# The Hahn-Banach Theorem for Real Vector Spaces

Gertrud Bauer
http://www.in.tum.de/~bauerg/

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**Abstract**

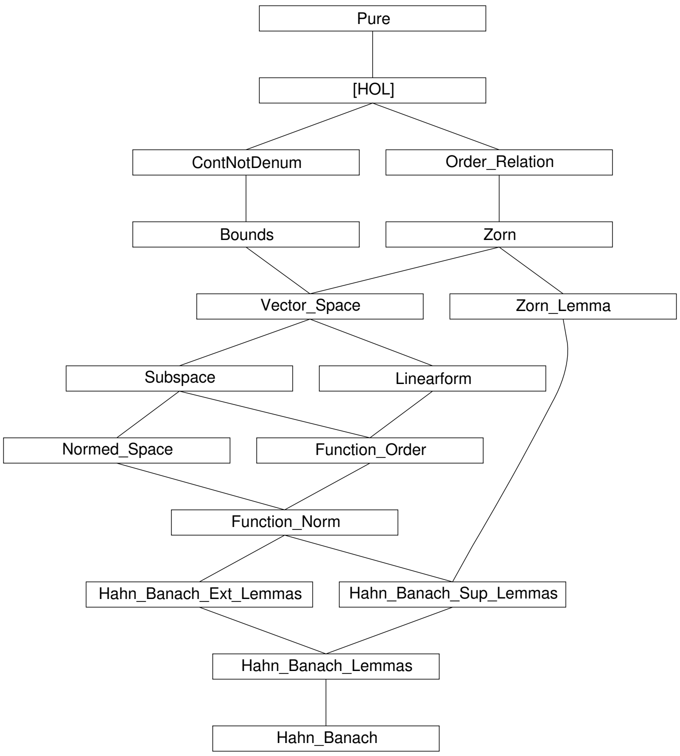
The Hahn-Banach Theorem is one of the most fundamental results in functional analysis. We present a fully formal proof of two versions of the theorem, one for general linear spaces and another for normed spaces. This development is based on simply-typed classical set-theory, as provided by Isabelle/HOL.

## Contents

# 1 Preface

This is a fully formal proof of the Hahn-Banach Theorem. It closely follows the informal presentation given in Heuser's textbook [1, § 36]. Another formal proof of the same theorem has been done in Mizar [3]. A general overview of the relevance and history of the Hahn-Banach Theorem is given by Narici and Beckenstein [2].

The document is structured as follows. The first part contains definitions of basic notions of linear algebra: vector spaces, subspaces, normed spaces, continuous linear-forms, norm of functions and an order on functions by domain extension. The second part contains some lemmas about the supremum (w.r.t. the function order) and extension of non-maximal functions. With these preliminaries, the main proof of the theorem (in its two versions) is conducted in the third part. The dependencies of individual theories are as follows.



# Part I

# Basic Notions

## 2 Bounds

**theory** *Bounds*
**imports** *Main ContNotDenum*
**begin**

**locale** *lub* =
  **fixes** $A$ **and** $x$
  **assumes** *least* [*intro?*]: $(\bigwedge a.\ a \in A \Longrightarrow a \leq b) \Longrightarrow x \leq b$
    **and** *upper* [*intro?*]: $a \in A \Longrightarrow a \leq x$

**lemmas** [*elim?*] = *lub.least lub.upper*

**definition**
  *the-lub* :: *'a::order set* ⇒ *'a* **where**
  *the-lub* $A$ = *The* (*lub* $A$)

**notation** (*xsymbols*)
  *the-lub* ($\bigsqcup$ - [*90*] *90*)

**lemma** *the-lub-equality* [*elim?*]:
  **assumes** *lub* $A$ $x$
  **shows** $\bigsqcup A = (x::'a::order)$
⟨*proof*⟩

**lemma** *the-lubI-ex*:
  **assumes** *ex*: $\exists x.$ *lub* $A$ $x$
  **shows** *lub* $A$ ($\bigsqcup A$)
⟨*proof*⟩

**lemma** *lub-compat*: *lub* $A$ $x$ = *isLub UNIV* $A$ $x$
⟨*proof*⟩

**lemma** *real-complete*:
  **fixes** $A$ :: *real set*
  **assumes** *nonempty*: $\exists a.\ a \in A$
    **and** *ex-upper*: $\exists y.\ \forall a \in A.\ a \leq y$
  **shows** $\exists x.$ *lub* $A$ $x$
⟨*proof*⟩

**end**

## 3 Vector spaces

**theory** *Vector-Space*
**imports** *Real Bounds Zorn*
**begin**

## 3.1 Signature

For the definition of real vector spaces a type $'a$ of the sort $\{plus, minus, zero\}$ is considered, on which a real scalar multiplication $\cdot$ is declared.

**consts**
  *prod* :: $real \Rightarrow 'a::\{plus, minus, zero\} \Rightarrow 'a$     (**infixr** $'(*')$ *70*)

**notation** (*xsymbols*)
  *prod* (**infixr** $\cdot$ *70*)
**notation** (*HTML* **output**)
  *prod* (**infixr** $\cdot$ *70*)

## 3.2 Vector space laws

A *vector space* is a non-empty set $V$ of elements from $'a$ with the following vector space laws: The set $V$ is closed under addition and scalar multiplication, addition is associative and commutative; $-\ x$ is the inverse of $x$ w. r. t. addition and $0$ is the neutral element of addition. Addition and multiplication are distributive; scalar multiplication is associative and the real number $1$ is the neutral element of scalar multiplication.

**locale** *var-V* = **fixes** $V$

**locale** *vectorspace* = *var-V* +
  **assumes** *non-empty* [*iff*, *intro?*]: $V \neq \{\}$
   **and** *add-closed* [*iff*]: $x \in V \Longrightarrow y \in V \Longrightarrow x + y \in V$
   **and** *mult-closed* [*iff*]: $x \in V \Longrightarrow a \cdot x \in V$
   **and** *add-assoc*: $x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x + y) + z = x + (y + z)$
   **and** *add-commute*: $x \in V \Longrightarrow y \in V \Longrightarrow x + y = y + x$
   **and** *diff-self* [*simp*]: $x \in V \Longrightarrow x - x = 0$
   **and** *add-zero-left* [*simp*]: $x \in V \Longrightarrow 0 + x = x$
   **and** *add-mult-distrib1*: $x \in V \Longrightarrow y \in V \Longrightarrow a \cdot (x + y) = a \cdot x + a \cdot y$
   **and** *add-mult-distrib2*: $x \in V \Longrightarrow (a + b) \cdot x = a \cdot x + b \cdot x$
   **and** *mult-assoc*: $x \in V \Longrightarrow (a * b) \cdot x = a \cdot (b \cdot x)$
   **and** *mult-1* [*simp*]: $x \in V \Longrightarrow 1 \cdot x = x$
   **and** *negate-eq1*: $x \in V \Longrightarrow -\ x = (-\ 1) \cdot x$
   **and** *diff-eq1*: $x \in V \Longrightarrow y \in V \Longrightarrow x - y = x + -\ y$

