Isabelle/HOL — Higher-Order Logic

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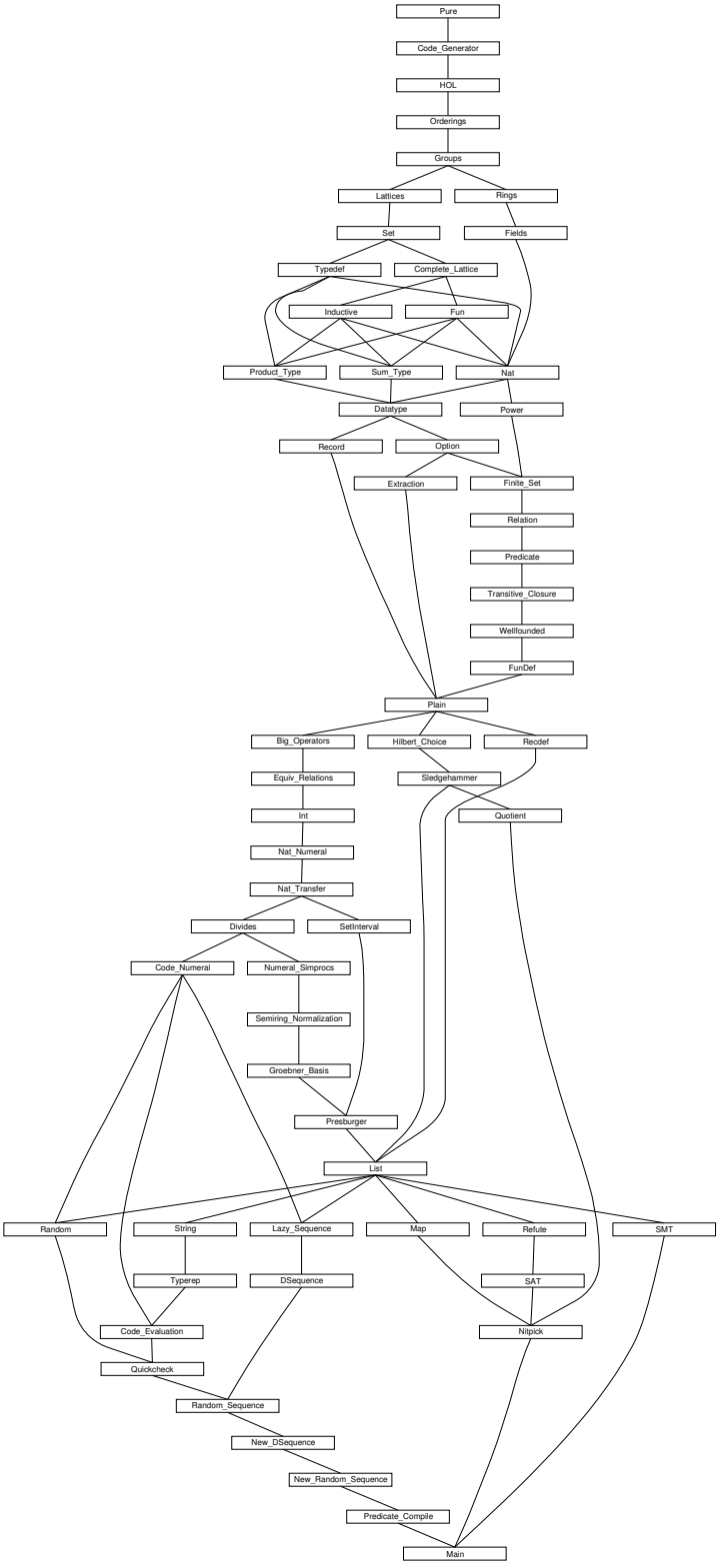
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Main

1 Code-Generator: Loading the code generator modules

theory *Code-Generator*
imports *Pure*
uses
~~/src/Tools/auto-solve.ML
~~/src/Tools/auto-counterexample.ML
~~/src/Tools/quickcheck.ML
~~/src/Tools/value.ML
~~/src/Tools/Code/code-preproc.ML
~~/src/Tools/Code/code-thingol.ML
~~/src/Tools/Code/code-printer.ML
~~/src/Tools/Code/code-target.ML
~~/src/Tools/Code/code-ml.ML
~~/src/Tools/Code/code-eval.ML
~~/src/Tools/Code/code-haskell.ML
~~/src/Tools/Code/code-scala.ML
~~/src/Tools/nbe.ML
begin

setup ⟪
Code-Preproc.setup
#> Code-ML.setup
#> Code-Eval.setup
#> Code-Haskell.setup
#> Code-Scala.setup
#> Nbe.setup
#> Quickcheck.setup
⟫

end

2 HOL: The basis of Higher-Order Logic

theory *HOL*
imports *Pure ~~/src/Tools/Code-Generator*
uses
(*Tools/hologic.ML*)
~~/src/Tools/IsaPlanner/zipper.ML
~~/src/Tools/IsaPlanner/isand.ML
~~/src/Tools/IsaPlanner/rw-tools.ML
~~/src/Tools/IsaPlanner/rw-inst.ML
~~/src/Tools/intuitionistic.ML
~~/src/Tools/project-rule.ML
~~/src/Tools/cong-tac.ML
~~/src/Provers/hypsubst.ML

~~/src/Provers/splitter.ML
~~/src/Provers/classical.ML
~~/src/Provers/blast.ML
~~/src/Provers/clasimp.ML
~~/src/Tools/coherent.ML
~~/src/Tools/eqsubst.ML
~~/src/Provers/quantifier1.ML
(Tools/simpdata.ML)
~~/src/Tools/random-word.ML
~~/src/Tools/atomize-elim.ML
~~/src/Tools/induct.ML
(~~/src/Tools/induct-tacs.ML)
(Tools/recfun-codegen.ML)
begin

setup ⟪ *Intuitionistic.method-setup* @{*binding iprover*} ⟫

2.1 Primitive logic

2.1.1 Core syntax

classes *type*
default-sort *type*
setup ⟪ *Object-Logic.add-base-sort* @{*sort type*} ⟫

arities
fun :: (*type*, *type*) *type*
itself :: (*type*) *type*

global

typedecl *bool*

judgment
Trueprop :: *bool* => *prop* ((-) 5)

consts
Not :: *bool* => *bool* (~ - [40] 40)
True :: *bool*
False :: *bool*

The :: ('a => *bool*) => 'a
All :: ('a => *bool*) => *bool* (**binder** *ALL* 10)
Ex :: ('a => *bool*) => *bool* (**binder** *EX* 10)
Ex1 :: ('a => *bool*) => *bool* (**binder** *EX!* 10)
Let :: ['a, 'a => 'b] => 'b

op = :: ['a, 'a] => *bool* (**infixl** = 50)
op & :: [*bool*, *bool*] => *bool* (**infixr** & 35)
op | :: [*bool*, *bool*] => *bool* (**infixr** | 30)

op −−> :: [*bool*, *bool*] => *bool* (**infixr** −−> *25*)

local

consts
If :: [*bool*, *'a*, *'a*] => *'a* ((*if* (-)/ *then* (-)/ *else* (-)) [*0*, *0*, *10*] *10*)

2.1.2 Additional concrete syntax

notation (**output**)
op = (**infix** = *50*)

abbreviation
not-equal :: [*'a*, *'a*] => *bool* (**infixl** ~= *50*) **where**
x ~= y == *~* (*x = y*)

notation (**output**)
not-equal (**infix** ~= *50*)

notation (*xsymbols*)
Not (¬ - [*40*] *40*) **and**
op & (**infixr** ∧ *35*) **and**
op | (**infixr** ∨ *30*) **and**
op −−> (**infixr** ⟶ *25*) **and**
not-equal (**infix** ≠ *50*)

notation (*HTML* **output**)
Not (¬ - [*40*] *40*) **and**
op & (**infixr** ∧ *35*) **and**
op | (**infixr** ∨ *30*) **and**
not-equal (**infix** ≠ *50*)

abbreviation (*iff*)
iff :: [*bool*, *bool*] => *bool* (**infixr** <−> *25*) **where**
A <−> B == *A = B*

notation (*xsymbols*)
iff (**infixr** ⟷ *25*)

nonterminals
letbinds *letbind*
case-syn *cases-syn*

syntax
-The :: [*pttrn*, *bool*] => *'a* (*3THE -./ -*) [*0*, *10*] *10*)

-bind :: [*pttrn*, *'a*] => *letbind* (*2- =/ -*) *10*)
:: *letbind* => *letbinds* (-)
-binds :: [*letbind*, *letbinds*] => *letbinds* (-;/ -)

-Let :: [*letbinds*, *'a*] => *'a* ((*let* (-)/ *in* (-)) [*0*, *10*] *10*)

-case-syntax:: [*'a*, *cases-syn*] => *'b* ((*case* - *of*/ -) *10*)
-case1 :: [*'a*, *'b*] => *case-syn* ((*2*- =>/ -) *10*)
:: *case-syn* => *cases-syn* (-)
-case2 :: [*case-syn*, *cases-syn*] => *cases-syn* (-/ | -)

translations
THE x. P == *CONST The* (%*x*. *P*)
-Let (*-binds b bs*) *e* == *-Let b* (*-Let bs e*)
let x = a in e == *CONST Let a* (%*x*. *e*)

print-translation ⟪
[(@{*const-syntax The*}, *fn* [*Abs abs*] =>
let val (*x*, *t*) = *atomic-abs-tr' abs*
in Syntax.const @{*syntax-const -The*} $ *x* $ *t end*)]
⟫ — To avoid eta-contraction of body

syntax (*xsymbols*)
-case1 :: [*'a*, *'b*] => *case-syn* ((*2*- ⇒/ -) *10*)

notation (*xsymbols*)
All (**binder** ∀ *10*) **and**
Ex (**binder** ∃ *10*) **and**
Ex1 (**binder** ∃! *10*)

notation (*HTML* **output**)
All (**binder** ∀ *10*) **and**
Ex (**binder** ∃ *10*) **and**
Ex1 (**binder** ∃! *10*)

notation (*HOL*)
All (**binder** ! *10*) **and**
Ex (**binder** *?* *10*) **and**
Ex1 (**binder** *?*! *10*)

2.1.3 Axioms and basic definitions

axioms
refl: *t* = (*t*::*'a*)
subst: *s* = *t* ⟹ *P s* ⟹ *P t*
ext: (!!*x*::*'a*. (*f x* ::*'b*) = *g x*) ==> (%*x*. *f x*) = (%*x*. *g x*)
— Extensionality is built into the meta-logic, and this rule expresses a related property. It is an eta-expanded version of the traditional rule, and similar to the ABS rule of HOL

the-eq-trivial: (*THE x*. *x* = *a*) = (*a*::*'a*)

impI: (*P* ==> *Q*) ==> *P*−−>*Q*

mp: [| P−−>Q; P |] ==> Q

defs
True-def: *True* == ((%x::*bool*. x) = (%x. x))
All-def: *All*(P) == (P = (%x. *True*))
Ex-def: *Ex*(P) == !Q. (!x. P x −−> Q) −−> Q
False-def: *False* == (!P. P)
not-def: ~ P == P−−>*False*
and-def: P & Q == !R. (P−−>Q−−>R) −−> R
or-def: P | Q == !R. (P−−>R) −−> (Q−−>R) −−> R
Ex1-def: *Ex1*(P) == ? x. P(x) & (! y. P(y) −−> y=x)

axioms
iff: (P−−>Q) −−> (Q−−>P) −−> (P=Q)
True-or-False: (P=*True*) | (P=*False*)

defs
Let-def [*code*]: *Let* s f == f(s)
if-def: *If* P x y == *THE* z::'a. (P=*True* −−> z=x) & (P=*False* −−> z=y)

finalconsts
op =
op −−>
The

axiomatization
undefined :: 'a

class *default* =
fixes *default* :: 'a

2.2 Fundamental rules

2.2.1 Equality

lemma *sym*: s = t ==> t = s
by (*erule subst*) (*rule refl*)

lemma *ssubst*: t = s ==> P s ==> P t
by (*drule sym*) (*erule subst*)

lemma *trans*: [| r=s; s=t |] ==> r=t
by (*erule subst*)

lemma *meta-eq-to-obj-eq*:
assumes *meq*: A == B
shows A = B
by (*unfold meq*) (*rule refl*)

Useful with *erule* for proving equalities from known equalities.

lemma *box-equals*: [| *a*=*b*; *a*=*c*; *b*=*d* |] ==> *c*=*d*
apply (*rule trans*)
apply (*rule trans*)
apply (*rule sym*)
apply *assumption*+
done

For calculational reasoning:

lemma *forw-subst*: *a* = *b* ==> *P b* ==> *P a*
by (*rule ssubst*)

lemma *back-subst*: *P a* ==> *a* = *b* ==> *P b*
by (*rule subst*)

2.2.2 Congruence rules for application

Similar to *AP-THM* in Gordon's HOL.

lemma *fun-cong*: (*f*::'*a*=>'*b*) = *g* ==> *f*(*x*)=*g*(*x*)
apply (*erule subst*)
apply (*rule refl*)
done

Similar to *AP-TERM* in Gordon's HOL and FOL's *subst-context*.

lemma *arg-cong*: *x*=*y* ==> *f*(*x*)=*f*(*y*)
apply (*erule subst*)
apply (*rule refl*)
done

lemma *arg-cong2*: ⟦ *a* = *b*; *c* = *d* ⟧ ⟹ *f a c* = *f b d*
apply (*erule ssubst*)+
apply (*rule refl*)
done

lemma *cong*: [| *f* = *g*; (*x*::'*a*) = *y* |] ==> *f x* = *g y*
apply (*erule subst*)+
apply (*rule refl*)
done

ML ⟪ *val cong-tac* = *Cong-Tac.cong-tac* @{*thm cong*} ⟫

2.2.3 Equality of booleans – iff

lemma *iffI*: **assumes** *P* ==> *Q* **and** *Q* ==> *P* **shows** *P*=*Q*
by (*iprover intro*: *iff* [*THEN mp*, *THEN mp*] *impI assms*)

lemma *iffD2*: [| *P*=*Q*; *Q* |] ==> *P*
by (*erule ssubst*)

lemma *rev-iffD2*: [| Q; P=Q |] ==> P
by (*erule iffD2*)

lemma *iffD1*: $Q = P \Longrightarrow Q \Longrightarrow P$
by (*drule sym*) (*rule iffD2*)

lemma *rev-iffD1*: $Q \Longrightarrow Q = P \Longrightarrow P$
by (*drule sym*) (*rule rev-iffD2*)

lemma *iffE*:
assumes *major*: P=Q
and *minor*: [| P --> Q; Q --> P |] ==> R
shows R
by (*iprover intro*: *minor impI major* [*THEN iffD2*] *major* [*THEN iffD1*])

2.2.4 True

lemma *TrueI*: *True*
unfolding *True-def* **by** (*rule refl*)

lemma *eqTrueI*: P ==> P = *True*
by (*iprover intro*: *iffI TrueI*)

lemma *eqTrueE*: P = *True* ==> P
by (*erule iffD2*) (*rule TrueI*)

2.2.5 Universal quantifier

lemma *allI*: **assumes** !!x::'a. $P(x)$ **shows** *ALL* x. $P(x)$
unfolding *All-def* **by** (*iprover intro*: *ext eqTrueI assms*)

lemma *spec*: *ALL* x::'a. $P(x)$ ==> $P(x)$
apply (*unfold All-def*)
apply (*rule eqTrueE*)
apply (*erule fun-cong*)
done

lemma *allE*:
assumes *major*: *ALL* x. $P(x)$
and *minor*: $P(x)$ ==> R
shows R
by (*iprover intro*: *minor major* [*THEN spec*])

lemma *all-dupE*:
assumes *major*: *ALL* x. $P(x)$
and *minor*: [| $P(x)$; *ALL* x. $P(x)$ |] ==> R
shows R
by (*iprover intro*: *minor major major* [*THEN spec*])

2.2.6 False

Depends upon *spec*; it is impossible to do propositional logic before quantifiers!

lemma *FalseE*: *False* ==> *P*
apply (*unfold False-def*)
apply (*erule spec*)
done

lemma *False-neq-True*: *False* = *True* ==> *P*
by (*erule eqTrueE* [*THEN FalseE*])

2.2.7 Negation

lemma *notI*:
assumes *P* ==> *False*
shows ~*P*
apply (*unfold not-def*)
apply (*iprover intro*: *impI assms*)
done

lemma *False-not-True*: *False* ~= *True*
apply (*rule notI*)
apply (*erule False-neq-True*)
done

lemma *True-not-False*: *True* ~= *False*
apply (*rule notI*)
apply (*drule sym*)
apply (*erule False-neq-True*)
done

lemma *notE*: [| ~*P*; *P* |] ==> *R*
apply (*unfold not-def*)
apply (*erule mp* [*THEN FalseE*])
apply *assumption*
done

lemma *notI2*: (*P* ⟹ ¬ *Pa*) ⟹ (*P* ⟹ *Pa*) ⟹ ¬ *P*
by (*erule notE* [*THEN notI*]) (*erule meta-mp*)

2.2.8 Implication

lemma *impE*:
assumes *P*−−>*Q* *P* *Q* ==> *R*
shows *R*
by (*iprover intro*: *assms mp*)

lemma *rev-mp*: [| *P*; *P* −−> *Q* |] ==> *Q*
by (*iprover intro*: *mp*)

lemma *contrapos-nn*:
assumes *major*: ~*Q*
and *minor*: *P*==>*Q*
shows ~*P*
by (*iprover intro*: *notI minor major* [*THEN notE*])

lemma *contrapos-pn*:
assumes *major*: *Q*
and *minor*: *P* ==> ~*Q*
shows ~*P*
by (*iprover intro*: *notI minor major notE*)

lemma *not-sym*: *t* ~= *s* ==> *s* ~= *t*
by (*erule contrapos-nn*) (*erule sym*)

lemma *eq-neq-eq-imp-neq*: [| *x* = *a* ; *a* ~= *b*; *b* = *y* |] ==> *x* ~= *y*
by (*erule subst*, *erule ssubst*, *assumption*)

lemma *rev-contrapos*:
assumes *pq*: *P* ==> *Q*
and *nq*: ~*Q*
shows ~*P*
apply (*rule nq* [*THEN contrapos-nn*])
apply (*erule pq*)
done

2.2.9 Existential quantifier

lemma *exI*: *P x* ==> *EX x*::'*a*. *P x*
apply (*unfold Ex-def*)
apply (*iprover intro*: *allI allE impI mp*)
done

lemma *exE*:
assumes *major*: *EX x*::'*a*. *P*(*x*)
and *minor*: !!*x*. *P*(*x*) ==> *Q*
shows *Q*
apply (*rule major* [*unfolded Ex-def*, *THEN spec*, *THEN mp*])
apply (*iprover intro*: *impI* [*THEN allI*] *minor*)
done

2.2.10 Conjunction

lemma *conjI*: [| *P*; *Q* |] ==> *P*&*Q*
apply (*unfold and-def*)

apply (*iprover intro*: *impI* [*THEN allI*] *mp*)
done

lemma *conjunct1*: [| *P* & *Q* |] ==> *P*
apply (*unfold and-def*)
apply (*iprover intro*: *impI dest*: *spec mp*)
done

lemma *conjunct2*: [| *P* & *Q* |] ==> *Q*
apply (*unfold and-def*)
apply (*iprover intro*: *impI dest*: *spec mp*)
done

lemma *conjE*:
assumes *major*: *P*&*Q*
and *minor*: [| *P*; *Q* |] ==> *R*
shows *R*
apply (*rule minor*)
apply (*rule major* [*THEN conjunct1*])
apply (*rule major* [*THEN conjunct2*])
done

lemma *context-conjI*:
assumes *P* *P* ==> *Q* **shows** *P* & *Q*
by (*iprover intro*: *conjI assms*)

2.2.11 Disjunction

lemma *disjI1*: *P* ==> *P*|*Q*
apply (*unfold or-def*)
apply (*iprover intro*: *allI impI mp*)
done

lemma *disjI2*: *Q* ==> *P*|*Q*
apply (*unfold or-def*)
apply (*iprover intro*: *allI impI mp*)
done

lemma *disjE*:
assumes *major*: *P*|*Q*
and *minorP*: *P* ==> *R*
and *minorQ*: *Q* ==> *R*
shows *R*
by (*iprover intro*: *minorP minorQ impI*
major [*unfolded or-def*, *THEN spec*, *THEN mp*, *THEN mp*])

2.2.12 Classical logic

lemma *classical*:
assumes *prem*: ~*P* ==> *P*

shows P
apply (*rule True-or-False* [*THEN disjE*, *THEN eqTrueE*])
apply *assumption*
apply (*rule notI* [*THEN prem*, *THEN eqTrueI*])
apply (*erule subst*)
apply *assumption*
done

lemmas *ccontr* = *FalseE* [*THEN classical*, *standard*]

lemma *rev-notE*:
assumes *premp*: P
and *premnot*: $\sim R \Longrightarrow \sim P$
shows R
apply (*rule ccontr*)
apply (*erule notE* [*OF premnot premp*])
done

lemma *notnotD*: $\sim\sim P \Longrightarrow P$
apply (*rule classical*)
apply (*erule notE*)
apply *assumption*
done

lemma *contrapos-pp*:
assumes *p1*: Q
and *p2*: $\sim P \Longrightarrow \sim Q$
shows P
by (*iprover intro*: *classical p1 p2 notE*)

2.2.13 Unique existence

lemma *ex1I*:
assumes $P\ a$ $!!x.\ P(x) \Longrightarrow x=a$
shows $EX!\ x.\ P(x)$
by (*unfold Ex1-def*, *iprover intro*: *assms exI conjI allI impI*)

Sometimes easier to use: the premises have no shared variables. Safe!

lemma *ex-ex1I*:
assumes *ex-prem*: $EX\ x.\ P(x)$
and *eq*: $!!x\ y.\ [|\ P(x);\ P(y)\ |] \Longrightarrow x=y$
shows $EX!\ x.\ P(x)$
by (*iprover intro*: *ex-prem* [*THEN exE*] *ex1I eq*)

lemma *ex1E*:
assumes *major*: $EX!\ x.\ P(x)$
and *minor*: $!!x.\ [|\ P(x);\ \ ALL\ y.\ P(y) \longrightarrow y=x\ |] \Longrightarrow R$

shows *R*
apply (*rule major* [*unfolded Ex1-def*, *THEN exE*])
apply (*erule conjE*)
apply (*iprover intro*: *minor*)
done

lemma *ex1-implies-ex*: *EX*! *x*. *P x* ==> *EX x*. *P x*
apply (*erule ex1E*)
apply (*rule exI*)
apply *assumption*
done

2.2.14 THE: definite description operator

lemma *the-equality*:
assumes *prema*: *P a*
and *premx*: !!*x*. *P x* ==> *x*=*a*
shows (*THE x*. *P x*) = *a*
apply (*rule trans* [*OF - the-eq-trivial*])
apply (*rule-tac f* = *The* **in** *arg-cong*)
apply (*rule ext*)
apply (*rule iffI*)
apply (*erule premx*)
apply (*erule ssubst*, *rule prema*)
done

lemma *theI*:
assumes *P a* **and** !!*x*. *P x* ==> *x*=*a*
shows *P* (*THE x*. *P x*)
by (*iprover intro*: *assms the-equality* [*THEN ssubst*])

lemma *theI'*: *EX*! *x*. *P x* ==> *P* (*THE x*. *P x*)
apply (*erule ex1E*)
apply (*erule theI*)
apply (*erule allE*)
apply (*erule mp*)
apply *assumption*
done

lemma *theI2*:
assumes *P a* !!*x*. *P x* ==> *x*=*a* !!*x*. *P x* ==> *Q x*
shows *Q* (*THE x*. *P x*)
by (*iprover intro*: *assms theI*)

lemma *the1I2*: **assumes** *EX*! *x*. *P x* $\bigwedge$*x*. *P x* $\Longrightarrow$ *Q x* **shows** *Q* (*THE x*. *P x*)
by(*iprover intro*:*assms*(*2*) *theI2*[**where** *P*=*P* **and** *Q*=*Q*] *ex1E*[*OF assms*(*1*)]
elim:*allE impE*)

lemma *the1-equality* [*elim?*]: [| *EX!x. P x*; *P a* |] ==> (*THE x. P x*) = *a*
apply (*rule the-equality*)
apply *assumption*
apply (*erule ex1E*)
apply (*erule all-dupE*)
apply (*drule mp*)
apply *assumption*
apply (*erule ssubst*)
apply (*erule allE*)
apply (*erule mp*)
apply *assumption*
done

lemma *the-sym-eq-trivial*: (*THE y. x=y*) = *x*
apply (*rule the-equality*)
apply (*rule refl*)
apply (*erule sym*)
done

2.2.15 Classical intro rules for disjunction and existential quantifiers

lemma *disjCI*:
assumes ~*Q* ==> *P* **shows** *P*|*Q*
apply (*rule classical*)
apply (*iprover intro*: *assms disjI1 disjI2 notI elim*: *notE*)
done

lemma *excluded-middle*: ~*P* | *P*
by (*iprover intro*: *disjCI*)

case distinction as a natural deduction rule. Note that ¬ *P* is the second case, not the first

lemma *case-split* [*case-names True False*]:
assumes *prem1*: *P* ==> *Q*
and *prem2*: ~*P* ==> *Q*
shows *Q*
apply (*rule excluded-middle* [*THEN disjE*])
apply (*erule prem2*)
apply (*erule prem1*)
done

lemma *impCE*:
assumes *major*: *P*−−>*Q*
and *minor*: ~*P* ==> *R* *Q* ==> *R*
shows *R*
apply (*rule excluded-middle* [*of P*, *THEN disjE*])
apply (*iprover intro*: *minor major* [*THEN mp*])+

done

lemma *impCE'*:
assumes *major*: $P-->Q$
and *minor*: Q ==> R ~P ==> R
shows R
apply (*rule excluded-middle* [*of* P, *THEN disjE*])
apply (*iprover intro*: *minor major* [*THEN mp*])+
done

lemma *iffCE*:
assumes *major*: $P=Q$
and *minor*: [| P; Q |] ==> R [| ~P; ~Q |] ==> R
shows R
apply (*rule major* [*THEN iffE*])
apply (*iprover intro*: *minor elim*: *impCE notE*)
done

lemma *exCI*:
assumes *ALL x*. ~$P(x)$ ==> $P(a)$
shows *EX x*. $P(x)$
apply (*rule ccontr*)
apply (*iprover intro*: *assms exI allI notI notE* [*of* $\exists x.\ P\ x$])
done

2.2.16 Intuitionistic Reasoning

lemma *impE'*:
assumes *1*: P --> Q
and *2*: Q ==> R
and *3*: P --> Q ==> P
shows R
proof −
from *3* **and** *1* **have** P **.**
with *1* **have** Q **by** (*rule impE*)
with *2* **show** R **.**
qed

lemma *allE'*:
assumes *1*: *ALL x*. $P\ x$
and *2*: $P\ x$ ==> *ALL x*. $P\ x$ ==> Q
shows Q
proof −
from *1* **have** $P\ x$ **by** (*rule spec*)
from *this* **and** *1* **show** Q **by** (*rule 2*)
qed

lemma *notE'*:
  assumes *1*: ~ *P*
    and *2*: ~ *P* ==> *P*
  shows *R*
proof −
  from *2* **and** *1* **have** *P* **.**
  with *1* **show** *R* **by** (*rule notE*)
qed

lemma *TrueE*: *True* ==> *P* ==> *P* **.**
lemma *notFalseE*: ~ *False* ==> *P* ==> *P* **.**

lemmas [*Pure.elim*!] = *disjE iffE FalseE conjE exE TrueE notFalseE*
  and [*Pure.intro*!] = *iffI conjI impI TrueI notI allI refl*
  and [*Pure.elim 2*] = *allE notE' impE'*
  and [*Pure.intro*] = *exI disjI2 disjI1*

lemmas [*trans*] = *trans*
  and [*sym*] = *sym not-sym*
  and [*Pure.elim?*] = *iffD1 iffD2 impE*

use *Tools/hologic.ML*

2.2.17 Atomizing meta-level connectives

axiomatization where
  eq-reflection: *x* = *y* ⟹ *x* ≡ *y*

lemma *atomize-all* [*atomize*]: (!!*x*. *P x*) == *Trueprop* (*ALL x*. *P x*)
proof
  assume !!*x*. *P x*
  then show *ALL x*. *P x* **..**
next
  assume *ALL x*. *P x*
  then show !!*x*. *P x* **by** (*rule allE*)
qed

lemma *atomize-imp* [*atomize*]: (*A* ==> *B*) == *Trueprop* (*A* −−> *B*)
proof
  assume *r*: *A* ==> *B*
  show *A* −−> *B* **by** (*rule impI*) (*rule r*)
next
  assume *A* −−> *B* **and** *A*
  then show *B* **by** (*rule mp*)
qed

lemma *atomize-not*: (*A* ==> *False*) == *Trueprop* (~*A*)
proof
  assume *r*: *A* ==> *False*

show ~*A* **by** (*rule notI*) (*rule r*)
next
assume ~*A* **and** *A*
then show *False* **by** (*rule notE*)
qed

lemma *atomize-eq* [*atomize*]: (*x* == *y*) == *Trueprop* (*x* = *y*)
proof
assume *x* == *y*
show *x* = *y* **by** (*unfold* ‹*x* == *y*›) (*rule refl*)
next
assume *x* = *y*
then show *x* == *y* **by** (*rule eq-reflection*)
qed

lemma *atomize-conj* [*atomize*]: (*A* &&& *B*) == *Trueprop* (*A* & *B*)
proof
assume *conj*: *A* &&& *B*
show *A* & *B*
proof (*rule conjI*)
from *conj* **show** *A* **by** (*rule conjunctionD1*)
from *conj* **show** *B* **by** (*rule conjunctionD2*)
qed
next
assume *conj*: *A* & *B*
show *A* &&& *B*
proof −
from *conj* **show** *A* **..**
from *conj* **show** *B* **..**
qed
qed

lemmas [*symmetric*, *rulify*] = *atomize-all atomize-imp*
and [*symmetric*, *defn*] = *atomize-all atomize-imp atomize-eq*

2.2.18 Atomizing elimination rules

setup *AtomizeElim.setup*

lemma *atomize-exL*[*atomize-elim*]: (!!*x*. *P x* ==> *Q*) == ((*EX x*. *P x*) ==> *Q*)
by *rule iprover+*

lemma *atomize-conjL*[*atomize-elim*]: (*A* ==> *B* ==> *C*) == (*A* & *B* ==> *C*)
by *rule iprover+*

lemma *atomize-disjL*[*atomize-elim*]: ((*A* ==> *C*) ==> (*B* ==> *C*) ==> *C*)
== ((*A* | *B* ==> *C*) ==> *C*)
by *rule iprover+*

lemma *atomize-elimL*[*atomize-elim*]: (!!*B*. (*A* ==> *B*) ==> *B*) == *Trueprop A*
..

2.3 Package setup

2.3.1 Sledgehammer setup

Theorems blacklisted to Sledgehammer. These theorems typically produce clauses that are prolific (match too many equality or membership literals) and relate to seldom-used facts. Some duplicate other rules.

