocale** UP_domain < "domain" P
  ⟨*proof*⟩

## 14.10 The Evaluation Homomorphism and Universal Property

**lemma** (**in** abelian_monoid) boundD_carrier:
  "[| bound **0** n f; n < m |] ==> f m ∈ carrier G"
  ⟨*proof*⟩

**context** ring
**begin**

**theorem** diagonal_sum:
  "[| f ∈ {..n + m::nat} -> carrier R; g ∈ {..n + m} -> carrier R |] ==>
  ($\bigoplus$k ∈ {..n + m}. $\bigoplus$i ∈ {..k}. f i ⊗ g (k - i)) =
  ($\bigoplus$k ∈ {..n + m}. $\bigoplus$i ∈ {..n + m - k}. f k ⊗ g i)"

⟨*proof*⟩

**theorem** cauchy_product:
  **assumes** bf: "bound **0** n f" **and** bg: "bound **0** m g"
    **and** Rf: "f ∈ {..n} -> carrier R" **and** Rg: "g ∈ {..m} -> carrier R"
  **shows** "(⨁k ∈ {..n + m}. ⨁i ∈ {..k}. f i ⊗ g (k - i)) =
    (⨁i ∈ {..n}. f i) ⊗ (⨁i ∈ {..m}. g i)"
⟨*proof*⟩

**end**

**lemma** (**in** UP_ring) const_ring_hom:
  "(%a. monom P a 0) ∈ ring_hom R P"
  ⟨*proof*⟩

**definition**
  eval :: "[('a, 'm) ring_scheme, ('b, 'n) ring_scheme,
          'a => 'b, 'b, nat => 'a] => 'b"
  **where** "eval R S phi s = (λp ∈ carrier (UP R).
    ⨁$_S$i ∈ {..deg R p}. phi (coeff (UP R) p i) ⊗$_S$ s (^)$_S$ i)"

**context** UP
**begin**

**lemma** eval_on_carrier:
  **fixes** S (**structure**)
  **shows** "p ∈ carrier P ==>
  eval R S phi s p = (⨁$_S$ i ∈ {..deg R p}. phi (coeff P p i) ⊗$_S$ s (^)$_S$
i)"
  ⟨*proof*⟩

**lemma** eval_extensional:
  "eval R S phi p ∈ extensional (carrier P)"
  ⟨*proof*⟩

**end**

The universal property of the polynomial ring

**locale** UP_pre_univ_prop = ring_hom_cring + UP_cring

**locale** UP_univ_prop = UP_pre_univ_prop +
  **fixes** s **and** Eval
  **assumes** indet_img_carrier [simp, intro]: "s ∈ carrier S"
  **defines** Eval_def: "Eval == eval R S h s"

JE: I have moved the following lemma from Ring.thy and lifted then to the locale ring_hom_ring from ring_hom_cring.

JE: I was considering using it in `eval_ring_hom`, but that property does not hold for non commutative rings, so maybe it is not that necessary.

**lemma (in** `ring_hom_ring`**)** `hom_finsum [simp]:`
```
  "[| finite A; f ∈ A -> carrier R |] ==>
  h (finsum R f A) = finsum S (h o f) A"
```
⟨*proof*⟩

**context** `UP_pre_univ_prop`
**begin**

**theorem** `eval_ring_hom:`
```
  assumes S: "s ∈ carrier S"
  shows "eval R S h s ∈ ring_hom P S"
```
⟨*proof*⟩

The following lemma could be proved in `UP_cring` with the additional assumption that `h` is closed.

**lemma (in** `UP_pre_univ_prop`**)** `eval_const:`
```
  "[| s ∈ carrier S; r ∈ carrier R |] ==> eval R S h s (monom P r 0) =
h r"
```
⟨*proof*⟩

Further properties of the evaluation homomorphism.

The following proof is complicated by the fact that in arbitrary rings one might have **1 = 0**.