**lemma** (**in** *vectorspace*) *negate-eq2*: $x \in V \Longrightarrow (-\ 1) \cdot x = -\ x$
  ⟨*proof*⟩

**lemma** (**in** *vectorspace*) *negate-eq2a*: $x \in V \Longrightarrow -1 \cdot x = -\ x$
  ⟨*proof*⟩

**lemma** (**in** *vectorspace*) *diff-eq2*: $x \in V \Longrightarrow y \in V \Longrightarrow x + -\ y = x - y$
  ⟨*proof*⟩

**lemma** (**in** *vectorspace*) *diff-closed* [*iff*]: $x \in V \Longrightarrow y \in V \Longrightarrow x - y \in V$
  ⟨*proof*⟩

**lemma** (**in** *vectorspace*) *neg-closed* [*iff*]: $x \in V \Longrightarrow -\ x \in V$
  ⟨*proof*⟩

**lemma** (**in** *vectorspace*) *add-left-commute*:

$x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow x + (y + z) = y + (x + z)$
⟨*proof*⟩

**theorems** (**in** *vectorspace*) *add-ac* =
*add-assoc add-commute add-left-commute*

The existence of the zero element of a vector space follows from the non-emptiness of carrier set.

**lemma** (**in** *vectorspace*) *zero* [*iff*]: $0 \in V$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *add-zero-right* [*simp*]:
$x \in V \Longrightarrow x + 0 = x$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *mult-assoc2*:
$x \in V \Longrightarrow a \cdot b \cdot x = (a * b) \cdot x$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *diff-mult-distrib1*:
$x \in V \Longrightarrow y \in V \Longrightarrow a \cdot (x - y) = a \cdot x - a \cdot y$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *diff-mult-distrib2*:
$x \in V \Longrightarrow (a - b) \cdot x = a \cdot x - (b \cdot x)$
⟨*proof*⟩

**lemmas** (**in** *vectorspace*) *distrib* =
*add-mult-distrib1 add-mult-distrib2*
*diff-mult-distrib1 diff-mult-distrib2*

Further derived laws:

**lemma** (**in** *vectorspace*) *mult-zero-left* [*simp*]:
$x \in V \Longrightarrow 0 \cdot x = 0$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *mult-zero-right* [*simp*]:
$a \cdot 0 = (0::'a)$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *minus-mult-cancel* [*simp*]:
$x \in V \Longrightarrow (- a) \cdot - x = a \cdot x$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *add-minus-left-eq-diff*:
$x \in V \Longrightarrow y \in V \Longrightarrow - x + y = y - x$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *add-minus* [*simp*]:
$x \in V \Longrightarrow x + - x = 0$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *add-minus-left* [*simp*]:

$x \in V \Longrightarrow - x + x = 0$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *minus-minus* [*simp*]:
$x \in V \Longrightarrow - (- x) = x$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *minus-zero* [*simp*]:
$- (0::'a) = 0$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *minus-zero-iff* [*simp*]:
$x \in V \Longrightarrow (- x = 0) = (x = 0)$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *add-minus-cancel* [*simp*]:
$x \in V \Longrightarrow y \in V \Longrightarrow x + (- x + y) = y$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *minus-add-cancel* [*simp*]:
$x \in V \Longrightarrow y \in V \Longrightarrow - x + (x + y) = y$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *minus-add-distrib* [*simp*]:
$x \in V \Longrightarrow y \in V \Longrightarrow - (x + y) = - x + - y$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *diff-zero* [*simp*]:
$x \in V \Longrightarrow x - 0 = x$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *diff-zero-right* [*simp*]:
$x \in V \Longrightarrow 0 - x = - x$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *add-left-cancel*:
$x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x + y = x + z) = (y = z)$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *add-right-cancel*:
$x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (y + x = z + x) = (y = z)$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *add-assoc-cong*:
$x \in V \Longrightarrow y \in V \Longrightarrow x' \in V \Longrightarrow y' \in V \Longrightarrow z \in V$
$\Longrightarrow x + y = x' + y' \Longrightarrow x + (y + z) = x' + (y' + z)$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *mult-left-commute*:
$x \in V \Longrightarrow a \cdot b \cdot x = b \cdot a \cdot x$
$\langle proof \rangle$

**lemma** (**in** *vectorspace*) *mult-zero-uniq*:
$x \in V \Longrightarrow x \neq 0 \Longrightarrow a \cdot x = 0 \Longrightarrow a = 0$

⟨*proof*⟩

**lemma** (**in** *vectorspace*) *mult-left-cancel*:
$x \in V \Longrightarrow y \in V \Longrightarrow a \neq 0 \Longrightarrow (a \cdot x = a \cdot y) = (x = y)$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *mult-right-cancel*:
$x \in V \Longrightarrow x \neq 0 \Longrightarrow (a \cdot x = b \cdot x) = (a = b)$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *eq-diff-eq*:
$x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x = z - y) = (x + y = z)$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *add-minus-eq-minus*:
$x \in V \Longrightarrow y \in V \Longrightarrow x + y = 0 \Longrightarrow x = - y$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *add-minus-eq*:
$x \in V \Longrightarrow y \in V \Longrightarrow x - y = 0 \Longrightarrow x = y$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *add-diff-swap*:
$a \in V \Longrightarrow b \in V \Longrightarrow c \in V \Longrightarrow d \in V \Longrightarrow a + b = c + d$
$\Longrightarrow a - c = d - b$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *vs-add-cancel-21*:
$x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow u \in V$
$\Longrightarrow (x + (y + z) = y + u) = (x + z = u)$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *add-cancel-end*:
$x \in V \Longrightarrow y \in V \Longrightarrow z \in V \Longrightarrow (x + (y + z) = y) = (x = - z)$
⟨*proof*⟩

**end**

# 4 Subspaces

**theory** *Subspace*
**imports** *Vector-Space*
**begin**

## 4.1 Definition

A non-empty subset $U$ of a vector space $V$ is a *subspace* of $V$, iff $U$ is closed under addition and scalar multiplication.