ML ⟪
structure No-ATPs = Named-Thms
(
  val name = no-atp
  val description = theorems that should be filtered out by Sledgehammer
)
⟫

setup ⟪ *No-ATPs.setup* ⟫

2.3.2 Classical Reasoner setup

lemma *imp-elim*: *P* −−> *Q* ==> (~ *R* ==> *P*) ==> (*Q* ==> *R*) ==> *R*
  by (*rule classical*) *iprover*

lemma *swap*: ~ *P* ==> (~ *R* ==> *P*) ==> *R*
  by (*rule classical*) *iprover*

lemma *thin-refl*:
  ⋀*X*. ⟦ *x*=*x*; *PROP W* ⟧ ⟹ *PROP W* **.**

ML ⟪
structure Hypsubst = HypsubstFun(
struct
  structure Simplifier = Simplifier
  val dest-eq = HOLogic.dest-eq
  val dest-Trueprop = HOLogic.dest-Trueprop
  val dest-imp = HOLogic.dest-imp
  val eq-reflection = @{thm eq-reflection}
  val rev-eq-reflection = @{thm meta-eq-to-obj-eq}
  val imp-intr = @{thm impI}
  val rev-mp = @{thm rev-mp}
  val subst = @{thm subst}
  val sym = @{thm sym}
  val thin-refl = @{thm thin-refl};
  val prop-subst = @{lemma PROP P t ==> PROP prop (x = t ==> PROP P x)
                by (unfold prop-def) (drule eq-reflection, unfold)}
end);

```
open Hypsubst;

structure Classical = ClassicalFun(
struct
  val imp-elim = @{thm imp-elim}
  val not-elim = @{thm notE}
  val swap = @{thm swap}
  val classical = @{thm classical}
  val sizef = Drule.size-of-thm
  val hyp-subst-tacs = [Hypsubst.hyp-subst-tac]
end);

structure Basic-Classical: BASIC-CLASSICAL = Classical;
open Basic-Classical;

ML-Antiquote.value claset
  (Scan.succeed Classical.claset-of (ML-Context.the-local-context ()));
⟩⟩

setup Classical.setup

setup ⟨⟨
let
  fun non-bool-eq (@{const-name op =}, Type (-, [T, -])) = T <> @{typ bool}
    | non-bool-eq - = false;
  val hyp-subst-tac' =
    SUBGOAL (fn (goal, i) =>
      if Term.exists-Const non-bool-eq goal
      then Hypsubst.hyp-subst-tac i
      else no-tac);
in
  Hypsubst.hypsubst-setup
  (*prevent substitution on bool*)
  #> Context-Rules.addSWrapper (fn tac => hyp-subst-tac' ORELSE' tac)
end
⟩⟩

declare iffI [intro!]
  and notI [intro!]
  and impI [intro!]
  and disjCI [intro!]
  and conjI [intro!]
  and TrueI [intro!]
  and refl [intro!]

declare iffCE [elim!]
  and FalseE [elim!]
  and impCE [elim!]
  and disjE [elim!]
```

and *conjE* [*elim*!]

declare *ex-ex1I* [*intro*!]
and *allI* [*intro*!]
and *the-equality* [*intro*]
and *exI* [*intro*]

declare *exE* [*elim*!]
allE [*elim*]

ML ⟪ *val HOL-cs* = @{*claset*} ⟫

lemma *contrapos-np*: ~ *Q* ==> (~ *P* ==> *Q*) ==> *P*
apply (*erule swap*)
apply (*erule* (*1*) *meta-mp*)
done

declare *ex-ex1I* [*rule del*, *intro*! *2*]
and *ex1I* [*intro*]

lemmas [*intro?*] = *ext*
and [*elim?*] = *ex1-implies-ex*

lemma *alt-ex1E* [*elim*!]:
assumes *major*: ∃!*x*. *P x*
and *prem*: ⋀*x*. ⟦ *P x*; ∀ *y y'*. *P y* ∧ *P y'* ⟶ *y* = *y'* ⟧ ⟹ *R*
shows *R*
apply (*rule ex1E* [*OF major*])
apply (*rule prem*)
apply (*tactic* ⟪ *ares-tac* @{*thms allI*} *1* ⟫)+
apply (*tactic* ⟪ *etac* (*Classical.dup-elim* @{*thm allE*}) *1* ⟫)
apply *iprover*
done

ML ⟪
structure Blast = *Blast*
(
val thy = @{*theory*}
type claset = *Classical.claset*
val equality-name = @{*const-name op* =}
val not-name = @{*const-name Not*}
val notE = @{*thm notE*}
val ccontr = @{*thm ccontr*}
val contr-tac = *Classical.contr-tac*
val dup-intr = *Classical.dup-intr*
val hyp-subst-tac = *Hypsubst.blast-hyp-subst-tac*
val rep-cs = *Classical.rep-cs*
val cla-modifiers = *Classical.cla-modifiers*

val cla-meth' = Classical.cla-meth'
);
val blast-tac = Blast.blast-tac;
⟫

setup *Blast.setup*

2.3.3 Simplifier

lemma *eta-contract-eq*: $(\%s.\ f\ s) = f$ **..**

lemma *simp-thms*:
shows *not-not*: $(\sim\ \sim\ P) = P$
and *Not-eq-iff*: $((\sim P) = (\sim Q)) = (P = Q)$
and
$(P\ \sim= Q) = (P = (\sim Q))$
$(P \mid \sim P) = True \quad (\sim P \mid P) = True$
$(x = x) = True$
and *not-True-eq-False* [*code*]: $(\neg\ True) = False$
and *not-False-eq-True* [*code*]: $(\neg\ False) = True$
and
$(\sim P)\ \sim= P \ \ P\ \sim= (\sim P)$
$(True=P) = P$
and *eq-True*: $(P = True) = P$
and $(False=P) = (\sim P)$
and *eq-False*: $(P = False) = (\neg\ P)$
and
$(True \longrightarrow P) = P \ \ (False \longrightarrow P) = True$
$(P \longrightarrow True) = True \ \ (P \longrightarrow P) = True$
$(P \longrightarrow False) = (\sim P) \ \ (P \longrightarrow \sim P) = (\sim P)$
$(P\ \&\ True) = P \ \ (True\ \&\ P) = P$
$(P\ \&\ False) = False \ \ (False\ \&\ P) = False$
$(P\ \&\ P) = P \ \ (P\ \&\ (P\ \&\ Q)) = (P\ \&\ Q)$
$(P\ \&\ \sim P) = False \quad (\sim P\ \&\ P) = False$
$(P \mid True) = True \ \ (True \mid P) = True$
$(P \mid False) = P \ \ (False \mid P) = P$
$(P \mid P) = P \ \ (P \mid (P \mid Q)) = (P \mid Q)$ **and**
$(ALL\ x.\ P) = P \ \ (EX\ x.\ P) = P \ \ EX\ x.\ x{=}t \ \ EX\ x.\ t{=}x$
and
$!!P.\ (EX\ x.\ x{=}t\ \&\ P(x)) = P(t)$
$!!P.\ (EX\ x.\ t{=}x\ \&\ P(x)) = P(t)$
$!!P.\ (ALL\ x.\ x{=}t \longrightarrow P(x)) = P(t)$
$!!P.\ (ALL\ x.\ t{=}x \longrightarrow P(x)) = P(t)$
by (*blast*, *blast*, *blast*, *blast*, *blast*, *iprover*+)

lemma *disj-absorb*: $(A \mid A) = A$
by *blast*

lemma *disj-left-absorb*: $(A \mid (A \mid B)) = (A \mid B)$

by *blast*

lemma *conj-absorb*: $(A \ \& \ A) = A$
by *blast*

lemma *conj-left-absorb*: $(A \ \& \ (A \ \& \ B)) = (A \ \& \ B)$
by *blast*

lemma *eq-ac*:
shows *eq-commute*: $(a{=}b) = (b{=}a)$
and *eq-left-commute*: $(P{=}(Q{=}R)) = (Q{=}(P{=}R))$
and *eq-assoc*: $((P{=}Q){=}R) = (P{=}(Q{=}R))$ **by** (*iprover*, *blast*+)
lemma *neq-commute*: $(a^{\sim}{=}b) = (b^{\sim}{=}a)$ **by** *iprover*

lemma *conj-comms*:
shows *conj-commute*: $(P\&Q) = (Q\&P)$
and *conj-left-commute*: $(P\&(Q\&R)) = (Q\&(P\&R))$ **by** *iprover*+
lemma *conj-assoc*: $((P\&Q)\&R) = (P\&(Q\&R))$ **by** *iprover*

lemmas *conj-ac* = *conj-commute conj-left-commute conj-assoc*

lemma *disj-comms*:
shows *disj-commute*: $(P|Q) = (Q|P)$
and *disj-left-commute*: $(P|(Q|R)) = (Q|(P|R))$ **by** *iprover*+
lemma *disj-assoc*: $((P|Q)|R) = (P|(Q|R))$ **by** *iprover*

lemmas *disj-ac* = *disj-commute disj-left-commute disj-assoc*

lemma *conj-disj-distribL*: $(P\&(Q|R)) = (P\&Q \mid P\&R)$ **by** *iprover*
lemma *conj-disj-distribR*: $((P|Q)\&R) = (P\&R \mid Q\&R)$ **by** *iprover*

lemma *disj-conj-distribL*: $(P|(Q\&R)) = ((P|Q) \ \& \ (P|R))$ **by** *iprover*
lemma *disj-conj-distribR*: $((P\&Q)|R) = ((P|R) \ \& \ (Q|R))$ **by** *iprover*

lemma *imp-conjR*: $(P \dashrightarrow (Q\&R)) = ((P\dashrightarrow Q) \ \& \ (P\dashrightarrow R))$ **by** *iprover*
lemma *imp-conjL*: $((P\&Q) \dashrightarrow R) \ = (P \dashrightarrow (Q \dashrightarrow R))$ **by** *iprover*
lemma *imp-disjL*: $((P|Q) \dashrightarrow R) = ((P\dashrightarrow R)\&(Q\dashrightarrow R))$ **by** *iprover*

These two are specialized, but *imp-disj-not1* is useful in *Auth/Yahalom*.

lemma *imp-disj-not1*: $(P \dashrightarrow Q \mid R) = (^{\sim}Q \dashrightarrow P \dashrightarrow R)$ **by** *blast*
lemma *imp-disj-not2*: $(P \dashrightarrow Q \mid R) = (^{\sim}R \dashrightarrow P \dashrightarrow Q)$ **by** *blast*

lemma *imp-disj1*: $((P\dashrightarrow Q)|R) = (P\dashrightarrow Q|R)$ **by** *blast*
lemma *imp-disj2*: $(Q|(P\dashrightarrow R)) = (P\dashrightarrow Q|R)$ **by** *blast*

lemma *imp-cong*: $(P = P') \Longrightarrow (P' \Longrightarrow (Q = Q')) \Longrightarrow ((P \dashrightarrow Q) = (P' \dashrightarrow Q'))$
by *iprover*

lemma *de-Morgan-disj*: $(\sim(P \mid Q)) = (\sim P \;\&\; \sim Q)$ **by** *iprover*
lemma *de-Morgan-conj*: $(\sim(P \;\&\; Q)) = (\sim P \mid \sim Q)$ **by** *blast*
lemma *not-imp*: $(\sim(P \dashrightarrow Q)) = (P \;\&\; \sim Q)$ **by** *blast*
lemma *not-iff*: $(P \sim= Q) = (P = (\sim Q))$ **by** *blast*
lemma *disj-not1*: $(\sim P \mid Q) = (P \dashrightarrow Q)$ **by** *blast*
lemma *disj-not2*: $(P \mid \sim Q) = (Q \dashrightarrow P)$ — changes orientation :-(
by *blast*
lemma *imp-conv-disj*: $(P \dashrightarrow Q) = ((\sim P) \mid Q)$ **by** *blast*

lemma *iff-conv-conj-imp*: $(P = Q) = ((P \dashrightarrow Q) \;\&\; (Q \dashrightarrow P))$ **by** *iprover*

lemma *cases-simp*: $((P \dashrightarrow Q) \;\&\; (\sim P \dashrightarrow Q)) = Q$
— Avoids duplication of subgoals after *split-if*, when the true and false
— cases boil down to the same thing.
by *blast*

lemma *not-all*: $(\sim\ (!\ x.\ P(x))) = (?\ x.\sim P(x))$ **by** *blast*
lemma *imp-all*: $((!\ x.\ P\ x) \dashrightarrow Q) = (?\ x.\ P\ x \dashrightarrow Q)$ **by** *blast*
lemma *not-ex*: $(\sim\ (?\ x.\ P(x))) = (!\ x.\sim P(x))$ **by** *iprover*
lemma *imp-ex*: $((?\ x.\ P\ x) \dashrightarrow Q) = (!\ x.\ P\ x \dashrightarrow Q)$ **by** *iprover*
lemma *all-not-ex*: $(ALL\ x.\ P\ x) = (\sim\ (EX\ x.\ \sim\ P\ x\ ))$ **by** *blast*

declare *All-def* [*no-atp*]

lemma *ex-disj-distrib*: $(?\ x.\ P(x) \mid Q(x)) = ((?\ x.\ P(x)) \mid (?\ x.\ Q(x)))$ **by** *iprover*
lemma *all-conj-distrib*: $(!x.\ P(x) \;\&\; Q(x)) = ((!\ x.\ P(x)) \;\&\; (!\ x.\ Q(x)))$ **by** *iprover*

The & congruence rule: not included by default! May slow rewrite proofs down by as much as 50%

lemma *conj-cong*:
$(P = P') \Longrightarrow (P' \Longrightarrow (Q = Q')) \Longrightarrow ((P \;\&\; Q) = (P' \;\&\; Q'))$
by *iprover*

lemma *rev-conj-cong*:
$(Q = Q') \Longrightarrow (Q' \Longrightarrow (P = P')) \Longrightarrow ((P \;\&\; Q) = (P' \;\&\; Q'))$
by *iprover*

The | congruence rule: not included by default!

lemma *disj-cong*:
$(P = P') \Longrightarrow (\sim P' \Longrightarrow (Q = Q')) \Longrightarrow ((P \mid Q) = (P' \mid Q'))$
by *blast*

if-then-else rules

lemma *if-True* [*code*]: $(\textit{if True then } x \textit{ else } y) = x$
by (*unfold if-def*) *blast*

lemma *if-False* [*code*]: $(\textit{if False then } x \textit{ else } y) = y$

by (*unfold if-def*) *blast*

lemma *if-P*: $P \Longrightarrow (\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ y) = x$
by (*unfold if-def*) *blast*

lemma *if-not-P*: $\sim P \Longrightarrow (\mathit{if}\ P\ \mathit{then}\ x\ \mathit{else}\ y) = y$
by (*unfold if-def*) *blast*

lemma *split-if*: $P\ (\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y) = ((Q \longrightarrow P(x)) \ \&\ (\sim Q \longrightarrow P(y)))$
apply (*rule case-split* [*of Q*])
apply (*simplesubst if-P*)
prefer *3* **apply** (*simplesubst if-not-P*, *blast*+)
done

lemma *split-if-asm*: $P\ (\mathit{if}\ Q\ \mathit{then}\ x\ \mathit{else}\ y) = (\sim((Q\ \&\ \sim P\ x) \mid (\sim Q\ \&\ \sim P\ y)))$
by (*simplesubst split-if*, *blast*)

lemmas *if-splits* [*no-atp*] = *split-if split-if-asm*

lemma *if-cancel*: $(\mathit{if}\ c\ \mathit{then}\ x\ \mathit{else}\ x) = x$
by (*simplesubst split-if*, *blast*)

lemma *if-eq-cancel*: $(\mathit{if}\ x = y\ \mathit{then}\ y\ \mathit{else}\ x) = x$
by (*simplesubst split-if*, *blast*)

lemma *if-bool-eq-conj*: $(\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ R) = ((P \longrightarrow Q)\ \&\ (\sim P \longrightarrow R))$
— This form is useful for expanding *if*s on the RIGHT of the ==> symbol.
by (*rule split-if*)

lemma *if-bool-eq-disj*: $(\mathit{if}\ P\ \mathit{then}\ Q\ \mathit{else}\ R) = ((P \& Q) \mid (\sim P \& R))$
— And this form is useful for expanding *if*s on the LEFT.
apply (*simplesubst split-if*, *blast*)
done

lemma *Eq-TrueI*: $P \Longrightarrow P \equiv \mathit{True}$ **by** (*unfold atomize-eq*) *iprover*
lemma *Eq-FalseI*: $\sim P \Longrightarrow P \equiv \mathit{False}$ **by** (*unfold atomize-eq*) *iprover*

let rules for simproc

lemma *Let-folded*: $f\ x \equiv g\ x \Longrightarrow \mathit{Let}\ x\ f \equiv \mathit{Let}\ x\ g$
by (*unfold Let-def*)

lemma *Let-unfold*: $f\ x \equiv g \Longrightarrow \mathit{Let}\ x\ f \equiv g$
by (*unfold Let-def*)

The following copy of the implication operator is useful for fine-tuning congruence rules. It instructs the simplifier to simplify its premise.

definition *simp-implies* :: [*prop*, *prop*] => *prop* (**infixr** =*simp*=> *1*) **where**
[*code del*]: *simp-implies* ≡ *op* ==>

lemma *simp-impliesI*:
assumes *PQ*: (*PROP P* $\Longrightarrow$ *PROP Q*)
shows *PROP P* =*simp*=> *PROP Q*
apply (*unfold simp-implies-def*)
apply (*rule PQ*)
apply *assumption*
done

lemma *simp-impliesE*:
assumes *PQ*: *PROP P* =*simp*=> *PROP Q*
and *P*: *PROP P*
and *QR*: *PROP Q* $\Longrightarrow$ *PROP R*
shows *PROP R*
apply (*rule QR*)
apply (*rule PQ* [*unfolded simp-implies-def*])
apply (*rule P*)
done

lemma *simp-implies-cong*:
assumes *PP'* :*PROP P* == *PROP P'*
and *P'QQ'*: *PROP P'* ==> (*PROP Q* == *PROP Q'*)
shows (*PROP P* =*simp*=> *PROP Q*) == (*PROP P'* =*simp*=> *PROP Q'*)
proof (*unfold simp-implies-def*, *rule equal-intr-rule*)
assume *PQ*: *PROP P* $\Longrightarrow$ *PROP Q*
and *P'*: *PROP P'*
from *PP'* [*symmetric*] **and** *P'* **have** *PROP P*
by (*rule equal-elim-rule1*)
then have *PROP Q* **by** (*rule PQ*)
with *P'QQ'* [*OF P'*] **show** *PROP Q'* **by** (*rule equal-elim-rule1*)
next
assume *P'Q'*: *PROP P'* $\Longrightarrow$ *PROP Q'*
and *P*: *PROP P*
from *PP'* **and** *P* **have** *P'*: *PROP P'* **by** (*rule equal-elim-rule1*)
then have *PROP Q'* **by** (*rule P'Q'*)
with *P'QQ'* [*OF P'*, *symmetric*] **show** *PROP Q*
by (*rule equal-elim-rule1*)
qed

lemma *uncurry*:
assumes $P \longrightarrow Q \longrightarrow R$
shows $P \wedge Q \longrightarrow R$
using *assms* **by** *blast*

lemma *iff-allI*:
assumes $\bigwedge x.\ P\ x = Q\ x$
shows $(\forall x.\ P\ x) = (\forall x.\ Q\ x)$
using *assms* **by** *blast*

lemma *iff-exI*:
assumes $\bigwedge x.\ P\ x = Q\ x$
shows $(\exists x.\ P\ x) = (\exists x.\ Q\ x)$
using *assms* **by** *blast*

lemma *all-comm*:
$(\forall x\ y.\ P\ x\ y) = (\forall y\ x.\ P\ x\ y)$
by *blast*

lemma *ex-comm*:
$(\exists x\ y.\ P\ x\ y) = (\exists y\ x.\ P\ x\ y)$
by *blast*

use *Tools/simpdata.ML*
ML ⟪ *open Simpdata* ⟫

setup ⟪
Simplifier.method-setup Splitter.split-modifiers
#> Simplifier.map-simpset (K Simpdata.simpset-simprocs)
#> Splitter.setup
#> clasimp-setup
#> EqSubst.setup
⟫

Simproc for proving $(y = x) == False$ from premise $\sim(x = y)$:

simproc-setup *neq* $(x = y)$ = ⟪ *fn - =>*
let
val neq-to-EQ-False = @{thm not-sym} RS @{thm Eq-FalseI};
fun is-neq eq lhs rhs thm =
(case Thm.prop-of thm of
- $ (Not $ (eq' $ l' $ r')) =>
Not = HOLogic.Not andalso eq' = eq andalso
r' aconv lhs andalso l' aconv rhs
| - => false);
fun proc ss ct =
(case Thm.term-of ct of
eq $ lhs $ rhs =>
(case find-first (is-neq eq lhs rhs) (Simplifier.prems-of-ss ss) of
SOME thm => SOME (thm RS neq-to-EQ-False)
| NONE => NONE)
| - => NONE);
in proc end;
⟫

simproc-setup *let-simp* $(Let\ x\ f)$ = ⟪
let
val (f-Let-unfold, x-Let-unfold) =
let val [(- $ (f $ x) $ -)] = prems-of @{thm Let-unfold}
in (cterm-of @{theory} f, cterm-of @{theory} x) end

```
  val (f-Let-folded, x-Let-folded) =
    let val [(- $ (f $ x) $ -)] = prems-of @{thm Let-folded}
    in (cterm-of @{theory} f, cterm-of @{theory} x) end;
  val g-Let-folded =
    let val [(- $ - $ (g $ -))] = prems-of @{thm Let-folded}
    in cterm-of @{theory} g end;
  fun count-loose (Bound i) k = if i >= k then 1 else 0
    | count-loose (s $ t) k = count-loose s k + count-loose t k
    | count-loose (Abs (-, -, t)) k = count-loose  t (k + 1)
    | count-loose - - = 0;
  fun is-trivial-let (Const (@{const-name Let}, -) $ x $ t) =
   case t
    of Abs (-, -, t') => count-loose t' 0 <= 1
     | - => true;
in fn - => fn ss => fn ct => if is-trivial-let (Thm.term-of ct)
  then SOME @{thm Let-def} (*no or one ocurrence of bound variable*)
  else let (*Norbert Schirmer's case*)
    val ctxt = Simplifier.the-context ss;
    val thy = ProofContext.theory-of ctxt;
    val t = Thm.term-of ct;
    val ([t'], ctxt') = Variable.import-terms false [t] ctxt;
  in Option.map (hd o Variable.export ctxt' ctxt o single)
    (case t' of Const (@{const-name Let},-) $ x $ f => (* x and f are already in
normal form *)
      if is-Free x orelse is-Bound x orelse is-Const x
      then SOME @{thm Let-def}
      else
        let
          val n = case f of (Abs (x, -, -)) => x | - => x;
          val cx = cterm-of thy x;
          val {T = xT, ...} = rep-cterm cx;
          val cf = cterm-of thy f;
          val fx-g = Simplifier.rewrite ss (Thm.capply cf cx);
          val (- $ - $ g) = prop-of fx-g;
          val g' = abstract-over (x,g);
        in (if (g aconv g')
            then
              let
                val rl =
                    cterm-instantiate [(f-Let-unfold, cf), (x-Let-unfold, cx)] @{thm
Let-unfold};
              in SOME (rl OF [fx-g]) end
              else if Term.betapply (f, x) aconv g then NONE (*avoid identity
conversion*)
            else let
                 val abs-g'= Abs (n,xT,g');
                 val g'x = abs-g'$x;
                 val g-g'x = Thm.symmetric (Thm.beta-conversion false (cterm-of
thy g'x));
```

```
                val rl = cterm-instantiate
                        [(f-Let-folded, cterm-of thy f), (x-Let-folded, cx),
                         (g-Let-folded, cterm-of thy abs-g')]
                        @{thm Let-folded};
              in SOME (rl OF [Thm.transitive fx-g g-g'x])
              end)
      end
  | - => NONE)
 end
end ⟩⟩
```

lemma *True-implies-equals*: $(True \Longrightarrow PROP\ P) \equiv PROP\ P$
proof
 assume $True \Longrightarrow PROP\ P$
 from *this* [*OF TrueI*] **show** $PROP\ P$ **.**
next
 assume $PROP\ P$
 then show $PROP\ P$ **.**
qed

lemma *ex-simps*:
 $!!P\ Q.\ (EX\ x.\ P\ x\ \&\ Q) = ((EX\ x.\ P\ x)\ \&\ Q)$
 $!!P\ Q.\ (EX\ x.\ P\ \&\ Q\ x) = (P\ \&\ (EX\ x.\ Q\ x))$
 $!!P\ Q.\ (EX\ x.\ P\ x \mid Q) = ((EX\ x.\ P\ x) \mid Q)$
 $!!P\ Q.\ (EX\ x.\ P \mid Q\ x) = (P \mid (EX\ x.\ Q\ x))$
 $!!P\ Q.\ (EX\ x.\ P\ x \longrightarrow Q) = ((ALL\ x.\ P\ x) \longrightarrow Q)$
 $!!P\ Q.\ (EX\ x.\ P \longrightarrow Q\ x) = (P \longrightarrow (EX\ x.\ Q\ x))$
 — Miniscoping: pushing in existential quantifiers.
 by (*iprover* | *blast*)+

lemma *all-simps*:
 $!!P\ Q.\ (ALL\ x.\ P\ x\ \&\ Q) = ((ALL\ x.\ P\ x)\ \&\ Q)$
 $!!P\ Q.\ (ALL\ x.\ P\ \&\ Q\ x) = (P\ \&\ (ALL\ x.\ Q\ x))$
 $!!P\ Q.\ (ALL\ x.\ P\ x \mid Q) = ((ALL\ x.\ P\ x) \mid Q)$
 $!!P\ Q.\ (ALL\ x.\ P \mid Q\ x) = (P \mid (ALL\ x.\ Q\ x))$
 $!!P\ Q.\ (ALL\ x.\ P\ x \longrightarrow Q) = ((EX\ x.\ P\ x) \longrightarrow Q)$
 $!!P\ Q.\ (ALL\ x.\ P \longrightarrow Q\ x) = (P \longrightarrow (ALL\ x.\ Q\ x))$
 — Miniscoping: pushing in universal quantifiers.
 by (*iprover* | *blast*)+

lemmas [*simp*] =
 triv-forall-equality
 True-implies-equals
 if-True
 if-False
 if-cancel
 if-eq-cancel
 imp-disjL

conj-assoc
disj-assoc
de-Morgan-conj
de-Morgan-disj
imp-disj1
imp-disj2
not-imp
disj-not1
not-all
not-ex
cases-simp
the-eq-trivial
the-sym-eq-trivial
ex-simps
all-simps
simp-thms

lemmas [*cong*] = *imp-cong simp-implies-cong*
lemmas [*split*] = *split-if*

ML ⟨⟨ *val HOL-ss* = @{*simpset*} ⟩⟩

Simplifies x assuming c and y assuming c

lemma *if-cong*:
assumes $b = c$
and $c \Longrightarrow x = u$
and $\neg\ c \Longrightarrow y = v$
shows (*if b then x else y*) = (*if c then u else v*)
unfolding *if-def* **using** *assms* **by** *simp*

Prevents simplification of x and y: faster and allows the execution of functional programs.

lemma *if-weak-cong* [*cong*]:
assumes $b = c$
shows (*if b then x else y*) = (*if c then x else y*)
using *assms* **by** (*rule arg-cong*)

Prevents simplification of t: much faster

lemma *let-weak-cong*:
assumes $a = b$
shows (*let x* = *a in t x*) = (*let x* = *b in t x*)
using *assms* **by** (*rule arg-cong*)

To tidy up the result of a simproc. Only the RHS will be simplified.

lemma *eq-cong2*:
assumes $u = u'$
shows $(t \equiv u) \equiv (t \equiv u')$
using *assms* **by** *simp*

lemma *if-distrib*:
f (*if c then x else y*) = (*if c then f x else f y*)
by *simp*

This lemma restricts the effect of the rewrite rule u=v to the left-hand side of an equality. Used in {*Integ*,*Real*}/*simproc.ML*

lemma *restrict-to-left*:
assumes $x = y$
shows $(x = z) = (y = z)$
using *assms* **by** *simp*

2.3.4 Generic cases and induction

Rule projections:

ML ⟪
structure Project-Rule = Project-Rule
(
val conjunct1 = @{*thm conjunct1*}
val conjunct2 = @{*thm conjunct2*}
val mp = @{*thm mp*}
)
⟫

definition *induct-forall* **where**
induct-forall P == ∀ *x*. *P x*

definition *induct-implies* **where**
induct-implies A B == *A* ⟶ *B*

definition *induct-equal* **where**
induct-equal x y == *x* = *y*

definition *induct-conj* **where**
induct-conj A B == *A* ∧ *B*

definition *induct-true* **where**
induct-true == *True*

definition *induct-false* **where**
induct-false == *False*

lemma *induct-forall-eq*: (!!*x*. *P x*) == *Trueprop* (*induct-forall* (λ*x*. *P x*))
by (*unfold atomize-all induct-forall-def*)

lemma *induct-implies-eq*: (*A* ==> *B*) == *Trueprop* (*induct-implies A B*)
by (*unfold atomize-imp induct-implies-def*)

lemma *induct-equal-eq*: $(x == y) == \mathit{Trueprop}\ (\mathit{induct\text{-}equal}\ x\ y)$
by (*unfold atomize-eq induct-equal-def*)

lemma *induct-conj-eq*: $(A\ \&\&\&\ B) == \mathit{Trueprop}\ (\mathit{induct\text{-}conj}\ A\ B)$
by (*unfold atomize-conj induct-conj-def*)

lemmas *induct-atomize'* = *induct-forall-eq induct-implies-eq induct-conj-eq*
lemmas *induct-atomize* = *induct-atomize' induct-equal-eq*
lemmas *induct-rulify'* [*symmetric*, *standard*] = *induct-atomize'*
lemmas *induct-rulify* [*symmetric*, *standard*] = *induct-atomize*
lemmas *induct-rulify-fallback* =
induct-forall-def induct-implies-def induct-equal-def induct-conj-def
induct-true-def induct-false-def

lemma *induct-forall-conj*: $\mathit{induct\text{-}forall}\ (\lambda x.\ \mathit{induct\text{-}conj}\ (A\ x)\ (B\ x)) =$
$\mathit{induct\text{-}conj}\ (\mathit{induct\text{-}forall}\ A)\ (\mathit{induct\text{-}forall}\ B)$
by (*unfold induct-forall-def induct-conj-def*) *iprover*

lemma *induct-implies-conj*: $\mathit{induct\text{-}implies}\ C\ (\mathit{induct\text{-}conj}\ A\ B) =$
$\mathit{induct\text{-}conj}\ (\mathit{induct\text{-}implies}\ C\ A)\ (\mathit{induct\text{-}implies}\ C\ B)$
by (*unfold induct-implies-def induct-conj-def*) *iprover*

lemma *induct-conj-curry*: $(\mathit{induct\text{-}conj}\ A\ B \Longrightarrow \mathit{PROP}\ C) == (A \Longrightarrow B \Longrightarrow \mathit{PROP}\ C)$
proof
assume *r*: $\mathit{induct\text{-}conj}\ A\ B \Longrightarrow \mathit{PROP}\ C$ **and** $A\ B$
show $\mathit{PROP}\ C$ **by** (*rule r*) (*simp add*: *induct-conj-def* ‹*A*› ‹*B*›)
next
assume *r*: $A \Longrightarrow B \Longrightarrow \mathit{PROP}\ C$ **and** $\mathit{induct\text{-}conj}\ A\ B$
show $\mathit{PROP}\ C$ **by** (*rule r*) (*simp-all add*: ‹$\mathit{induct\text{-}conj}\ A\ B$› [*unfolded induct-conj-def*])
qed

lemmas *induct-conj* = *induct-forall-conj induct-implies-conj induct-conj-curry*

lemma *induct-trueI*: *induct-true*
by (*simp add*: *induct-true-def*)

Method setup.

ML ⟪
structure Induct = *Induct*
(
val cases-default = @{*thm case-split*}
val atomize = @{*thms induct-atomize*}
val rulify = @{*thms induct-rulify'*}
val rulify-fallback = @{*thms induct-rulify-fallback*}
val equal-def = @{*thm induct-equal-def*}
fun dest-def (*Const* (@{*const-name induct-equal*}, -) $ *t* $ *u*) = *SOME* (*t*, *u*)
| *dest-def* - = *NONE*

```
  val trivial-tac = match-tac @{thms induct-trueI}
)
⟫

setup ⟪
  Induct.setup #>
  Context.theory-map (Induct.map-simpset (fn ss => ss
    setmksimps (fn ss => Simpdata.mksimps Simpdata.mksimps-pairs ss #>
      map (Simplifier.rewrite-rule (map Thm.symmetric
        @{thms induct-rulify-fallback}))))
    addsimprocs
      [Simplifier.simproc @{theory} swap-induct-false
        [induct-false ==> PROP P ==> PROP Q]
        (fn - => fn - =>
          (fn - $ (P as - $ @{const induct-false}) $ (- $ Q $ -) =>
              if P <> Q then SOME Drule.swap-prems-eq else NONE
            | - => NONE)),
       Simplifier.simproc @{theory} induct-equal-conj-curry
        [induct-conj P Q ==> PROP R]
        (fn - => fn - =>
          (fn - $ (- $ P) $ - =>
              let
                fun is-conj (@{const induct-conj} $ P $ Q) =
                      is-conj P andalso is-conj Q
                  | is-conj (Const (@{const-name induct-equal}, -) $ - $ -) = true
                  | is-conj @{const induct-true} = true
                  | is-conj @{const induct-false} = true
                  | is-conj - = false
              in if is-conj P then SOME @{thm induct-conj-curry} else NONE end
            | - => NONE))]))
⟫
```

Pre-simplification of induction and cases rules

```
lemma [induct-simp]: (!!x. induct-equal x t ==> PROP P x) == PROP P t
  unfolding induct-equal-def
proof
  assume R: !!x. x = t ==> PROP P x
  show PROP P t by (rule R [OF refl])
next
  fix x assume PROP P t x = t
  then show PROP P x by simp
qed

lemma [induct-simp]: (!!x. induct-equal t x ==> PROP P x) == PROP P t
  unfolding induct-equal-def
proof
  assume R: !!x. t = x ==> PROP P x
  show PROP P t by (rule R [OF refl])
next
```

fix x **assume** *PROP P t t* = x
then show *PROP P x* **by** *simp*
qed

lemma [*induct-simp*]: (*induct-false* ==> *P*) == *Trueprop induct-true*
unfolding *induct-false-def induct-true-def*
by (*iprover intro*: *equal-intr-rule*)

lemma [*induct-simp*]: (*induct-true* ==> *PROP P*) == *PROP P*
unfolding *induct-true-def*
proof
assume *R*: *True* ⟹ *PROP P*
from *TrueI* **show** *PROP P* **by** (*rule R*)
next
assume *PROP P*
then show *PROP P* **.**
qed

lemma [*induct-simp*]: (*PROP P* ==> *induct-true*) == *Trueprop induct-true*
unfolding *induct-true-def*
by (*iprover intro*: *equal-intr-rule*)

lemma [*induct-simp*]: (!!*x*. *induct-true*) == *Trueprop induct-true*
unfolding *induct-true-def*
by (*iprover intro*: *equal-intr-rule*)

lemma [*induct-simp*]: *induct-implies induct-true P* == *P*
by (*simp add*: *induct-implies-def induct-true-def*)

lemma [*induct-simp*]: (*x* = *x*) = *True*
by (*rule simp-thms*)

hide-const *induct-forall induct-implies induct-equal induct-conj induct-true induct-false*

use ~~/*src*/*Tools*/*induct-tacs.ML*
setup *InductTacs.setup*

2.3.5 Coherent logic

ML ⟪
structure Coherent = *Coherent*
(
val atomize-elimL = @{*thm atomize-elimL*}
val atomize-exL = @{*thm atomize-exL*}
val atomize-conjL = @{*thm atomize-conjL*}
val atomize-disjL = @{*thm atomize-disjL*}
val operator-names =
[@{*const-name op* |}, @{*const-name op* &}, @{*const-name Ex*}]
);

⟩⟩

setup *Coherent.setup*

2.3.6 Reorienting equalities

ML ⟨⟨
```
signature REORIENT-PROC =
sig
  val add : (term -> bool) -> theory -> theory
  val proc : morphism -> simpset -> cterm -> thm option
end;

structure Reorient-Proc : REORIENT-PROC =
struct
  structure Data = Theory-Data
  (
    type T = ((term -> bool) * stamp) list;
    val empty = [];
    val extend = I;
    fun merge data : T = Library.merge (eq-snd op =) data;
  );
  fun add m = Data.map (cons (m, stamp ()));
  fun matches thy t = exists (fn (m, -) => m t) (Data.get thy);

  val meta-reorient = @{thm eq-commute [THEN eq-reflection]};
  fun proc phi ss ct =
    let
      val ctxt = Simplifier.the-context ss;
      val thy = ProofContext.theory-of ctxt;
    in
      case Thm.term-of ct of
        (- $ t $ u) => if matches thy u then NONE else SOME meta-reorient
      | - => NONE
    end;
end;
```
⟩⟩

2.4 Other simple lemmas and lemma duplicates

lemma *ex1-eq* [*iff*]: *EX*! *x*. *x* = *t* *EX*! *x*. *t* = *x*
by *blast*+

lemma *choice-eq*: (*ALL x*. *EX*! *y*. *P x y*) = (*EX*! *f*. *ALL x*. *P x* (*f x*))
apply (*rule iffI*)
apply (*rule-tac a* = %*x*. *THE y*. *P x y* **in** *ex1I*)
apply (*fast dest*!: *theI'*)
apply (*fast intro*: *ext the1-equality* [*symmetric*])
apply (*erule ex1E*)
apply (*rule allI*)

apply (*rule ex1I*)
apply (*erule spec*)
apply (*erule-tac x = %z. if z = x then y else f z* **in** *allE*)
apply (*erule impE*)
apply (*rule allI*)
apply (*case-tac xa = x*)
apply (*drule-tac [3] x = x* **in** *fun-cong*, *simp-all*)
done

lemmas *eq-sym-conv = eq-commute*

lemma *nnf-simps*:
(¬(*P* ∧ *Q*)) = (¬ *P* ∨ ¬ *Q*) (¬ (*P* ∨ *Q*)) = (¬ *P* ∧ ¬*Q*) (*P* ⟶ *Q*) = (¬*P* ∨ *Q*)
(*P* = *Q*) = ((*P* ∧ *Q*) ∨ (¬*P* ∧ ¬ *Q*)) (¬(*P* = *Q*)) = ((*P* ∧ ¬ *Q*) ∨ (¬*P* ∧ *Q*))
(¬ ¬(*P*)) = *P*
by *blast+*

2.5 Basic ML bindings

ML ⟪
val FalseE = @{thm FalseE}
val Let-def = @{thm Let-def}
val TrueI = @{thm TrueI}
val allE = @{thm allE}
val allI = @{thm allI}
val all-dupE = @{thm all-dupE}
val arg-cong = @{thm arg-cong}
val box-equals = @{thm box-equals}
val ccontr = @{thm ccontr}
val classical = @{thm classical}
val conjE = @{thm conjE}
val conjI = @{thm conjI}
val conjunct1 = @{thm conjunct1}
val conjunct2 = @{thm conjunct2}
val disjCI = @{thm disjCI}
val disjE = @{thm disjE}
val disjI1 = @{thm disjI1}
val disjI2 = @{thm disjI2}
val eq-reflection = @{thm eq-reflection}
val ex1E = @{thm ex1E}
val ex1I = @{thm ex1I}
val ex1-implies-ex = @{thm ex1-implies-ex}
val exE = @{thm exE}
val exI = @{thm exI}
val excluded-middle = @{thm excluded-middle}
val ext = @{thm ext}
val fun-cong = @{thm fun-cong}
val iffD1 = @{thm iffD1}

val iffD2 = @{thm iffD2}
val iffI = @{thm iffI}
val impE = @{thm impE}
val impI = @{thm impI}
val meta-eq-to-obj-eq = @{thm meta-eq-to-obj-eq}
val mp = @{thm mp}
val notE = @{thm notE}
val notI = @{thm notI}
val not-all = @{thm not-all}
val not-ex = @{thm not-ex}
val not-iff = @{thm not-iff}
val not-not = @{thm not-not}
val not-sym = @{thm not-sym}
val refl = @{thm refl}
val rev-mp = @{thm rev-mp}
val spec = @{thm spec}
val ssubst = @{thm ssubst}
val subst = @{thm subst}
val sym = @{thm sym}
val trans = @{thm trans}
⟩⟩