**lemma (in** `UP_pre_univ_prop`**)** `eval_monom1:`
```
  assumes S: "s ∈ carrier S"
  shows "eval R S h s (monom P 1 1) = s"
```
⟨*proof*⟩

**end**

Interpretation of ring homomorphism lemmas.

**sublocale** `UP_univ_prop < ring_hom_cring P S Eval`
⟨*proof*⟩

**lemma (in** `UP_cring`**)** `monom_pow:`
```
  assumes R: "a ∈ carrier R"
  shows "(monom P a n) (^)_P m = monom P (a (^) m) (n * m)"
```
⟨*proof*⟩

**lemma (in** `ring_hom_cring`**)** `hom_pow [simp]:`
```
  "x ∈ carrier R ==> h (x (^) n) = h x (^)_S (n::nat)"
```
⟨*proof*⟩

**lemma (in** `UP_univ_prop`**)** `Eval_monom:`
```
  "r ∈ carrier R ==> Eval (monom P r n) = h r ⊗_S s (^)_S n"
```

⟨*proof*⟩

**lemma (in** UP_pre_univ_prop**)** eval_monom:
  **assumes** R: "r ∈ carrier R" **and** S: "s ∈ carrier S"
  **shows** "eval R S h s (monom P r n) = h r ⊗$_S$ s (^)$_S$ n"
⟨*proof*⟩

**lemma (in** UP_univ_prop**)** Eval_smult:
  "[| r ∈ carrier R; p ∈ carrier P |] ==> Eval (r ⊙$_P$ p) = h r ⊗$_S$ Eval p"
⟨*proof*⟩

**lemma** ring_hom_cringI:
  **assumes** "cring R"
    **and** "cring S"
    **and** "h ∈ ring_hom R S"
  **shows** "ring_hom_cring R S h"
  ⟨*proof*⟩

**context** UP_pre_univ_prop
**begin**

**lemma** UP_hom_unique:
  **assumes** "ring_hom_cring P S Phi"
  **assumes** Phi: "Phi (monom P **1** (Suc 0)) = s"
      "!!r. r ∈ carrier R ==> Phi (monom P r 0) = h r"
  **assumes** "ring_hom_cring P S Psi"
  **assumes** Psi: "Psi (monom P **1** (Suc 0)) = s"
      "!!r. r ∈ carrier R ==> Psi (monom P r 0) = h r"
    **and** P: "p ∈ carrier P" **and** S: "s ∈ carrier S"
  **shows** "Phi p = Psi p"
⟨*proof*⟩

**lemma** ring_homD:
  **assumes** Phi: "Phi ∈ ring_hom P S"
  **shows** "ring_hom_cring P S Phi"
  ⟨*proof*⟩

**theorem** UP_universal_property:
  **assumes** S: "s ∈ carrier S"
  **shows** "EX! Phi. Phi ∈ ring_hom P S ∩ extensional (carrier P) &
    Phi (monom P **1** 1) = s &
    (ALL r : carrier R. Phi (monom P r 0) = h r)"
  ⟨*proof*⟩

**end**

JE: The following lemma was added by me; it might be even lifted to a simpler locale

**context** monoid
**begin**

**lemma** nat_pow_eone[simp]: **assumes** x_in_G: "x ∈ carrier G" **shows** "x (^) (1::nat) = x"
⟨*proof*⟩

**end**

**context** UP_ring
**begin**

**abbreviation** lcoeff :: "(nat =>'a) => 'a" **where** "lcoeff p == coeff P p (deg R p)"

**lemma** lcoeff_nonzero2: **assumes** p_in_R: "p ∈ carrier P" **and** p_not_zero: "p ≠ 0$_P$" **shows** "lcoeff p ≠ 0"
⟨*proof*⟩

## 14.11 The long division algorithm: some previous facts.

**lemma** coeff_minus [simp]:
**assumes** p: "p ∈ carrier P" **and** q: "q ∈ carrier P" **shows** "coeff P (p ⊖$_P$ q) n = coeff P p n ⊖ coeff P q n"
⟨*proof*⟩