**locale** *subspace* =
**fixes** $U$ :: $'a$::{*minus*, *plus*, *zero*, *uminus*} *set* **and** $V$
**assumes** *non-empty* [*iff*, *intro*]: $U \neq \{\}$
**and** *subset* [*iff*]: $U \subseteq V$

**and** *add-closed* [*iff*]: $x \in U \Longrightarrow y \in U \Longrightarrow x + y \in U$
**and** *mult-closed* [*iff*]: $x \in U \Longrightarrow a \cdot x \in U$

**notation** (*symbols*)
*subspace* (**infix** $\trianglelefteq$ *50*)

**declare** *vectorspace.intro* [*intro?*] *subspace.intro* [*intro?*]

**lemma** *subspace-subset* [*elim*]: $U \trianglelefteq V \Longrightarrow U \subseteq V$
⟨*proof*⟩

**lemma** (**in** *subspace*) *subsetD* [*iff*]: $x \in U \Longrightarrow x \in V$
⟨*proof*⟩

**lemma** *subspaceD* [*elim*]: $U \trianglelefteq V \Longrightarrow x \in U \Longrightarrow x \in V$
⟨*proof*⟩

**lemma** *rev-subspaceD* [*elim?*]: $x \in U \Longrightarrow U \trianglelefteq V \Longrightarrow x \in V$
⟨*proof*⟩

**lemma** (**in** *subspace*) *diff-closed* [*iff*]:
**assumes** *vectorspace V*
**assumes** $x$: $x \in U$ **and** $y$: $y \in U$
**shows** $x - y \in U$
⟨*proof*⟩

Similar as for linear spaces, the existence of the zero element in every subspace follows from the non-emptiness of the carrier set and by vector space laws.

**lemma** (**in** *subspace*) *zero* [*intro*]:
**assumes** *vectorspace V*
**shows** $0 \in U$
⟨*proof*⟩

**lemma** (**in** *subspace*) *neg-closed* [*iff*]:
**assumes** *vectorspace V*
**assumes** $x$: $x \in U$
**shows** $- x \in U$
⟨*proof*⟩

Further derived laws: every subspace is a vector space.

**lemma** (**in** *subspace*) *vectorspace* [*iff*]:
**assumes** *vectorspace V*
**shows** *vectorspace U*
⟨*proof*⟩

The subspace relation is reflexive.

**lemma** (**in** *vectorspace*) *subspace-refl* [*intro*]: $V \trianglelefteq V$
⟨*proof*⟩

The subspace relation is transitive.

**lemma** (**in** *vectorspace*) *subspace-trans* [*trans*]:
$U \trianglelefteq V \Longrightarrow V \trianglelefteq W \Longrightarrow U \trianglelefteq W$
⟨*proof*⟩

## 4.2 Linear closure

The *linear closure* of a vector $x$ is the set of all scalar multiples of $x$.

**definition**
$lin :: ('a::\{minus, plus, zero\}) \Rightarrow 'a\ set$ **where**
$lin\ x = \{a \cdot x \mid a.\ True\}$

**lemma** *linI* [*intro*]: $y = a \cdot x \Longrightarrow y \in lin\ x$
⟨*proof*⟩

**lemma** *linI'* [*iff*]: $a \cdot x \in lin\ x$
⟨*proof*⟩

**lemma** *linE* [*elim*]: $x \in lin\ v \Longrightarrow (\bigwedge a::real.\ x = a \cdot v \Longrightarrow C) \Longrightarrow C$
⟨*proof*⟩

Every vector is contained in its linear closure.

**lemma** (**in** *vectorspace*) *x-lin-x* [*iff*]: $x \in V \Longrightarrow x \in lin\ x$
⟨*proof*⟩

**lemma** (**in** *vectorspace*) *0-lin-x* [*iff*]: $x \in V \Longrightarrow 0 \in lin\ x$
⟨*proof*⟩

Any linear closure is a subspace.

**lemma** (**in** *vectorspace*) *lin-subspace* [*intro*]:
$x \in V \Longrightarrow lin\ x \trianglelefteq V$
⟨*proof*⟩

Any linear closure is a vector space.

**lemma** (**in** *vectorspace*) *lin-vectorspace* [*intro*]:
**assumes** $x \in V$
**shows** *vectorspace* (*lin* $x$)
⟨*proof*⟩

## 4.3 Sum of two vectorspaces

The *sum* of two vectorspaces $U$ and $V$ is the set of all sums of elements from $U$ and $V$.

**instantiation** *fun* :: (*type*, *type*) *plus*
**begin**

**definition**
*sum-def*: $plus\text{-}fun\ U\ V = \{u + v \mid u\ v.\ u \in U \land v \in V\}$

**instance** ⟨*proof*⟩

**end**

**lemma** *sumE* [*elim*]:
$x \in U + V \Longrightarrow (\bigwedge u\ v.\ x = u + v \Longrightarrow u \in U \Longrightarrow v \in V \Longrightarrow C) \Longrightarrow C$
⟨*proof*⟩

**lemma** *sumI* [*intro*]:
  $u \in U \Longrightarrow v \in V \Longrightarrow x = u + v \Longrightarrow x \in U + V$
  ⟨*proof*⟩

**lemma** *sumI'* [*intro*]:
  $u \in U \Longrightarrow v \in V \Longrightarrow u + v \in U + V$
  ⟨*proof*⟩

$U$ is a subspace of $U + V$.

**lemma** *subspace-sum1* [*iff*]:
  **assumes** *vectorspace U vectorspace V*
  **shows** $U \trianglelefteq U + V$
⟨*proof*⟩

The sum of two subspaces is again a subspace.

**lemma** *sum-subspace* [*intro?*]:
  **assumes** *subspace U E vectorspace E subspace V E*
  **shows** $U + V \trianglelefteq E$
⟨*proof*⟩

The sum of two subspaces is a vectorspace.

**lemma** *sum-vs* [*intro?*]:
  $U \trianglelefteq E \Longrightarrow V \trianglelefteq E \Longrightarrow \textit{vectorspace } E \Longrightarrow \textit{vectorspace } (U + V)$
  ⟨*proof*⟩

## 4.4 Direct sums

The sum of $U$ and $V$ is called *direct*, iff the zero element is the only common element of $U$ and $V$. For every element $x$ of the direct sum of $U$ and $V$ the decomposition in $x = u + v$ with $u \in U$ and $v \in V$ is unique.

**lemma** *decomp*:
  **assumes** *vectorspace E subspace U E subspace V E*
  **assumes** *direct*: $U \cap V = \{0\}$
    **and** *u1*: $u1 \in U$ **and** *u2*: $u2 \in U$
    **and** *v1*: $v1 \in V$ **and** *v2*: $v2 \in V$
    **and** *sum*: $u1 + v1 = u2 + v2$
  **shows** $u1 = u2 \wedge v1 = v2$
⟨*proof*⟩

An application of the previous lemma will be used in the proof of the Hahn-Banach Theorem (see page **??**): for any element $y + a \cdot x_0$ of the direct sum of a vectorspace $H$ and the linear closure of $x_0$ the components $y \in H$ and $a$ are uniquely determined.