2.6 Code generator setup

2.6.1 SML code generator setup

use *Tools/recfun-codegen.ML*

setup ⟨⟨
Codegen.setup
#> RecfunCodegen.setup
#> Codegen.map-unfold (K HOL-basic-ss)
⟩⟩

types-code
bool (bool)
attach (*term-of*) ⟨⟨
fun term-of-bool b = if b then HOLogic.true-const else HOLogic.false-const;
⟩⟩
attach (*test*) ⟨⟨
fun gen-bool i =
let val b = one-of [false, true]
in (b, fn () => term-of-bool b) end;
⟩⟩
prop (bool)
attach (*term-of*) ⟨⟨
fun term-of-prop b =
HOLogic.mk-Trueprop (if b then HOLogic.true-const else HOLogic.false-const);
⟩⟩

consts-code
Trueprop ((-))
True (*true*)
False (*false*)
Not (*Bool.not*)
op | ((- *orelse*/ -))
op & ((- *andalso*/ -))
If ((*if* -/ *then* -/ *else* -))

setup ⟪
let

fun eq-codegen thy defs dep thyname b t gr =
(*case strip-comb t of*
(*Const* (@{*const-name op* =}, *Type* (-, [*Type* (*fun*, -), -])), -) => *NONE*
| (*Const* (@{*const-name op* =}, -), [*t*, *u*]) =>
let
val (*pt*, *gr'*) = *Codegen.invoke-codegen thy defs dep thyname false t gr*;
val (*pu*, *gr''*) = *Codegen.invoke-codegen thy defs dep thyname false u gr'*;
val (-, *gr'''*) = *Codegen.invoke-tycodegen thy defs dep thyname false HOLogic.boolT gr''*;
in
SOME (*Codegen.parens*
(*Pretty.block* [*pt*, *Codegen.str* =, *Pretty.brk 1*, *pu*]), *gr'''*)
end
| (*t as Const* (@{*const-name op* =}, -), *ts*) => *SOME* (*Codegen.invoke-codegen thy defs dep thyname b* (*Codegen.eta-expand t ts 2*) *gr*)
| - => *NONE*);

in
Codegen.add-codegen eq-codegen eq-codegen
end
⟫

2.6.2 Generic code generator preprocessor setup

setup ⟪
Code-Preproc.map-pre (*K HOL-basic-ss*)
#> *Code-Preproc.map-post* (*K HOL-basic-ss*)
⟫

2.6.3 Equality

class *eq* =
fixes *eq* :: *'a* ⇒ *'a* ⇒ *bool*
assumes *eq-equals*: *eq x y* ⟷ *x* = *y*
begin

lemma *eq* [*code-unfold*, *code-inline del*]: *eq* = (*op* =)
by (*rule ext eq-equals*)+

lemma *eq-refl*: *eq x x* ⟷ *True*
unfolding *eq* **by** *rule+*

lemma *equals-eq*: (*op* =) ≡ *eq*
by (*rule eq-reflection*) (*rule ext*, *rule ext*, *rule sym*, *rule eq-equals*)

declare *equals-eq* [*symmetric*, *code-post*]

end

declare *equals-eq* [*code*]

setup ⟪
Code-Preproc.map-pre (*fn simpset* =>
simpset addsimprocs [*Simplifier.simproc-i* @{*theory*} *eq* [@{*term op* =}]
(*fn thy* => *fn* - => *fn t as Const* (-, *T*) => *case strip-type T*
of ((*T as Type* -) :: -, -) => *SOME* @{*thm equals-eq*}
| - => *NONE*)])
⟫

2.6.4 Generic code generator foundation

Datatypes

code-datatype *True False*

code-datatype *TYPE*('*a*::{})

code-datatype *prop Trueprop*

Code equations

lemma [*code*]:
shows (*True* ⟹ *PROP Q*) ≡ *PROP Q*
and (*PROP Q* ⟹ *True*) ≡ *Trueprop True*
and (*P* ⟹ *R*) ≡ *Trueprop* (*P* ⟶ *R*) **by** (*auto intro*!: *equal-intr-rule*)

lemma [*code*]:
shows *False* ∧ *P* ⟷ *False*
and *True* ∧ *P* ⟷ *P*
and *P* ∧ *False* ⟷ *False*
and *P* ∧ *True* ⟷ *P* **by** *simp-all*

lemma [*code*]:
shows *False* ∨ *P* ⟷ *P*
and *True* ∨ *P* ⟷ *True*
and *P* ∨ *False* ⟷ *P*
and *P* ∨ *True* ⟷ *True* **by** *simp-all*

lemma [*code*]:

shows $(False \longrightarrow P) \longleftrightarrow True$
and $(True \longrightarrow P) \longleftrightarrow P$
and $(P \longrightarrow False) \longleftrightarrow \neg P$
and $(P \longrightarrow True) \longleftrightarrow True$ **by** *simp-all*

instantiation *itself* :: (*type*) *eq*
begin

definition *eq-itself* :: $'a$ *itself* $\Rightarrow$ $'a$ *itself* $\Rightarrow$ *bool* **where**
eq-itself x y $\longleftrightarrow$ $x = y$

instance proof
qed (*fact eq-itself-def*)

end

lemma *eq-itself-code* [*code*]:
eq-class.eq TYPE($'a$) *TYPE*($'a$) $\longleftrightarrow$ *True*
by (*simp add*: *eq*)

Equality

declare *simp-thms*(6) [*code nbe*]

setup ⟪
Sign.add-const-constraint (@{*const-name eq*}, *SOME* @{*typ* $'a$::*type* $\Rightarrow$ $'a$ $\Rightarrow$ *bool*})
⟫

lemma *equals-alias-cert*: *OFCLASS*($'a$, *eq-class*) $\equiv$ ((*op* = :: $'a$ $\Rightarrow$ $'a$ $\Rightarrow$ *bool*) $\equiv$ *eq*) (**is** *?ofclass* $\equiv$ *?eq*)
proof
assume *PROP ?ofclass*
show *PROP ?eq*
by (*tactic* ⟪ *ALLGOALS* (*rtac* (*Thm.unconstrainT* @{*thm equals-eq*})) ⟫)
(*fact* ‹*PROP ?ofclass*›)
next
assume *PROP ?eq*
show *PROP ?ofclass* **proof**
qed (*simp add*: ‹*PROP ?eq*›)
qed

setup ⟪
Sign.add-const-constraint (@{*const-name eq*}, *SOME* @{*typ* $'a$::*eq* $\Rightarrow$ $'a$ $\Rightarrow$ *bool*})
⟫

setup ⟪
Nbe.add-const-alias @{*thm equals-alias-cert*}
⟫

hide-const (**open**) *eq*

Cases

lemma *Let-case-cert*:
  assumes *CASE* ≡ (λ*x*. *Let x f*)
  shows *CASE x* ≡ *f x*
  using *assms* **by** *simp-all*

lemma *If-case-cert*:
  assumes *CASE* ≡ (λ*b*. *If b f g*)
  shows (*CASE True* ≡ *f*) &&& (*CASE False* ≡ *g*)
  using *assms* **by** *simp-all*

setup ⟪
  Code.add-case @{*thm Let-case-cert*}
  #> *Code.add-case* @{*thm If-case-cert*}
  #> *Code.add-undefined* @{*const-name undefined*}
⟫

code-abort *undefined*

2.6.5 Generic code generator target languages

type bool

code-type *bool*
  (*SML bool*)
  (*OCaml bool*)
  (*Haskell Bool*)
  (*Scala Boolean*)

code-const *True* **and** *False* **and** *Not* **and** *op &* **and** *op |* **and** *If*
  (*SML true* **and** *false* **and** *not*
    and infixl *1 andalso* **and infixl** *0 orelse*
    and !(*if* (-)/ *then* (-)/ *else* (-)))
  (*OCaml true* **and** *false* **and** *not*
    and infixl *4* && **and infixl** *2* ||
    and !(*if* (-)/ *then* (-)/ *else* (-)))
  (*Haskell True* **and** *False* **and** *not*
    and infixl *3* && **and infixl** *2* ||
    and !(*if* (-)/ *then* (-)/ *else* (-)))
  (*Scala true* **and** *false* **and** *'*! -
    and infixl *3* && **and infixl** *1* ||
    and !(*if* ((-))/ (-)/ *else* (-)))

code-reserved *SML*
  bool true false not

code-reserved *OCaml*
  bool not

code-reserved *Scala*
Boolean

using built-in Haskell equality

code-class *eq*
(*Haskell Eq*)

code-const *eq-class.eq*
(*Haskell* **infixl** *4* ==)

code-const *op* =
(*Haskell* **infixl** *4* ==)

undefined

code-const *undefined*
(*SML* !(*raise*/ *Fail*/ *undefined*))
(*OCaml failwith*/ *undefined*)
(*Haskell error*/ *undefined*)
(*Scala* !*error*(*undefined*))

2.6.6 Evaluation and normalization by evaluation

Avoid some named infixes in evaluation environment

code-reserved *Eval oo ooo oooo upto downto orf andf*

setup ⟪
Value.add-evaluator (*SML*, *Codegen.eval-term o ProofContext.theory-of*)
⟫

ML ⟪
structure Eval-Method =
struct

val eval-ref : (*unit* −> *bool*) *option Unsynchronized.ref* = *Unsynchronized.ref NONE*;

end;
⟫

oracle *eval-oracle* = ⟪ *fn ct* =>
let
val thy = *Thm.theory-of-cterm ct*;
val t = *Thm.term-of ct*;
val dummy = @{*cprop True*};
in case try HOLogic.dest-Trueprop t
of SOME t' => *if Code-Eval.eval NONE*
(*Eval-Method.eval-ref*, *Eval-Method.eval-ref*) (*K I*) *thy t'* []

```
      then Thm.capply (Thm.capply @{cterm op ≡ :: prop ⇒ prop ⇒ prop} ct)
dummy
      else dummy
  | NONE => dummy
 end
⟩⟩

ML ⟨⟨
fun gen-eval-method conv ctxt = SIMPLE-METHOD'
  (CONVERSION (Conv.params-conv (~1) (K (Conv.concl-conv (~1) conv))
ctxt)
   THEN' rtac TrueI)
⟩⟩

method-setup eval = ⟨⟨ Scan.succeed (gen-eval-method eval-oracle) ⟩⟩
 solve goal by evaluation

method-setup evaluation = ⟨⟨ Scan.succeed (gen-eval-method Codegen.evaluation-conv)
⟩⟩
 solve goal by evaluation

method-setup normalization = ⟨⟨
 Scan.succeed (K (SIMPLE-METHOD' (CONVERSION Nbe.norm-conv THEN'
(fn k => TRY (rtac TrueI k)))))
⟩⟩ solve goal by normalization
```

2.7 Counterexample Search Units

2.7.1 Quickcheck

quickcheck-params [*size* = 5, *iterations* = 50]

2.7.2 Nitpick setup

```
ML ⟨⟨
structure Nitpick-Defs = Named-Thms
(
 val name = nitpick-def
 val description = alternative definitions of constants as needed by Nitpick
)
structure Nitpick-Simps = Named-Thms
(
 val name = nitpick-simp
 val description = equational specification of constants as needed by Nitpick
)
structure Nitpick-Psimps = Named-Thms
(
 val name = nitpick-psimp
 val description = partial equational specification of constants as needed by Nitpick
)
```

```
structure Nitpick-Choice-Specs = Named-Thms
(
  val name = nitpick-choice-spec
  val description = choice specification of constants as needed by Nitpick
)
⟩⟩

setup ⟨⟨
  Nitpick-Defs.setup
  #> Nitpick-Simps.setup
  #> Nitpick-Psimps.setup
  #> Nitpick-Choice-Specs.setup
⟩⟩
```

2.8 Preprocessing for the predicate compiler

```
ML ⟨⟨
structure Predicate-Compile-Alternative-Defs = Named-Thms
(
  val name = code-pred-def
  val description = alternative definitions of constants for the Predicate Compiler
)
structure Predicate-Compile-Inline-Defs = Named-Thms
(
  val name = code-pred-inline
  val description = inlining definitions for the Predicate Compiler
)
structure Predicate-Compile-Simps = Named-Thms
(
  val name = code-pred-simp
  val description = simplification rules for the optimisations in the Predicate Compiler
)
⟩⟩

setup ⟨⟨
  Predicate-Compile-Alternative-Defs.setup
  #> Predicate-Compile-Inline-Defs.setup
  #> Predicate-Compile-Simps.setup
⟩⟩
```

2.9 Legacy tactics and ML bindings

```
ML ⟨⟨
fun strip-tac i = REPEAT (resolve-tac [impI, allI] i);

(* combination of (spec RS spec RS ...(j times) ... spec RS mp) *)
local
  fun wrong-prem (Const (@{const-name All}, -) $ Abs (-, -, t)) = wrong-prem t
    | wrong-prem (Bound -) = true
    | wrong-prem - = false;
```

```
  val filter-right = filter (not o wrong-prem o HOLogic.dest-Trueprop o hd o Thm.prems-of);
in
  fun smp i = funpow i (fn m => filter-right ([spec] RL m)) ([mp]);
  fun smp-tac j = EVERY'[dresolve-tac (smp j), atac];
end;

val all-conj-distrib = thm all-conj-distrib;
val all-simps = thms all-simps;
val atomize-not = thm atomize-not;
val case-split = thm case-split;
val cases-simp = thm cases-simp;
val choice-eq = thm choice-eq
val cong = thm cong
val conj-comms = thms conj-comms;
val conj-cong = thm conj-cong;
val de-Morgan-conj = thm de-Morgan-conj;
val de-Morgan-disj = thm de-Morgan-disj;
val disj-assoc = thm disj-assoc;
val disj-comms = thms disj-comms;
val disj-cong = thm disj-cong;
val eq-ac = thms eq-ac;
val eq-cong2 = thm eq-cong2
val Eq-FalseI = thm Eq-FalseI;
val Eq-TrueI = thm Eq-TrueI;
val Ex1-def = thm Ex1-def
val ex-disj-distrib = thm ex-disj-distrib;
val ex-simps = thms ex-simps;
val if-cancel = thm if-cancel;
val if-eq-cancel = thm if-eq-cancel;
val if-False = thm if-False;
val iff-conv-conj-imp = thm iff-conv-conj-imp;
val iff = thm iff
val if-splits = thms if-splits;
val if-True = thm if-True;
val if-weak-cong = thm if-weak-cong
val imp-all = thm imp-all;
val imp-cong = thm imp-cong;
val imp-conjL = thm imp-conjL;
val imp-conjR = thm imp-conjR;
val imp-conv-disj = thm imp-conv-disj;
val simp-implies-def = thm simp-implies-def;
val simp-thms = thms simp-thms;
val split-if = thm split-if;
val the1-equality = thm the1-equality
val theI = thm theI
val theI' = thm theI'
val True-implies-equals = thm True-implies-equals;
val nnf-conv = Simplifier.rewrite (HOL-basic-ss addsimps simp-thms @ @{thms
nnf-simps})
```

⟩⟩

end

3 Orderings: Abstract orderings

theory *Orderings*
imports *HOL*
uses
~~/*src*/*Provers*/*order.ML*
~~/*src*/*Provers*/*quasi.ML*
begin

3.1 Syntactic orders

class *ord* =
fixes *less-eq* :: *'a* ⇒ *'a* ⇒ *bool*
and *less* :: *'a* ⇒ *'a* ⇒ *bool*
begin

notation
less-eq (*op* <=) **and**
less-eq ((-/ <= -) [*51*, *51*] *50*) **and**
less (*op* <) **and**
less ((-/ < -) [*51*, *51*] *50*)

notation (*xsymbols*)
less-eq (*op* ≤) **and**
less-eq ((-/ ≤ -) [*51*, *51*] *50*)

notation (*HTML* **output**)
less-eq (*op* ≤) **and**
less-eq ((-/ ≤ -) [*51*, *51*] *50*)

abbreviation (*input*)
greater-eq (**infix** >= *50*) **where**
x >= *y* ≡ *y* <= *x*

notation (*input*)
greater-eq (**infix** ≥ *50*)

abbreviation (*input*)
greater (**infix** > *50*) **where**
x > *y* ≡ *y* < *x*

end

3.2 Quasi orders

class *preorder* = *ord* +
  assumes *less-le-not-le*: $x < y \longleftrightarrow x \leq y \wedge \neg (y \leq x)$
  and *order-refl* [*iff*]: $x \leq x$
  and *order-trans*: $x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
begin

Reflexivity.

lemma *eq-refl*: $x = y \Longrightarrow x \leq y$
  — This form is useful with the classical reasoner.
by (*erule ssubst*) (*rule order-refl*)

lemma *less-irrefl* [*iff*]: $\neg\ x < x$
by (*simp add*: *less-le-not-le*)

lemma *less-imp-le*: $x < y \Longrightarrow x \leq y$
unfolding *less-le-not-le* **by** *blast*

Asymmetry.

lemma *less-not-sym*: $x < y \Longrightarrow \neg (y < x)$
by (*simp add*: *less-le-not-le*)

lemma *less-asym*: $x < y \Longrightarrow (\neg\ P \Longrightarrow y < x) \Longrightarrow P$
by (*drule less-not-sym*, *erule contrapos-np*) *simp*

Transitivity.

lemma *less-trans*: $x < y \Longrightarrow y < z \Longrightarrow x < z$
by (*auto simp add*: *less-le-not-le intro*: *order-trans*)

lemma *le-less-trans*: $x \leq y \Longrightarrow y < z \Longrightarrow x < z$
by (*auto simp add*: *less-le-not-le intro*: *order-trans*)

lemma *less-le-trans*: $x < y \Longrightarrow y \leq z \Longrightarrow x < z$
by (*auto simp add*: *less-le-not-le intro*: *order-trans*)

Useful for simplification, but too risky to include by default.

lemma *less-imp-not-less*: $x < y \Longrightarrow (\neg\ y < x) \longleftrightarrow \mathit{True}$
by (*blast elim*: *less-asym*)

lemma *less-imp-triv*: $x < y \Longrightarrow (y < x \longrightarrow P) \longleftrightarrow \mathit{True}$
by (*blast elim*: *less-asym*)

Transitivity rules for calculational reasoning

lemma *less-asym'*: $a < b \Longrightarrow b < a \Longrightarrow P$
by (*rule less-asym*)

Dual order

lemma *dual-preorder*:
class.preorder $(op \geq)$ $(op >)$
proof qed (*auto simp add*: *less-le-not-le intro*: *order-trans*)

end

3.3 Partial orders

class *order* = *preorder* +
assumes *antisym*: $x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
begin

Reflexivity.

lemma *less-le*: $x < y \longleftrightarrow x \leq y \wedge x \neq y$
by (*auto simp add*: *less-le-not-le intro*: *antisym*)

lemma *le-less*: $x \leq y \longleftrightarrow x < y \vee x = y$
— NOT suitable for iff, since it can cause PROOF FAILED.
by (*simp add*: *less-le*) *blast*

lemma *le-imp-less-or-eq*: $x \leq y \Longrightarrow x < y \vee x = y$
unfolding *less-le* **by** *blast*

Useful for simplification, but too risky to include by default.

lemma *less-imp-not-eq*: $x < y \Longrightarrow (x = y) \longleftrightarrow False$
by *auto*

lemma *less-imp-not-eq2*: $x < y \Longrightarrow (y = x) \longleftrightarrow False$
by *auto*

Transitivity rules for calculational reasoning

lemma *neq-le-trans*: $a \neq b \Longrightarrow a \leq b \Longrightarrow a < b$
by (*simp add*: *less-le*)

lemma *le-neq-trans*: $a \leq b \Longrightarrow a \neq b \Longrightarrow a < b$
by (*simp add*: *less-le*)

Asymmetry.

lemma *eq-iff*: $x = y \longleftrightarrow x \leq y \wedge y \leq x$
by (*blast intro*: *antisym*)

lemma *antisym-conv*: $y \leq x \Longrightarrow x \leq y \longleftrightarrow x = y$
by (*blast intro*: *antisym*)

lemma *less-imp-neq*: $x < y \Longrightarrow x \neq y$
by (*erule contrapos-pn*, *erule subst*, *rule less-irrefl*)

Least value operator

definition (**in** *ord*)
  Least :: (*'a* ⇒ *bool*) ⇒ *'a* (**binder** *LEAST* *10*) **where**
  Least P = (*THE x*. *P x* ∧ (∀ *y*. *P y* ⟶ *x* ≤ *y*))

lemma *Least-equality*:
  assumes *P x*
    and ⋀*y*. *P y* ⟹ *x* ≤ *y*
  shows *Least P* = *x*
unfolding *Least-def* **by** (*rule the-equality*)
  (*blast intro*: *assms antisym*)+

lemma *LeastI2-order*:
  assumes *P x*
    and ⋀*y*. *P y* ⟹ *x* ≤ *y*
    and ⋀*x*. *P x* ⟹ ∀ *y*. *P y* ⟶ *x* ≤ *y* ⟹ *Q x*
  shows *Q* (*Least P*)
unfolding *Least-def* **by** (*rule theI2*)
  (*blast intro*: *assms antisym*)+

Dual order

lemma *dual-order*:
  class.order (*op* ≥) (*op* >)
by (*intro-locales*, *rule dual-preorder*) (*unfold-locales*, *rule antisym*)

end

3.4 Linear (total) orders

class *linorder* = *order* +
  assumes *linear*: *x* ≤ *y* ∨ *y* ≤ *x*
begin

lemma *less-linear*: *x* < *y* ∨ *x* = *y* ∨ *y* < *x*
unfolding *less-le* **using** *less-le linear* **by** *blast*

lemma *le-less-linear*: *x* ≤ *y* ∨ *y* < *x*
by (*simp add*: *le-less less-linear*)

lemma *le-cases* [*case-names le ge*]:
  (*x* ≤ *y* ⟹ *P*) ⟹ (*y* ≤ *x* ⟹ *P*) ⟹ *P*
using *linear* **by** *blast*

lemma *linorder-cases* [*case-names less equal greater*]:
  (*x* < *y* ⟹ *P*) ⟹ (*x* = *y* ⟹ *P*) ⟹ (*y* < *x* ⟹ *P*) ⟹ *P*
using *less-linear* **by** *blast*

lemma *not-less*: ¬ *x* < *y* ⟷ *y* ≤ *x*
apply (*simp add*: *less-le*)
using *linear* **apply** (*blast intro*: *antisym*)

done

lemma *not-less-iff-gr-or-eq*:
$\neg(x < y) \longleftrightarrow (x > y \mid x = y)$
apply(*simp add*:*not-less le-less*)
apply *blast*
done

lemma *not-le*: $\neg\ x \leq y \longleftrightarrow y < x$
apply (*simp add*: *less-le*)
using *linear* **apply** (*blast intro*: *antisym*)
done

lemma *neq-iff*: $x \neq y \longleftrightarrow x < y \vee y < x$
by (*cut-tac* $x = x$ **and** $y = y$ **in** *less-linear*, *auto*)

lemma *neqE*: $x \neq y \Longrightarrow (x < y \Longrightarrow R) \Longrightarrow (y < x \Longrightarrow R) \Longrightarrow R$
by (*simp add*: *neq-iff*) *blast*

lemma *antisym-conv1*: $\neg\ x < y \Longrightarrow x \leq y \longleftrightarrow x = y$
by (*blast intro*: *antisym dest*: *not-less* [*THEN iffD1*])

lemma *antisym-conv2*: $x \leq y \Longrightarrow \neg\ x < y \longleftrightarrow x = y$
by (*blast intro*: *antisym dest*: *not-less* [*THEN iffD1*])

lemma *antisym-conv3*: $\neg\ y < x \Longrightarrow \neg\ x < y \longleftrightarrow x = y$
by (*blast intro*: *antisym dest*: *not-less* [*THEN iffD1*])

lemma *leI*: $\neg\ x < y \Longrightarrow y \leq x$
unfolding *not-less* **.**

lemma *leD*: $y \leq x \Longrightarrow \neg\ x < y$
unfolding *not-less* **.**

lemma *not-leE*: $\neg\ y \leq x \Longrightarrow x < y$
unfolding *not-le* **.**

Dual order

lemma *dual-linorder*:
class.linorder (*op* $\geq$) (*op* $>$)
by (*rule class.linorder.intro*, *rule dual-order*) (*unfold-locales*, *rule linear*)

min/max

definition (**in** *ord*) *min* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
[*code del*]: *min a b* = (*if a* $\leq$ *b then a else b*)

definition (**in** *ord*) *max* :: $'a \Rightarrow 'a \Rightarrow 'a$ **where**
[*code del*]: *max a b* = (*if a* $\leq$ *b then b else a*)

lemma *min-le-iff-disj*:
$min\ x\ y \leq z \longleftrightarrow x \leq z \vee y \leq z$
unfolding *min-def* **using** *linear* **by** (*auto intro*: *order-trans*)

lemma *le-max-iff-disj*:
$z \leq max\ x\ y \longleftrightarrow z \leq x \vee z \leq y$
unfolding *max-def* **using** *linear* **by** (*auto intro*: *order-trans*)

lemma *min-less-iff-disj*:
$min\ x\ y < z \longleftrightarrow x < z \vee y < z$
unfolding *min-def le-less* **using** *less-linear* **by** (*auto intro*: *less-trans*)

lemma *less-max-iff-disj*:
$z < max\ x\ y \longleftrightarrow z < x \vee z < y$
unfolding *max-def le-less* **using** *less-linear* **by** (*auto intro*: *less-trans*)

lemma *min-less-iff-conj* [*simp*]:
$z < min\ x\ y \longleftrightarrow z < x \wedge z < y$
unfolding *min-def le-less* **using** *less-linear* **by** (*auto intro*: *less-trans*)

lemma *max-less-iff-conj* [*simp*]:
$max\ x\ y < z \longleftrightarrow x < z \wedge y < z$
unfolding *max-def le-less* **using** *less-linear* **by** (*auto intro*: *less-trans*)

lemma *split-min* [*no-atp*]:
$P\ (min\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ i) \wedge (\neg\ i \leq j \longrightarrow P\ j)$
by (*simp add*: *min-def*)

lemma *split-max* [*no-atp*]:
$P\ (max\ i\ j) \longleftrightarrow (i \leq j \longrightarrow P\ j) \wedge (\neg\ i \leq j \longrightarrow P\ i)$
by (*simp add*: *max-def*)

end

Explicit dictionaries for code generation

lemma *min-ord-min* [*code*, *code-unfold*, *code-inline del*]:
$min = ord.min\ (op \leq)$
by (*rule ext*)+ (*simp add*: *min-def ord.min-def*)

declare *ord.min-def* [*code*]

lemma *max-ord-max* [*code*, *code-unfold*, *code-inline del*]:
$max = ord.max\ (op \leq)$
by (*rule ext*)+ (*simp add*: *max-def ord.max-def*)

declare *ord.max-def* [*code*]

3.5 Reasoning tools setup

ML ⟪

```
signature ORDERS =
sig
  val print-structures: Proof.context -> unit
  val setup: theory -> theory
  val order-tac: Proof.context -> thm list -> int -> tactic
end;

structure Orders: ORDERS =
struct

(** Theory and context data **)

fun struct-eq ((s1: string, ts1), (s2, ts2)) =
  (s1 = s2) andalso eq-list (op aconv) (ts1, ts2);

structure Data = Generic-Data
(
  type T = ((string * term list) * Order-Tac.less-arith) list;
    (* Order structures:
       identifier of the structure, list of operations and record of theorems
       needed to set up the transitivity reasoner,
       identifier and operations identify the structure uniquely. *)
  val empty = [];
  val extend = I;
  fun merge data = AList.join struct-eq (K fst) data;
);

fun print-structures ctxt =
  let
    val structs = Data.get (Context.Proof ctxt);
    fun pretty-term t = Pretty.block
      [Pretty.quote (Syntax.pretty-term ctxt t), Pretty.brk 1,
       Pretty.str ::, Pretty.brk 1,
       Pretty.quote (Syntax.pretty-typ ctxt (type-of t))];
    fun pretty-struct ((s, ts), -) = Pretty.block
      [Pretty.str s, Pretty.str :, Pretty.brk 1,
       Pretty.enclose ( ) (Pretty.breaks (map pretty-term ts))];
  in
    Pretty.writeln (Pretty.big-list Order structures: (map pretty-struct structs))
  end;


(** Method **)

fun struct-tac ((s, [eq, le, less]), thms) ctxt prems =
  let
```

```
    fun decomp thy (@{const Trueprop} $ t) =
      let
        fun excluded t =
          (* exclude numeric types: linear arithmetic subsumes transitivity *)
          let val T = type-of t
          in
            T = HOLogic.natT orelse T = HOLogic.intT orelse T = HOLogic.realT
          end;
        fun rel (bin-op $ t1 $ t2) =
              if excluded t1 then NONE
              else if Pattern.matches thy (eq, bin-op) then SOME (t1, =, t2)
              else if Pattern.matches thy (le, bin-op) then SOME (t1, <=, t2)
              else if Pattern.matches thy (less, bin-op) then SOME (t1, <, t2)
              else NONE
          | rel - = NONE;
        fun dec (Const (@{const-name Not}, -) $ t) = (case rel t
              of NONE => NONE
               | SOME (t1, rel, t2) => SOME (t1, ~ ^ rel, t2))
          | dec x = rel x;
      in dec t end
      | decomp thy - = NONE;
  in
    case s of
      order => Order-Tac.partial-tac decomp thms ctxt prems
    | linorder => Order-Tac.linear-tac decomp thms ctxt prems
      | - => error (Unknown kind of order ' ^ s ^ ' encountered in transitivity
reasoner.)
  end

fun order-tac ctxt prems =
  FIRST' (map (fn s => CHANGED o struct-tac s ctxt prems) (Data.get (Context.Proof
ctxt)));


(** Attribute **)

fun add-struct-thm s tag =
  Thm.declaration-attribute
    (fn thm => Data.map (AList.map-default struct-eq (s, Order-Tac.empty TrueI)
(Order-Tac.update tag thm)));
fun del-struct s =
  Thm.declaration-attribute
    (fn - => Data.map (AList.delete struct-eq s));

val attrib-setup =
  Attrib.setup @{binding order}
    (Scan.lift ((Args.add -- Args.name >> (fn (-, s) => SOME s) || Args.del
>> K NONE) --|
      Args.colon (* FIXME || Scan.succeed true *) ) -- Scan.lift Args.name --
```

```
    Scan.repeat Args.term
    >> (fn ((SOME tag, n), ts) => add-struct-thm (n, ts) tag
        | ((NONE, n), ts) => del-struct (n, ts)))
  theorems controlling transitivity reasoner;


(** Diagnostic command **)

val - =
  Outer-Syntax.improper-command print-orders
    print order structures available to transitivity reasoner Keyword.diag
    (Scan.succeed (Toplevel.no-timing o Toplevel.unknown-context o
      Toplevel.keep (print-structures o Toplevel.context-of)));


(** Setup **)

val setup =
  Method.setup @{binding order} (Scan.succeed (fn ctxt => SIMPLE-METHOD'
(order-tac ctxt [])))
    transitivity reasoner #>
  attrib-setup;

end;

⟩⟩
```

setup *Orders.setup*

Declarations to set up transitivity reasoner of partial and linear orders.

context *order*
begin

declare *less-irrefl* [*THEN notE*, *order add less-reflE*: *order op* = :: *'a* ⇒ *'a* ⇒ *bool op* <= *op* <]

declare *order-refl* [*order add le-refl*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *less-imp-le* [*order add less-imp-le*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *antisym* [*order add eqI*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *eq-refl* [*order add eqD1*: *order op* = :: *'a* => *'a* => *bool op* <= *op* <]

declare *sym* [*THEN eq-refl*, *order add eqD2*: *order op* = :: *'a* => *'a* => *bool op*

<= op <]

declare *less-trans* [*order add less-trans*: *order op = :: 'a => 'a => bool op <= op <*]

declare *less-le-trans* [*order add less-le-trans*: *order op = :: 'a => 'a => bool op <= op <*]

declare *le-less-trans* [*order add le-less-trans*: *order op = :: 'a => 'a => bool op <= op <*]

declare *order-trans* [*order add le-trans*: *order op = :: 'a => 'a => bool op <= op <*]

declare *le-neq-trans* [*order add le-neq-trans*: *order op = :: 'a => 'a => bool op <= op <*]

declare *neq-le-trans* [*order add neq-le-trans*: *order op = :: 'a => 'a => bool op <= op <*]

declare *less-imp-neq* [*order add less-imp-neq*: *order op = :: 'a => 'a => bool op <= op <*]

declare *eq-neq-eq-imp-neq* [*order add eq-neq-eq-imp-neq*: *order op = :: 'a => 'a => bool op <= op <*]

declare *not-sym* [*order add not-sym*: *order op = :: 'a => 'a => bool op <= op <*]

end

context *linorder*
begin

declare [[*order del*: *order op = :: 'a => 'a => bool op <= op <*]]

declare *less-irrefl* [*THEN notE*, *order add less-reflE*: *linorder op = :: 'a => 'a => bool op <= op <*]

declare *order-refl* [*order add le-refl*: *linorder op = :: 'a => 'a => bool op <= op <*]

declare *less-imp-le* [*order add less-imp-le*: *linorder op = :: 'a => 'a => bool op <= op <*]

declare *not-less* [*THEN iffD2*, *order add not-lessI*: *linorder op = :: 'a => 'a => bool op <= op <*]

declare *not-le* [*THEN iffD2*, *order add not-leI*: *linorder op = :: 'a => 'a => bool*

op <= op <]

declare *not-less* [*THEN iffD1*, *order add not-lessD*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *not-le* [*THEN iffD1*, *order add not-leD*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *antisym* [*order add eqI*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *eq-refl* [*order add eqD1*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *sym* [*THEN eq-refl*, *order add eqD2*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *less-trans* [*order add less-trans*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *less-le-trans* [*order add less-le-trans*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *le-less-trans* [*order add le-less-trans*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *order-trans* [*order add le-trans*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *le-neq-trans* [*order add le-neq-trans*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *neq-le-trans* [*order add neq-le-trans*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *less-imp-neq* [*order add less-imp-neq*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *eq-neq-eq-imp-neq* [*order add eq-neq-eq-imp-neq*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

declare *not-sym* [*order add not-sym*: *linorder op* = :: *'a* => *'a* => *bool op <= op <*]

end

setup ⟪
let

fun prp t thm = (#*prop* (*rep-thm thm*) = *t*);

```
fun prove-antisym-le sg ss ((le as Const(-,T)) $ r $ s) =
  let val prems = prems-of-ss ss;
      val less = Const (@{const-name less}, T);
      val t = HOLogic.mk-Trueprop(le $ s $ r);
  in case find-first (prp t) prems of
       NONE =>
        let val t = HOLogic.mk-Trueprop(HOLogic.Not $ (less $ r $ s))
        in case find-first (prp t) prems of
             NONE => NONE
           | SOME thm => SOME(mk-meta-eq(thm RS @{thm linorder-class.antisym-conv1}))
        end
   | SOME thm => SOME(mk-meta-eq(thm RS @{thm order-class.antisym-conv}))
  end
  handle THM - => NONE;

fun prove-antisym-less sg ss (NotC $ ((less as Const(-,T)) $ r $ s)) =
  let val prems = prems-of-ss ss;
      val le = Const (@{const-name less-eq}, T);
      val t = HOLogic.mk-Trueprop(le $ r $ s);
  in case find-first (prp t) prems of
       NONE =>
        let val t = HOLogic.mk-Trueprop(NotC $ (less $ s $ r))
        in case find-first (prp t) prems of
             NONE => NONE
           | SOME thm => SOME(mk-meta-eq(thm RS @{thm linorder-class.antisym-conv3}))
        end
   | SOME thm => SOME(mk-meta-eq(thm RS @{thm linorder-class.antisym-conv2}))
  end
  handle THM - => NONE;

fun add-simprocs procs thy =
  Simplifier.map-simpset (fn ss => ss
    addsimprocs (map (fn (name, raw-ts, proc) =>
      Simplifier.simproc thy name raw-ts proc) procs)) thy;
fun add-solver name tac =
  Simplifier.map-simpset (fn ss => ss addSolver
   mk-solver' name (fn ss => tac (Simplifier.the-context ss) (Simplifier.prems-of-ss
ss)));

in
  add-simprocs [
      (antisym le, [(x::'a::order) <= y], prove-antisym-le),
      (antisym less, [~ (x::'a::linorder) < y], prove-antisym-less)
    ]
  #> add-solver Transitivity Orders.order-tac
  (* Adding the transitivity reasoners also as safe solvers showed a slight
     speed up, but the reasoning strength appears to be not higher (at least
     no breaking of additional proofs in the entire HOL distribution, as
```

of 5 March 2004, was observed). ∗)
end
⟩⟩

3.6 Bounded quantifiers

syntax
-All-less :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<-./ -*) [*0*, *0*, *10*] *10*)
-All-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<=-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<=-./ -*) [*0*, *0*, *10*] *10*)