**lemma** lcoeff_closed [simp]: **assumes** p: "p ∈ carrier P" **shows** "lcoeff p ∈ carrier R"
⟨*proof*⟩

**lemma** deg_smult_decr: **assumes** a_in_R: "a ∈ carrier R" **and** f_in_P: "f ∈ carrier P" **shows** "deg R (a ⊙$_P$ f) ≤ deg R f"
⟨*proof*⟩

**lemma** coeff_monom_mult: **assumes** R: "c ∈ carrier R" **and** P: "p ∈ carrier P"
**shows** "coeff P (monom P c n ⊗$_P$ p) (m + n) = c ⊗ (coeff P p m)"
⟨*proof*⟩

**lemma** deg_lcoeff_cancel:
**assumes** p_in_P: "p ∈ carrier P" **and** q_in_P: "q ∈ carrier P" **and** r_in_P: "r ∈ carrier P"
**and** deg_r_nonzero: "deg R r ≠ 0"
**and** deg_R_p: "deg R p ≤ deg R r" **and** deg_R_q: "deg R q ≤ deg R r"

**and** coeff_R_p_eq_q: "coeff P p (deg R r) = ⊖$_R$ (coeff P q (deg R r))"
**shows** "deg R (p ⊕$_P$ q) < deg R r"
⟨*proof*⟩

**lemma** monom_deg_mult:
  **assumes** f_in_P: "f ∈ carrier P" **and** g_in_P: "g ∈ carrier P" **and** deg_le: "deg R g ≤ deg R f"
  **and** a_in_R: "a ∈ carrier R"
  **shows** "deg R (g ⊗$_P$ monom P a (deg R f - deg R g)) ≤ deg R f"
  ⟨*proof*⟩

**lemma** deg_zero_impl_monom:
  **assumes** f_in_P: "f ∈ carrier P" **and** deg_f: "deg R f = 0"
  **shows** "f = monom P (coeff P f 0) 0"
  ⟨*proof*⟩

**end**

## 14.12 The long division proof for commutative rings

**context** UP_cring
**begin**

**lemma** exI3: **assumes** exist: "Pred x y z"
  **shows** "∃ x y z. Pred x y z"
  ⟨*proof*⟩

Jacobson's Theorem 2.14

**lemma** long_div_theorem:
  **assumes** g_in_P [simp]: "g ∈ carrier P" **and** f_in_P [simp]: "f ∈ carrier P"
  **and** g_not_zero: "g ≠ **0**$_P$"
  **shows** "∃ q r (k::nat). (q ∈ carrier P) ∧ (r ∈ carrier P) ∧ (lcoeff g)(^)$_R$k ⊙$_P$ f = g ⊗$_P$ q ⊕$_P$ r ∧ (r = **0**$_P$ | deg R r < deg R g)"
⟨*proof*⟩

**end**

The remainder theorem as corollary of the long division theorem.

**context** UP_cring
**begin**

**lemma** deg_minus_monom:
  **assumes** a: "a ∈ carrier R"
  **and** R_not_trivial: "(carrier R ≠ {**0**})"
  **shows** "deg R (monom P **1**$_R$ 1 ⊖$_P$ monom P a 0) = 1"
  (**is** "deg R ?g = 1")
⟨*proof*⟩

**lemma** lcoeff_monom:
  **assumes** a: "a ∈ carrier R" **and** R_not_trivial: "(carrier R ≠ {**0**})"
  **shows** "lcoeff (monom P **1**$_R$ 1 ⊖$_P$ monom P a 0) = **1**"
  ⟨*proof*⟩

**lemma** deg_nzero_nzero:
**assumes** deg_p_nzero: "deg R p ≠ 0"
**shows** "p ≠ $\mathbf{0}_P$"
⟨*proof*⟩

**lemma** deg_monom_minus:
**assumes** a: "a ∈ carrier R"
**and** R_not_trivial: "carrier R ≠ {0}"
**shows** "deg R (monom P $\mathbf{1}_R$ 1 ⊖$_P$ monom P a 0) = 1"
(**is** "deg R ?g = 1")
⟨*proof*⟩