**lemma** *decomp-H'*:
  **assumes** *vectorspace E subspace H E*
  **assumes** *y1*: $y1 \in H$ **and** *y2*: $y2 \in H$
    **and** $x'$: $x' \notin H \;\; x' \in E \;\; x' \neq 0$
    **and** *eq*: $y1 + a1 \cdot x' = y2 + a2 \cdot x'$
  **shows** $y1 = y2 \wedge a1 = a2$
⟨*proof*⟩

Since for any element $y + a \cdot x'$ of the direct sum of a vectorspace $H$ and the linear closure of $x'$ the components $y \in H$ and $a$ are unique, it follows from $y \in H$ that $a = 0$.

**lemma** *decomp-H′-H*:
**assumes** *vectorspace E subspace H E*
**assumes** *t*: $t \in H$
**and** *x′*: $x' \notin H$ $x' \in E$ $x' \neq 0$
**shows** $(SOME\ (y, a).\ t = y + a \cdot x' \wedge y \in H) = (t, 0)$
⟨*proof*⟩

The components $y \in H$ and $a$ in $y + a \cdot x'$ are unique, so the function $h'$ defined by $h'\ (y + a \cdot x') = h\ y + a \cdot \xi$ is definite.

**lemma** *h′-definite*:
**fixes** *H*
**assumes** *h′-def*:
$h' \equiv (\lambda x.\ let\ (y, a) = SOME\ (y, a).\ (x = y + a \cdot x' \wedge y \in H)$
$in\ (h\ y) + a * xi)$
**and** *x*: $x = y + a \cdot x'$
**assumes** *vectorspace E subspace H E*
**assumes** *y*: $y \in H$
**and** *x′*: $x' \notin H$ $x' \in E$ $x' \neq 0$
**shows** $h'\ x = h\ y + a * xi$
⟨*proof*⟩

**end**

# 5 Normed vector spaces

**theory** *Normed-Space*
**imports** *Subspace*
**begin**

## 5.1 Quasinorms

A *seminorm* $\|\cdot\|$ is a function on a real vector space into the reals that has the following properties: it is positive definite, absolute homogenous and subadditive.

**locale** *norm-syntax* =
**fixes** *norm* :: $'a \Rightarrow real$ (‖-‖)

**locale** *seminorm* = *var-V* + *norm-syntax* +
**constrains** $V$ :: $'a$::{*minus*, *plus*, *zero*, *uminus*} *set*
**assumes** *ge-zero* [*iff?*]: $x \in V \Longrightarrow 0 \leq \|x\|$
**and** *abs-homogenous* [*iff?*]: $x \in V \Longrightarrow \|a \cdot x\| = |a| * \|x\|$
**and** *subadditive* [*iff?*]: $x \in V \Longrightarrow y \in V \Longrightarrow \|x + y\| \leq \|x\| + \|y\|$

**declare** *seminorm.intro* [*intro?*]

**lemma** (**in** *seminorm*) *diff-subadditive*:
**assumes** *vectorspace V*

**shows** $x \in V \Longrightarrow y \in V \Longrightarrow \|x - y\| \leq \|x\| + \|y\|$
$\langle proof \rangle$

**lemma** (**in** *seminorm*) *minus*:
**assumes** *vectorspace V*
**shows** $x \in V \Longrightarrow \|- x\| = \|x\|$
$\langle proof \rangle$

## 5.2 Norms

A *norm* $\|\cdot\|$ is a seminorm that maps only the *0* vector to *0*.

**locale** *norm* = *seminorm* +
**assumes** *zero-iff* [*iff*]: $x \in V \Longrightarrow (\|x\| = 0) = (x = 0)$

## 5.3 Normed vector spaces

A vector space together with a norm is called a *normed space*.

**locale** *normed-vectorspace* = *vectorspace* + *norm*

**declare** *normed-vectorspace.intro* [*intro?*]

**lemma** (**in** *normed-vectorspace*) *gt-zero* [*intro?*]:
$x \in V \Longrightarrow x \neq 0 \Longrightarrow 0 < \|x\|$
$\langle proof \rangle$

Any subspace of a normed vector space is again a normed vectorspace.

**lemma** *subspace-normed-vs* [*intro?*]:
**fixes** *F E norm*
**assumes** *subspace F E normed-vectorspace E norm*
**shows** *normed-vectorspace F norm*
$\langle proof \rangle$

**end**

# 6 Linearforms

**theory** *Linearform*
**imports** *Vector-Space*
**begin**

A *linear form* is a function on a vector space into the reals that is additive and multiplicative.

**locale** *linearform* =
**fixes** $V :: 'a::\{minus, plus, zero, uminus\}$ *set* **and** $f$
**assumes** *add* [*iff*]: $x \in V \Longrightarrow y \in V \Longrightarrow f\ (x + y) = f\,x + f\,y$
**and** *mult* [*iff*]: $x \in V \Longrightarrow f\ (a \cdot x) = a * f\,x$

**declare** *linearform.intro* [*intro?*]

**lemma** (**in** *linearform*) *neg* [*iff*]:

**assumes** *vectorspace V*
**shows** $x \in V \Longrightarrow f\ (-\ x) = -\ f\ x$
⟨*proof*⟩

**lemma** (**in** *linearform*) *diff* [*iff*]:
**assumes** *vectorspace V*
**shows** $x \in V \Longrightarrow y \in V \Longrightarrow f\ (x - y) = f\ x - f\ y$
⟨*proof*⟩

Every linear form yields *0* for the *0* vector.

**lemma** (**in** *linearform*) *zero* [*iff*]:
**assumes** *vectorspace V*
**shows** $f\ 0 = 0$
⟨*proof*⟩

**end**

# 7 An order on functions

**theory** *Function-Order*
**imports** *Subspace Linearform*
**begin**

## 7.1 The graph of a function

We define the *graph* of a (real) function $f$ with domain $F$ as the set

$$\{(x, f\ x).\ x \in F\}$$

So we are modeling partial functions by specifying the domain and the mapping function. We use the term "function" also for its graph.