-All-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL ->-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3EX ->-./ -*) [*0*, *0*, *10*] *10*)
-All-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL ->=-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3EX ->=-./ -*) [*0*, *0*, *10*] *10*)

syntax (*xsymbols*)
-All-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-<-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-<-./ -*) [*0*, *0*, *10*] *10*)
-All-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≤*-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≤*-./ -*) [*0*, *0*, *10*] *10*)

-All-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*->-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*->-./ -*) [*0*, *0*, *10*] *10*)
-All-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≥*-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≥*-./ -*) [*0*, *0*, *10*] *10*)

syntax (*HOL*)
-All-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*! *-<-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less :: [*idt*, *'a*, *bool*] => *bool* ((*3?* *-<-./ -*) [*0*, *0*, *10*] *10*)
-All-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*! *-<=-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3?* *-<=-./ -*) [*0*, *0*, *10*] *10*)

syntax (*HTML* **output**)
-All-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-<-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-<-./ -*) [*0*, *0*, *10*] *10*)
-All-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≤*-./ -*) [*0*, *0*, *10*] *10*)
-Ex-less-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≤*-./ -*) [*0*, *0*, *10*] *10*)

-All-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*->-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*->-./ -*) [*0*, *0*, *10*] *10*)
-All-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀*-*≥*-./ -*) [*0*, *0*, *10*] *10*)
-Ex-greater-eq :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃*-*≥*-./ -*) [*0*, *0*, *10*] *10*)

translations
ALL x<y. P => *ALL x. x* < *y* ⟶ *P*
EX x<y. P => *EX x. x* < *y* ∧ *P*
ALL x<=y. P => *ALL x. x* <= *y* ⟶ *P*

```
  EX x<=y. P   =>  EX x. x <= y ∧ P
  ALL x>y. P   =>  ALL x. x > y ⟶ P
  EX x>y. P    =>  EX x. x > y ∧ P
  ALL x>=y. P  =>  ALL x. x >= y ⟶ P
  EX x>=y. P   =>  EX x. x >= y ∧ P

print-translation ⟪
let
  val All-binder = Syntax.binder-name @{const-syntax All};
  val Ex-binder = Syntax.binder-name @{const-syntax Ex};
  val impl = @{const-syntax op -->};
  val conj = @{const-syntax op &};
  val less = @{const-syntax less};
  val less-eq = @{const-syntax less-eq};

  val trans =
   [((All-binder, impl, less),
    (@{syntax-const -All-less}, @{syntax-const -All-greater})),
    ((All-binder, impl, less-eq),
    (@{syntax-const -All-less-eq}, @{syntax-const -All-greater-eq})),
    ((Ex-binder, conj, less),
    (@{syntax-const -Ex-less}, @{syntax-const -Ex-greater})),
    ((Ex-binder, conj, less-eq),
    (@{syntax-const -Ex-less-eq}, @{syntax-const -Ex-greater-eq}))];

  fun matches-bound v t =
    (case t of
      Const (@{syntax-const -bound}, -) $ Free (v', -) => v = v'
    | - => false);
  fun contains-var v = Term.exists-subterm (fn Free (x, -) => x = v | - => false);
  fun mk v c n P = Syntax.const c $ Syntax.mark-bound v $ n $ P;

  fun tr' q = (q,
    fn [Const (@{syntax-const -bound}, -) $ Free (v, -),
        Const (c, -) $ (Const (d, -) $ t $ u) $ P] =>
        (case ALista.lookup (op =) trans (q, c, d) of
          NONE => raise Match
        | SOME (l, g) =>
            if matches-bound v t andalso not (contains-var v u) then mk v l u P
            else if matches-bound v u andalso not (contains-var v t) then mk v g t P
            else raise Match)
    | - => raise Match);
in [tr' All-binder, tr' Ex-binder] end
⟫
```

3.7 Transitivity reasoning

context *ord*
begin

lemma *ord-le-eq-trans*: $a \leq b \Longrightarrow b = c \Longrightarrow a \leq c$
by (*rule subst*)

lemma *ord-eq-le-trans*: $a = b \Longrightarrow b \leq c \Longrightarrow a \leq c$
by (*rule ssubst*)

lemma *ord-less-eq-trans*: $a < b \Longrightarrow b = c \Longrightarrow a < c$
by (*rule subst*)

lemma *ord-eq-less-trans*: $a = b \Longrightarrow b < c \Longrightarrow a < c$
by (*rule ssubst*)

end

lemma *order-less-subst2*: $(a::'a::order) < b \Longrightarrow f\ b < (c::'c::order) \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow f\ a < c$
proof −
assume *r*: $!!x\ y.\ x < y \Longrightarrow f\ x < f\ y$
assume $a < b$ **hence** $f\ a < f\ b$ **by** (*rule r*)
also assume $f\ b < c$
finally (*less-trans*) **show** *?thesis* **.**
qed

lemma *order-less-subst1*: $(a::'a::order) < f\ b \Longrightarrow (b::'b::order) < c \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow a < f\ c$
proof −
assume *r*: $!!x\ y.\ x < y \Longrightarrow f\ x < f\ y$
assume $a < f\ b$
also assume $b < c$ **hence** $f\ b < f\ c$ **by** (*rule r*)
finally (*less-trans*) **show** *?thesis* **.**
qed

lemma *order-le-less-subst2*: $(a::'a::order) \leq b \Longrightarrow f\ b < (c::'c::order) \Longrightarrow$
$(!!x\ y.\ x \leq y \Longrightarrow f\ x \leq f\ y) \Longrightarrow f\ a < c$
proof −
assume *r*: $!!x\ y.\ x \leq y \Longrightarrow f\ x \leq f\ y$
assume $a \leq b$ **hence** $f\ a \leq f\ b$ **by** (*rule r*)
also assume $f\ b < c$
finally (*le-less-trans*) **show** *?thesis* **.**
qed

lemma *order-le-less-subst1*: $(a::'a::order) \leq f\ b \Longrightarrow (b::'b::order) < c \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow a < f\ c$
proof −
assume *r*: $!!x\ y.\ x < y \Longrightarrow f\ x < f\ y$
assume $a \leq f\ b$
also assume $b < c$ **hence** $f\ b < f\ c$ **by** (*rule r*)
finally (*le-less-trans*) **show** *?thesis* **.**

qed

lemma *order-less-le-subst2*: $(a::'a::order) < b \Longrightarrow f\ b <= (c::'c::order) \Longrightarrow$
$(!!x\ y.\ x < y \Longrightarrow f\ x < f\ y) \Longrightarrow f\ a < c$
proof −
assume *r*: $!!x\ y.\ x < y \Longrightarrow f\ x < f\ y$
assume $a < b$ **hence** $f\ a < f\ b$ **by** (*rule r*)
also assume $f\ b <= c$
finally (*less-le-trans*) **show** *?thesis* **.**
qed

lemma *order-less-le-subst1*: $(a::'a::order) < f\ b \Longrightarrow (b::'b::order) <= c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow a < f\ c$
proof −
assume *r*: $!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y$
assume $a < f\ b$
also assume $b <= c$ **hence** $f\ b <= f\ c$ **by** (*rule r*)
finally (*less-le-trans*) **show** *?thesis* **.**
qed

lemma *order-subst1*: $(a::'a::order) <= f\ b \Longrightarrow (b::'b::order) <= c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow a <= f\ c$
proof −
assume *r*: $!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y$
assume $a <= f\ b$
also assume $b <= c$ **hence** $f\ b <= f\ c$ **by** (*rule r*)
finally (*order-trans*) **show** *?thesis* **.**
qed

lemma *order-subst2*: $(a::'a::order) <= b \Longrightarrow f\ b <= (c::'c::order) \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow f\ a <= c$
proof −
assume *r*: $!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y$
assume $a <= b$ **hence** $f\ a <= f\ b$ **by** (*rule r*)
also assume $f\ b <= c$
finally (*order-trans*) **show** *?thesis* **.**
qed

lemma *ord-le-eq-subst*: $a <= b \Longrightarrow f\ b = c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow f\ a <= c$
proof −
assume *r*: $!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y$
assume $a <= b$ **hence** $f\ a <= f\ b$ **by** (*rule r*)
also assume $f\ b = c$
finally (*ord-le-eq-trans*) **show** *?thesis* **.**
qed

lemma *ord-eq-le-subst*: $a = f\ b \Longrightarrow b <= c \Longrightarrow$
$(!!x\ y.\ x <= y \Longrightarrow f\ x <= f\ y) \Longrightarrow a <= f\ c$

proof −
assume *r*: !!*x y*. *x* <= *y* ==> *f x* <= *f y*
assume *a* = *f b*
also assume *b* <= *c* **hence** *f b* <= *f c* **by** (*rule r*)
finally (*ord-eq-le-trans*) **show** *?thesis* **.**
qed

lemma *ord-less-eq-subst*: *a* < *b* ==> *f b* = *c* ==>
(!!*x y*. *x* < *y* ==> *f x* < *f y*) ==> *f a* < *c*
proof −
assume *r*: !!*x y*. *x* < *y* ==> *f x* < *f y*
assume *a* < *b* **hence** *f a* < *f b* **by** (*rule r*)
also assume *f b* = *c*
finally (*ord-less-eq-trans*) **show** *?thesis* **.**
qed

lemma *ord-eq-less-subst*: *a* = *f b* ==> *b* < *c* ==>
(!!*x y*. *x* < *y* ==> *f x* < *f y*) ==> *a* < *f c*
proof −
assume *r*: !!*x y*. *x* < *y* ==> *f x* < *f y*
assume *a* = *f b*
also assume *b* < *c* **hence** *f b* < *f c* **by** (*rule r*)
finally (*ord-eq-less-trans*) **show** *?thesis* **.**
qed

Note that this list of rules is in reverse order of priorities.

lemmas [*trans*] =
order-less-subst2
order-less-subst1
order-le-less-subst2
order-le-less-subst1
order-less-le-subst2
order-less-le-subst1
order-subst2
order-subst1
ord-le-eq-subst
ord-eq-le-subst
ord-less-eq-subst
ord-eq-less-subst
forw-subst
back-subst
rev-mp
mp

lemmas (**in** *order*) [*trans*] =
neq-le-trans
le-neq-trans

lemmas (**in** *preorder*) [*trans*] =

less-trans
less-asym'
le-less-trans
less-le-trans
order-trans

lemmas (**in** *order*) [*trans*] =
antisym

lemmas (**in** *ord*) [*trans*] =
ord-le-eq-trans
ord-eq-le-trans
ord-less-eq-trans
ord-eq-less-trans

lemmas [*trans*] =
trans

lemmas *order-trans-rules* =
order-less-subst2
order-less-subst1
order-le-less-subst2
order-le-less-subst1
order-less-le-subst2
order-less-le-subst1
order-subst2
order-subst1
ord-le-eq-subst
ord-eq-le-subst
ord-less-eq-subst
ord-eq-less-subst
forw-subst
back-subst
rev-mp
mp
neq-le-trans
le-neq-trans
less-trans
less-asym'
le-less-trans
less-le-trans
order-trans
antisym
ord-le-eq-trans
ord-eq-le-trans
ord-less-eq-trans
ord-eq-less-trans
trans

These support proving chains of decreasing inequalities a ¿= b ¿= c ... in

Isar proofs.

lemma *xt1*:
$a = b \Longrightarrow b > c \Longrightarrow a > c$
$a > b \Longrightarrow b = c \Longrightarrow a > c$
$a = b \Longrightarrow b >= c \Longrightarrow a >= c$
$a >= b \Longrightarrow b = c \Longrightarrow a >= c$
$(x::'a::order) >= y \Longrightarrow y >= x \Longrightarrow x = y$
$(x::'a::order) >= y \Longrightarrow y >= z \Longrightarrow x >= z$
$(x::'a::order) > y \Longrightarrow y >= z \Longrightarrow x > z$
$(x::'a::order) >= y \Longrightarrow y > z \Longrightarrow x > z$
$(a::'a::order) > b \Longrightarrow b > a \Longrightarrow P$
$(x::'a::order) > y \Longrightarrow y > z \Longrightarrow x > z$
$(a::'a::order) >= b \Longrightarrow a \sim= b \Longrightarrow a > b$
$(a::'a::order) \sim= b \Longrightarrow a >= b \Longrightarrow a > b$
$a = f\ b \Longrightarrow b > c \Longrightarrow (!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow a > f\ c$
$a > b \Longrightarrow f\ b = c \Longrightarrow (!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow f\ a > c$
$a = f\ b \Longrightarrow b >= c \Longrightarrow (!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow a >= f\ c$
$a >= b \Longrightarrow f\ b = c \Longrightarrow (!!\ x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow f\ a >= c$
by *auto*

lemma *xt2*:
$(a::'a::order) >= f\ b \Longrightarrow b >= c \Longrightarrow (!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow a >= f\ c$
by (*subgoal-tac* $f\ b >= f\ c$, *force*, *force*)

lemma *xt3*: $(a::'a::order) >= b \Longrightarrow (f\ b::'b::order) >= c \Longrightarrow$
$(!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow f\ a >= c$
by (*subgoal-tac* $f\ a >= f\ b$, *force*, *force*)

lemma *xt4*: $(a::'a::order) > f\ b \Longrightarrow (b::'b::order) >= c \Longrightarrow$
$(!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow a > f\ c$
by (*subgoal-tac* $f\ b >= f\ c$, *force*, *force*)

lemma *xt5*: $(a::'a::order) > b \Longrightarrow (f\ b::'b::order) >= c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow f\ a > c$
by (*subgoal-tac* $f\ a > f\ b$, *force*, *force*)

lemma *xt6*: $(a::'a::order) >= f\ b \Longrightarrow b > c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow a > f\ c$
by (*subgoal-tac* $f\ b > f\ c$, *force*, *force*)

lemma *xt7*: $(a::'a::order) >= b \Longrightarrow (f\ b::'b::order) > c \Longrightarrow$
$(!!x\ y.\ x >= y \Longrightarrow f\ x >= f\ y) \Longrightarrow f\ a > c$
by (*subgoal-tac* $f\ a >= f\ b$, *force*, *force*)

lemma *xt8*: $(a::'a::order) > f\ b \Longrightarrow (b::'b::order) > c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow a > f\ c$
by (*subgoal-tac* $f\ b > f\ c$, *force*, *force*)

lemma *xt9*: $(a::'a::order) > b \Longrightarrow (f\ b::'b::order) > c \Longrightarrow$
$(!!x\ y.\ x > y \Longrightarrow f\ x > f\ y) \Longrightarrow f\ a > c$
by (*subgoal-tac* $f\ a > f\ b$, *force*, *force*)

lemmas *xtrans* = *xt1 xt2 xt3 xt4 xt5 xt6 xt7 xt8 xt9*

3.8 Monotonicity, least value operator and min/max

context *order*
begin

definition *mono* :: $('a \Rightarrow 'b::order) \Rightarrow bool$ **where**
$mono\ f \longleftrightarrow (\forall x\ y.\ x \leq y \longrightarrow f\ x \leq f\ y)$

lemma *monoI* [*intro?*]:
fixes $f :: 'a \Rightarrow 'b::order$
shows $(\bigwedge x\ y.\ x \leq y \Longrightarrow f\ x \leq f\ y) \Longrightarrow mono\ f$
unfolding *mono-def* **by** *iprover*

lemma *monoD* [*dest?*]:
fixes $f :: 'a \Rightarrow 'b::order$
shows $mono\ f \Longrightarrow x \leq y \Longrightarrow f\ x \leq f\ y$
unfolding *mono-def* **by** *iprover*

definition *strict-mono* :: $('a \Rightarrow 'b::order) \Rightarrow bool$ **where**
$strict\text{-}mono\ f \longleftrightarrow (\forall x\ y.\ x < y \longrightarrow f\ x < f\ y)$

lemma *strict-monoI* [*intro?*]:
assumes $\bigwedge x\ y.\ x < y \Longrightarrow f\ x < f\ y$
shows *strict-mono f*
using *assms* **unfolding** *strict-mono-def* **by** *auto*

lemma *strict-monoD* [*dest?*]:
$strict\text{-}mono\ f \Longrightarrow x < y \Longrightarrow f\ x < f\ y$
unfolding *strict-mono-def* **by** *auto*

lemma *strict-mono-mono* [*dest?*]:
assumes *strict-mono f*
shows *mono f*
proof (*rule monoI*)
fix *x y*
assume $x \leq y$
show $f\ x \leq f\ y$
proof (*cases* $x = y$)
case *True* **then show** *?thesis* **by** *simp*
next
case *False* **with** ‹$x \leq y$› **have** $x < y$ **by** *simp*
with *assms strict-monoD* **have** $f\ x < f\ y$ **by** *auto*
then show *?thesis* **by** *simp*

```
  qed
qed

end

context linorder
begin

lemma strict-mono-eq:
  assumes strict-mono f
  shows f x = f y ⟷ x = y
proof
  assume f x = f y
  show x = y proof (cases x y rule: linorder-cases)
    case less with assms strict-monoD have f x < f y by auto
    with ‹f x = f y› show ?thesis by simp
  next
    case equal then show ?thesis .
  next
    case greater with assms strict-monoD have f y < f x by auto
    with ‹f x = f y› show ?thesis by simp
  qed
qed simp

lemma strict-mono-less-eq:
  assumes strict-mono f
  shows f x ≤ f y ⟷ x ≤ y
proof
  assume x ≤ y
  with assms strict-mono-mono monoD show f x ≤ f y by auto
next
  assume f x ≤ f y
  show x ≤ y proof (rule ccontr)
    assume ¬ x ≤ y then have y < x by simp
    with assms strict-monoD have f y < f x by auto
    with ‹f x ≤ f y› show False by simp
  qed
qed

lemma strict-mono-less:
  assumes strict-mono f
  shows f x < f y ⟷ x < y
  using assms
    by (auto simp add: less-le Orderings.less-le strict-mono-eq strict-mono-less-eq)

lemma min-of-mono:
  fixes f :: 'a ⇒ 'b::linorder
  shows mono f ⟹ min (f m) (f n) = f (min m n)
  by (auto simp: mono-def Orderings.min-def min-def intro: Orderings.antisym)
```

lemma *max-of-mono*:
fixes $f :: 'a \Rightarrow 'b$::*linorder*
shows *mono* $f \Longrightarrow$ *max* $(f\ m)\ (f\ n) = f\ (max\ m\ n)$
by (*auto simp*: *mono-def Orderings.max-def max-def intro*: *Orderings.antisym*)

end

lemma *min-leastL*: $(!!x.\ least <= x) ==> min\ least\ x = least$
by (*simp add*: *min-def*)

lemma *max-leastL*: $(!!x.\ least <= x) ==> max\ least\ x = x$
by (*simp add*: *max-def*)

lemma *min-leastR*: $(\bigwedge x::'a::order.\ least \leq x) \Longrightarrow min\ x\ least = least$
apply (*simp add*: *min-def*)
apply (*blast intro*: *antisym*)
done

lemma *max-leastR*: $(\bigwedge x::'a::order.\ least \leq x) \Longrightarrow max\ x\ least = x$
apply (*simp add*: *max-def*)
apply (*blast intro*: *antisym*)
done

3.9 Top and bottom elements

class *top* = *preorder* +
fixes *top* :: $'a$
assumes *top-greatest* [*simp*]: $x \leq top$

class *bot* = *preorder* +
fixes *bot* :: $'a$
assumes *bot-least* [*simp*]: $bot \leq x$

3.10 Dense orders

class *dense-linorder* = *linorder* +
assumes *gt-ex*: $\exists y.\ x < y$
and *lt-ex*: $\exists y.\ y < x$
and *dense*: $x < y \Longrightarrow (\exists z.\ x < z \wedge z < y)$
begin

lemma *dense-le*:
fixes $y\ z :: 'a$
assumes $\bigwedge x.\ x < y \Longrightarrow x \leq z$
shows $y \leq z$
proof (*rule ccontr*)
assume $\neg$ *?thesis*
hence $z < y$ **by** *simp*
from *dense*[*OF this*]

obtain x **where** $x < y$ **and** $z < x$ **by** *safe*
moreover have $x \leq z$ **using** *assms*[*OF* ‹$x < y$›] **.**
ultimately show *False* **by** *auto*
qed

lemma *dense-le-bounded*:
fixes $x\ y\ z :: 'a$
assumes $x < y$
assumes $*$: $\bigwedge w.\ [\![\ x < w\ ;\ w < y\ ]\!] \Longrightarrow w \leq z$
shows $y \leq z$
proof (*rule dense-le*)
fix w **assume** $w < y$
from *dense*[*OF* ‹$x < y$›] **obtain** u **where** $x < u\ u < y$ **by** *safe*
from *linear*[*of u w*]
show $w \leq z$
proof (*rule disjE*)
assume $u \leq w$
from *less-le-trans*[*OF* ‹$x < u$› ‹$u \leq w$›] ‹$w < y$›
show $w \leq z$ **by** (*rule* $*$)
next
assume $w \leq u$
from ‹$w \leq u$› $*$[*OF* ‹$x < u$› ‹$u < y$›]
show $w \leq z$ **by** (*rule order-trans*)
qed
qed

end

3.11 Wellorders

class *wellorder* = *linorder* +
assumes *less-induct* [*case-names less*]: $(\bigwedge x.\ (\bigwedge y.\ y < x \Longrightarrow P\ y) \Longrightarrow P\ x) \Longrightarrow P\ a$
begin

lemma *wellorder-Least-lemma*:
fixes $k :: 'a$
assumes $P\ k$
shows *LeastI*: $P\ (LEAST\ x.\ P\ x)$ **and** *Least-le*: $(LEAST\ x.\ P\ x) \leq k$
proof −
have $P\ (LEAST\ x.\ P\ x) \wedge (LEAST\ x.\ P\ x) \leq k$
using *assms* **proof** (*induct k rule*: *less-induct*)
case (*less x*) **then have** $P\ x$ **by** *simp*
show *?case* **proof** (*rule classical*)
assume *assm*: $\neg\ (P\ (LEAST\ a.\ P\ a) \wedge (LEAST\ a.\ P\ a) \leq x)$
have $\bigwedge y.\ P\ y \Longrightarrow x \leq y$
proof (*rule classical*)
fix y
assume $P\ y$ **and** $\neg\ x \leq y$

with *less* **have** P $(LEAST\ a.\ P\ a)$ **and** $(LEAST\ a.\ P\ a) \leq y$
by (*auto simp add*: *not-le*)
with *assm* **have** $x < (LEAST\ a.\ P\ a)$ **and** $(LEAST\ a.\ P\ a) \leq y$
by *auto*
then show $x \leq y$ **by** *auto*
qed
with ‹$P\ x$› **have** *Least*: $(LEAST\ a.\ P\ a) = x$
by (*rule Least-equality*)
with ‹$P\ x$› **show** *?thesis* **by** *simp*
qed
qed
then show P $(LEAST\ x.\ P\ x)$ **and** $(LEAST\ x.\ P\ x) \leq k$ **by** *auto*
qed

— The following 3 lemmas are due to Brian Huffman
lemma *LeastI-ex*: $\exists\, x.\ P\ x \Longrightarrow P\ (Least\ P)$
by (*erule exE*) (*erule LeastI*)

lemma *LeastI2*:
$P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (Least\ P)$
by (*blast intro*: *LeastI*)

lemma *LeastI2-ex*:
$\exists\, a.\ P\ a \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow Q\ (Least\ P)$
by (*blast intro*: *LeastI-ex*)

lemma *not-less-Least*: $k < (LEAST\ x.\ P\ x) \Longrightarrow \neg\ P\ k$
apply (*simp* (*no-asm-use*) *add*: *not-le* [*symmetric*])
apply (*erule contrapos-nn*)
apply (*erule Least-le*)
done

end

3.12 Order on bool

instantiation *bool* :: {*order*, *top*, *bot*}
begin

definition
le-bool-def [*code del*]: $P \leq Q \longleftrightarrow P \longrightarrow Q$

definition
less-bool-def [*code del*]: $(P{::}bool) < Q \longleftrightarrow \neg\ P \wedge Q$

definition
top-bool-eq: $top = True$

definition

bot-bool-eq: $bot = False$

instance proof
qed (*auto simp add*: *bot-bool-eq top-bool-eq less-bool-def*, *auto simp add*: *le-bool-def*)

end

lemma *le-boolI*: $(P \Longrightarrow Q) \Longrightarrow P \leq Q$
by (*simp add*: *le-bool-def*)

lemma *le-boolI'*: $P \longrightarrow Q \Longrightarrow P \leq Q$
by (*simp add*: *le-bool-def*)

lemma *le-boolE*: $P \leq Q \Longrightarrow P \Longrightarrow (Q \Longrightarrow R) \Longrightarrow R$
by (*simp add*: *le-bool-def*)

lemma *le-boolD*: $P \leq Q \Longrightarrow P \longrightarrow Q$
by (*simp add*: *le-bool-def*)

lemma *bot-boolE*: $bot \Longrightarrow P$
by (*simp add*: *bot-bool-eq*)

lemma *top-boolI*: *top*
by (*simp add*: *top-bool-eq*)

lemma [*code*]:
$False \leq b \longleftrightarrow True$
$True \leq b \longleftrightarrow b$
$False < b \longleftrightarrow b$
$True < b \longleftrightarrow False$
unfolding *le-bool-def less-bool-def* **by** *simp-all*

3.13 Order on functions

instantiation *fun* :: (*type*, *ord*) *ord*
begin

definition
le-fun-def [*code del*]: $f \leq g \longleftrightarrow (\forall x.\ f\ x \leq g\ x)$

definition
less-fun-def [*code del*]: $(f::'a \Rightarrow 'b) < g \longleftrightarrow f \leq g \wedge \neg\ (g \leq f)$

instance ..

end

instance *fun* :: (*type*, *preorder*) *preorder* **proof**
qed (*auto simp add*: *le-fun-def less-fun-def*

intro: *order-trans antisym intro*!: *ext*)

instance *fun* :: (*type*, *order*) *order* **proof**
qed (*auto simp add*: *le-fun-def intro*: *antisym ext*)

instantiation *fun* :: (*type*, *top*) *top*
begin

definition
top-fun-eq: $top = (\lambda x.\ top)$

instance proof
qed (*simp add*: *top-fun-eq le-fun-def*)

end

instantiation *fun* :: (*type*, *bot*) *bot*
begin

definition
bot-fun-eq: $bot = (\lambda x.\ bot)$

instance proof
qed (*simp add*: *bot-fun-eq le-fun-def*)

end

lemma *le-funI*: $(\bigwedge x.\ f\ x \leq g\ x) \Longrightarrow f \leq g$
unfolding *le-fun-def* **by** *simp*

lemma *le-funE*: $f \leq g \Longrightarrow (f\ x \leq g\ x \Longrightarrow P) \Longrightarrow P$
unfolding *le-fun-def* **by** *simp*

lemma *le-funD*: $f \leq g \Longrightarrow f\ x \leq g\ x$
unfolding *le-fun-def* **by** *simp*

3.14 Name duplicates

lemmas *order-eq-refl* = *preorder-class.eq-refl*
lemmas *order-less-irrefl* = *preorder-class.less-irrefl*
lemmas *order-less-imp-le* = *preorder-class.less-imp-le*
lemmas *order-less-not-sym* = *preorder-class.less-not-sym*
lemmas *order-less-asym* = *preorder-class.less-asym*
lemmas *order-less-trans* = *preorder-class.less-trans*
lemmas *order-le-less-trans* = *preorder-class.le-less-trans*
lemmas *order-less-le-trans* = *preorder-class.less-le-trans*
lemmas *order-less-imp-not-less* = *preorder-class.less-imp-not-less*
lemmas *order-less-imp-triv* = *preorder-class.less-imp-triv*
lemmas *order-less-asym'* = *preorder-class.less-asym'*

lemmas *order-less-le = order-class.less-le*
lemmas *order-le-less = order-class.le-less*
lemmas *order-le-imp-less-or-eq = order-class.le-imp-less-or-eq*
lemmas *order-less-imp-not-eq = order-class.less-imp-not-eq*
lemmas *order-less-imp-not-eq2 = order-class.less-imp-not-eq2*
lemmas *order-neq-le-trans = order-class.neq-le-trans*
lemmas *order-le-neq-trans = order-class.le-neq-trans*
lemmas *order-antisym = order-class.antisym*
lemmas *order-eq-iff = order-class.eq-iff*
lemmas *order-antisym-conv = order-class.antisym-conv*

lemmas *linorder-linear = linorder-class.linear*
lemmas *linorder-less-linear = linorder-class.less-linear*
lemmas *linorder-le-less-linear = linorder-class.le-less-linear*
lemmas *linorder-le-cases = linorder-class.le-cases*
lemmas *linorder-not-less = linorder-class.not-less*
lemmas *linorder-not-le = linorder-class.not-le*
lemmas *linorder-neq-iff = linorder-class.neq-iff*
lemmas *linorder-neqE = linorder-class.neqE*
lemmas *linorder-antisym-conv1 = linorder-class.antisym-conv1*
lemmas *linorder-antisym-conv2 = linorder-class.antisym-conv2*
lemmas *linorder-antisym-conv3 = linorder-class.antisym-conv3*

end

4 Groups: Groups, also combined with orderings

theory *Groups*
imports *Orderings*
uses (~~/*src*/*Provers*/*Arith*/*abel-cancel.ML*)
begin

4.1 Fact collections

ML ⟪
structure Ac-Simps = Named-Thms(
  val name = ac-simps
  val description = associativity and commutativity simplification rules
)
⟫

setup *Ac-Simps.setup*

The rewrites accumulated in *algebra-simps* deal with the classical algebraic structures of groups, rings and family. They simplify terms by multiplying everything out (in case of a ring) and bringing sums and products into a canonical form (by ordered rewriting). As a result it decides group and ring

equalities but also helps with inequalities.

Of course it also works for fields, but it knows nothing about multiplicative inverses or division. This is catered for by *field-simps*.

ML ⟪
structure Algebra-Simps = Named-Thms(
  val name = algebra-simps
  val description = algebra simplification rules
)
⟫

setup *Algebra-Simps.setup*

Lemmas *field-simps* multiply with denominators in (in)equations if they can be proved to be non-zero (for equations) or positive/negative (for inequations). Can be too aggressive and is therefore separate from the more benign *algebra-simps*.

ML ⟪
structure Field-Simps = Named-Thms(
  val name = field-simps
  val description = algebra simplification rules for fields
)
⟫

setup *Field-Simps.setup*

4.2 Abstract structures

These locales provide basic structures for interpretation into bigger structures; extensions require careful thinking, otherwise undesired effects may occur due to interpretation.

locale *semigroup* =
  fixes $f :: 'a \Rightarrow 'a \Rightarrow 'a$ (**infixl** $*$ *70*)
  assumes *assoc* [*ac-simps*]: $a * b * c = a * (b * c)$

locale *abel-semigroup* = *semigroup* +
  assumes *commute* [*ac-simps*]: $a * b = b * a$
begin

lemma *left-commute* [*ac-simps*]:
  $b * (a * c) = a * (b * c)$
proof −
  have $(b * a) * c = (a * b) * c$
    by (*simp only*: *commute*)
  then show *?thesis*
    by (*simp only*: *assoc*)
qed

end

locale *monoid* = *semigroup* +
  fixes *z* :: *'a* (*1*)
  assumes *left-neutral* [*simp*]: *1 * a = a*
  assumes *right-neutral* [*simp*]: *a * 1 = a*

locale *comm-monoid* = *abel-semigroup* +
  fixes *z* :: *'a* (*1*)
  assumes *comm-neutral*: *a * 1 = a*

sublocale *comm-monoid* < *monoid* **proof**
qed (*simp-all add*: *commute comm-neutral*)