**lemma** eval_monom_expr:
**assumes** a: "a ∈ carrier R"
**shows** "eval R R id a (monom P $\mathbf{1}_R$ 1 ⊖$_P$ monom P a 0) = **0**"
(**is** "eval R R id a ?g = _")
⟨*proof*⟩

**lemma** remainder_theorem_exist:
**assumes** f: "f ∈ carrier P" **and** a: "a ∈ carrier R"
**and** R_not_trivial: "carrier R ≠ {0}"
**shows** "∃ q r. (q ∈ carrier P) ∧ (r ∈ carrier P) ∧ f = (monom P $\mathbf{1}_R$ 1 ⊖$_P$ monom P a 0) ⊗$_P$ q ⊕$_P$ r ∧ (deg R r = 0)"
(**is** "∃ q r. (q ∈ carrier P) ∧ (r ∈ carrier P) ∧ f = ?g ⊗$_P$ q ⊕$_P$ r ∧ (deg R r = 0)")
⟨*proof*⟩

**lemma** remainder_theorem_expression:
**assumes** f [simp]: "f ∈ carrier P" **and** a [simp]: "a ∈ carrier R"
**and** q [simp]: "q ∈ carrier P" **and** r [simp]: "r ∈ carrier P"
**and** R_not_trivial: "carrier R ≠ {0}"
**and** f_expr: "f = (monom P $\mathbf{1}_R$ 1 ⊖$_P$ monom P a 0) ⊗$_P$ q ⊕$_P$ r"
(**is** "f = ?g ⊗$_P$ q ⊕$_P$ r" **is** "f = ?gq ⊕$_P$ r")
**and** deg_r_0: "deg R r = 0"
**shows** "r = monom P (eval R R id a f) 0"
⟨*proof*⟩

**corollary** remainder_theorem:
**assumes** f [simp]: "f ∈ carrier P" **and** a [simp]: "a ∈ carrier R"
**and** R_not_trivial: "carrier R ≠ {0}"
**shows** "∃ q r. (q ∈ carrier P) ∧ (r ∈ carrier P) ∧
f = (monom P $\mathbf{1}_R$ 1 ⊖$_P$ monom P a 0) ⊗$_P$ q ⊕$_P$ monom P (eval R R id a f) 0"
(**is** "∃ q r. (q ∈ carrier P) ∧ (r ∈ carrier P) ∧ f = ?g ⊗$_P$ q ⊕$_P$ monom P (eval R R id a f) 0")
⟨*proof*⟩

**end**

### 14.13 Sample Application of Evaluation Homomorphism

**lemma** UP_pre_univ_propI:
  **assumes** "cring R"
    **and** "cring S"
    **and** "h ∈ ring_hom R S"
  **shows** "UP_pre_univ_prop R S h"
  ⟨*proof*⟩

**definition**
  INTEG :: "int ring"
  **where** "INTEG = (| carrier = UNIV, mult = op *, one = 1, zero = 0, add = op + |)"

**lemma** INTEG_cring: "cring INTEG"
  ⟨*proof*⟩

**lemma** INTEG_id_eval:
  "UP_pre_univ_prop INTEG INTEG id"
  ⟨*proof*⟩

Interpretation now enables to import all theorems and lemmas valid in the context of homomorphisms between INTEG and UP INTEG globally.

**interpretation** INTEG: UP_pre_univ_prop INTEG INTEG id "UP INTEG"
  ⟨*proof*⟩

**lemma** INTEG_closed [intro, simp]:
  "z ∈ carrier INTEG"
  ⟨*proof*⟩

**lemma** INTEG_mult [simp]:
  "mult INTEG z w = z * w"
  ⟨*proof*⟩

**lemma** INTEG_pow [simp]:
  "pow INTEG z n = z ^ n"
  ⟨*proof*⟩

**lemma** "eval INTEG INTEG id 10 (monom (UP INTEG) 5 2) = 500"
  ⟨*proof*⟩

**end**

## References

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