**types** $'a\ graph = ('a \times real)\ set$

**definition**
$graph :: 'a\ set \Rightarrow ('a \Rightarrow real) \Rightarrow 'a\ graph$ **where**
$graph\ F\ f = \{(x, f\ x) \mid x.\ x \in F\}$

**lemma** *graphI* [*intro*]: $x \in F \Longrightarrow (x, f\ x) \in graph\ F\ f$
⟨*proof*⟩

**lemma** *graphI2* [*intro?*]: $x \in F \Longrightarrow \exists\, t \in graph\ F\ f.\ t = (x, f\ x)$
⟨*proof*⟩

**lemma** *graphE* [*elim?*]:
$(x, y) \in graph\ F\ f \Longrightarrow (x \in F \Longrightarrow y = f\ x \Longrightarrow C) \Longrightarrow C$
⟨*proof*⟩

## 7.2 Functions ordered by domain extension

A function $h'$ is an extension of $h$, iff the graph of $h$ is a subset of the graph of $h'$.

**lemma** *graph-extI*:
$(\bigwedge x.\ x \in H \Longrightarrow h\ x = h'\ x) \Longrightarrow H \subseteq H'$
$\Longrightarrow graph\ H\ h \subseteq graph\ H'\ h'$
$\langle proof \rangle$

**lemma** *graph-extD1* [*dest?*]:
$graph\ H\ h \subseteq graph\ H'\ h' \Longrightarrow x \in H \Longrightarrow h\ x = h'\ x$
$\langle proof \rangle$

**lemma** *graph-extD2* [*dest?*]:
$graph\ H\ h \subseteq graph\ H'\ h' \Longrightarrow H \subseteq H'$
$\langle proof \rangle$

## 7.3 Domain and function of a graph

The inverse functions to *graph* are *domain* and *funct*.

**definition**
$domain :: 'a\ graph \Rightarrow 'a\ set$ **where**
$domain\ g = \{x.\ \exists\, y.\ (x,\ y) \in g\}$

**definition**
$funct :: 'a\ graph \Rightarrow ('a \Rightarrow real)$ **where**
$funct\ g = (\lambda x.\ (SOME\ y.\ (x,\ y) \in g))$

The following lemma states that $g$ is the graph of a function if the relation induced by $g$ is unique.

**lemma** *graph-domain-funct*:
**assumes** *uniq*: $\bigwedge x\ y\ z.\ (x,\ y) \in g \Longrightarrow (x,\ z) \in g \Longrightarrow z = y$
**shows** $graph\ (domain\ g)\ (funct\ g) = g$
$\langle proof \rangle$

## 7.4 Norm-preserving extensions of a function

Given a linear form $f$ on the space $F$ and a seminorm $p$ on $E$. The set of all linear extensions of $f$, to superspaces $H$ of $F$, which are bounded by $p$, is defined as follows.

**definition**
$norm\text{-}pres\text{-}extensions ::$
$'a::\{plus,\ minus,\ uminus,\ zero\}\ set \Rightarrow ('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow ('a \Rightarrow real)$
$\Rightarrow 'a\ graph\ set$ **where**
$norm\text{-}pres\text{-}extensions\ E\ p\ F\ f$
$= \{g.\ \exists\, H\ h.\ g = graph\ H\ h$
$\wedge\ linearform\ H\ h$
$\wedge\ H \trianglelefteq E$
$\wedge\ F \trianglelefteq H$
$\wedge\ graph\ F\ f \subseteq graph\ H\ h$
$\wedge\ (\forall\, x \in H.\ h\ x \leq p\ x)\}$

**lemma** *norm-pres-extensionE* [*elim*]:
$g \in norm\text{-}pres\text{-}extensions\ E\ p\ F\ f$
$\Longrightarrow (\bigwedge H\ h.\ g = graph\ H\ h \Longrightarrow linearform\ H\ h$
$\Longrightarrow H \trianglelefteq E \Longrightarrow F \trianglelefteq H \Longrightarrow graph\ F\ f \subseteq graph\ H\ h$

$\Longrightarrow \forall\, x \in H.\ h\ x \leq p\ x \Longrightarrow C) \Longrightarrow C$
⟨*proof*⟩

**lemma** *norm-pres-extensionI2* [*intro*]:
*linearform* $H\ h \Longrightarrow H \trianglelefteq E \Longrightarrow F \trianglelefteq H$
$\Longrightarrow$ *graph* $F\ f \subseteq$ *graph* $H\ h \Longrightarrow \forall\, x \in H.\ h\ x \leq p\ x$
$\Longrightarrow$ *graph* $H\ h \in$ *norm-pres-extensions* $E\ p\ F\ f$
⟨*proof*⟩

**lemma** *norm-pres-extensionI*:
$\exists\, H\ h.\ g =$ *graph* $H\ h$
$\wedge$ *linearform* $H\ h$
$\wedge\ H \trianglelefteq E$
$\wedge\ F \trianglelefteq H$
$\wedge$ *graph* $F\ f \subseteq$ *graph* $H\ h$
$\wedge\ (\forall\, x \in H.\ h\ x \leq p\ x) \Longrightarrow g \in$ *norm-pres-extensions* $E\ p\ F\ f$
⟨*proof*⟩

**end**

# 8 The norm of a function

**theory** *Function-Norm*
**imports** *Normed-Space Function-Order*
**begin**

## 8.1 Continuous linear forms

A linear form $f$ on a normed vector space $(V, \|\cdot\|)$ is *continuous*, iff it is bounded, i.e.

$$\exists\, c \in R.\ \forall\, x \in V.\ |f\ x| \leq c \cdot \|x\|$$

In our application no other functions than linear forms are considered, so we can define continuous linear forms as bounded linear forms:

**locale** *continuous* = *var-V* + *norm-syntax* + *linearform* +
**assumes** *bounded*: $\exists\, c.\ \forall\, x \in V.\ |f\ x| \leq c * \|x\|$

**declare** *continuous.intro* [*intro?*] *continuous-axioms.intro* [*intro?*]

**lemma** *continuousI* [*intro*]:
**fixes** *norm* :: - $\Rightarrow$ *real* (‖-‖)
**assumes** *linearform* $V\ f$
**assumes** $r$: $\bigwedge x.\ x \in V \Longrightarrow |f\ x| \leq c * \|x\|$
**shows** *continuous* $V$ *norm* $f$
⟨*proof*⟩

## 8.2 The norm of a linear form

The least real number $c$ for which holds

$$\forall\, x \in V.\ |f\ x| \leq c \cdot \|x\|$$

is called the *norm* of *f*.
For non-trivial vector spaces $V \neq \{0\}$ the norm can be defined as

$$\|f\| = \sup x \neq 0.\ |f\ x|\ /\ \|x\|$$

For the case $V = \{0\}$ the supremum would be taken from an empty set. Since $\mathbb{R}$ is unbounded, there would be no supremum. To avoid this situation it must be guaranteed that there is an element in this set. This element must be $\{\} \geq 0$ so that *fn-norm* has the norm properties. Furthermore it does not have to change the norm in all other cases, so it must be $0$, as all other elements are $\{\} \geq 0$.
Thus we define the set $B$ where the supremum is taken from as follows:

$$\{0\} \cup \{|f\ x|\ /\ \|x\|.\ x \neq 0 \wedge x \in F\}$$

*fn-norm* is equal to the supremum of $B$, if the supremum exists (otherwise it is undefined).