4.3 Generic operations

class *zero* =
  fixes *zero* :: *'a* (*0*)

class *one* =
  fixes *one* :: *'a* (*1*)

hide-const (**open**) *zero one*

syntax
  -index1 :: *index* ($_1$)
translations
  (*index*) $_1$ => (*index*) $_\diamond$

lemma *Let-0* [*simp*]: *Let 0 f = f 0*
  unfolding *Let-def* **..**

lemma *Let-1* [*simp*]: *Let 1 f = f 1*
  unfolding *Let-def* **..**

setup ⟪
  Reorient-Proc.add
    (*fn Const*(@{*const-name Groups.zero*}, -) => *true*
      | *Const*(@{*const-name Groups.one*}, -) => *true*
      | - => *false*)
⟫

simproc-setup *reorient-zero* (*0 = x*) = *Reorient-Proc.proc*
simproc-setup *reorient-one* (*1 = x*) = *Reorient-Proc.proc*

typed-print-translation ⟪
let
  fun tr' c = (*c*, *fn show-sorts* => *fn T* => *fn ts* =>
    if (*not o null*) *ts orelse T = dummyT*

orelse not (! *show-types*) *andalso can Term.dest-Type T*
then raise Match
else Syntax.const Syntax.constrainC $ *Syntax.const c* $ *Syntax.term-of-typ show-sorts T*);
in map tr' [@{*const-syntax Groups.one*}, @{*const-syntax Groups.zero*}] *end*;
⟩⟩ — show types that are presumably too general

class *plus* =
fixes *plus* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** + *65*)

class *minus* =
fixes *minus* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** − *65*)

class *uminus* =
fixes *uminus* :: *'a* ⇒ *'a* (− - [*81*] *80*)

class *times* =
fixes *times* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** ∗ *70*)

use ~~/*src/Provers/Arith/abel-cancel.ML*

4.4 Semigroups and Monoids

class *semigroup-add* = *plus* +
assumes *add-assoc* [*algebra-simps*, *field-simps*]: (*a* + *b*) + *c* = *a* + (*b* + *c*)

sublocale *semigroup-add* < *add*!: *semigroup plus* **proof**
qed (*fact add-assoc*)

class *ab-semigroup-add* = *semigroup-add* +
assumes *add-commute* [*algebra-simps*, *field-simps*]: *a* + *b* = *b* + *a*

sublocale *ab-semigroup-add* < *add*!: *abel-semigroup plus* **proof**
qed (*fact add-commute*)

context *ab-semigroup-add*
begin

lemmas *add-left-commute* [*algebra-simps*, *field-simps*] = *add.left-commute*

theorems *add-ac* = *add-assoc add-commute add-left-commute*

end

theorems *add-ac* = *add-assoc add-commute add-left-commute*

class *semigroup-mult* = *times* +
assumes *mult-assoc* [*algebra-simps*, *field-simps*]: (*a* ∗ *b*) ∗ *c* = *a* ∗ (*b* ∗ *c*)

sublocale *semigroup-mult* < *mult*!: *semigroup times* **proof**
qed (*fact mult-assoc*)

class *ab-semigroup-mult* = *semigroup-mult* +
  assumes *mult-commute* [*algebra-simps*, *field-simps*]: $a * b = b * a$

sublocale *ab-semigroup-mult* < *mult*!: *abel-semigroup times* **proof**
qed (*fact mult-commute*)

context *ab-semigroup-mult*
begin

lemmas *mult-left-commute* [*algebra-simps*, *field-simps*] = *mult.left-commute*

theorems *mult-ac* = *mult-assoc mult-commute mult-left-commute*

end

theorems *mult-ac* = *mult-assoc mult-commute mult-left-commute*

class *monoid-add* = *zero* + *semigroup-add* +
  assumes *add-0-left*: $0 + a = a$
    and *add-0-right*: $a + 0 = a$

sublocale *monoid-add* < *add*!: *monoid plus 0* **proof**
qed (*fact add-0-left add-0-right*)+

lemma *zero-reorient*: $0 = x \longleftrightarrow x = 0$
by (*rule eq-commute*)

class *comm-monoid-add* = *zero* + *ab-semigroup-add* +
  assumes *add-0*: $0 + a = a$

sublocale *comm-monoid-add* < *add*!: *comm-monoid plus 0* **proof**
qed (*insert add-0*, *simp add*: *ac-simps*)

subclass (**in** *comm-monoid-add*) *monoid-add* **proof**
qed (*fact add.left-neutral add.right-neutral*)+

class *monoid-mult* = *one* + *semigroup-mult* +
  assumes *mult-1-left*: $1 * a = a$
    and *mult-1-right*: $a * 1 = a$

sublocale *monoid-mult* < *mult*!: *monoid times 1* **proof**
qed (*fact mult-1-left mult-1-right*)+

lemma *one-reorient*: $1 = x \longleftrightarrow x = 1$
by (*rule eq-commute*)

class *comm-monoid-mult* = *one* + *ab-semigroup-mult* +
  assumes *mult-1*: *1* ∗ *a* = *a*

sublocale *comm-monoid-mult* < *mult*!: *comm-monoid times 1* **proof**
qed (*insert mult-1*, *simp add*: *ac-simps*)

subclass (**in** *comm-monoid-mult*) *monoid-mult* **proof**
qed (*fact mult.left-neutral mult.right-neutral*)+

class *cancel-semigroup-add* = *semigroup-add* +
  assumes *add-left-imp-eq*: *a* + *b* = *a* + *c* ⟹ *b* = *c*
  assumes *add-right-imp-eq*: *b* + *a* = *c* + *a* ⟹ *b* = *c*
begin

lemma *add-left-cancel* [*simp*]:
  a + *b* = *a* + *c* ⟷ *b* = *c*
by (*blast dest*: *add-left-imp-eq*)

lemma *add-right-cancel* [*simp*]:
  b + *a* = *c* + *a* ⟷ *b* = *c*
by (*blast dest*: *add-right-imp-eq*)

end

class *cancel-ab-semigroup-add* = *ab-semigroup-add* +
  assumes *add-imp-eq*: *a* + *b* = *a* + *c* ⟹ *b* = *c*
begin

subclass *cancel-semigroup-add*
proof
  fix *a b c* :: *'a*
  assume *a* + *b* = *a* + *c*
  then show *b* = *c* **by** (*rule add-imp-eq*)
next
  fix *a b c* :: *'a*
  assume *b* + *a* = *c* + *a*
  then have *a* + *b* = *a* + *c* **by** (*simp only*: *add-commute*)
  then show *b* = *c* **by** (*rule add-imp-eq*)
qed

end

class *cancel-comm-monoid-add* = *cancel-ab-semigroup-add* + *comm-monoid-add*

4.5 Groups

class *group-add* = *minus* + *uminus* + *monoid-add* +
  assumes *left-minus* [*simp*]: − *a* + *a* = *0*
  assumes *diff-minus*: *a* − *b* = *a* + (− *b*)

begin

lemma *minus-unique*:
  assumes $a + b = 0$ **shows** $- a = b$
proof −
  have $- a = - a + (a + b)$ **using** *assms* **by** *simp*
  also have $\ldots = b$ **by** (*simp add*: *add-assoc* [*symmetric*])
  finally show *?thesis* **.**
qed

lemmas *equals-zero-I* = *minus-unique*

lemma *minus-zero* [*simp*]: $- 0 = 0$
proof −
  have $0 + 0 = 0$ **by** (*rule add-0-right*)
  thus $- 0 = 0$ **by** (*rule minus-unique*)
qed

lemma *minus-minus* [*simp*]: $- (- a) = a$
proof −
  have $- a + a = 0$ **by** (*rule left-minus*)
  thus $- (- a) = a$ **by** (*rule minus-unique*)
qed

lemma *right-minus* [*simp*]: $a + - a = 0$
proof −
  have $a + - a = - (- a) + - a$ **by** *simp*
  also have $\ldots = 0$ **by** (*rule left-minus*)
  finally show *?thesis* **.**
qed

lemma *minus-add-cancel*: $- a + (a + b) = b$
by (*simp add*: *add-assoc* [*symmetric*])

lemma *add-minus-cancel*: $a + (- a + b) = b$
by (*simp add*: *add-assoc* [*symmetric*])

lemma *minus-add*: $- (a + b) = - b + - a$
proof −
  have $(a + b) + (- b + - a) = 0$
    by (*simp add*: *add-assoc add-minus-cancel*)
  thus $- (a + b) = - b + - a$
    by (*rule minus-unique*)
qed

lemma *right-minus-eq*: $a - b = 0 \longleftrightarrow a = b$
proof
  assume $a - b = 0$
  have $a = (a - b) + b$ **by** (*simp add*:*diff-minus add-assoc*)

also have $\ldots = b$ **using** ‹$a - b = 0$› **by** *simp*
finally show $a = b$ **.**
next
assume $a = b$ **thus** $a - b = 0$ **by** (*simp add*: *diff-minus*)
qed

lemma *diff-self* [*simp*]: $a - a = 0$
by (*simp add*: *diff-minus*)

lemma *diff-0* [*simp*]: $0 - a = - a$
by (*simp add*: *diff-minus*)

lemma *diff-0-right* [*simp*]: $a - 0 = a$
by (*simp add*: *diff-minus*)

lemma *diff-minus-eq-add* [*simp*]: $a - - b = a + b$
by (*simp add*: *diff-minus*)

lemma *neg-equal-iff-equal* [*simp*]:
$- a = - b \longleftrightarrow a = b$
proof
assume $- a = - b$
hence $- (- a) = - (- b)$ **by** *simp*
thus $a = b$ **by** *simp*
next
assume $a = b$
thus $- a = - b$ **by** *simp*
qed

lemma *neg-equal-0-iff-equal* [*simp*]:
$- a = 0 \longleftrightarrow a = 0$
by (*subst neg-equal-iff-equal* [*symmetric*], *simp*)

lemma *neg-0-equal-iff-equal* [*simp*]:
$0 = - a \longleftrightarrow 0 = a$
by (*subst neg-equal-iff-equal* [*symmetric*], *simp*)

The next two equations can make the simplifier loop!

lemma *equation-minus-iff*:
$a = - b \longleftrightarrow b = - a$
proof −
have $- (- a) = - b \longleftrightarrow - a = b$ **by** (*rule neg-equal-iff-equal*)
thus *?thesis* **by** (*simp add*: *eq-commute*)
qed

lemma *minus-equation-iff*:
$- a = b \longleftrightarrow - b = a$
proof −
have $- a = - (- b) \longleftrightarrow a = -b$ **by** (*rule neg-equal-iff-equal*)

thus *?thesis* **by** (*simp add*: *eq-commute*)
qed

lemma *diff-add-cancel*: $a - b + b = a$
by (*simp add*: *diff-minus add-assoc*)

lemma *add-diff-cancel*: $a + b - b = a$
by (*simp add*: *diff-minus add-assoc*)

declare *diff-minus*[*symmetric*, *algebra-simps*, *field-simps*]

lemma *eq-neg-iff-add-eq-0*: $a = - b \longleftrightarrow a + b = 0$
proof
assume $a = - b$ **then show** $a + b = 0$ **by** *simp*
next
assume $a + b = 0$
moreover have $a + (b + - b) = (a + b) + - b$
by (*simp only*: *add-assoc*)
ultimately show $a = - b$ **by** *simp*
qed

end

class *ab-group-add* = *minus* + *uminus* + *comm-monoid-add* +
assumes *ab-left-minus*: $- a + a = 0$
assumes *ab-diff-minus*: $a - b = a + (- b)$
begin

subclass *group-add*
proof qed (*simp-all add*: *ab-left-minus ab-diff-minus*)

subclass *cancel-comm-monoid-add*
proof
fix $a\ b\ c :: 'a$
assume $a + b = a + c$
then have $- a + a + b = - a + a + c$
unfolding *add-assoc* **by** *simp*
then show $b = c$ **by** *simp*
qed

lemma *uminus-add-conv-diff*[*algebra-simps*, *field-simps*]:
$- a + b = b - a$
by (*simp add*:*diff-minus add-commute*)

lemma *minus-add-distrib* [*simp*]:
$- (a + b) = - a + - b$
by (*rule minus-unique*) (*simp add*: *add-ac*)

lemma *minus-diff-eq* [*simp*]:

$-(a - b) = b - a$
by (*simp add*: *diff-minus add-commute*)

lemma *add-diff-eq*[*algebra-simps*, *field-simps*]: $a + (b - c) = (a + b) - c$
by (*simp add*: *diff-minus add-ac*)

lemma *diff-add-eq*[*algebra-simps*, *field-simps*]: $(a - b) + c = (a + c) - b$
by (*simp add*: *diff-minus add-ac*)

lemma *diff-eq-eq*[*algebra-simps*, *field-simps*]: $a - b = c \longleftrightarrow a = c + b$
by (*auto simp add*: *diff-minus add-assoc*)

lemma *eq-diff-eq*[*algebra-simps*, *field-simps*]: $a = c - b \longleftrightarrow a + b = c$
by (*auto simp add*: *diff-minus add-assoc*)

lemma *diff-diff-eq*[*algebra-simps*, *field-simps*]: $(a - b) - c = a - (b + c)$
by (*simp add*: *diff-minus add-ac*)

lemma *diff-diff-eq2*[*algebra-simps*, *field-simps*]: $a - (b - c) = (a + c) - b$
by (*simp add*: *diff-minus add-ac*)

lemma *eq-iff-diff-eq-0*: $a = b \longleftrightarrow a - b = 0$
by (*simp add*: *algebra-simps*)

lemma *diff-eq-0-iff-eq* [*simp*, *no-atp*]: $a - b = 0 \longleftrightarrow a = b$
by (*simp add*: *algebra-simps*)

end

4.6 (Partially) Ordered Groups

The theory of partially ordered groups is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer.

class *ordered-ab-semigroup-add* = *order* + *ab-semigroup-add* +
assumes *add-left-mono*: $a \leq b \Longrightarrow c + a \leq c + b$
begin

lemma *add-right-mono*:
  $a \leq b \Longrightarrow a + c \leq b + c$
by (*simp add*: *add-commute* [*of* - *c*] *add-left-mono*)

non-strict, in both arguments

lemma *add-mono*:
  $a \leq b \Longrightarrow c \leq d \Longrightarrow a + c \leq b + d$
  apply (*erule add-right-mono* [*THEN order-trans*])
  apply (*simp add*: *add-commute add-left-mono*)
  done

end

class *ordered-cancel-ab-semigroup-add* =
  ordered-ab-semigroup-add + *cancel-ab-semigroup-add*
begin

lemma *add-strict-left-mono*:
  $a < b \Longrightarrow c + a < c + b$
by (*auto simp add*: *less-le add-left-mono*)

lemma *add-strict-right-mono*:
  $a < b \Longrightarrow a + c < b + c$
by (*simp add*: *add-commute* [*of* - *c*] *add-strict-left-mono*)

Strict monotonicity in both arguments

lemma *add-strict-mono*:
  $a < b \Longrightarrow c < d \Longrightarrow a + c < b + d$
apply (*erule add-strict-right-mono* [*THEN less-trans*])
apply (*erule add-strict-left-mono*)
done

lemma *add-less-le-mono*:
  $a < b \Longrightarrow c \leq d \Longrightarrow a + c < b + d$
apply (*erule add-strict-right-mono* [*THEN less-le-trans*])
apply (*erule add-left-mono*)
done

lemma *add-le-less-mono*:
  $a \leq b \Longrightarrow c < d \Longrightarrow a + c < b + d$
apply (*erule add-right-mono* [*THEN le-less-trans*])
apply (*erule add-strict-left-mono*)
done

end

class *ordered-ab-semigroup-add-imp-le* =
  ordered-cancel-ab-semigroup-add +

assumes *add-le-imp-le-left*: $c + a \leq c + b \Longrightarrow a \leq b$
begin

lemma *add-less-imp-less-left*:
assumes *less*: $c + a < c + b$ **shows** $a < b$
proof −
from *less* **have** *le*: $c + a <= c + b$ **by** (*simp add*: *order-le-less*)
have $a <= b$
apply (*insert le*)
apply (*drule add-le-imp-le-left*)
by (*insert le*, *drule add-le-imp-le-left*, *assumption*)
moreover have $a \neq b$
proof (*rule ccontr*)
assume $^{\sim}(a \neq b)$
then have $a = b$ **by** *simp*
then have $c + a = c + b$ **by** *simp*
with *less* **show** *False***by** *simp*
qed
ultimately show $a < b$ **by** (*simp add*: *order-le-less*)
qed

lemma *add-less-imp-less-right*:
$a + c < b + c \Longrightarrow a < b$
apply (*rule add-less-imp-less-left* [*of c*])
apply (*simp add*: *add-commute*)
done

lemma *add-less-cancel-left* [*simp*]:
$c + a < c + b \longleftrightarrow a < b$
by (*blast intro*: *add-less-imp-less-left add-strict-left-mono*)

lemma *add-less-cancel-right* [*simp*]:
$a + c < b + c \longleftrightarrow a < b$
by (*blast intro*: *add-less-imp-less-right add-strict-right-mono*)

lemma *add-le-cancel-left* [*simp*]:
$c + a \leq c + b \longleftrightarrow a \leq b$
by (*auto*, *drule add-le-imp-le-left*, *simp-all add*: *add-left-mono*)

lemma *add-le-cancel-right* [*simp*]:
$a + c \leq b + c \longleftrightarrow a \leq b$
by (*simp add*: *add-commute* [*of a c*] *add-commute* [*of b c*])

lemma *add-le-imp-le-right*:
$a + c \leq b + c \Longrightarrow a \leq b$
by *simp*

lemma *max-add-distrib-left*:
$max\ x\ y + z = max\ (x + z)\ (y + z)$

unfolding *max-def* **by** *auto*

lemma *min-add-distrib-left*:
min x y + z = min (x + z) (y + z)
unfolding *min-def* **by** *auto*

end

4.7 Support for reasoning about signs

class *ordered-comm-monoid-add* =
ordered-cancel-ab-semigroup-add + comm-monoid-add
begin

lemma *add-pos-nonneg*:
assumes *0 < a* **and** *0 ≤ b* **shows** *0 < a + b*
proof −
have *0 + 0 < a + b*
using *assms* **by** (*rule add-less-le-mono*)
then show *?thesis* **by** *simp*
qed

lemma *add-pos-pos*:
assumes *0 < a* **and** *0 < b* **shows** *0 < a + b*
by (*rule add-pos-nonneg*) (*insert assms*, *auto*)

lemma *add-nonneg-pos*:
assumes *0 ≤ a* **and** *0 < b* **shows** *0 < a + b*
proof −
have *0 + 0 < a + b*
using *assms* **by** (*rule add-le-less-mono*)
then show *?thesis* **by** *simp*
qed

lemma *add-nonneg-nonneg* [*simp*]:
assumes *0 ≤ a* **and** *0 ≤ b* **shows** *0 ≤ a + b*
proof −
have *0 + 0 ≤ a + b*
using *assms* **by** (*rule add-mono*)
then show *?thesis* **by** *simp*
qed

lemma *add-neg-nonpos*:
assumes *a < 0* **and** *b ≤ 0* **shows** *a + b < 0*
proof −
have *a + b < 0 + 0*
using *assms* **by** (*rule add-less-le-mono*)
then show *?thesis* **by** *simp*
qed

lemma *add-neg-neg*:
assumes $a < 0$ **and** $b < 0$ **shows** $a + b < 0$
by (*rule add-neg-nonpos*) (*insert assms*, *auto*)

lemma *add-nonpos-neg*:
assumes $a \leq 0$ **and** $b < 0$ **shows** $a + b < 0$
proof −
have $a + b < 0 + 0$
using *assms* **by** (*rule add-le-less-mono*)
then show *?thesis* **by** *simp*
qed

lemma *add-nonpos-nonpos*:
assumes $a \leq 0$ **and** $b \leq 0$ **shows** $a + b \leq 0$
proof −
have $a + b \leq 0 + 0$
using *assms* **by** (*rule add-mono*)
then show *?thesis* **by** *simp*
qed

lemmas *add-sign-intros* =
add-pos-nonneg add-pos-pos add-nonneg-pos add-nonneg-nonneg
add-neg-nonpos add-neg-neg add-nonpos-neg add-nonpos-nonpos

lemma *add-nonneg-eq-0-iff*:
assumes x: $0 \leq x$ **and** y: $0 \leq y$
shows $x + y = 0 \longleftrightarrow x = 0 \wedge y = 0$
proof (*intro iffI conjI*)
have $x = x + 0$ **by** *simp*
also have $x + 0 \leq x + y$ **using** y **by** (*rule add-left-mono*)
also assume $x + y = 0$
also have $0 \leq x$ **using** x **.**
finally show $x = 0$ **.**
next
have $y = 0 + y$ **by** *simp*
also have $0 + y \leq x + y$ **using** x **by** (*rule add-right-mono*)
also assume $x + y = 0$
also have $0 \leq y$ **using** y **.**
finally show $y = 0$ **.**
next
assume $x = 0 \wedge y = 0$
then show $x + y = 0$ **by** *simp*
qed

end

class *ordered-ab-group-add* =
ab-group-add + *ordered-ab-semigroup-add*

begin

subclass *ordered-cancel-ab-semigroup-add* **..**

subclass *ordered-ab-semigroup-add-imp-le*
proof
fix a b c :: $'a$
assume $c + a \leq c + b$
hence $(-c) + (c + a) \leq (-c) + (c + b)$ **by** (*rule add-left-mono*)
hence $((-c) + c) + a \leq ((-c) + c) + b$ **by** (*simp only*: *add-assoc*)
thus $a \leq b$ **by** *simp*
qed

subclass *ordered-comm-monoid-add* **..**

lemma *max-diff-distrib-left*:
shows $max\ x\ y - z = max\ (x - z)\ (y - z)$
by (*simp add*: *diff-minus*, *rule max-add-distrib-left*)

lemma *min-diff-distrib-left*:
shows $min\ x\ y - z = min\ (x - z)\ (y - z)$
by (*simp add*: *diff-minus*, *rule min-add-distrib-left*)

lemma *le-imp-neg-le*:
assumes $a \leq b$ **shows** $-b \leq -a$
proof −
have $-a+a \leq -a+b$ **using** ‹$a \leq b$› **by** (*rule add-left-mono*)
hence $0 \leq -a+b$ **by** *simp*
hence $0 + (-b) \leq (-a + b) + (-b)$ **by** (*rule add-right-mono*)
thus *?thesis* **by** (*simp add*: *add-assoc*)
qed

lemma *neg-le-iff-le* [*simp*]: $- b \leq - a \longleftrightarrow a \leq b$
proof
assume $- b \leq - a$
hence $- (- a) \leq - (- b)$ **by** (*rule le-imp-neg-le*)
thus $a \leq b$ **by** *simp*
next
assume $a \leq b$
thus $-b \leq -a$ **by** (*rule le-imp-neg-le*)
qed

lemma *neg-le-0-iff-le* [*simp*]: $- a \leq 0 \longleftrightarrow 0 \leq a$
by (*subst neg-le-iff-le* [*symmetric*], *simp*)

lemma *neg-0-le-iff-le* [*simp*]: $0 \leq - a \longleftrightarrow a \leq 0$
by (*subst neg-le-iff-le* [*symmetric*], *simp*)

lemma *neg-less-iff-less* [*simp*]: $- b < - a \longleftrightarrow a < b$

by (*force simp add*: *less-le*)

lemma *neg-less-0-iff-less* [*simp*]: $- a < 0 \longleftrightarrow 0 < a$
by (*subst neg-less-iff-less* [*symmetric*], *simp*)

lemma *neg-0-less-iff-less* [*simp*]: $0 < - a \longleftrightarrow a < 0$
by (*subst neg-less-iff-less* [*symmetric*], *simp*)

The next several equations can make the simplifier loop!

lemma *less-minus-iff*: $a < - b \longleftrightarrow b < - a$
proof −
  have $(- (-a) < - b) = (b < - a)$ **by** (*rule neg-less-iff-less*)
  thus *?thesis* **by** *simp*
qed

lemma *minus-less-iff*: $- a < b \longleftrightarrow - b < a$
proof −
  have $(- a < - (-b)) = (- b < a)$ **by** (*rule neg-less-iff-less*)
  thus *?thesis* **by** *simp*
qed

lemma *le-minus-iff*: $a \leq - b \longleftrightarrow b \leq - a$
proof −
 have *mm*: !! a (b::'a). $(-(-a)) < -b \Longrightarrow -(-b) < -a$ **by** (*simp only*: *minus-less-iff*)
  have $(- (- a) <= -b) = (b <= - a)$
    apply (*auto simp only*: *le-less*)
    apply (*drule mm*)
    apply (*simp-all*)
    apply (*drule mm*[*simplified*], *assumption*)
    done
  then show *?thesis* **by** *simp*
qed

lemma *minus-le-iff*: $- a \leq b \longleftrightarrow - b \leq a$
by (*auto simp add*: *le-less minus-less-iff*)

lemma *less-iff-diff-less-0*: $a < b \longleftrightarrow a - b < 0$
proof −
  have $(a < b) = (a + (- b) < b + (-b))$
    by (*simp only*: *add-less-cancel-right*)
  also have ... = $(a - b < 0)$ **by** (*simp add*: *diff-minus*)
  finally show *?thesis* **.**
qed

lemma *diff-less-eq*[*algebra-simps*, *field-simps*]: $a - b < c \longleftrightarrow a < c + b$
apply (*subst less-iff-diff-less-0* [*of a*])
apply (*rule less-iff-diff-less-0* [*of - c*, *THEN ssubst*])
apply (*simp add*: *diff-minus add-ac*)
done

lemma *less-diff-eq*[*algebra-simps*, *field-simps*]: $a < c - b \longleftrightarrow a + b < c$
apply (*subst less-iff-diff-less-0* [*of* $a + b$])
apply (*subst less-iff-diff-less-0* [*of* a])
apply (*simp add*: *diff-minus add-ac*)
done

lemma *diff-le-eq*[*algebra-simps*, *field-simps*]: $a - b \leq c \longleftrightarrow a \leq c + b$
by (*auto simp add*: *le-less diff-less-eq diff-add-cancel add-diff-cancel*)

lemma *le-diff-eq*[*algebra-simps*, *field-simps*]: $a \leq c - b \longleftrightarrow a + b \leq c$
by (*auto simp add*: *le-less less-diff-eq diff-add-cancel add-diff-cancel*)

lemma *le-iff-diff-le-0*: $a \leq b \longleftrightarrow a - b \leq 0$
by (*simp add*: *algebra-simps*)

end

class *linordered-ab-semigroup-add* =
  linorder + *ordered-ab-semigroup-add*

class *linordered-cancel-ab-semigroup-add* =
  linorder + *ordered-cancel-ab-semigroup-add*
begin

subclass *linordered-ab-semigroup-add* **..**

subclass *ordered-ab-semigroup-add-imp-le*
proof
  fix a b c :: $'a$
  assume *le*: $c + a <= c + b$
  show $a <= b$
  proof (*rule ccontr*)
    assume w: $\sim a \leq b$
    hence $b <= a$ **by** (*simp add*: *linorder-not-le*)
    hence *le2*: $c + b <= c + a$ **by** (*rule add-left-mono*)
    have $a = b$
      apply (*insert le*)
      apply (*insert le2*)
      apply (*drule antisym*, *simp-all*)
      done
    with w **show** *False*
      by (*simp add*: *linorder-not-le* [*symmetric*])
  qed
qed

end

class *linordered-ab-group-add* = *linorder* + *ordered-ab-group-add*

begin

subclass *linordered-cancel-ab-semigroup-add* **..**

lemma *neg-less-eq-nonneg* [*simp*]:
$- a \leq a \longleftrightarrow 0 \leq a$
proof
assume *A*: $- a \leq a$ **show** $0 \leq a$
proof (*rule classical*)
assume $\neg\ 0 \leq a$
then have $a < 0$ **by** *auto*
with *A* **have** $- a < 0$ **by** (*rule le-less-trans*)
then show *?thesis* **by** *auto*
qed
next
assume *A*: $0 \leq a$ **show** $- a \leq a$
proof (*rule order-trans*)
show $- a \leq 0$ **using** *A* **by** (*simp add*: *minus-le-iff*)
next
show $0 \leq a$ **using** *A* **.**
qed
qed

lemma *neg-less-nonneg* [*simp*]:
$- a < a \longleftrightarrow 0 < a$
proof
assume *A*: $- a < a$ **show** $0 < a$
proof (*rule classical*)
assume $\neg\ 0 < a$
then have $a \leq 0$ **by** *auto*
with *A* **have** $- a < 0$ **by** (*rule less-le-trans*)
then show *?thesis* **by** *auto*
qed
next
assume *A*: $0 < a$ **show** $- a < a$
proof (*rule less-trans*)
show $- a < 0$ **using** *A* **by** (*simp add*: *minus-le-iff*)
next
show $0 < a$ **using** *A* **.**
qed
qed

lemma *less-eq-neg-nonpos* [*simp*]:
$a \leq - a \longleftrightarrow a \leq 0$
proof
assume *A*: $a \leq - a$ **show** $a \leq 0$
proof (*rule classical*)
assume $\neg\ a \leq 0$
then have $0 < a$ **by** *auto*

```
    then have 0 < − a using A by (rule less-le-trans)
    then show ?thesis by auto
  qed
next
  assume A: a ≤ 0 show a ≤ − a
  proof (rule order-trans)
    show 0 ≤ − a using A by (simp add: minus-le-iff)
  next
    show a ≤ 0 using A .
  qed
qed

lemma equal-neg-zero [simp]:
  a = − a ⟷ a = 0
proof
  assume a = 0 then show a = − a by simp
next
  assume A: a = − a show a = 0
  proof (cases 0 ≤ a)
    case True with A have 0 ≤ − a by auto
    with le-minus-iff have a ≤ 0 by simp
    with True show ?thesis by (auto intro: order-trans)
  next
    case False then have B: a ≤ 0 by auto
    with A have − a ≤ 0 by auto
    with B show ?thesis by (auto intro: order-trans)
  qed
qed

lemma neg-equal-zero [simp]:
  − a = a ⟷ a = 0
  by (auto dest: sym)

lemma double-zero [simp]:
  a + a = 0 ⟷ a = 0
proof
  assume assm: a + a = 0
  then have a: − a = a by (rule minus-unique)
  then show a = 0 by (simp only: neg-equal-zero)
qed simp

lemma double-zero-sym [simp]:
  0 = a + a ⟷ a = 0
  by (rule, drule sym) simp-all

lemma zero-less-double-add-iff-zero-less-single-add [simp]:
  0 < a + a ⟷ 0 < a
proof
  assume 0 < a + a
```

```
  then have 0 − a < a by (simp only: diff-less-eq)
  then have − a < a by simp
  then show 0 < a by (simp only: neg-less-nonneg)
next
  assume 0 < a
  with this have 0 + 0 < a + a
    by (rule add-strict-mono)
  then show 0 < a + a by simp
qed

lemma zero-le-double-add-iff-zero-le-single-add [simp]:
  0 ≤ a + a ⟷ 0 ≤ a
  by (auto simp add: le-less)

lemma double-add-less-zero-iff-single-add-less-zero [simp]:
  a + a < 0 ⟷ a < 0
proof −
  have ¬ a + a < 0 ⟷ ¬ a < 0
    by (simp add: not-less)
  then show ?thesis by simp
qed

lemma double-add-le-zero-iff-single-add-le-zero [simp]:
  a + a ≤ 0 ⟷ a ≤ 0
proof −
  have ¬ a + a ≤ 0 ⟷ ¬ a ≤ 0
    by (simp add: not-le)
  then show ?thesis by simp
qed

lemma le-minus-self-iff:
  a ≤ − a ⟷ a ≤ 0
proof −
  from add-le-cancel-left [of − a a + a 0]
  have a ≤ − a ⟷ a + a ≤ 0
    by (simp add: add-assoc [symmetric])
  thus ?thesis by simp
qed

lemma minus-le-self-iff:
  − a ≤ a ⟷ 0 ≤ a
proof −
  from add-le-cancel-left [of − a 0 a + a]
  have − a ≤ a ⟷ 0 ≤ a + a
    by (simp add: add-assoc [symmetric])
  thus ?thesis by simp
qed

lemma minus-max-eq-min:
```

$- \ max\ x\ y = min\ (-x)\ (-y)$
by (*auto simp add*: *max-def min-def*)

lemma *minus-min-eq-max*:
$- \ min\ x\ y = max\ (-x)\ (-y)$
by (*auto simp add*: *max-def min-def*)

end

context *ordered-comm-monoid-add*
begin

lemma *add-increasing*:
$0 \leq a \Longrightarrow b \leq c \Longrightarrow b \leq a + c$
by (*insert add-mono* [*of 0 a b c*], *simp*)

lemma *add-increasing2*:
$0 \leq c \Longrightarrow b \leq a \Longrightarrow b \leq a + c$
by (*simp add*: *add-increasing add-commute* [*of a*])

lemma *add-strict-increasing*:
$0 < a \Longrightarrow b \leq c \Longrightarrow b < a + c$
by (*insert add-less-le-mono* [*of 0 a b c*], *simp*)

lemma *add-strict-increasing2*:
$0 \leq a \Longrightarrow b < c \Longrightarrow b < a + c$
by (*insert add-le-less-mono* [*of 0 a b c*], *simp*)

end

class *abs* =
fixes *abs* :: $'a \Rightarrow 'a$
begin

notation (*xsymbols*)
abs (|-|)

notation (*HTML* **output**)
abs (|-|)

end

class *sgn* =
fixes *sgn* :: $'a \Rightarrow 'a$

class *abs-if* = *minus* + *uminus* + *ord* + *zero* + *abs* +
assumes *abs-if*: $|a| = (if\ a < 0\ then\ -\ a\ else\ a)$

class *sgn-if* = *minus* + *uminus* + *zero* + *one* + *ord* + *sgn* +

assumes *sgn-if*: $sgn\ x = (if\ x = 0\ then\ 0\ else\ if\ 0 < x\ then\ 1\ else\ -1)$
begin

lemma *sgn0* [*simp*]: $sgn\ 0 = 0$
by (*simp add*:*sgn-if*)

end

class *ordered-ab-group-add-abs* = *ordered-ab-group-add* + *abs* +
assumes *abs-ge-zero* [*simp*]: $|a| \geq 0$
and *abs-ge-self*: $a \leq |a|$
and *abs-leI*: $a \leq b \Longrightarrow -a \leq b \Longrightarrow |a| \leq b$
and *abs-minus-cancel* [*simp*]: $|-a| = |a|$
and *abs-triangle-ineq*: $|a + b| \leq |a| + |b|$
begin

lemma *abs-minus-le-zero*: $-|a| \leq 0$
unfolding *neg-le-0-iff-le* **by** *simp*

lemma *abs-of-nonneg* [*simp*]:
assumes *nonneg*: $0 \leq a$ **shows** $|a| = a$
proof (*rule antisym*)
from *nonneg le-imp-neg-le* **have** $-a \leq 0$ **by** *simp*
from *this nonneg* **have** $-a \leq a$ **by** (*rule order-trans*)
then show $|a| \leq a$ **by** (*auto intro*: *abs-leI*)
qed (*rule abs-ge-self*)

lemma *abs-idempotent* [*simp*]: $||a|| = |a|$
by (*rule antisym*)
(*auto intro*!: *abs-ge-self abs-leI order-trans* [*of* $-|a|$ *0* $|a|$])

lemma *abs-eq-0* [*simp*]: $|a| = 0 \longleftrightarrow a = 0$
proof −
have $|a| = 0 \Longrightarrow a = 0$
proof (*rule antisym*)
assume *zero*: $|a| = 0$
with *abs-ge-self* **show** $a \leq 0$ **by** *auto*
from *zero* **have** $|-a| = 0$ **by** *simp*
with *abs-ge-self* [*of* $-a$] **have** $-a \leq 0$ **by** *auto*
with *neg-le-0-iff-le* **show** $0 \leq a$ **by** *auto*
qed
then show *?thesis* **by** *auto*
qed

lemma *abs-zero* [*simp*]: $|0| = 0$
by *simp*

lemma *abs-0-eq* [*simp*, *no-atp*]: $0 = |a| \longleftrightarrow a = 0$
proof −

have $0 = |a| \longleftrightarrow |a| = 0$ **by** (*simp only*: *eq-ac*)
thus *?thesis* **by** *simp*
qed

lemma *abs-le-zero-iff* [*simp*]: $|a| \leq 0 \longleftrightarrow a = 0$
proof
assume $|a| \leq 0$
then have $|a| = 0$ **by** (*rule antisym*) *simp*
thus $a = 0$ **by** *simp*
next
assume $a = 0$
thus $|a| \leq 0$ **by** *simp*
qed

lemma *zero-less-abs-iff* [*simp*]: $0 < |a| \longleftrightarrow a \neq 0$
by (*simp add*: *less-le*)

lemma *abs-not-less-zero* [*simp*]: $\neg\ |a| < 0$
proof −
have *a*: $\bigwedge x\ y.\ x \leq y \Longrightarrow \neg\ y < x$ **by** *auto*
show *?thesis* **by** (*simp add*: *a*)
qed

lemma *abs-ge-minus-self*: $-\ a \leq |a|$
proof −
have $-\ a \leq |-a|$ **by** (*rule abs-ge-self*)
then show *?thesis* **by** *simp*
qed

lemma *abs-minus-commute*:
$|a - b| = |b - a|$
proof −
have $|a - b| = |-\ (a - b)|$ **by** (*simp only*: *abs-minus-cancel*)
also have ... $= |b - a|$ **by** *simp*
finally show *?thesis* **.**
qed

lemma *abs-of-pos*: $0 < a \Longrightarrow |a| = a$
by (*rule abs-of-nonneg*, *rule less-imp-le*)

lemma *abs-of-nonpos* [*simp*]:
assumes $a \leq 0$ **shows** $|a| = -\ a$
proof −
let *?b* $= -\ a$
have $-\ ?b \leq 0 \Longrightarrow |-\ ?b| = -\ (-\ ?b)$
unfolding *abs-minus-cancel* [*of ?b*]
unfolding *neg-le-0-iff-le* [*of ?b*]
unfolding *minus-minus* **by** (*erule abs-of-nonneg*)
then show *?thesis* **using** *assms* **by** *auto*

qed

lemma *abs-of-neg*: $a < 0 \Longrightarrow |a| = - a$
by (*rule abs-of-nonpos*, *rule less-imp-le*)

lemma *abs-le-D1*: $|a| \leq b \Longrightarrow a \leq b$
by (*insert abs-ge-self*, *blast intro*: *order-trans*)

lemma *abs-le-D2*: $|a| \leq b \Longrightarrow - a \leq b$
by (*insert abs-le-D1* [*of* $- a$], *simp*)

lemma *abs-le-iff*: $|a| \leq b \longleftrightarrow a \leq b \wedge - a \leq b$
by (*blast intro*: *abs-leI dest*: *abs-le-D1 abs-le-D2*)

lemma *abs-triangle-ineq2*: $|a| - |b| \leq |a - b|$
proof −
  have $|a| = |b + (a - b)|$
    by (*simp add*: *algebra-simps add-diff-cancel*)
  then have $|a| \leq |b| + |a - b|$
    by (*simp add*: *abs-triangle-ineq*)
  then show *?thesis*
    by (*simp add*: *algebra-simps*)
qed

lemma *abs-triangle-ineq2-sym*: $|a| - |b| \leq |b - a|$
  by (*simp only*: *abs-minus-commute* [*of b*] *abs-triangle-ineq2*)

lemma *abs-triangle-ineq3*: $||a| - |b|| \leq |a - b|$
  by (*simp add*: *abs-le-iff abs-triangle-ineq2 abs-triangle-ineq2-sym*)

lemma *abs-triangle-ineq4*: $|a - b| \leq |a| + |b|$
proof −
  have $|a - b| = |a + - b|$ **by** (*subst diff-minus*, *rule refl*)
  also have $... \leq |a| + |- b|$ **by** (*rule abs-triangle-ineq*)
  finally show *?thesis* **by** *simp*
qed

lemma *abs-diff-triangle-ineq*: $|a + b - (c + d)| \leq |a - c| + |b - d|$
proof −
  have $|a + b - (c+d)| = |(a-c) + (b-d)|$ **by** (*simp add*: *diff-minus add-ac*)
  also have $... \leq |a-c| + |b-d|$ **by** (*rule abs-triangle-ineq*)
  finally show *?thesis* **.**
qed

lemma *abs-add-abs* [*simp*]:
  $||a| + |b|| = |a| + |b|$ (**is** *?L* = *?R*)
proof (*rule antisym*)
  show *?L* $\geq$ *?R* **by**(*rule abs-ge-self*)
next

have *?L* $\leq ||a|| + ||b||$ **by**(*rule abs-triangle-ineq*)
also have $\ldots = ?R$ **by** *simp*
finally show *?L* $\leq$ *?R* **.**
qed

end

Needed for abelian cancellation simprocs:

lemma *add-cancel-21*: $((x::'a::ab\text{-}group\text{-}add) + (y + z) = y + u) = (x + z = u)$
apply (*subst add-left-commute*)
apply (*subst add-left-cancel*)
apply *simp*
done

lemma *add-cancel-end*: $(x + (y + z) = y) = (x = -\ (z::'a::ab\text{-}group\text{-}add))$
apply (*subst add-cancel-21*[*of - - - 0*, *simplified*])
apply (*simp add*: *add-right-cancel*[*symmetric*, *of x* $-z$ *z*, *simplified*])
done

lemma *less-eqI*: $(x::'a::ordered\text{-}ab\text{-}group\text{-}add) - y = x' - y' \Longrightarrow (x < y) = (x' < y')$
by (*simp add*: *less-iff-diff-less-0*[*of x y*] *less-iff-diff-less-0*[*of x' y'*])

lemma *le-eqI*: $(x::'a::ordered\text{-}ab\text{-}group\text{-}add) - y = x' - y' \Longrightarrow (y <= x) = (y' <= x')$
apply (*simp add*: *le-iff-diff-le-0*[*of y x*] *le-iff-diff-le-0*[*of y' x'*])
apply (*simp add*: *neg-le-iff-le*[*symmetric*, *of* $y-x$ *0*] *neg-le-iff-le*[*symmetric*, *of* $y'-x'$ *0*])
done

lemma *eq-eqI*: $(x::'a::ab\text{-}group\text{-}add) - y = x' - y' \Longrightarrow (x = y) = (x' = y')$
by (*simp only*: *eq-iff-diff-eq-0*[*of x y*] *eq-iff-diff-eq-0*[*of x' y'*])

lemma *diff-def*: $(x::'a::ab\text{-}group\text{-}add) - y == x + (-y)$
by (*simp add*: *diff-minus*)

lemma *le-add-right-mono*:
assumes
$a <= b + (c::'a::ordered\text{-}ab\text{-}group\text{-}add)$
$c <= d$
shows $a <= b + d$
apply (*rule-tac order-trans*[**where** $y = b+c$])
apply (*simp-all add*: *prems*)
done

4.8 Tools setup

lemma *add-mono-thms-linordered-semiring* [*no-atp*]:
fixes *i j k* :: $'a::ordered\text{-}ab\text{-}semigroup\text{-}add$

shows $i \leq j \wedge k \leq l \Longrightarrow i + k \leq j + l$
and $i = j \wedge k \leq l \Longrightarrow i + k \leq j + l$
and $i \leq j \wedge k = l \Longrightarrow i + k \leq j + l$
and $i = j \wedge k = l \Longrightarrow i + k = j + l$
by (*rule add-mono*, *clarify*+)+

lemma *add-mono-thms-linordered-field* [*no-atp*]:
fixes $i\ j\ k :: 'a$::*ordered-cancel-ab-semigroup-add*
shows $i < j \wedge k = l \Longrightarrow i + k < j + l$
and $i = j \wedge k < l \Longrightarrow i + k < j + l$
and $i < j \wedge k \leq l \Longrightarrow i + k < j + l$
and $i \leq j \wedge k < l \Longrightarrow i + k < j + l$
and $i < j \wedge k < l \Longrightarrow i + k < j + l$
by (*auto intro*: *add-strict-right-mono add-strict-left-mono*
add-less-le-mono add-le-less-mono add-strict-mono)

Simplification of $x - y < (0::'a)$, etc.

lemmas *diff-less-0-iff-less* [*simp*, *no-atp*] = *less-iff-diff-less-0* [*symmetric*]
lemmas *diff-le-0-iff-le* [*simp*, *no-atp*] = *le-iff-diff-le-0* [*symmetric*]

ML ⟪
structure ab-group-add-cancel = Abel-Cancel
(

(* *term order for abelian groups* *)

fun agrp-ord (Const (a, -)) = find-index (fn a' => a = a')
[@{const-name Groups.zero}, @{const-name Groups.plus},
@{const-name Groups.uminus}, @{const-name Groups.minus}]
| agrp-ord - = ~1;

fun termless-agrp (a, b) = (Term-Ord.term-lpo agrp-ord (a, b) = LESS);

local
val ac1 = mk-meta-eq @{thm add-assoc};
val ac2 = mk-meta-eq @{thm add-commute};
val ac3 = mk-meta-eq @{thm add-left-commute};
fun solve-add-ac thy - (- $ (Const (@{const-name Groups.plus},-) $ - $ -) $ -) =
SOME ac1
| solve-add-ac thy - (- $ x $ (Const (@{const-name Groups.plus},-) $ y $ z)) =
if termless-agrp (y, x) then SOME ac3 else NONE
| solve-add-ac thy - (- $ x $ y) =
if termless-agrp (y, x) then SOME ac2 else NONE
| solve-add-ac thy - - = NONE
in
val add-ac-proc = Simplifier.simproc @{theory}
add-ac-proc [x + y::'a::ab-semigroup-add] solve-add-ac;
end;

val eq-reflection = @{*thm eq-reflection*};

val T = @{*typ 'a*::*ab-group-add*};

val cancel-ss = *HOL-basic-ss settermless termless-agrp*
addsimprocs [*add-ac-proc*] *addsimps*
[@{*thm add-0-left*}, @{*thm add-0-right*}, @{*thm diff-def*},
@{*thm minus-add-distrib*}, @{*thm minus-minus*}, @{*thm minus-zero*},
@{*thm right-minus*}, @{*thm left-minus*}, @{*thm add-minus-cancel*},
@{*thm minus-add-cancel*}];

val sum-pats = [@{*cterm x* + *y*::*'a*::*ab-group-add*}, @{*cterm x* − *y*::*'a*::*ab-group-add*}];

val eqI-rules = [@{*thm less-eqI*}, @{*thm le-eqI*}, @{*thm eq-eqI*}];

val dest-eqI =
fst o HOLogic.dest-bin @{*const-name op* =} *HOLogic.boolT o HOLogic.dest-Trueprop*
o concl-of;

);
⟩⟩

ML ⟨⟨
Addsimprocs [*ab-group-add-cancel.sum-conv*, *ab-group-add-cancel.rel-conv*];
⟩⟩

code-modulename *SML*
Groups Arith

code-modulename *OCaml*
Groups Arith

code-modulename *Haskell*
Groups Arith

end

5 Lattices: Abstract lattices

theory *Lattices*
imports *Orderings Groups*
begin

5.1 Abstract semilattice

This locales provide a basic structure for interpretation into bigger structures; extensions require careful thinking, otherwise undesired effects may

occur due to interpretation.

locale *semilattice* = *abel-semigroup* +
  assumes *idem* [*simp*]: *f a a* = *a*
begin

lemma *left-idem* [*simp*]:
  f a (*f a b*) = *f a b*
  by (*simp add*: *assoc* [*symmetric*])

end

5.2 Idempotent semigroup

class *ab-semigroup-idem-mult* = *ab-semigroup-mult* +
  assumes *mult-idem*: *x* ∗ *x* = *x*

sublocale *ab-semigroup-idem-mult* < *times*!: *semilattice times* **proof**
qed (*fact mult-idem*)

context *ab-semigroup-idem-mult*
begin

lemmas *mult-left-idem* = *times.left-idem*

end

5.3 Concrete lattices

notation
  less-eq (**infix** ⊑ *50*) **and**
  less (**infix** ⊏ *50*) **and**
  top (⊤) **and**
  bot (⊥)

class *semilattice-inf* = *order* +
  fixes *inf* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** ⊓ *70*)
  assumes *inf-le1* [*simp*]: *x* ⊓ *y* ⊑ *x*
  and *inf-le2* [*simp*]: *x* ⊓ *y* ⊑ *y*
  and *inf-greatest*: *x* ⊑ *y* ⟹ *x* ⊑ *z* ⟹ *x* ⊑ *y* ⊓ *z*

class *semilattice-sup* = *order* +
  fixes *sup* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** ⊔ *65*)
  assumes *sup-ge1* [*simp*]: *x* ⊑ *x* ⊔ *y*
  and *sup-ge2* [*simp*]: *y* ⊑ *x* ⊔ *y*
  and *sup-least*: *y* ⊑ *x* ⟹ *z* ⊑ *x* ⟹ *y* ⊔ *z* ⊑ *x*
begin

Dual lattice

lemma *dual-semilattice*:

class.semilattice-inf (*op* ≥) (*op* >) *sup*
by (*rule class.semilattice-inf.intro*, *rule dual-order*)
(*unfold-locales*, *simp-all add*: *sup-least*)

end

class *lattice* = *semilattice-inf* + *semilattice-sup*

5.3.1 Intro and elim rules

context *semilattice-inf*
begin

lemma *le-infI1*:
$a \sqsubseteq x \Longrightarrow a \sqcap b \sqsubseteq x$
by (*rule order-trans*) *auto*

lemma *le-infI2*:
$b \sqsubseteq x \Longrightarrow a \sqcap b \sqsubseteq x$
by (*rule order-trans*) *auto*

lemma *le-infI*: $x \sqsubseteq a \Longrightarrow x \sqsubseteq b \Longrightarrow x \sqsubseteq a \sqcap b$
by (*rule inf-greatest*)

lemma *le-infE*: $x \sqsubseteq a \sqcap b \Longrightarrow (x \sqsubseteq a \Longrightarrow x \sqsubseteq b \Longrightarrow P) \Longrightarrow P$
by (*blast intro*: *order-trans inf-le1 inf-le2*)

lemma *le-inf-iff* [*simp*]:
$x \sqsubseteq y \sqcap z \longleftrightarrow x \sqsubseteq y \wedge x \sqsubseteq z$
by (*blast intro*: *le-infI elim*: *le-infE*)

lemma *le-iff-inf*:
$x \sqsubseteq y \longleftrightarrow x \sqcap y = x$
by (*auto intro*: *le-infI1 antisym dest*: *eq-iff* [*THEN iffD1*])

lemma *inf-mono*: $a \sqsubseteq c \Longrightarrow b \leq d \Longrightarrow a \sqcap b \sqsubseteq c \sqcap d$
by (*fast intro*: *inf-greatest le-infI1 le-infI2*)

lemma *mono-inf*:
fixes $f :: 'a \Rightarrow 'b$::*semilattice-inf*
shows *mono* $f \Longrightarrow f\ (A \sqcap B) \sqsubseteq f\ A \sqcap f\ B$
by (*auto simp add*: *mono-def intro*: *Lattices.inf-greatest*)

end

context *semilattice-sup*
begin

lemma *le-supI1*:

$x \sqsubseteq a \Longrightarrow x \sqsubseteq a \sqcup b$
by (*rule order-trans*) *auto*

lemma *le-supI2*:
$x \sqsubseteq b \Longrightarrow x \sqsubseteq a \sqcup b$
by (*rule order-trans*) *auto*

lemma *le-supI*:
$a \sqsubseteq x \Longrightarrow b \sqsubseteq x \Longrightarrow a \sqcup b \sqsubseteq x$
by (*rule sup-least*)

lemma *le-supE*:
$a \sqcup b \sqsubseteq x \Longrightarrow (a \sqsubseteq x \Longrightarrow b \sqsubseteq x \Longrightarrow P) \Longrightarrow P$
by (*blast intro*: *order-trans sup-ge1 sup-ge2*)

lemma *le-sup-iff* [*simp*]:
$x \sqcup y \sqsubseteq z \longleftrightarrow x \sqsubseteq z \wedge y \sqsubseteq z$
by (*blast intro*: *le-supI elim*: *le-supE*)

lemma *le-iff-sup*:
$x \sqsubseteq y \longleftrightarrow x \sqcup y = y$
by (*auto intro*: *le-supI2 antisym dest*: *eq-iff* [*THEN iffD1*])

lemma *sup-mono*: $a \sqsubseteq c \Longrightarrow b \leq d \Longrightarrow a \sqcup b \sqsubseteq c \sqcup d$
by (*fast intro*: *sup-least le-supI1 le-supI2*)

lemma *mono-sup*:
fixes $f :: 'a \Rightarrow 'b$::*semilattice-sup*
shows *mono* $f \Longrightarrow f\ A \sqcup f\ B \sqsubseteq f\ (A \sqcup B)$
by (*auto simp add*: *mono-def intro*: *Lattices.sup-least*)

end

5.3.2 Equational laws

sublocale *semilattice-inf* $<$ *inf*!: *semilattice inf*
proof
fix *a b c*
show $(a \sqcap b) \sqcap c = a \sqcap (b \sqcap c)$
by (*rule antisym*) (*auto intro*: *le-infI1 le-infI2*)
show $a \sqcap b = b \sqcap a$
by (*rule antisym*) *auto*
show $a \sqcap a = a$
by (*rule antisym*) *auto*
qed

context *semilattice-inf*
begin

lemma *inf-assoc*: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
by (*fact inf.assoc*)

lemma *inf-commute*: $(x \sqcap y) = (y \sqcap x)$
by (*fact inf.commute*)

lemma *inf-left-commute*: $x \sqcap (y \sqcap z) = y \sqcap (x \sqcap z)$
by (*fact inf.left-commute*)

lemma *inf-idem*: $x \sqcap x = x$
by (*fact inf.idem*)

lemma *inf-left-idem*: $x \sqcap (x \sqcap y) = x \sqcap y$
by (*fact inf.left-idem*)

lemma *inf-absorb1*: $x \sqsubseteq y \Longrightarrow x \sqcap y = x$
by (*rule antisym*) *auto*

lemma *inf-absorb2*: $y \sqsubseteq x \Longrightarrow x \sqcap y = y$
by (*rule antisym*) *auto*

lemmas *inf-aci* = *inf-commute inf-assoc inf-left-commute inf-left-idem*

end

sublocale *semilattice-sup* < *sup*!: *semilattice sup*
proof
fix *a b c*
show $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$
by (*rule antisym*) (*auto intro*: *le-supI1 le-supI2*)
show $a \sqcup b = b \sqcup a$
by (*rule antisym*) *auto*
show $a \sqcup a = a$
by (*rule antisym*) *auto*
qed

context *semilattice-sup*
begin

lemma *sup-assoc*: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
by (*fact sup.assoc*)

lemma *sup-commute*: $(x \sqcup y) = (y \sqcup x)$
by (*fact sup.commute*)

lemma *sup-left-commute*: $x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
by (*fact sup.left-commute*)

lemma *sup-idem*: $x \sqcup x = x$

by (*fact sup.idem*)

lemma *sup-left-idem*: $x \sqcup (x \sqcup y) = x \sqcup y$
by (*fact sup.left-idem*)

lemma *sup-absorb1*: $y \sqsubseteq x \Longrightarrow x \sqcup y = x$
by (*rule antisym*) *auto*

lemma *sup-absorb2*: $x \sqsubseteq y \Longrightarrow x \sqcup y = y$
by (*rule antisym*) *auto*

lemmas *sup-aci* = *sup-commute sup-assoc sup-left-commute sup-left-idem*

end

context *lattice*
begin

lemma *dual-lattice*:
class.lattice (*op* $\geq$) (*op* $>$) *sup inf*
by (*rule class.lattice.intro*, *rule dual-semilattice*, *rule class.semilattice-sup.intro*, *rule dual-order*)
(*unfold-locales*, *auto*)

lemma *inf-sup-absorb*: $x \sqcap (x \sqcup y) = x$
by (*blast intro*: *antisym inf-le1 inf-greatest sup-ge1*)

lemma *sup-inf-absorb*: $x \sqcup (x \sqcap y) = x$
by (*blast intro*: *antisym sup-ge1 sup-least inf-le1*)

lemmas *inf-sup-aci* = *inf-aci sup-aci*

lemmas *inf-sup-ord* = *inf-le1 inf-le2 sup-ge1 sup-ge2*

Towards distributivity

lemma *distrib-sup-le*: $x \sqcup (y \sqcap z) \sqsubseteq (x \sqcup y) \sqcap (x \sqcup z)$
by (*auto intro*: *le-infI1 le-infI2 le-supI1 le-supI2*)

lemma *distrib-inf-le*: $(x \sqcap y) \sqcup (x \sqcap z) \sqsubseteq x \sqcap (y \sqcup z)$
by (*auto intro*: *le-infI1 le-infI2 le-supI1 le-supI2*)

If you have one of them, you have them all.

lemma *distrib-imp1*:
assumes *D*: $!!x\ y\ z.\ x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
shows $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
proof−
have $x \sqcup (y \sqcap z) = (x \sqcup (x \sqcap z)) \sqcup (y \sqcap z)$ **by**(*simp add*:*sup-inf-absorb*)
also have $\ldots = x \sqcup (z \sqcap (x \sqcup y))$ **by**(*simp add*:*D inf-commute sup-assoc*)
also have $\ldots = ((x \sqcup y) \sqcap x) \sqcup ((x \sqcup y) \sqcap z)$

by(*simp add*:*inf-sup-absorb inf-commute*)
also have ... = $(x \sqcup y) \sqcap (x \sqcup z)$ **by**(*simp add*:*D*)
finally show *?thesis* **.**
qed

lemma *distrib-imp2*:
assumes *D*: !!*x y z*. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
shows $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
proof−
have $x \sqcap (y \sqcup z) = (x \sqcap (x \sqcup z)) \sqcap (y \sqcup z)$ **by**(*simp add*:*inf-sup-absorb*)
also have ... = $x \sqcap (z \sqcup (x \sqcap y))$ **by**(*simp add*:*D sup-commute inf-assoc*)
also have ... = $((x \sqcap y) \sqcup x) \sqcap ((x \sqcap y) \sqcup z)$
by(*simp add*:*sup-inf-absorb sup-commute*)
also have ... = $(x \sqcap y) \sqcup (x \sqcap z)$ **by**(*simp add*:*D*)
finally show *?thesis* **.**
qed

end

5.3.3 Strict order

context *semilattice-inf*
begin

lemma *less-infI1*:
$a \sqsubset x \Longrightarrow a \sqcap b \sqsubset x$
by (*auto simp add*: *less-le inf-absorb1 intro*: *le-infI1*)

lemma *less-infI2*:
$b \sqsubset x \Longrightarrow a \sqcap b \sqsubset x$
by (*auto simp add*: *less-le inf-absorb2 intro*: *le-infI2*)

end

context *semilattice-sup*
begin

lemma *less-supI1*:
$x \sqsubset a \Longrightarrow x \sqsubset a \sqcup b$
proof −
interpret *dual*: *semilattice-inf op* $\geq$ *op* $>$ *sup*
by (*fact dual-semilattice*)
assume $x \sqsubset a$
then show $x \sqsubset a \sqcup b$
by (*fact dual.less-infI1*)
qed

lemma *less-supI2*:
$x \sqsubset b \Longrightarrow x \sqsubset a \sqcup b$

proof −
interpret *dual*: *semilattice-inf op* ≥ *op* > *sup*
by (*fact dual-semilattice*)
assume $x \sqsubset b$
then show $x \sqsubset a \sqcup b$
by (*fact dual.less-infI2*)
qed

end

5.4 Distributive lattices

class *distrib-lattice* = *lattice* +
assumes *sup-inf-distrib1*: $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$

context *distrib-lattice*
begin

lemma *sup-inf-distrib2*:
$(y \sqcap z) \sqcup x = (y \sqcup x) \sqcap (z \sqcup x)$
by(*simp add*: *inf-sup-aci sup-inf-distrib1*)

lemma *inf-sup-distrib1*:
$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
by(*rule distrib-imp2*[*OF sup-inf-distrib1*])

lemma *inf-sup-distrib2*:
$(y \sqcup z) \sqcap x = (y \sqcap x) \sqcup (z \sqcap x)$
by(*simp add*: *inf-sup-aci inf-sup-distrib1*)

lemma *dual-distrib-lattice*:
class.distrib-lattice (*op* ≥) (*op* >) *sup inf*
by (*rule class.distrib-lattice.intro*, *rule dual-lattice*)
(*unfold-locales*, *fact inf-sup-distrib1*)

lemmas *sup-inf-distrib* =
sup-inf-distrib1 sup-inf-distrib2

lemmas *inf-sup-distrib* =
inf-sup-distrib1 inf-sup-distrib2

lemmas *distrib* =
sup-inf-distrib1 sup-inf-distrib2 inf-sup-distrib1 inf-sup-distrib2

end

5.5 Bounded lattices and boolean algebras

class *bounded-lattice-bot* = *lattice* + *bot*
begin

lemma *inf-bot-left* [*simp*]:
$\bot \sqcap x = \bot$
by (*rule inf-absorb1*) *simp*

lemma *inf-bot-right* [*simp*]:
$x \sqcap \bot = \bot$
by (*rule inf-absorb2*) *simp*

lemma *sup-bot-left* [*simp*]:
$\bot \sqcup x = x$
by (*rule sup-absorb2*) *simp*

lemma *sup-bot-right* [*simp*]:
$x \sqcup \bot = x$
by (*rule sup-absorb1*) *simp*

lemma *sup-eq-bot-iff* [*simp*]:
$x \sqcup y = \bot \longleftrightarrow x = \bot \wedge y = \bot$
by (*simp add*: *eq-iff*)

end

class *bounded-lattice-top* = *lattice* + *top*
begin

lemma *sup-top-left* [*simp*]:
$\top \sqcup x = \top$
by (*rule sup-absorb1*) *simp*

lemma *sup-top-right* [*simp*]:
$x \sqcup \top = \top$
by (*rule sup-absorb2*) *simp*

lemma *inf-top-left* [*simp*]:
$\top \sqcap x = x$
by (*rule inf-absorb2*) *simp*

lemma *inf-top-right* [*simp*]:
$x \sqcap \top = x$
by (*rule inf-absorb1*) *simp*

lemma *inf-eq-top-iff* [*simp*]:
$x \sqcap y = \top \longleftrightarrow x = \top \wedge y = \top$
by (*simp add*: *eq-iff*)

end

class *bounded-lattice* = *bounded-lattice-bot* + *bounded-lattice-top*

begin

lemma *dual-bounded-lattice*:
class.bounded-lattice $(op \geq)$ $(op >)$ $(op \sqcup)$ $(op \sqcap)$ $\top$ $\bot$
by *unfold-locales* (*auto simp add*: *less-le-not-le*)

end

class *boolean-algebra* = *distrib-lattice* + *bounded-lattice* + *minus* + *uminus* +
assumes *inf-compl-bot*: $x \sqcap - x = \bot$
and *sup-compl-top*: $x \sqcup - x = \top$
assumes *diff-eq*: $x - y = x \sqcap - y$
begin

lemma *dual-boolean-algebra*:
class.boolean-algebra $(\lambda x\ y.\ x \sqcup - y)$ *uminus* $(op \geq)$ $(op >)$ $(op \sqcup)$ $(op \sqcap)$ $\top$ $\bot$
by (*rule class.boolean-algebra.intro*, *rule dual-bounded-lattice*, *rule dual-distrib-lattice*)
(*unfold-locales*, *auto simp add*: *inf-compl-bot sup-compl-top diff-eq*)

lemma *compl-inf-bot*:
$- x \sqcap x = \bot$
by (*simp add*: *inf-commute inf-compl-bot*)

lemma *compl-sup-top*:
$- x \sqcup x = \top$
by (*simp add*: *sup-commute sup-compl-top*)

lemma *compl-unique*:
assumes $x \sqcap y = \bot$
and $x \sqcup y = \top$
shows $- x = y$
proof −
have $(x \sqcap - x) \sqcup (- x \sqcap y) = (x \sqcap y) \sqcup (- x \sqcap y)$
using *inf-compl-bot assms*(*1*) **by** *simp*
then have $(- x \sqcap x) \sqcup (- x \sqcap y) = (y \sqcap x) \sqcup (y \sqcap - x)$
by (*simp add*: *inf-commute*)
then have $- x \sqcap (x \sqcup y) = y \sqcap (x \sqcup - x)$
by (*simp add*: *inf-sup-distrib1*)
then have $- x \sqcap \top = y \sqcap \top$
using *sup-compl-top assms*(*2*) **by** *simp*
then show $- x = y$ **by** *simp*
qed

lemma *double-compl* [*simp*]:
$- (- x) = x$
using *compl-inf-bot compl-sup-top* **by** (*rule compl-unique*)

lemma *compl-eq-compl-iff* [*simp*]:
$- x = - y \longleftrightarrow x = y$

proof
assume $- x = - y$
then have $- (- x) = - (- y)$ **by** (*rule arg-cong*)
then show $x = y$ **by** *simp*
next
assume $x = y$
then show $- x = - y$ **by** *simp*
qed

lemma *compl-bot-eq* [*simp*]:
$- \bot = \top$
proof −
from *sup-compl-top* **have** $\bot \sqcup - \bot = \top$ **.**
then show *?thesis* **by** *simp*
qed

lemma *compl-top-eq* [*simp*]:
$- \top = \bot$
proof −
from *inf-compl-bot* **have** $\top \sqcap - \top = \bot$ **.**
then show *?thesis* **by** *simp*
qed

lemma *compl-inf* [*simp*]:
$- (x \sqcap y) = - x \sqcup - y$
proof (*rule compl-unique*)
have $(x \sqcap y) \sqcap (- x \sqcup - y) = (y \sqcap (x \sqcap - x)) \sqcup (x \sqcap (y \sqcap - y))$
by (*simp only*: *inf-sup-distrib inf-aci*)
then show $(x \sqcap y) \sqcap (- x \sqcup - y) = \bot$
by (*simp add*: *inf-compl-bot*)
next
have $(x \sqcap y) \sqcup (- x \sqcup - y) = (- y \sqcup (x \sqcup - x)) \sqcap (- x \sqcup (y \sqcup - y))$
by (*simp only*: *sup-inf-distrib sup-aci*)
then show $(x \sqcap y) \sqcup (- x \sqcup - y) = \top$
by (*simp add*: *sup-compl-top*)
qed

lemma *compl-sup* [*simp*]:
$- (x \sqcup y) = - x \sqcap - y$
proof −
interpret *boolean-algebra* $\lambda x\ y.\ x \sqcup - y$ *uminus op* $\geq$ *op* $>$ *op* $\sqcup$ *op* $\sqcap$ $\top$ $\bot$
by (*rule dual-boolean-algebra*)
then show *?thesis* **by** *simp*
qed

lemma *compl-mono*:
$x \sqsubseteq y \Longrightarrow - y \sqsubseteq - x$
proof −
assume $x \sqsubseteq y$

then have $x \sqcup y = y$ **by** (*simp only*: *le-iff-sup*)
then have $- (x \sqcup y) = - y$ **by** *simp*
then have $- x \sqcap - y = - y$ **by** *simp*
then have $- y \sqcap - x = - y$ **by** (*simp only*: *inf-commute*)
then show $- y \sqsubseteq - x$ **by** (*simp only*: *le-iff-inf*)
qed

lemma *compl-le-compl-iff*:
$- x \leq - y \longleftrightarrow y \leq x$
by (*auto dest*: *compl-mono*)

end

5.6 Uniqueness of inf and sup

lemma (**in** *semilattice-inf*) *inf-unique*:
fixes *f* (**infixl** $\triangle$ *70*)
assumes *le1*: $\bigwedge x\ y.\ x \triangle y \sqsubseteq x$ **and** *le2*: $\bigwedge x\ y.\ x \triangle y \sqsubseteq y$
and *greatest*: $\bigwedge x\ y\ z.\ x \sqsubseteq y \Longrightarrow x \sqsubseteq z \Longrightarrow x \sqsubseteq y \triangle z$
shows $x \sqcap y = x \triangle y$
proof (*rule antisym*)
show $x \triangle y \sqsubseteq x \sqcap y$ **by** (*rule le-infI*) (*rule le1*, *rule le2*)
next
have *leI*: $\bigwedge x\ y\ z.\ x \sqsubseteq y \Longrightarrow x \sqsubseteq z \Longrightarrow x \sqsubseteq y \triangle z$ **by** (*blast intro*: *greatest*)
show $x \sqcap y \sqsubseteq x \triangle y$ **by** (*rule leI*) *simp-all*
qed

lemma (**in** *semilattice-sup*) *sup-unique*:
fixes *f* (**infixl** ∇ *70*)
assumes *ge1* [*simp*]: $\bigwedge x\ y.\ x \sqsubseteq x \nabla y$ **and** *ge2*: $\bigwedge x\ y.\ y \sqsubseteq x \nabla y$
and *least*: $\bigwedge x\ y\ z.\ y \sqsubseteq x \Longrightarrow z \sqsubseteq x \Longrightarrow y \nabla z \sqsubseteq x$
shows $x \sqcup y = x \nabla y$
proof (*rule antisym*)
show $x \sqcup y \sqsubseteq x \nabla y$ **by** (*rule le-supI*) (*rule ge1*, *rule ge2*)
next
have *leI*: $\bigwedge x\ y\ z.\ x \sqsubseteq z \Longrightarrow y \sqsubseteq z \Longrightarrow x \nabla y \sqsubseteq z$ **by** (*blast intro*: *least*)
show $x \nabla y \sqsubseteq x \sqcup y$ **by** (*rule leI*) *simp-all*
qed

5.7 *min*/*max* on linear orders as special case of *op* $\sqcap$/*op* $\sqcup$

sublocale *linorder* $<$ *min-max*!: *distrib-lattice less-eq less min max*
proof
fix *x y z*
show *max x* (*min y z*) = *min* (*max x y*) (*max x z*)
by (*auto simp add*: *min-def max-def*)
qed (*auto simp add*: *min-def max-def not-le less-imp-le*)

lemma *inf-min*: *inf* = (*min* :: *'a*::{*semilattice-inf*, *linorder*} $\Rightarrow$ *'a* $\Rightarrow$ *'a*)
by (*rule ext*)+ (*auto intro*: *antisym*)

lemma *sup-max*: *sup* = (*max* :: *'a*::{*semilattice-sup*, *linorder*} ⇒ *'a* ⇒ *'a*)
by (*rule ext*)+ (*auto intro*: *antisym*)

lemmas *le-maxI1* = *min-max.sup-ge1*
lemmas *le-maxI2* = *min-max.sup-ge2*

lemmas *min-ac* = *min-max.inf-assoc min-max.inf-commute*
min-max.inf.left-commute

lemmas *max-ac* = *min-max.sup-assoc min-max.sup-commute*
min-max.sup.left-commute

5.8 Bool as lattice

instantiation *bool* :: *boolean-algebra*
begin

definition
bool-Compl-def: *uminus* = *Not*

definition
bool-diff-def: *A* − *B* ⟷ *A* ∧ ¬ *B*

definition
inf-bool-eq: *P* ⊓ *Q* ⟷ *P* ∧ *Q*

definition
sup-bool-eq: *P* ⊔ *Q* ⟷ *P* ∨ *Q*

instance proof
qed (*simp-all add*: *inf-bool-eq sup-bool-eq le-bool-def*
bot-bool-eq top-bool-eq bool-Compl-def bool-diff-def, *auto*)

end

lemma *sup-boolI1*:
P ⟹ *P* ⊔ *Q*
by (*simp add*: *sup-bool-eq*)

lemma *sup-boolI2*:
Q ⟹ *P* ⊔ *Q*
by (*simp add*: *sup-bool-eq*)

lemma *sup-boolE*:
P ⊔ *Q* ⟹ (*P* ⟹ *R*) ⟹ (*Q* ⟹ *R*) ⟹ *R*
by (*auto simp add*: *sup-bool-eq*)

5.9 Fun as lattice

instantiation *fun* :: (*type*, *lattice*) *lattice*
begin

definition
inf-fun-eq [*code del*]: $f \sqcap g = (\lambda x.\ f\ x \sqcap g\ x)$

definition
sup-fun-eq [*code del*]: $f \sqcup g = (\lambda x.\ f\ x \sqcup g\ x)$

instance proof
qed (*simp-all add*: *le-fun-def inf-fun-eq sup-fun-eq*)

end

instance *fun* :: (*type*, *distrib-lattice*) *distrib-lattice*
proof
qed (*simp-all add*: *inf-fun-eq sup-fun-eq sup-inf-distrib1*)

instance *fun* :: (*type*, *bounded-lattice*) *bounded-lattice* **..**

instantiation *fun* :: (*type*, *uminus*) *uminus*
begin

definition
fun-Compl-def: $- A = (\lambda x.\ -\ A\ x)$

instance ..

end

instantiation *fun* :: (*type*, *minus*) *minus*
begin

definition
fun-diff-def: $A - B = (\lambda x.\ A\ x - B\ x)$

instance ..

end

instance *fun* :: (*type*, *boolean-algebra*) *boolean-algebra*
proof
qed (*simp-all add*: *inf-fun-eq sup-fun-eq bot-fun-eq top-fun-eq fun-Compl-def fun-diff-def*
inf-compl-bot sup-compl-top diff-eq)

no-notation
less-eq (**infix** $\sqsubseteq$ *50*) **and**

less (**infix** $\sqsubset$ *50*) **and**
inf (**infixl** $\sqcap$ *70*) **and**
sup (**infixl** $\sqcup$ *65*) **and**
top ($\top$) **and**
bot ($\bot$)

end

6 Set: Set theory for higher-order logic

theory *Set*
imports *Lattices*
begin

6.1 Sets as predicates

global

types *'a set* = *'a => bool*

consts
Collect :: (*'a => bool*) *=> 'a set* — comprehension
op : :: *'a => 'a set => bool* — membership

local

notation
op : (*op :*) **and**
op : ((*-/ : -*) [*50*, *51*] *50*)

defs
mem-def [*code*]: *x : S == S x*
Collect-def [*code*]: *Collect P == P*

abbreviation
not-mem x A == $\sim$ (*x : A*) — non-membership

notation
not-mem (*op* $\sim$:) **and**
not-mem ((*-/* $\sim$: *-*) [*50*, *51*] *50*)

notation (*xsymbols*)
op : (*op* $\in$) **and**
op : ((*-/* $\in$ *-*) [*50*, *51*] *50*) **and**
not-mem (*op* $\notin$) **and**
not-mem ((*-/* $\notin$ *-*) [*50*, *51*] *50*)

notation (*HTML* **output**)

op : (*op* ∈) **and**
op : ((-/ ∈ -) [*50*, *51*] *50*) **and**
not-mem (*op* ∉) **and**
not-mem ((-/ ∉ -) [*50*, *51*] *50*)

Set comprehensions

syntax
-*Coll* :: *pttrn* => *bool* => *'a set* ((*1*{-./ -}))
translations
{*x*. *P*} == *CONST Collect* (%*x*. *P*)

syntax
-*Collect* :: *idt* => *'a set* => *bool* => *'a set* ((*1*{- :/ -./ -}))
syntax (*xsymbols*)
-*Collect* :: *idt* => *'a set* => *bool* => *'a set* ((*1*{- ∈/ -./ -}))
translations
{*x*:*A*. *P*} => {*x*. *x*:*A* & *P*}

lemma *mem-Collect-eq* [*iff*]: (*a* : {*x*. *P*(*x*)}) = *P*(*a*)
by (*simp add*: *Collect-def mem-def*)

lemma *Collect-mem-eq* [*simp*]: {*x*. *x*:*A*} = *A*
by (*simp add*: *Collect-def mem-def*)

lemma *CollectI*: *P*(*a*) ==> *a* : {*x*. *P*(*x*)}
by *simp*

lemma *CollectD*: *a* : {*x*. *P*(*x*)} ==> *P*(*a*)
by *simp*

lemma *Collect-cong*: (!!*x*. *P x* = *Q x*) ==> {*x*. *P*(*x*)} = {*x*. *Q*(*x*)}
by *simp*