**locale** *fn-norm* = *norm-syntax* +
  **fixes** $B$ **defines** $B\ V\ f \equiv \{0\} \cup \{|f\ x|\ /\ \|x\|\ |\ x.\ x \neq 0 \wedge x \in V\}$
  **fixes** *fn-norm* ($\|$-$\|$-- [$0$, $1000$] $999$)
  **defines** $\|f\|$-$V \equiv \bigsqcup(B\ V\ f)$

**locale** *normed-vectorspace-with-fn-norm* = *normed-vectorspace* + *fn-norm*

**lemma** (**in** *fn-norm*) *B-not-empty* [*intro*]: $0 \in B\ V\ f$
  ⟨*proof*⟩

The following lemma states that every continuous linear form on a normed space $(V, \|\cdot\|)$ has a function norm.

**lemma** (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-works*:
  **assumes** *continuous* $V$ *norm* $f$
  **shows** *lub* ($B\ V\ f$) ($\|f\|$-$V$)
⟨*proof*⟩

**lemma** (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-ub* [*iff?*]:
  **assumes** *continuous* $V$ *norm* $f$
  **assumes** *b*: $b \in B\ V\ f$
  **shows** $b \leq \|f\|$-$V$
⟨*proof*⟩

**lemma** (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-leastB*:
  **assumes** *continuous* $V$ *norm* $f$
  **assumes** *b*: $\bigwedge b.\ b \in B\ V\ f \Longrightarrow b \leq y$
  **shows** $\|f\|$-$V \leq y$
⟨*proof*⟩

The norm of a continuous function is always $\geq 0$.

**lemma** (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-ge-zero* [*iff*]:
  **assumes** *continuous* $V$ *norm* $f$
  **shows** $0 \leq \|f\|$-$V$
⟨*proof*⟩

The fundamental property of function norms is:

$$|f\ x| \le \|f\| \cdot \|x\|$$

**lemma** (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-le-cong*:
  **assumes** *continuous V norm f linearform V f*
  **assumes** $x$: $x \in V$
  **shows** $|f\ x| \le \|f\|\text{-}V * \|x\|$
⟨*proof*⟩

The function norm is the least positive real number for which the following inequation holds:

$$|f\ x| \le c \cdot \|x\|$$

**lemma** (**in** *normed-vectorspace-with-fn-norm*) *fn-norm-least* [*intro?*]:
  **assumes** *continuous V norm f*
  **assumes** *ineq*: $\forall x \in V.\ |f\ x| \le c * \|x\|$ **and** *ge*: $0 \le c$
  **shows** $\|f\|\text{-}V \le c$
⟨*proof*⟩

**end**

# 9 Zorn's Lemma

**theory** *Zorn-Lemma*
**imports** *Zorn*
**begin**

Zorn's Lemmas states: if every linear ordered subset of an ordered set $S$ has an upper bound in $S$, then there exists a maximal element in $S$. In our application, $S$ is a set of sets ordered by set inclusion. Since the union of a chain of sets is an upper bound for all elements of the chain, the conditions of Zorn's lemma can be modified: if $S$ is non-empty, it suffices to show that for every non-empty chain $c$ in $S$ the union of $c$ also lies in $S$.

**theorem** *Zorn's-Lemma*:
  **assumes** $r$: $\bigwedge c.\ c \in chain\ S \Longrightarrow \exists x.\ x \in c \Longrightarrow \bigcup c \in S$
    **and** $aS$: $a \in S$
  **shows** $\exists y \in S.\ \forall z \in S.\ y \subseteq z \longrightarrow y = z$
⟨*proof*⟩

**end**

# Part II

# Lemmas for the Proof

## 10 The supremum w.r.t. the function order

**theory** *Hahn-Banach-Sup-Lemmas*
**imports** *Function-Norm Zorn-Lemma*
**begin**

This section contains some lemmas that will be used in the proof of the Hahn-Banach Theorem. In this section the following context is presumed. Let $E$ be a real vector space with a seminorm $p$ on $E$. $F$ is a subspace of $E$ and $f$ a linear form on $F$. We consider a chain $c$ of norm-preserving extensions of $f$, such that $\bigcup c = \mathit{graph}\ H\ h$. We will show some properties about the limit function $h$, i.e. the supremum of the chain $c$.

Let $c$ be a chain of norm-preserving extensions of the function $f$ and let *graph* $H$ $h$ be the supremum of $c$. Every element in $H$ is member of one of the elements of the chain.

**lemmas** [*dest?*] = *chainD*
**lemmas** *chainE2* [*elim?*] = *chainD2* [*elim-format*, *standard*]

**lemma** *some-H′h′t*:
  **assumes** $M$: $M = \mathit{norm\text{-}pres\text{-}extensions}\ E\ p\ F\ f$
    **and** $cM$: $c \in \mathit{chain}\ M$
    **and** $u$: $\mathit{graph}\ H\ h = \bigcup c$
    **and** $x$: $x \in H$
  **shows** $\exists H'\ h'.\ \mathit{graph}\ H'\ h' \in c$
    $\wedge\ (x,\ h\ x) \in \mathit{graph}\ H'\ h'$
    $\wedge\ \mathit{linearform}\ H'\ h' \wedge H' \trianglelefteq E$
    $\wedge\ F \trianglelefteq H' \wedge \mathit{graph}\ F\ f \subseteq \mathit{graph}\ H'\ h'$
    $\wedge\ (\forall x \in H'.\ h'\ x \le p\ x)$
⟨*proof*⟩

Let $c$ be a chain of norm-preserving extensions of the function $f$ and let *graph* $H$ $h$ be the supremum of $c$. Every element in the domain $H$ of the supremum function is member of the domain $H'$ of some function $h'$, such that $h$ extends $h'$.