Simproc for pulling *x*=*t* in {*x*. ... & *x*=*t* & ...} to the front (and similarly for *t*=*x*):

setup ⟪
let
val Coll-perm-tac = *rtac* @{*thm Collect-cong*} *1 THEN rtac* @{*thm iffI*} *1 THEN*
ALLGOALS(*EVERY'*[*REPEAT-DETERM o* (*etac* @{*thm conjE*}),
DEPTH-SOLVE-1 o (*ares-tac* [@{*thm conjI*}])])
val defColl-regroup = *Simplifier.simproc* @{*theory*}
defined Collect [{*x*. *P x* & *Q x*}]
(*Quantifier1.rearrange-Coll Coll-perm-tac*)
in
Simplifier.map-simpset (*fn ss* => *ss addsimprocs* [*defColl-regroup*])
end
⟫

lemmas *CollectE* = *CollectD* [*elim-format*]

Set enumerations

abbreviation *empty* :: *'a set* ({}) **where**
  $\{\} \equiv bot$

definition *insert* :: *'a* ⇒ *'a set* ⇒ *'a set* **where**
  insert-compr: $insert\ a\ B = \{x.\ x = a \vee x \in B\}$

syntax
  -Finset :: *args* => *'a set* ({(-)})
translations
  $\{x, xs\} == CONST\ insert\ x\ \{xs\}$
  $\{x\} == CONST\ insert\ x\ \{\}$

6.2 Subsets and bounded quantifiers

abbreviation
  subset :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
  $subset \equiv less$

abbreviation
  subset-eq :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
  $subset\text{-}eq \equiv less\text{-}eq$

notation (**output**)
  $subset\ \ (op <)$ **and**
  $subset\ \ ((-/ < -)\ [50, 51]\ 50)$ **and**
  $subset\text{-}eq\ \ (op <=)$ **and**
  $subset\text{-}eq\ \ ((-/ <= -)\ [50, 51]\ 50)$

notation (*xsymbols*)
  $subset\ \ (op \subset)$ **and**
  $subset\ \ ((-/ \subset -)\ [50, 51]\ 50)$ **and**
  $subset\text{-}eq\ \ (op \subseteq)$ **and**
  $subset\text{-}eq\ \ ((-/ \subseteq -)\ [50, 51]\ 50)$

notation (*HTML* **output**)
  $subset\ \ (op \subset)$ **and**
  $subset\ \ ((-/ \subset -)\ [50, 51]\ 50)$ **and**
  $subset\text{-}eq\ \ (op \subseteq)$ **and**
  $subset\text{-}eq\ \ ((-/ \subseteq -)\ [50, 51]\ 50)$

abbreviation (*input*)
  supset :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
  $supset \equiv greater$

abbreviation (*input*)
  supset-eq :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
  $supset\text{-}eq \equiv greater\text{-}eq$

notation (*xsymbols*)
supset (*op* ⊃) **and**
supset ((-/ ⊃ -) [*50*, *51*] *50*) **and**
supset-eq (*op* ⊇) **and**
supset-eq ((-/ ⊇ -) [*50*, *51*] *50*)

global

consts
Ball :: *'a set => ('a => bool) => bool* — bounded universal quantifiers

Bex :: *'a set => ('a => bool) => bool* — bounded existential quantifiers

local

defs
Ball-def: *Ball A P* == *ALL x. x:A --> P(x)*
Bex-def: *Bex A P* == *EX x. x:A & P(x)*

syntax
-Ball :: *pttrn => 'a set => bool => bool* ((*3ALL* -:-./ -) [*0*, *0*, *10*] *10*)
-Bex :: *pttrn => 'a set => bool => bool* ((*3EX* -:-./ -) [*0*, *0*, *10*] *10*)
-Bex1 :: *pttrn => 'a set => bool => bool* ((*3EX!* -:-./ -) [*0*, *0*, *10*] *10*)
-Bleast :: *id => 'a set => bool => 'a* ((*3LEAST* -:-./ -) [*0*, *0*, *10*] *10*)

syntax (*HOL*)
-Ball :: *pttrn => 'a set => bool => bool* ((*3!* -:-./ -) [*0*, *0*, *10*] *10*)
-Bex :: *pttrn => 'a set => bool => bool* ((*3?* -:-./ -) [*0*, *0*, *10*] *10*)
-Bex1 :: *pttrn => 'a set => bool => bool* ((*3?!* -:-./ -) [*0*, *0*, *10*] *10*)

syntax (*xsymbols*)
-Ball :: *pttrn => 'a set => bool => bool* ((*3∀* -∈-./ -) [*0*, *0*, *10*] *10*)
-Bex :: *pttrn => 'a set => bool => bool* ((*3∃* -∈-./ -) [*0*, *0*, *10*] *10*)
-Bex1 :: *pttrn => 'a set => bool => bool* ((*3∃!* -∈-./ -) [*0*, *0*, *10*] *10*)
-Bleast :: *id => 'a set => bool => 'a* ((*3LEAST* -∈-./ -) [*0*, *0*, *10*] *10*)

syntax (*HTML* **output**)
-Ball :: *pttrn => 'a set => bool => bool* ((*3∀* -∈-./ -) [*0*, *0*, *10*] *10*)
-Bex :: *pttrn => 'a set => bool => bool* ((*3∃* -∈-./ -) [*0*, *0*, *10*] *10*)
-Bex1 :: *pttrn => 'a set => bool => bool* ((*3∃!* -∈-./ -) [*0*, *0*, *10*] *10*)

translations
ALL x:A. P == *CONST Ball A (%x. P)*
EX x:A. P == *CONST Bex A (%x. P)*
EX! x:A. P => *EX! x. x:A & P*
LEAST x:A. P => *LEAST x. x:A & P*

syntax (**output**)
-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<-./ -*) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<-./ -*) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3ALL -<=-./ -*) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3EX -<=-./ -*) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3EX*! *-<=-./ -*) [*0*, *0*, *10*] *10*)

syntax (*xsymbols*)
-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ *-⊂-./ -*) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ *-⊂-./ -*) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ *-⊆-./ -*) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ *-⊆-./ -*) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃!*-⊆-./ -*) [*0*, *0*, *10*] *10*)

syntax (*HOL* **output**)
-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*! *-<-./ -*) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3?* *-<-./ -*) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*! *-<=-./ -*) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3?* *-<=-./ -*) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3?*! *-<=-./ -*) [*0*, *0*, *10*] *10*)

syntax (*HTML* **output**)
-setlessAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ *-⊂-./ -*) [*0*, *0*, *10*] *10*)
-setlessEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ *-⊂-./ -*) [*0*, *0*, *10*] *10*)
-setleAll :: [*idt*, *'a*, *bool*] => *bool* ((*3*∀ *-⊆-./ -*) [*0*, *0*, *10*] *10*)
-setleEx :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃ *-⊆-./ -*) [*0*, *0*, *10*] *10*)
-setleEx1 :: [*idt*, *'a*, *bool*] => *bool* ((*3*∃!*-⊆-./ -*) [*0*, *0*, *10*] *10*)

translations
∀ *A*⊂*B*. *P* => *ALL A*. *A* ⊂ *B* --> *P*
∃ *A*⊂*B*. *P* => *EX A*. *A* ⊂ *B* & *P*
∀ *A*⊆*B*. *P* => *ALL A*. *A* ⊆ *B* --> *P*
∃ *A*⊆*B*. *P* => *EX A*. *A* ⊆ *B* & *P*
∃!*A*⊆*B*. *P* => *EX*! *A*. *A* ⊆ *B* & *P*

print-translation ⟪
let
val Type (*set-type*, *-*) = @{*typ 'a set*}; (∗ *FIXME 'a => bool* (!*?*!) ∗)
val All-binder = *Syntax.binder-name* @{*const-syntax All*};
val Ex-binder = *Syntax.binder-name* @{*const-syntax Ex*};
val impl = @{*const-syntax op* -->};
val conj = @{*const-syntax op* &};
val sbset = @{*const-syntax subset*};
val sbset-eq = @{*const-syntax subset-eq*};

val trans =
[((*All-binder*, *impl*, *sbset*), @{*syntax-const -setlessAll*}),
((*All-binder*, *impl*, *sbset-eq*), @{*syntax-const -setleAll*}),

```
    ((Ex-binder, conj, sbset), @{syntax-const -setlessEx}),
    ((Ex-binder, conj, sbset-eq), @{syntax-const -setleEx})];

  fun mk v v' c n P =
    if v = v' andalso not (Term.exists-subterm (fn Free (x, -) => x = v | - =>
false) n)
    then Syntax.const c $ Syntax.mark-bound v' $ n $ P else raise Match;

  fun tr' q = (q,
        fn [Const (@{syntax-const -bound}, -) $ Free (v, Type (T, -)),
            Const (c, -) $
              (Const (d, -) $ (Const (@{syntax-const -bound}, -) $ Free (v', -)) $ n)
$ P] =>
            if T = set-type then
              (case AList.lookup (op =) trans (q, c, d) of
                NONE => raise Match
              | SOME l => mk v v' l n P)
            else raise Match
        | - => raise Match);
in
  [tr' All-binder, tr' Ex-binder]
end
⟫
```

Translate between $\{e \mid x1...xn.\ P\}$ and $\{u.\ EX\ x1..xn.\ u = e\ \&\ P\}$; $\{y.\ EX\ x1..xn.\ y = e\ \&\ P\}$ is only translated if $[0..n]$ *subset* $bvs(e)$.

syntax

```
  -Setcompr :: 'a => idts => bool => 'a set    ((1{- |/-./ -}))
```

parse-translation ⟪

```
  let
    val ex-tr = snd (mk-binder-tr (EX , @{const-syntax Ex}));

    fun nvars (Const (@{syntax-const -idts}, -) $ - $ idts) = nvars idts + 1
      | nvars - = 1;

    fun setcompr-tr [e, idts, b] =
      let
        val eq = Syntax.const @{const-syntax op =} $ Bound (nvars idts) $ e;
        val P = Syntax.const @{const-syntax op &} $ eq $ b;
        val exP = ex-tr [idts, P];
      in Syntax.const @{const-syntax Collect} $ Term.absdummy (dummyT, exP)
end;

  in [(@{syntax-const -Setcompr}, setcompr-tr)] end;
⟫
```

print-translation ⟪

```
 [Syntax.preserve-binder-abs2-tr' @{const-syntax Ball} @{syntax-const -Ball},
```

```
  Syntax.preserve-binder-abs2-tr' @{const-syntax Bex} @{syntax-const -Bex}]
⟩⟩ — to avoid eta-contraction of body

print-translation ⟨⟨
let
  val ex-tr' = snd (mk-binder-tr' (@{const-syntax Ex}, DUMMY));

  fun setcompr-tr' [Abs (abs as (-, -, P))] =
    let
      fun check (Const (@{const-syntax Ex}, -) $ Abs (-, -, P), n) = check (P, n
+ 1)
        | check (Const (@{const-syntax op &}, -) $
              (Const (@{const-syntax op =}, -) $ Bound m $ e) $ P, n) =
            n > 0 andalso m = n andalso not (loose-bvar1 (P, n)) andalso
            subset (op =) (0 upto (n − 1), add-loose-bnos (e, 0, [])))
        | check - = false;

        fun tr' (- $ abs) =
          let val - $ idts $ (- $ (- $ - $ e) $ Q) = ex-tr' [abs]
          in Syntax.const @{syntax-const -Setcompr} $ e $ idts $ Q end;
    in
      if check (P, 0) then tr' P
      else
        let
          val (x as - $ Free(xN, -), t) = atomic-abs-tr' abs;
          val M = Syntax.const @{syntax-const -Coll} $ x $ t;
        in
          case t of
            Const (@{const-syntax op &}, -) $
              (Const (@{const-syntax op :}, -) $
                (Const (@{syntax-const -bound}, -) $ Free (yN, -)) $ A) $ P =>
            if xN = yN then Syntax.const @{syntax-const -Collect} $ x $ A $ P else
M
          | - => M
        end
    end;
  in [(@{const-syntax Collect}, setcompr-tr')] end;
⟩⟩

setup ⟨⟨
let
  val unfold-bex-tac = unfold-tac @{thms Bex-def};
 fun prove-bex-tac ss = unfold-bex-tac ss THEN Quantifier1.prove-one-point-ex-tac;
  val rearrange-bex = Quantifier1.rearrange-bex prove-bex-tac;
  val unfold-ball-tac = unfold-tac @{thms Ball-def};
 fun prove-ball-tac ss = unfold-ball-tac ss THEN Quantifier1.prove-one-point-all-tac;
  val rearrange-ball = Quantifier1.rearrange-ball prove-ball-tac;
  val defBEX-regroup = Simplifier.simproc @{theory}
    defined BEX [EX x:A. P x & Q x] rearrange-bex;
```

```
  val defBALL-regroup = Simplifier.simproc @{theory}
    defined BALL [ALL x:A. P x --> Q x] rearrange-ball;
in
 Simplifier.map-simpset (fn ss => ss addsimprocs [defBALL-regroup, defBEX-regroup])
end
```

lemma *ballI* [*intro*!]: (!!*x*. *x*:*A* ==> *P x*) ==> *ALL x*:*A*. *P x*
by (*simp add*: *Ball-def*)

lemmas *strip* = *impI allI ballI*

lemma *bspec* [*dest?*]: *ALL x*:*A*. *P x* ==> *x*:*A* ==> *P x*
by (*simp add*: *Ball-def*)

Gives better instantiation for bound:

```
declaration ⟪ fn - =>
  Classical.map-cs (fn cs => cs addbefore (bspec, datac @{thm bspec} 1))
⟫

ML ⟪
structure Simpdata =
struct

open Simpdata;

val mksimps-pairs = [(@{const-name Ball}, @{thms bspec})] @ mksimps-pairs;

end;

open Simpdata;
⟫

declaration ⟪ fn - =>
  Simplifier.map-ss (fn ss => ss setmksimps (mksimps mksimps-pairs))
⟫
```

lemma *ballE* [*elim*]: *ALL x*:*A*. *P x* ==> (*P x* ==> *Q*) ==> (*x* ~: *A* ==> *Q*)
==> *Q*
by (*unfold Ball-def*) *blast*

lemma *bexI* [*intro*]: *P x* ==> *x*:*A* ==> *EX x*:*A*. *P x*
— Normally the best argument order: *P x* constrains the choice of $x \in A$.
by (*unfold Bex-def*) *blast*

lemma *rev-bexI* [*intro?*]: *x*:*A* ==> *P x* ==> *EX x*:*A*. *P x*
— The best argument order when there is only one $x \in A$.
by (*unfold Bex-def*) *blast*

lemma *bexCI*: $(ALL\ x{:}A.\ \sim P\ x\ ==>\ P\ a)\ ==>\ a{:}A\ ==>\ EX\ x{:}A.\ P\ x$
by (*unfold Bex-def*) *blast*

lemma *bexE* [*elim*!]: $EX\ x{:}A.\ P\ x\ ==>\ (!!x.\ x{:}A\ ==>\ P\ x\ ==>\ Q)\ ==>\ Q$
by (*unfold Bex-def*) *blast*

lemma *ball-triv* [*simp*]: $(ALL\ x{:}A.\ P) = ((EX\ x.\ x{:}A)\ -->\ P)$
— Trival rewrite rule.
by (*simp add*: *Ball-def*)

lemma *bex-triv* [*simp*]: $(EX\ x{:}A.\ P) = ((EX\ x.\ x{:}A)\ \&\ P)$
— Dual form for existentials.
by (*simp add*: *Bex-def*)

lemma *bex-triv-one-point1* [*simp*]: $(EX\ x{:}A.\ x = a) = (a{:}A)$
by *blast*

lemma *bex-triv-one-point2* [*simp*]: $(EX\ x{:}A.\ a = x) = (a{:}A)$
by *blast*

lemma *bex-one-point1* [*simp*]: $(EX\ x{:}A.\ x = a\ \&\ P\ x) = (a{:}A\ \&\ P\ a)$
by *blast*

lemma *bex-one-point2* [*simp*]: $(EX\ x{:}A.\ a = x\ \&\ P\ x) = (a{:}A\ \&\ P\ a)$
by *blast*

lemma *ball-one-point1* [*simp*]: $(ALL\ x{:}A.\ x = a\ -->\ P\ x) = (a{:}A\ -->\ P\ a)$
by *blast*

lemma *ball-one-point2* [*simp*]: $(ALL\ x{:}A.\ a = x\ -->\ P\ x) = (a{:}A\ -->\ P\ a)$
by *blast*

Congruence rules

lemma *ball-cong*:
$A = B\ ==>\ (!!x.\ x{:}B\ ==>\ P\ x = Q\ x)\ ==>$
$(ALL\ x{:}A.\ P\ x) = (ALL\ x{:}B.\ Q\ x)$
by (*simp add*: *Ball-def*)

lemma *strong-ball-cong* [*cong*]:
$A = B\ ==>\ (!!x.\ x{:}B\ =simp=>\ P\ x = Q\ x)\ ==>$
$(ALL\ x{:}A.\ P\ x) = (ALL\ x{:}B.\ Q\ x)$
by (*simp add*: *simp-implies-def Ball-def*)

lemma *bex-cong*:
$A = B\ ==>\ (!!x.\ x{:}B\ ==>\ P\ x = Q\ x)\ ==>$
$(EX\ x{:}A.\ P\ x) = (EX\ x{:}B.\ Q\ x)$
by (*simp add*: *Bex-def cong*: *conj-cong*)

lemma *strong-bex-cong* [*cong*]:

$A = B$ ==> (!!x. x:B =*simp*=> $P\ x = Q\ x$) ==>
($EX\ x$:A. $P\ x$) = ($EX\ x$:B. $Q\ x$)
by (*simp add*: *simp-implies-def Bex-def cong*: *conj-cong*)

6.3 Basic operations

6.3.1 Subsets

lemma *subsetI* [*intro*!]: $(\bigwedge x.\ x \in A \Longrightarrow x \in B) \Longrightarrow A \subseteq B$
unfolding *mem-def* **by** (*rule le-funI*, *rule le-boolI*)

Map the type $'a\ set$ => *anything* to just $'a$; for overloading constants whose first argument has type $'a\ set$.

lemma *subsetD* [*elim*, *intro?*]: $A \subseteq B$ ==> $c \in A$ ==> $c \in B$
unfolding *mem-def* **by** (*erule le-funE*, *erule le-boolE*)
— Rule in Modus Ponens style.

lemma *rev-subsetD* [*no-atp*,*intro?*]: $c \in A$ ==> $A \subseteq B$ ==> $c \in B$
— The same, with reversed premises for use with *erule* – cf *rev-mp*.
by (*rule subsetD*)

Converts $A \subseteq B$ to $x \in A \Longrightarrow x \in B$.

lemma *subsetCE* [*no-atp*,*elim*]: $A \subseteq B$ ==> ($c \notin A$ ==> P) ==> ($c \in B$ ==> P) ==> P
— Classical elimination rule.
unfolding *mem-def* **by** (*blast dest*: *le-funE le-boolE*)

lemma *subset-eq* [*no-atp*]: $A \leq B = (\forall x{\in}A.\ x \in B)$ **by** *blast*

lemma *contra-subsetD* [*no-atp*]: $A \subseteq B$ ==> $c \notin B$ ==> $c \notin A$
by *blast*

lemma *subset-refl* [*simp*]: $A \subseteq A$
by (*fact order-refl*)

lemma *subset-trans*: $A \subseteq B$ ==> $B \subseteq C$ ==> $A \subseteq C$
by (*fact order-trans*)

lemma *set-rev-mp*: x:A ==> $A \subseteq B$ ==> x:B
by (*rule subsetD*)

lemma *set-mp*: $A \subseteq B$ ==> x:A ==> x:B
by (*rule subsetD*)

lemma *eq-mem-trans*: a=b ==> $b \in A$ ==> $a \in A$
by *simp*

lemmas *basic-trans-rules* [*trans*] =
order-trans-rules set-rev-mp set-mp eq-mem-trans

6.3.2 Equality

lemma *set-ext*: **assumes** *prem*: (!!*x*. (*x*:*A*) = (*x*:*B*)) **shows** *A* = *B*
apply (*rule prem* [*THEN ext*, *THEN arg-cong*, *THEN box-equals*])
apply (*rule Collect-mem-eq*)
apply (*rule Collect-mem-eq*)
done

lemma *expand-set-eq*: (*A* = *B*) = (*ALL x*. (*x*:*A*) = (*x*:*B*))
by(*auto intro*:*set-ext*)

lemma *subset-antisym* [*intro*!]: *A* ⊆ *B* ==> *B* ⊆ *A* ==> *A* = *B*
— Anti-symmetry of the subset relation.
by (*iprover intro*: *set-ext subsetD*)

Equality rules from ZF set theory – are they appropriate here?

lemma *equalityD1*: *A* = *B* ==> *A* ⊆ *B*
by *simp*

lemma *equalityD2*: *A* = *B* ==> *B* ⊆ *A*
by *simp*

Be careful when adding this to the claset as *subset-empty* is in the simpset: *A* = {} goes to {} ⊆ *A* and *A* ⊆ {} and then back to *A* = {}!

lemma *equalityE*: *A* = *B* ==> (*A* ⊆ *B* ==> *B* ⊆ *A* ==> *P*) ==> *P*
by *simp*

lemma *equalityCE* [*elim*]:
A = *B* ==> (*c* ∈ *A* ==> *c* ∈ *B* ==> *P*) ==> (*c* ∉ *A* ==> *c* ∉ *B* ==> *P*)
==> *P*
by *blast*

lemma *eqset-imp-iff*: *A* = *B* ==> (*x* : *A*) = (*x* : *B*)
by *simp*

lemma *eqelem-imp-iff*: *x* = *y* ==> (*x* : *A*) = (*y* : *A*)
by *simp*

6.3.3 The universal set – UNIV

abbreviation *UNIV* :: *'a set* **where**
UNIV ≡ *top*

lemma *UNIV-def*:
UNIV = {*x*. *True*}
by (*simp add*: *top-fun-eq top-bool-eq Collect-def*)

lemma *UNIV-I* [*simp*]: *x* : *UNIV*

by (*simp add*: *UNIV-def*)

declare *UNIV-I* [*intro*] — unsafe makes it less likely to cause problems

lemma *UNIV-witness* [*intro?*]: *EX x. x : UNIV*
by *simp*

lemma *subset-UNIV* [*simp*]: $A \subseteq UNIV$
by (*rule subsetI*) (*rule UNIV-I*)

Eta-contracting these two rules (to remove P) causes them to be ignored because of their interaction with congruence rules.

lemma *ball-UNIV* [*simp*]: *Ball UNIV P = All P*
by (*simp add*: *Ball-def*)

lemma *bex-UNIV* [*simp*]: *Bex UNIV P = Ex P*
by (*simp add*: *Bex-def*)

lemma *UNIV-eq-I*: $(\bigwedge x.\ x \in A) \Longrightarrow UNIV = A$
by *auto*

6.3.4 The empty set

lemma *empty-def*:
$\{\} = \{x.\ False\}$
by (*simp add*: *bot-fun-eq bot-bool-eq Collect-def*)

lemma *empty-iff* [*simp*]: $(c : \{\}) = False$
by (*simp add*: *empty-def*)

lemma *emptyE* [*elim!*]: $a : \{\} \Longrightarrow P$
by *simp*

lemma *empty-subsetI* [*iff*]: $\{\} \subseteq A$
— One effect is to delete the ASSUMPTION $\{\} \subseteq A$
by *blast*

lemma *equals0I*: $(!!y.\ y \in A \Longrightarrow False) \Longrightarrow A = \{\}$
by *blast*

lemma *equals0D*: $A = \{\} \Longrightarrow a \notin A$
— Use for reasoning about disjointness: *A Int B* $= \{\}$
by *blast*

lemma *ball-empty* [*simp*]: *Ball* $\{\}$ *P = True*
by (*simp add*: *Ball-def*)

lemma *bex-empty* [*simp*]: *Bex* $\{\}$ *P = False*
by (*simp add*: *Bex-def*)

lemma *UNIV-not-empty* [*iff*]: $UNIV \sim= \{\}$
by (*blast elim*: *equalityE*)

6.3.5 The Powerset operator – Pow

definition $Pow :: 'a\ set => 'a\ set\ set$ **where**
Pow-def: $Pow\ A = \{B.\ B \leq A\}$

lemma *Pow-iff* [*iff*]: $(A \in Pow\ B) = (A \subseteq B)$
by (*simp add*: *Pow-def*)

lemma *PowI*: $A \subseteq B ==> A \in Pow\ B$
by (*simp add*: *Pow-def*)

lemma *PowD*: $A \in Pow\ B ==> A \subseteq B$
by (*simp add*: *Pow-def*)

lemma *Pow-bottom*: $\{\} \in Pow\ B$
by *simp*

lemma *Pow-top*: $A \in Pow\ A$
by *simp*

6.3.6 Set complement

lemma *Compl-iff* [*simp*]: $(c \in -A) = (c \notin A)$
by (*simp add*: *mem-def fun-Compl-def bool-Compl-def*)

lemma *ComplI* [*intro!*]: $(c \in A ==> False) ==> c \in -A$
by (*unfold mem-def fun-Compl-def bool-Compl-def*) *blast*

This form, with negated conclusion, works well with the Classical prover. Negated assumptions behave like formulae on the right side of the notional turnstile ...

lemma *ComplD* [*dest!*]: $c : -A ==> c\sim:A$
by (*simp add*: *mem-def fun-Compl-def bool-Compl-def*)

lemmas *ComplE* = *ComplD* [*elim-format*]

lemma *Compl-eq*: $-A = \{x.\ \sim x : A\}$ **by** *blast*

6.3.7 Binary union – Un

abbreviation *union* :: $'a\ set \Rightarrow 'a\ set \Rightarrow 'a\ set$ (**infixl** *Un* 65) **where**
op Un ≡ *sup*

notation (*xsymbols*)
union (**infixl** ∪ 65)

notation (*HTML* **output**)
union (**infixl** $\cup$ *65*)

lemma *Un-def*:
$A \cup B = \{x.\ x \in A \vee x \in B\}$
by (*simp add*: *sup-fun-eq sup-bool-eq Collect-def mem-def*)

lemma *Un-iff* [*simp*]: $(c : A\ Un\ B) = (c{:}A \mid c{:}B)$
by (*unfold Un-def*) *blast*

lemma *UnI1* [*elim?*]: $c{:}A \Longrightarrow c : A\ Un\ B$
by *simp*

lemma *UnI2* [*elim?*]: $c{:}B \Longrightarrow c : A\ Un\ B$
by *simp*

Classical introduction rule: no commitment to A vs B.

lemma *UnCI* [*intro*!]: $(c^{\sim}{:}B \Longrightarrow c{:}A) \Longrightarrow c : A\ Un\ B$
by *auto*

lemma *UnE* [*elim*!]: $c : A\ Un\ B \Longrightarrow (c{:}A \Longrightarrow P) \Longrightarrow (c{:}B \Longrightarrow P) \Longrightarrow P$
by (*unfold Un-def*) *blast*

lemma *insert-def*: $insert\ a\ B = \{x.\ x = a\} \cup B$
by (*simp add*: *Collect-def mem-def insert-compr Un-def*)

lemma *mono-Un*: $mono\ f \Longrightarrow f\ A \cup f\ B \subseteq f\ (A \cup B)$
by (*fact mono-sup*)

6.3.8 Binary intersection – Int

abbreviation *inter* :: *'a set* $\Rightarrow$ *'a set* $\Rightarrow$ *'a set* (**infixl** *Int 70*) **where**
op Int $\equiv$ *inf*

notation (*xsymbols*)
inter (**infixl** $\cap$ *70*)

notation (*HTML* **output**)
inter (**infixl** $\cap$ *70*)

lemma *Int-def*:
$A \cap B = \{x.\ x \in A \wedge x \in B\}$
by (*simp add*: *inf-fun-eq inf-bool-eq Collect-def mem-def*)

lemma *Int-iff* [*simp*]: $(c : A\ Int\ B) = (c{:}A\ \&\ c{:}B)$
by (*unfold Int-def*) *blast*

lemma *IntI* [*intro*!]: *c*:*A* ==> *c*:*B* ==> *c* : *A Int B*
by *simp*

lemma *IntD1*: *c* : *A Int B* ==> *c*:*A*
by *simp*

lemma *IntD2*: *c* : *A Int B* ==> *c*:*B*
by *simp*

lemma *IntE* [*elim*!]: *c* : *A Int B* ==> (*c*:*A* ==> *c*:*B* ==> *P*) ==> *P*
by *simp*

lemma *mono-Int*: *mono f* ⟹ *f* (*A* ∩ *B*) ⊆ *f A* ∩ *f B*
by (*fact mono-inf*)

6.3.9 Set difference

lemma *Diff-iff* [*simp*]: (*c* : *A* − *B*) = (*c*:*A* & *c*~:*B*)
by (*simp add*: *mem-def fun-diff-def bool-diff-def*)

lemma *DiffI* [*intro*!]: *c* : *A* ==> *c* ~: *B* ==> *c* : *A* − *B*
by *simp*

lemma *DiffD1*: *c* : *A* − *B* ==> *c* : *A*
by *simp*

lemma *DiffD2*: *c* : *A* − *B* ==> *c* : *B* ==> *P*
by *simp*

lemma *DiffE* [*elim*!]: *c* : *A* − *B* ==> (*c*:*A* ==> *c*~:*B* ==> *P*) ==> *P*
by *simp*

lemma *set-diff-eq*: *A* − *B* = {*x*. *x* : *A* & ~ *x* : *B*} **by** *blast*

lemma *Compl-eq-Diff-UNIV*: −*A* = (*UNIV* − *A*)
by *blast*

6.3.10 Augmenting a set − *insert*

lemma *insert-iff* [*simp*]: (*a* : *insert b A*) = (*a* = *b* | *a*:*A*)
by (*unfold insert-def*) *blast*

lemma *insertI1*: *a* : *insert a B*
by *simp*

lemma *insertI2*: *a* : *B* ==> *a* : *insert b B*
by *simp*

lemma *insertE* [*elim*!]: *a* : *insert b A* ==> (*a* = *b* ==> *P*) ==> (*a*:*A* ==> *P*)
==> *P*

by (*unfold insert-def*) *blast*

lemma *insertCI* [*intro*!]: (*a*~:*B* ==> *a* = *b*) ==> *a*: *insert b B*
— Classical introduction rule.
by *auto*

lemma *subset-insert-iff*: (*A* ⊆ *insert x B*) = (*if x*:*A then A* − {*x*} ⊆ *B else A* ⊆ *B*)
by *auto*

lemma *set-insert*:
assumes *x* ∈ *A*
obtains *B* **where** *A* = *insert x B* **and** *x* ∉ *B*
proof
from *assms* **show** *A* = *insert x* (*A* − {*x*}) **by** *blast*
next
show *x* ∉ *A* − {*x*} **by** *blast*
qed

lemma *insert-ident*: *x* ~: *A* ==> *x* ~: *B* ==> (*insert x A* = *insert x B*) = (*A* = *B*)
by *auto*

6.3.11 Singletons, using insert

lemma *singletonI* [*intro*!,*no-atp*]: *a* : {*a*}
— Redundant? But unlike *insertCI*, it proves the subgoal immediately!
by (*rule insertI1*)

lemma *singletonD* [*dest*!,*no-atp*]: *b* : {*a*} ==> *b* = *a*
by *blast*

lemmas *singletonE* = *singletonD* [*elim-format*]

lemma *singleton-iff*: (*b* : {*a*}) = (*b* = *a*)
by *blast*

lemma *singleton-inject* [*dest*!]: {*a*} = {*b*} ==> *a* = *b*
by *blast*

lemma *singleton-insert-inj-eq* [*iff*,*no-atp*]:
(*{b} = insert a A*) = (*a = b & A ⊆ {b}*)
by *blast*

lemma *singleton-insert-inj-eq′* [*iff*,*no-atp*]:
(*insert a A = {b}*) = (*a = b & A ⊆ {b}*)
by *blast*

lemma *subset-singletonD*: *A* ⊆ {*x*} ==> *A* = {} | *A* = {*x*}

by *fast*

lemma *singleton-conv* [*simp*]: $\{x.\ x = a\} = \{a\}$
by *blast*

lemma *singleton-conv2* [*simp*]: $\{x.\ a = x\} = \{a\}$
by *blast*

lemma *diff-single-insert*: $A - \{x\} \subseteq B \Longrightarrow x \in A \Longrightarrow A \subseteq \mathit{insert}\ x\ B$
by *blast*

lemma *doubleton-eq-iff*: $(\{a,b\} = \{c,d\}) = (a=c\ \&\ b=d \mid a=d\ \&\ b=c)$
by (*blast elim*: *equalityE*)

6.3.12 Image of a set under a function

Frequently b does not have the syntactic form of $f\ x$.

definition *image* :: $('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'b\ set$ (**infixr** ‘ *90*) **where**
image-def [*no-atp*]: $f\ `\ A = \{y.\ EX\ x{:}A.\ y = f(x)\}$

abbreviation
range :: $('a \Rightarrow 'b) \Rightarrow 'b\ set$ **where** — of function
$\mathit{range}\ f == f\ `\ UNIV$

lemma *image-eqI* [*simp*, *intro*]: $b = f\ x \Longrightarrow x{:}A \Longrightarrow b : f`A$
by (*unfold image-def*) *blast*

lemma *imageI*: $x : A \Longrightarrow f\ x : f\ `\ A$
by (*rule image-eqI*) (*rule refl*)

lemma *rev-image-eqI*: $x{:}A \Longrightarrow b = f\ x \Longrightarrow b : f`A$
— This version's more effective when we already have the required x.
by (*unfold image-def*) *blast*

lemma *imageE* [*elim*!]:
$b : (\%x.\ f\ x)`A \Longrightarrow (!!x.\ b = f\ x \Longrightarrow x{:}A \Longrightarrow P) \Longrightarrow P$
— The eta-expansion gives variable-name preservation.
by (*unfold image-def*) *blast*

lemma *image-Un*: $f`(A\ Un\ B) = f`A\ Un\ f`B$
by *blast*

lemma *image-iff*: $(z : f`A) = (EX\ x{:}A.\ z = f\ x)$
by *blast*

lemma *image-subset-iff*: $(f`A \subseteq B) = (\forall x{\in}A.\ f\ x \in B)$
— This rewrite rule would confuse users if made default.
by *blast*

lemma *subset-image-iff*: $(B \subseteq f`A) = (EX\ AA.\ AA \subseteq A\ \&\ B = f`AA)$
apply *safe*
prefer *2* **apply** *fast*
apply (*rule-tac* $x = \{a.\ a : A\ \&\ f\ a : B\}$ **in** *exI*, *fast*)
done

lemma *image-subsetI*: $(!!x.\ x \in A \Longrightarrow f\ x \in B) \Longrightarrow f`A \subseteq B$
— Replaces the three steps *subsetI*, *imageE*, *hypsubst*, but breaks too many existing proofs.
by *blast*

Range of a function – just a translation for image!

lemma *range-eqI*: $b = f\ x \Longrightarrow b \in range\ f$
by *simp*

lemma *rangeI*: $f\ x \in range\ f$
by *simp*

lemma *rangeE* [*elim?*]: $b \in range\ (\lambda x.\ f\ x) \Longrightarrow (!!x.\ b = f\ x \Longrightarrow P) \Longrightarrow P$
by *blast*

6.3.13 Some rules with *if*

Elimination of $\{x.\ \ldots\ \&\ x{=}t\ \&\ \ldots\}$.

lemma *Collect-conv-if*: $\{x.\ x{=}a\ \&\ P\ x\} = (if\ P\ a\ then\ \{a\}\ else\ \{\})$
by *auto*

lemma *Collect-conv-if2*: $\{x.\ a{=}x\ \&\ P\ x\} = (if\ P\ a\ then\ \{a\}\ else\ \{\})$
by *auto*

Rewrite rules for boolean case-splitting: faster than *split-if* [*split*].

lemma *split-if-eq1*: $((if\ Q\ then\ x\ else\ y) = b) = ((Q \dashrightarrow x = b)\ \&\ (\sim Q \dashrightarrow y = b))$
by (*rule split-if*)

lemma *split-if-eq2*: $(a = (if\ Q\ then\ x\ else\ y)) = ((Q \dashrightarrow a = x)\ \&\ (\sim Q \dashrightarrow a = y))$
by (*rule split-if*)

Split ifs on either side of the membership relation. Not for [*simp*] – can cause goals to blow up!