**lemma** *some-H′h′*:
  **assumes** $M$: $M = \mathit{norm\text{-}pres\text{-}extensions}\ E\ p\ F\ f$
    **and** $cM$: $c \in \mathit{chain}\ M$
    **and** $u$: $\mathit{graph}\ H\ h = \bigcup c$
    **and** $x$: $x \in H$
  **shows** $\exists H'\ h'.\ x \in H' \wedge \mathit{graph}\ H'\ h' \subseteq \mathit{graph}\ H\ h$
    $\wedge\ \mathit{linearform}\ H'\ h' \wedge H' \trianglelefteq E \wedge F \trianglelefteq H'$
    $\wedge\ \mathit{graph}\ F\ f \subseteq \mathit{graph}\ H'\ h' \wedge (\forall x \in H'.\ h'\ x \le p\ x)$
⟨*proof*⟩

Any two elements $x$ and $y$ in the domain $H$ of the supremum function $h$ are both in the domain $H'$ of some function $h'$, such that $h$ extends $h'$.

**lemma** *some-H'h'2*:
  **assumes** $M$: $M = \textit{norm-pres-extensions } E\ p\ F\ f$
    **and** $cM$: $c \in \textit{chain } M$
    **and** $u$: $\textit{graph } H\ h = \bigcup c$
    **and** $x$: $x \in H$
    **and** $y$: $y \in H$
  **shows** $\exists H'\ h'.\ x \in H' \wedge y \in H'$
    $\wedge\ \textit{graph } H'\ h' \subseteq \textit{graph } H\ h$
    $\wedge\ \textit{linearform } H'\ h' \wedge H' \trianglelefteq E \wedge F \trianglelefteq H'$
    $\wedge\ \textit{graph } F\ f \subseteq \textit{graph } H'\ h' \wedge (\forall x \in H'.\ h'\ x \leq p\ x)$
⟨*proof*⟩

The relation induced by the graph of the supremum of a chain $c$ is definite, i. e. t is the graph of a function.

**lemma** *sup-definite*:
  **assumes** $M\text{-}def$: $M \equiv \textit{norm-pres-extensions } E\ p\ F\ f$
    **and** $cM$: $c \in \textit{chain } M$
    **and** $xy$: $(x,\ y) \in \bigcup c$
    **and** $xz$: $(x,\ z) \in \bigcup c$
  **shows** $z = y$
⟨*proof*⟩

The limit function $h$ is linear. Every element $x$ in the domain of $h$ is in the domain of a function $h'$ in the chain of norm preserving extensions. Furthermore, $h$ is an extension of $h'$ so the function values of $x$ are identical for $h'$ and $h$. Finally, the function $h'$ is linear by construction of $M$.

**lemma** *sup-lf*:
  **assumes** $M$: $M = \textit{norm-pres-extensions } E\ p\ F\ f$
    **and** $cM$: $c \in \textit{chain } M$
    **and** $u$: $\textit{graph } H\ h = \bigcup c$
  **shows** $\textit{linearform } H\ h$
⟨*proof*⟩

The limit of a non-empty chain of norm preserving extensions of $f$ is an extension of $f$, since every element of the chain is an extension of $f$ and the supremum is an extension for every element of the chain.

**lemma** *sup-ext*:
  **assumes** *graph*: $\textit{graph } H\ h = \bigcup c$
    **and** $M$: $M = \textit{norm-pres-extensions } E\ p\ F\ f$
    **and** $cM$: $c \in \textit{chain } M$
    **and** $ex$: $\exists x.\ x \in c$
  **shows** $\textit{graph } F\ f \subseteq \textit{graph } H\ h$
⟨*proof*⟩

The domain $H$ of the limit function is a superspace of $F$, since $F$ is a subset of $H$. The existence of the $\mathit{0}$ element in $F$ and the closure properties follow from the fact that $F$ is a vector space.

**lemma** *sup-supF*:
  **assumes** *graph*: $\textit{graph } H\ h = \bigcup c$
    **and** $M$: $M = \textit{norm-pres-extensions } E\ p\ F\ f$

**and** *cM*: $c \in$ *chain M*
**and** *ex*: $\exists\, x.\ x \in c$
**and** *FE*: $F \trianglelefteq E$
**shows** $F \trianglelefteq H$
⟨*proof*⟩

The domain $H$ of the limit function is a subspace of $E$.

**lemma** *sup-subE*:
**assumes** *graph*: *graph H h* $= \bigcup c$
**and** *M*: $M =$ *norm-pres-extensions E p F f*
**and** *cM*: $c \in$ *chain M*
**and** *ex*: $\exists\, x.\ x \in c$
**and** *FE*: $F \trianglelefteq E$
**and** *E*: *vectorspace E*
**shows** $H \trianglelefteq E$
⟨*proof*⟩

The limit function is bounded by the norm $p$ as well, since all elements in the chain are bounded by $p$.

**lemma** *sup-norm-pres*:
**assumes** *graph*: *graph H h* $= \bigcup c$
**and** *M*: $M =$ *norm-pres-extensions E p F f*
**and** *cM*: $c \in$ *chain M*
**shows** $\forall\, x \in H.\ h\ x \le p\ x$
⟨*proof*⟩

The following lemma is a property of linear forms on real vector spaces. It will be used for the lemma *abs-Hahn-Banach* (see page **??**). For real vector spaces the following inequations are equivalent:

$$\forall\, x \in H.\ |h\ x| \le p\ x \quad \text{and} \quad \forall\, x \in H.\ h\ x \le p\ x$$

**lemma** *abs-ineq-iff*:
**assumes** *subspace H E* **and** *vectorspace E* **and** *seminorm E p*
**and** *linearform H h*
**shows** $(\forall\, x \in H.\ |h\ x| \le p\ x) = (\forall\, x \in H.\ h\ x \le p\ x)$ (**is** *?L = ?R*)
⟨*proof*⟩

**end**

# 11 Extending non-maximal functions

**theory** *Hahn-Banach-Ext-Lemmas*
**imports** *Function-Norm*
**begin**

In this section the following context is presumed. Let $E$ be a real vector space with a seminorm $q$ on $E$. $F$ is a subspace of $E$ and $f$ a linear function on $F$. We consider a subspace $H$ of $E$ that is a superspace of $F$ and a linear form $h$ on $H$. $H$ is a not equal to $E$ and $x_0$ is an element in $E - H$. $H$ is extended to the

direct sum $H' = H + lin\ x_0$, so for any $x \in H'$ the decomposition of $x = y + a \cdot x$ with $y \in H$ is unique. $h'$ is defined on $H'$ by $h'\ x = h\ y + a \cdot \xi$ for a certain $\xi$.
Subsequently we show some properties of this extension $h'$ of $h$.

This lemma will be used to show the existence of a linear extension of $f$ (see page **??**). It is a consequence of the completeness of $\mathbb{R}$. To show

$$\exists \xi.\ \forall y \in F.\ a\ y \leq \xi \wedge \xi \leq b\ y$$

it suffices to show that

$$\forall u \in F.\ \forall v \in F.\ a\ u \leq b\ v$$

**lemma** *ex-xi*:
  **assumes** *vectorspace F*
  **assumes** *r*: $\bigwedge u\ v.\ u \in F \Longrightarrow v \in F \Longrightarrow a\ u \leq b\ v$
  **shows** $\exists\ xi::real.\ \forall y \in F.\ a\ y \leq xi \wedge xi \leq b\ y$
⟨*proof*⟩

The function $h'$ is defined as a $h'\ x = h\ y + a \cdot \xi$ where $x = y + a \cdot \xi$ is a linear extension of $h$ to $H'$.

**lemma** *h'-lf*:
  **assumes** *h'-def*: $h' \equiv \lambda x.\ let\ (y,\ a) =$
    $SOME\ (y,\ a).\ x = y + a \cdot x0 \wedge y \in H\ in\ h\ y + a * xi$
   **and** *H'-def*: $H' \equiv H + lin\ x0$
   **and** *HE*: $H \trianglelefteq E$
  **assumes** *linearform H h*
  **assumes** *x0*: $x0 \notin H\ \ x0 \in E\ \ x0 \neq 0$
  **assumes** *E*: *vectorspace E*
  **shows** *linearform H' h'*
⟨*proof*⟩

The linear extension $h'$ of $h$ is bounded by the seminorm $p$.

**lemma** *h'-norm-pres*:
  **assumes** *h'-def*: $h' \equiv \lambda x.\ let\ (y,\ a) =$
    $SOME\ (y,\ a).\ x = y + a \cdot x0 \wedge y \in H\ in\ h\ y + a * xi$
   **and** *H'-def*: $H' \equiv H + lin\ x0$
   **and** *x0*: $x0 \notin H\ \ x0 \in E\ \ x0 \neq 0$
  **assumes** *E*: *vectorspace E* **and** *HE*: *subspace H E*
   **and** *seminorm E p* **and** *linearform H h*
  **assumes** *a*: $\forall y \in H.\ h\ y \leq p\ y$
   **and** *a'*: $\forall y \in H.\ -\ p\ (y + x0) - h\ y \leq xi \wedge xi \leq p\ (y + x0) - h\ y$
  **shows** $\forall x \in H'.\ h'\ x \leq p\ x$
⟨*proof*⟩

**end**

# Part III
# The Main Proof

## 12 The Hahn-Banach Theorem

**theory** *Hahn-Banach*
**imports** *Hahn-Banach-Lemmas*
**begin**

We present the proof of two different versions of the Hahn-Banach Theorem, closely following [1, §36].

### 12.1 The Hahn-Banach Theorem for vector spaces

**Hahn-Banach Theorem.** Let $F$ be a subspace of a real vector space $E$, let $p$ be a semi-norm on $E$, and $f$ be a linear form defined on $F$ such that $f$ is bounded by $p$, i.e. $\forall x \in F. f\, x \leq p\, x$. Then $f$ can be extended to a linear form $h$ on $E$ such that $h$ is norm-preserving, i.e. $h$ is also bounded by $p$.

**Proof Sketch.**

1. Define $M$ as the set of norm-preserving extensions of $f$ to subspaces of $E$. The linear forms in $M$ are ordered by domain extension.
2. We show that every non-empty chain in $M$ has an upper bound in $M$.
3. With Zorn's Lemma we conclude that there is a maximal function $g$ in $M$.
4. The domain $H$ of $g$ is the whole space $E$, as shown by classical contradiction:
   - Assuming $g$ is not defined on whole $E$, it can still be extended in a norm-preserving way to a super-space $H'$ of $H$.
   - Thus $g$ can not be maximal. Contradiction!

**theorem** *Hahn-Banach*:
  **assumes** *E*: *vectorspace* $E$ **and** *subspace* $F$ $E$
    **and** *seminorm* $E$ $p$ **and** *linearform* $F$ $f$
  **assumes** *fp*: $\forall x \in F.\ f\ x \leq p\ x$
  **shows** $\exists h.$ *linearform* $E\ h \wedge (\forall x \in F.\ h\ x = f\ x) \wedge (\forall x \in E.\ h\ x \leq p\ x)$
    — Let $E$ be a vector space, $F$ a subspace of $E$, $p$ a seminorm on $E$,
    — and $f$ a linear form on $F$ such that $f$ is bounded by $p$,
    — then $f$ can be extended to a linear form $h$ on $E$ in a norm-preserving way.
⟨*proof*⟩

### 12.2 Alternative formulation

The following alternative formulation of the Hahn-Banach Theorem uses the fact that for a real linear form $f$ and a seminorm $p$ the following inequations are equivalent:[1]

[1]This was shown in lemma *abs-ineq-iff* (see page 21).

$$\forall x \in H.\ |h\ x| \leq p\ x \quad \text{and} \quad \forall x \in H.\ h\ x \leq p\ x$$

**theorem** *abs-Hahn-Banach*:
**assumes** *E*: *vectorspace E* **and** *FE*: *subspace F E*
**and** *lf*: *linearform F f* **and** *sn*: *seminorm E p*
**assumes** *fp*: $\forall x \in F.\ |f\ x| \leq p\ x$
**shows** $\exists g.$ *linearform E g*
$\wedge (\forall x \in F.\ g\ x = f\ x)$
$\wedge (\forall x \in E.\ |g\ x| \leq p\ x)$
⟨*proof*⟩

## 12.3 The Hahn-Banach Theorem for normed spaces

Every continuous linear form $f$ on a subspace $F$ of a norm space $E$, can be extended to a continuous linear form $g$ on $E$ such that $\|f\| = \|g\|$.

**theorem** *norm-Hahn-Banach*:
**fixes** *V* **and** *norm* (‖-‖)
**fixes** *B* **defines** $\bigwedge V f.\ B\ V\ f \equiv \{0\} \cup \{|f\ x|\ /\ \|x\|\ |\ x.\ x \neq 0 \wedge x \in V\}$
**fixes** *fn-norm* (‖-‖-- [*0*, *1000*] *999*)
**defines** $\bigwedge V f.\ \|f\|\text{-}V \equiv \bigsqcup(B\ V\ f)$
**assumes** *E-norm*: *normed-vectorspace E norm* **and** *FE*: *subspace F E*
**and** *linearform*: *linearform F f* **and** *continuous F norm f*
**shows** $\exists g.$ *linearform E g*
$\wedge$ *continuous E norm g*
$\wedge (\forall x \in F.\ g\ x = f\ x)$
$\wedge \|g\|\text{-}E = \|f\|\text{-}F$
⟨*proof*⟩

**end**

# References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