lemma *split-if-mem1*: $((if\ Q\ then\ x\ else\ y) : b) = ((Q \dashrightarrow x : b)\ \&\ (\sim Q \dashrightarrow y : b))$
by (*rule split-if*)

lemma *split-if-mem2*: $(a : (if\ Q\ then\ x\ else\ y)) = ((Q \dashrightarrow a : x)\ \&\ (\sim Q \dashrightarrow a : y))$

by (*rule split-if* [**where** P=%S. a : S])

lemmas *split-ifs* = *if-bool-eq-conj split-if-eq1 split-if-eq2 split-if-mem1 split-if-mem2*

6.4 Further operations and lemmas

6.4.1 The "proper subset" relation

lemma *psubsetI* [*intro*!,*no-atp*]: $A \subseteq B \Longrightarrow A \neq B \Longrightarrow A \subset B$
by (*unfold less-le*) *blast*

lemma *psubsetE* [*elim*!,*no-atp*]:
$[|A \subset B;\ [|A \subseteq B; \sim (B \subseteq A)|] \Longrightarrow R|] \Longrightarrow R$
by (*unfold less-le*) *blast*

lemma *psubset-insert-iff*:
$(A \subset \mathit{insert}\ x\ B) = (\mathit{if}\ x \in B\ \mathit{then}\ A \subset B\ \mathit{else}\ \mathit{if}\ x \in A\ \mathit{then}\ A - \{x\} \subset B\ \mathit{else}\ A \subseteq B)$
by (*auto simp add*: *less-le subset-insert-iff*)

lemma *psubset-eq*: $(A \subset B) = (A \subseteq B \ \&\ A \neq B)$
by (*simp only*: *less-le*)

lemma *psubset-imp-subset*: $A \subset B \Longrightarrow A \subseteq B$
by (*simp add*: *psubset-eq*)

lemma *psubset-trans*: $[|\ A \subset B;\ B \subset C\ |] \Longrightarrow A \subset C$
apply (*unfold less-le*)
apply (*auto dest*: *subset-antisym*)
done

lemma *psubsetD*: $[|\ A \subset B;\ c \in A\ |] \Longrightarrow c \in B$
apply (*unfold less-le*)
apply (*auto dest*: *subsetD*)
done

lemma *psubset-subset-trans*: $A \subset B \Longrightarrow B \subseteq C \Longrightarrow A \subset C$
by (*auto simp add*: *psubset-eq*)

lemma *subset-psubset-trans*: $A \subseteq B \Longrightarrow B \subset C \Longrightarrow A \subset C$
by (*auto simp add*: *psubset-eq*)

lemma *psubset-imp-ex-mem*: $A \subset B \Longrightarrow \exists\, b.\ b \in (B - A)$
by (*unfold less-le*) *blast*

lemma *atomize-ball*:
$(!!x.\ x \in A \Longrightarrow P\ x) == \mathit{Trueprop}\ (\forall\, x \in A.\ P\ x)$
by (*simp only*: *Ball-def atomize-all atomize-imp*)

lemmas [*symmetric*, *rulify*] = *atomize-ball*

and [*symmetric*, *defn*] = *atomize-ball*

6.4.2 Derived rules involving subsets.

insert.

lemma *subset-insertI*: $B \subseteq \mathit{insert}\ a\ B$
by (*rule subsetI*) (*erule insertI2*)

lemma *subset-insertI2*: $A \subseteq B \Longrightarrow A \subseteq \mathit{insert}\ b\ B$
by *blast*

lemma *subset-insert*: $x \notin A \Longrightarrow (A \subseteq \mathit{insert}\ x\ B) = (A \subseteq B)$
by *blast*

Finite Union – the least upper bound of two sets.

lemma *Un-upper1*: $A \subseteq A \cup B$
by (*fact sup-ge1*)

lemma *Un-upper2*: $B \subseteq A \cup B$
by (*fact sup-ge2*)

lemma *Un-least*: $A \subseteq C \Longrightarrow B \subseteq C \Longrightarrow A \cup B \subseteq C$
by (*fact sup-least*)

Finite Intersection – the greatest lower bound of two sets.

lemma *Int-lower1*: $A \cap B \subseteq A$
by (*fact inf-le1*)

lemma *Int-lower2*: $A \cap B \subseteq B$
by (*fact inf-le2*)

lemma *Int-greatest*: $C \subseteq A \Longrightarrow C \subseteq B \Longrightarrow C \subseteq A \cap B$
by (*fact inf-greatest*)

Set difference.

lemma *Diff-subset*: $A - B \subseteq A$
by *blast*

lemma *Diff-subset-conv*: $(A - B \subseteq C) = (A \subseteq B \cup C)$
by *blast*

6.4.3 Equalities involving union, intersection, inclusion, etc.

{}.

lemma *Collect-const* [*simp*]: $\{s.\ P\} = (\mathit{if}\ P\ \mathit{then}\ \mathit{UNIV}\ \mathit{else}\ \{\})$
— supersedes *Collect-False-empty*

by *auto*

lemma *subset-empty* [*simp*]: $(A \subseteq \{\}) = (A = \{\})$
by *blast*

lemma *not-psubset-empty* [*iff*]: $\neg\ (A < \{\})$
by (*unfold less-le*) *blast*

lemma *Collect-empty-eq* [*simp*]: $(Collect\ P = \{\}) = (\forall x.\ \neg\ P\ x)$
by *blast*

lemma *empty-Collect-eq* [*simp*]: $(\{\} = Collect\ P) = (\forall x.\ \neg\ P\ x)$
by *blast*

lemma *Collect-neg-eq*: $\{x.\ \neg\ P\ x\} = -\ \{x.\ P\ x\}$
by *blast*

lemma *Collect-disj-eq*: $\{x.\ P\ x \mid Q\ x\} = \{x.\ P\ x\} \cup \{x.\ Q\ x\}$
by *blast*

lemma *Collect-imp-eq*: $\{x.\ P\ x\ -->\ Q\ x\} = -\{x.\ P\ x\} \cup \{x.\ Q\ x\}$
by *blast*

lemma *Collect-conj-eq*: $\{x.\ P\ x\ \&\ Q\ x\} = \{x.\ P\ x\} \cap \{x.\ Q\ x\}$
by *blast*

insert.

lemma *insert-is-Un*: *insert a A* = $\{a\}$ *Un A*
— NOT SUITABLE FOR REWRITING since $\{a\}$ == *insert a* $\{\}$
by *blast*

lemma *insert-not-empty* [*simp*]: *insert a A* $\neq \{\}$
by *blast*

lemmas *empty-not-insert* = *insert-not-empty* [*symmetric*, *standard*]
declare *empty-not-insert* [*simp*]

lemma *insert-absorb*: $a \in A$ ==> *insert a A* = *A*
— [*simp*] causes recursive calls when there are nested inserts
— with *quadratic* running time
by *blast*

lemma *insert-absorb2* [*simp*]: *insert x* (*insert x A*) = *insert x A*
by *blast*

lemma *insert-commute*: *insert x* (*insert y A*) = *insert y* (*insert x A*)
by *blast*

lemma *insert-subset* [*simp*]: (*insert x A* $\subseteq$ *B*) = ($x \in B$ & $A \subseteq B$)

by *blast*

lemma *mk-disjoint-insert*: $a \in A \Longrightarrow \exists B.\ A = insert\ a\ B\ \&\ a \notin B$
— use new B rather than $A - \{a\}$ to avoid infinite unfolding
apply (*rule-tac* $x = A - \{a\}$ **in** *exI*, *blast*)
done

lemma *insert-Collect*: $insert\ a\ (Collect\ P) = \{u.\ u \neq a \longrightarrow P\ u\}$
by *auto*

lemma *insert-inter-insert*[*simp*]: $insert\ a\ A \cap insert\ a\ B = insert\ a\ (A \cap B)$
by *blast*

lemma *insert-disjoint* [*simp*,*no-atp*]:
$(insert\ a\ A \cap B = \{\}) = (a \notin B \wedge A \cap B = \{\})$
$(\{\} = insert\ a\ A \cap B) = (a \notin B \wedge \{\} = A \cap B)$
by *auto*

lemma *disjoint-insert* [*simp*,*no-atp*]:
$(B \cap insert\ a\ A = \{\}) = (a \notin B \wedge B \cap A = \{\})$
$(\{\} = A \cap insert\ b\ B) = (b \notin A \wedge \{\} = A \cap B)$
by *auto*

image.

lemma *image-empty* [*simp*]: $f`\{\} = \{\}$
by *blast*

lemma *image-insert* [*simp*]: $f\ `\ insert\ a\ B = insert\ (f\ a)\ (f`B)$
by *blast*

lemma *image-constant*: $x \in A \Longrightarrow (\lambda x.\ c)\ `\ A = \{c\}$
by *auto*

lemma *image-constant-conv*: $(\%x.\ c)\ `\ A = (if\ A = \{\}\ then\ \{\}\ else\ \{c\})$
by *auto*

lemma *image-image*: $f\ `\ (g\ `\ A) = (\lambda x.\ f\ (g\ x))\ `\ A$
by *blast*

lemma *insert-image* [*simp*]: $x \in A \Longrightarrow insert\ (f\ x)\ (f`A) = f`A$
by *blast*

lemma *image-is-empty* [*iff*]: $(f`A = \{\}) = (A = \{\})$
by *blast*

lemma *empty-is-image*[*iff*]: $(\{\} = f\ `\ A) = (A = \{\})$
by *blast*

lemma *image-Collect* [*no-atp*]: f ‘ $\{x.\ P\ x\} = \{f\ x \mid x.\ P\ x\}$
— NOT suitable as a default simprule: the RHS isn't simpler than the LHS, with its implicit quantifier and conjunction. Also image enjoys better equational properties than does the RHS.
by *blast*

lemma *if-image-distrib* [*simp*]:
$(\lambda x.\ if\ P\ x\ then\ f\ x\ else\ g\ x)$ ‘ S
$= (f$ ‘ $(S \cap \{x.\ P\ x\})) \cup (g$ ‘ $(S \cap \{x.\ \neg\ P\ x\}))$
by (*auto simp add*: *image-def*)

lemma *image-cong*: $M = N \Longrightarrow (!!x.\ x \in N \Longrightarrow f\ x = g\ x) \Longrightarrow f`M = g`N$
by (*simp add*: *image-def*)

range.

lemma *full-SetCompr-eq* [*no-atp*]: $\{u.\ \exists x.\ u = f\ x\} = range\ f$
by *auto*

lemma *range-composition*: $range\ (\lambda x.\ f\ (g\ x)) = f`range\ g$
by (*subst image-image*, *simp*)

Int

lemma *Int-absorb* [*simp*]: $A \cap A = A$
by (*fact inf-idem*)

lemma *Int-left-absorb*: $A \cap (A \cap B) = A \cap B$
by (*fact inf-left-idem*)

lemma *Int-commute*: $A \cap B = B \cap A$
by (*fact inf-commute*)

lemma *Int-left-commute*: $A \cap (B \cap C) = B \cap (A \cap C)$
by (*fact inf-left-commute*)

lemma *Int-assoc*: $(A \cap B) \cap C = A \cap (B \cap C)$
by (*fact inf-assoc*)

lemmas *Int-ac* = *Int-assoc Int-left-absorb Int-commute Int-left-commute*
— Intersection is an AC-operator

lemma *Int-absorb1*: $B \subseteq A \Longrightarrow A \cap B = B$
by (*fact inf-absorb2*)

lemma *Int-absorb2*: $A \subseteq B \Longrightarrow A \cap B = A$
by (*fact inf-absorb1*)

lemma *Int-empty-left* [*simp*]: $\{\} \cap B = \{\}$
by (*fact inf-bot-left*)

lemma *Int-empty-right* [*simp*]: $A \cap \{\} = \{\}$
by (*fact inf-bot-right*)

lemma *disjoint-eq-subset-Compl*: $(A \cap B = \{\}) = (A \subseteq -B)$
by *blast*

lemma *disjoint-iff-not-equal*: $(A \cap B = \{\}) = (\forall x{\in}A.\ \forall y{\in}B.\ x \neq y)$
by *blast*

lemma *Int-UNIV-left* [*simp*]: $UNIV \cap B = B$
by (*fact inf-top-left*)

lemma *Int-UNIV-right* [*simp*]: $A \cap UNIV = A$
by (*fact inf-top-right*)

lemma *Int-Un-distrib*: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
by (*fact inf-sup-distrib1*)

lemma *Int-Un-distrib2*: $(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$
by (*fact inf-sup-distrib2*)

lemma *Int-UNIV* [*simp*,*no-atp*]: $(A \cap B = UNIV) = (A = UNIV \ \&\ B = UNIV)$
by (*fact inf-eq-top-iff*)

lemma *Int-subset-iff* [*simp*]: $(C \subseteq A \cap B) = (C \subseteq A \ \&\ C \subseteq B)$
by (*fact le-inf-iff*)

lemma *Int-Collect*: $(x \in A \cap \{x.\ P\ x\}) = (x \in A \ \&\ P\ x)$
by *blast*

Un.

lemma *Un-absorb* [*simp*]: $A \cup A = A$
by (*fact sup-idem*)

lemma *Un-left-absorb*: $A \cup (A \cup B) = A \cup B$
by (*fact sup-left-idem*)

lemma *Un-commute*: $A \cup B = B \cup A$
by (*fact sup-commute*)

lemma *Un-left-commute*: $A \cup (B \cup C) = B \cup (A \cup C)$
by (*fact sup-left-commute*)

lemma *Un-assoc*: $(A \cup B) \cup C = A \cup (B \cup C)$
by (*fact sup-assoc*)

lemmas *Un-ac = Un-assoc Un-left-absorb Un-commute Un-left-commute*
— Union is an AC-operator

lemma *Un-absorb1*: $A \subseteq B \Longrightarrow A \cup B = B$
by (*fact sup-absorb2*)

lemma *Un-absorb2*: $B \subseteq A \Longrightarrow A \cup B = A$
by (*fact sup-absorb1*)

lemma *Un-empty-left* [*simp*]: $\{\} \cup B = B$
by (*fact sup-bot-left*)

lemma *Un-empty-right* [*simp*]: $A \cup \{\} = A$
by (*fact sup-bot-right*)

lemma *Un-UNIV-left* [*simp*]: $UNIV \cup B = UNIV$
by (*fact sup-top-left*)

lemma *Un-UNIV-right* [*simp*]: $A \cup UNIV = UNIV$
by (*fact sup-top-right*)

lemma *Un-insert-left* [*simp*]: $(insert\ a\ B) \cup C = insert\ a\ (B \cup C)$
by *blast*

lemma *Un-insert-right* [*simp*]: $A \cup (insert\ a\ B) = insert\ a\ (A \cup B)$
by *blast*

lemma *Int-insert-left*:
$(insert\ a\ B)\ Int\ C = (if\ a \in C\ then\ insert\ a\ (B \cap C)\ else\ B \cap C)$
by *auto*

lemma *Int-insert-left-if0*[*simp*]:
$a \notin C \Longrightarrow (insert\ a\ B)\ Int\ C = B \cap C$
by *auto*

lemma *Int-insert-left-if1*[*simp*]:
$a \in C \Longrightarrow (insert\ a\ B)\ Int\ C = insert\ a\ (B\ Int\ C)$
by *auto*

lemma *Int-insert-right*:
$A \cap (insert\ a\ B) = (if\ a \in A\ then\ insert\ a\ (A \cap B)\ else\ A \cap B)$
by *auto*

lemma *Int-insert-right-if0*[*simp*]:
$a \notin A \Longrightarrow A\ Int\ (insert\ a\ B) = A\ Int\ B$
by *auto*

lemma *Int-insert-right-if1*[*simp*]:
$a \in A \Longrightarrow A\ Int\ (insert\ a\ B) = insert\ a\ (A\ Int\ B)$
by *auto*

lemma *Un-Int-distrib*: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
by (*fact sup-inf-distrib1*)

lemma *Un-Int-distrib2*: $(B \cap C) \cup A = (B \cup A) \cap (C \cup A)$
by (*fact sup-inf-distrib2*)

lemma *Un-Int-crazy*:
$(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$
by *blast*

lemma *subset-Un-eq*: $(A \subseteq B) = (A \cup B = B)$
by (*fact le-iff-sup*)

lemma *Un-empty* [*iff*]: $(A \cup B = \{\}) = (A = \{\} \ \& \ B = \{\})$
by (*fact sup-eq-bot-iff*)

lemma *Un-subset-iff* [*simp*]: $(A \cup B \subseteq C) = (A \subseteq C \ \& \ B \subseteq C)$
by (*fact le-sup-iff*)

lemma *Un-Diff-Int*: $(A - B) \cup (A \cap B) = A$
by *blast*

lemma *Diff-Int2*: $A \cap C - B \cap C = A \cap C - B$
by *blast*

Set complement

lemma *Compl-disjoint* [*simp*]: $A \cap -A = \{\}$
by (*fact inf-compl-bot*)

lemma *Compl-disjoint2* [*simp*]: $-A \cap A = \{\}$
by (*fact compl-inf-bot*)

lemma *Compl-partition*: $A \cup -A = UNIV$
by (*fact sup-compl-top*)

lemma *Compl-partition2*: $-A \cup A = UNIV$
by (*fact compl-sup-top*)

lemma *double-complement* [*simp*]: $-(-A) = (A::'a\ set)$
by (*fact double-compl*)

lemma *Compl-Un* [*simp*]: $-(A \cup B) = (-A) \cap (-B)$
by (*fact compl-sup*)

lemma *Compl-Int* [*simp*]: $-(A \cap B) = (-A) \cup (-B)$
by (*fact compl-inf*)

lemma *subset-Compl-self-eq*: $(A \subseteq -A) = (A = \{\})$
by *blast*

lemma *Un-Int-assoc-eq*: $((A \cap B) \cup C = A \cap (B \cup C)) = (C \subseteq A)$
— Halmos, Naive Set Theory, page 16.
by *blast*

lemma *Compl-UNIV-eq* [*simp*]: $-UNIV = \{\}$
by (*fact compl-top-eq*)

lemma *Compl-empty-eq* [*simp*]: $-\{\} = UNIV$
by (*fact compl-bot-eq*)

lemma *Compl-subset-Compl-iff* [*iff*]: $(-A \subseteq -B) = (B \subseteq A)$
by (*fact compl-le-compl-iff*)

lemma *Compl-eq-Compl-iff* [*iff*]: $(-A = -B) = (A = (B::'a\ set))$
by (*fact compl-eq-compl-iff*)

Bounded quantifiers.
The following are not added to the default simpset because (a) they duplicate the body and (b) there are no similar rules for *Int*.

lemma *ball-Un*: $(\forall x \in A \cup B.\ P\ x) = ((\forall x{\in}A.\ P\ x)\ \&\ (\forall x{\in}B.\ P\ x))$
by *blast*

lemma *bex-Un*: $(\exists x \in A \cup B.\ P\ x) = ((\exists x{\in}A.\ P\ x) \mid (\exists x{\in}B.\ P\ x))$
by *blast*

Set difference.

lemma *Diff-eq*: $A - B = A \cap (-B)$
by *blast*

lemma *Diff-eq-empty-iff* [*simp*,*no-atp*]: $(A - B = \{\}) = (A \subseteq B)$
by *blast*

lemma *Diff-cancel* [*simp*]: $A - A = \{\}$
by *blast*

lemma *Diff-idemp* [*simp*]: $(A - B) - B = A - (B::'a\ set)$
by *blast*

lemma *Diff-triv*: $A \cap B = \{\} \Longrightarrow A - B = A$
by (*blast elim*: *equalityE*)

lemma *empty-Diff* [*simp*]: $\{\} - A = \{\}$
by *blast*

lemma *Diff-empty* [*simp*]: $A - \{\} = A$
by *blast*

lemma *Diff-UNIV* [*simp*]: $A - UNIV = \{\}$
by *blast*

lemma *Diff-insert0* [*simp*,*no-atp*]: $x \notin A \Longrightarrow A - insert\ x\ B = A - B$
by *blast*

lemma *Diff-insert*: $A - insert\ a\ B = A - B - \{a\}$
— NOT SUITABLE FOR REWRITING since $\{a\} == insert\ a\ 0$
by *blast*

lemma *Diff-insert2*: $A - insert\ a\ B = A - \{a\} - B$
— NOT SUITABLE FOR REWRITING since $\{a\} == insert\ a\ 0$
by *blast*

lemma *insert-Diff-if*: $insert\ x\ A - B = (if\ x \in B\ then\ A - B\ else\ insert\ x\ (A - B))$
by *auto*

lemma *insert-Diff1* [*simp*]: $x \in B \Longrightarrow insert\ x\ A - B = A - B$
by *blast*

lemma *insert-Diff-single*[*simp*]: $insert\ a\ (A - \{a\}) = insert\ a\ A$
by *blast*

lemma *insert-Diff*: $a \in A \Longrightarrow insert\ a\ (A - \{a\}) = A$
by *blast*

lemma *Diff-insert-absorb*: $x \notin A \Longrightarrow (insert\ x\ A) - \{x\} = A$
by *auto*

lemma *Diff-disjoint* [*simp*]: $A \cap (B - A) = \{\}$
by *blast*

lemma *Diff-partition*: $A \subseteq B \Longrightarrow A \cup (B - A) = B$
by *blast*

lemma *double-diff*: $A \subseteq B \Longrightarrow B \subseteq C \Longrightarrow B - (C - A) = A$
by *blast*

lemma *Un-Diff-cancel* [*simp*]: $A \cup (B - A) = A \cup B$
by *blast*

lemma *Un-Diff-cancel2* [*simp*]: $(B - A) \cup A = B \cup A$
by *blast*

lemma *Diff-Un*: $A - (B \cup C) = (A - B) \cap (A - C)$
by *blast*

lemma *Diff-Int*: $A - (B \cap C) = (A - B) \cup (A - C)$

by *blast*

lemma *Un-Diff*: $(A \cup B) - C = (A - C) \cup (B - C)$
by *blast*

lemma *Int-Diff*: $(A \cap B) - C = A \cap (B - C)$
by *blast*

lemma *Diff-Int-distrib*: $C \cap (A - B) = (C \cap A) - (C \cap B)$
by *blast*

lemma *Diff-Int-distrib2*: $(A - B) \cap C = (A \cap C) - (B \cap C)$
by *blast*

lemma *Diff-Compl* [*simp*]: $A - (- B) = A \cap B$
by *auto*

lemma *Compl-Diff-eq* [*simp*]: $- (A - B) = -A \cup B$
by *blast*

Quantification over type *bool*.

lemma *bool-induct*: $P\ True \Longrightarrow P\ False \Longrightarrow P\ x$
by (*cases x*) *auto*

lemma *all-bool-eq*: $(\forall b.\ P\ b) \longleftrightarrow P\ True \wedge P\ False$
by (*auto intro*: *bool-induct*)

lemma *bool-contrapos*: $P\ x \Longrightarrow \neg\ P\ False \Longrightarrow P\ True$
by (*cases x*) *auto*

lemma *ex-bool-eq*: $(\exists b.\ P\ b) \longleftrightarrow P\ True \vee P\ False$
by (*auto intro*: *bool-contrapos*)

Pow

lemma *Pow-empty* [*simp*]: $Pow\ \{\} = \{\{\}\}$
by (*auto simp add*: *Pow-def*)

lemma *Pow-insert*: $Pow\ (insert\ a\ A) = Pow\ A \cup (insert\ a\ `\ Pow\ A)$
by (*blast intro*: *image-eqI* [**where** *?x* = $u - \{a\}$, *standard*])

lemma *Pow-Compl*: $Pow\ (- A) = \{-B \mid B.\ A \in Pow\ B\}$
by (*blast intro*: *exI* [**where** *?x* = $- u$, *standard*])

lemma *Pow-UNIV* [*simp*]: $Pow\ UNIV = UNIV$
by *blast*

lemma *Un-Pow-subset*: $Pow\ A \cup Pow\ B \subseteq Pow\ (A \cup B)$
by *blast*

lemma *Pow-Int-eq* [*simp*]: $Pow\ (A \cap B) = Pow\ A \cap Pow\ B$
by *blast*

Miscellany.

lemma *set-eq-subset*: $(A = B) = (A \subseteq B\ \&\ B \subseteq A)$
by *blast*

lemma *subset-iff*: $(A \subseteq B) = (\forall t.\ t \in A \longrightarrow t \in B)$
by *blast*

lemma *subset-iff-psubset-eq*: $(A \subseteq B) = ((A \subset B) \mid (A = B))$
by (*unfold less-le*) *blast*

lemma *all-not-in-conv* [*simp*]: $(\forall x.\ x \notin A) = (A = \{\})$
by *blast*

lemma *ex-in-conv*: $(\exists x.\ x \in A) = (A \neq \{\})$
by *blast*

lemma *distinct-lemma*: $f\ x \neq f\ y \Longrightarrow x \neq y$
by *iprover*

6.4.4 Monotonicity of various operations

lemma *image-mono*: $A \subseteq B \Longrightarrow f`A \subseteq f`B$
by *blast*

lemma *Pow-mono*: $A \subseteq B \Longrightarrow Pow\ A \subseteq Pow\ B$
by *blast*

lemma *insert-mono*: $C \subseteq D \Longrightarrow insert\ a\ C \subseteq insert\ a\ D$
by *blast*

lemma *Un-mono*: $A \subseteq C \Longrightarrow B \subseteq D \Longrightarrow A \cup B \subseteq C \cup D$
by (*fact sup-mono*)

lemma *Int-mono*: $A \subseteq C \Longrightarrow B \subseteq D \Longrightarrow A \cap B \subseteq C \cap D$
by (*fact inf-mono*)

lemma *Diff-mono*: $A \subseteq C \Longrightarrow D \subseteq B \Longrightarrow A - B \subseteq C - D$
by *blast*

lemma *Compl-anti-mono*: $A \subseteq B \Longrightarrow -B \subseteq -A$
by (*fact compl-mono*)

Monotonicity of implications.

lemma *in-mono*: $A \subseteq B \Longrightarrow x \in A \longrightarrow x \in B$

apply (*rule impI*)
apply (*erule subsetD*, *assumption*)
done

lemma *conj-mono*: *P1* −−> *Q1* ==> *P2* −−> *Q2* ==> (*P1* & *P2*) −−> (*Q1* & *Q2*)
by *iprover*

lemma *disj-mono*: *P1* −−> *Q1* ==> *P2* −−> *Q2* ==> (*P1* | *P2*) −−> (*Q1* | *Q2*)
by *iprover*

lemma *imp-mono*: *Q1* −−> *P1* ==> *P2* −−> *Q2* ==> (*P1* −−> *P2*) −−> (*Q1* −−> *Q2*)
by *iprover*

lemma *imp-refl*: *P* −−> *P* **..**

lemma *not-mono*: *Q* −−> *P* ==> ~ *P* −−> ~ *Q*
by *iprover*

lemma *ex-mono*: (!!*x*. *P x* −−> *Q x*) ==> (*EX x*. *P x*) −−> (*EX x*. *Q x*)
by *iprover*

lemma *all-mono*: (!!*x*. *P x* −−> *Q x*) ==> (*ALL x*. *P x*) −−> (*ALL x*. *Q x*)
by *iprover*

lemma *Collect-mono*: (!!*x*. *P x* −−> *Q x*) ==> *Collect P* ⊆ *Collect Q*
by *blast*

lemma *Int-Collect-mono*:
A ⊆ *B* ==> (!!*x*. *x* ∈ *A* ==> *P x* −−> *Q x*) ==> *A* ∩ *Collect P* ⊆ *B* ∩ *Collect Q*
by *blast*

lemmas *basic-monos* =
subset-refl imp-refl disj-mono conj-mono
ex-mono Collect-mono in-mono

lemma *eq-to-mono*: *a* = *b* ==> *c* = *d* ==> *b* −−> *d* ==> *a* −−> *c*
by *iprover*

6.4.5 Inverse image of a function

definition *vimage* :: ('*a* => '*b*) => '*b set* => '*a set* (**infixr** −‘ *90*) **where**
[*code del*]: *f* −‘ *B* == {*x*. *f x* : *B*}

lemma *vimage-eq* [*simp*]: (*a* : *f* −‘ *B*) = (*f a* : *B*)
by (*unfold vimage-def*) *blast*

lemma *vimage-singleton-eq*: $(a : f -` \{b\}) = (f\ a = b)$
by *simp*

lemma *vimageI* [*intro*]: $f\ a = b \Longrightarrow b{:}B \Longrightarrow a : f -` B$
by (*unfold vimage-def*) *blast*

lemma *vimageI2*: $f\ a : A \Longrightarrow a : f -` A$
by (*unfold vimage-def*) *fast*

lemma *vimageE* [*elim*!]: $a{:}\ f -` B \Longrightarrow (!!x.\ f\ a = x \Longrightarrow x{:}B \Longrightarrow P) \Longrightarrow P$
by (*unfold vimage-def*) *blast*

lemma *vimageD*: $a : f -` A \Longrightarrow f\ a : A$
by (*unfold vimage-def*) *fast*

lemma *vimage-empty* [*simp*]: $f -` \{\} = \{\}$
by *blast*

lemma *vimage-Compl*: $f -` (-A) = -(f -` A)$
by *blast*

lemma *vimage-Un* [*simp*]: $f -` (A\ Un\ B) = (f -` A)\ Un\ (f -` B)$
by *blast*

lemma *vimage-Int* [*simp*]: $f -` (A\ Int\ B) = (f -` A)\ Int\ (f -` B)$
by *fast*

lemma *vimage-Collect-eq* [*simp*]: $f -` Collect\ P = \{y.\ P\ (f\ y)\}$
by *blast*

lemma *vimage-Collect*: $(!!x.\ P\ (f\ x) = Q\ x) \Longrightarrow f -` (Collect\ P) = Collect\ Q$
by *blast*

lemma *vimage-insert*: $f-`(insert\ a\ B) = (f-`\{a\})\ Un\ (f-`B)$
— NOT suitable for rewriting because of the recurrence of $\{a\}$.
by *blast*

lemma *vimage-Diff*: $f -` (A - B) = (f -` A) - (f -` B)$
by *blast*

lemma *vimage-UNIV* [*simp*]: $f -` UNIV = UNIV$
by *blast*

lemma *vimage-mono*: $A \subseteq B \Longrightarrow f -` A \subseteq f -` B$
— monotonicity
by *blast*

lemma *vimage-image-eq* [*no-atp*]: $f -` (f ` A) = \{y.\ EX\ x{:}A.\ f\ x = f\ y\}$
by (*blast intro*: *sym*)

lemma *image-vimage-subset*: $f ` (f -` A) <= A$
by *blast*

lemma *image-vimage-eq* [*simp*]: $f ` (f -` A) = A\ Int\ range\ f$
by *blast*

lemma *vimage-const* [*simp*]: $((\lambda x.\ c) -` A) = (if\ c \in A\ then\ UNIV\ else\ \{\})$
by *auto*

lemma *vimage-if* [*simp*]: $((\lambda x.\ if\ x \in B\ then\ c\ else\ d) -` A) =$
$(if\ c \in A\ then\ (if\ d \in A\ then\ UNIV\ else\ B)$
$else\ if\ d \in A\ then\ -B\ else\ \{\})$
by (*auto simp add*: *vimage-def*)

lemma *vimage-inter-cong*:
$(\bigwedge w.\ w \in S \Longrightarrow f\ w = g\ w) \Longrightarrow f -` y \cap S = g -` y \cap S$
by *auto*

lemma *image-Int-subset*: $f`(A\ Int\ B) <= f`A\ Int\ f`B$
by *blast*

lemma *image-diff-subset*: $f`A - f`B <= f`(A - B)$
by *blast*

6.4.6 Getting the Contents of a Singleton Set

definition *contents* :: $'a\ set \Rightarrow 'a$ **where**
[*code del*]: $contents\ X = (THE\ x.\ X = \{x\})$

lemma *contents-eq* [*simp*]: $contents\ \{x\} = x$
by (*simp add*: *contents-def*)

6.4.7 Least value operator

lemma *Least-mono*:
$mono\ (f{::}'a{::}order => 'b{::}order) ==> EX\ x{:}S.\ ALL\ y{:}S.\ x <= y$
$==> (LEAST\ y.\ y : f ` S) = f\ (LEAST\ x.\ x : S)$
— Courtesy of Stephan Merz
apply *clarify*
apply (*erule-tac* $P = \%x.\ x : S$ **in** *LeastI2-order*, *fast*)
apply (*rule LeastI2-order*)
apply (*auto elim*: *monoD intro*!: *order-antisym*)
done

6.5 Misc

Rudimentary code generation

lemma *insert-code* [*code*]: *insert y A x* $\longleftrightarrow$ *y = x* $\vee$ *A x*
by (*auto simp add*: *insert-compr Collect-def mem-def*)

lemma *vimage-code* [*code*]: (*f −‘ A*) *x* = *A* (*f x*)
by (*simp add*: *vimage-def Collect-def mem-def*)

Misc theorem and ML bindings

lemmas *equalityI* = *subset-antisym*

ML ⟪
val Ball-def = @{*thm Ball-def*}
val Bex-def = @{*thm Bex-def*}
val CollectD = @{*thm CollectD*}
val CollectE = @{*thm CollectE*}
val CollectI = @{*thm CollectI*}
val Collect-conj-eq = @{*thm Collect-conj-eq*}
val Collect-mem-eq = @{*thm Collect-mem-eq*}
val IntD1 = @{*thm IntD1*}
val IntD2 = @{*thm IntD2*}
val IntE = @{*thm IntE*}
val IntI = @{*thm IntI*}
val Int-Collect = @{*thm Int-Collect*}
val UNIV-I = @{*thm UNIV-I*}
val UNIV-witness = @{*thm UNIV-witness*}
val UnE = @{*thm UnE*}
val UnI1 = @{*thm UnI1*}
val UnI2 = @{*thm UnI2*}
val ballE = @{*thm ballE*}
val ballI = @{*thm ballI*}
val bexCI = @{*thm bexCI*}
val bexE = @{*thm bexE*}
val bexI = @{*thm bexI*}
val bex-triv = @{*thm bex-triv*}
val bspec = @{*thm bspec*}
val contra-subsetD = @{*thm contra-subsetD*}
val distinct-lemma = @{*thm distinct-lemma*}
val eq-to-mono = @{*thm eq-to-mono*}
val equalityCE = @{*thm equalityCE*}
val equalityD1 = @{*thm equalityD1*}
val equalityD2 = @{*thm equalityD2*}
val equalityE = @{*thm equalityE*}
val equalityI = @{*thm equalityI*}
val imageE = @{*thm imageE*}
val imageI = @{*thm imageI*}
val image-Un = @{*thm image-Un*}
val image-insert = @{*thm image-insert*}
val insert-commute = @{*thm insert-commute*}
val insert-iff = @{*thm insert-iff*}
val mem-Collect-eq = @{*thm mem-Collect-eq*}

val rangeE = @{thm rangeE}
val rangeI = @{thm rangeI}
val range-eqI = @{thm range-eqI}
val subsetCE = @{thm subsetCE}
val subsetD = @{thm subsetD}
val subsetI = @{thm subsetI}
val subset-refl = @{thm subset-refl}
val subset-trans = @{thm subset-trans}
val vimageD = @{thm vimageD}
val vimageE = @{thm vimageE}
val vimageI = @{thm vimageI}
val vimageI2 = @{thm vimageI2}
val vimage-Collect = @{thm vimage-Collect}
val vimage-Int = @{thm vimage-Int}
val vimage-Un = @{thm vimage-Un}
⟩⟩

end

7 Typedef: HOL type definitions

theory *Typedef*
imports *Set*
uses
(*Tools/typedef.ML*)
(*Tools/typecopy.ML*)
(*Tools/typedef-codegen.ML*)
begin

ML ⟨⟨
structure HOL = struct val thy = theory HOL end;
⟩⟩ — belongs to theory HOL

locale *type-definition* =
fixes *Rep* **and** *Abs* **and** *A*
assumes *Rep*: *Rep x* ∈ *A*
and *Rep-inverse*: *Abs* (*Rep x*) = *x*
and *Abs-inverse*: *y* ∈ *A* ==> *Rep* (*Abs y*) = *y*
— This will be axiomatized for each typedef!
begin

lemma *Rep-inject*:
(*Rep x* = *Rep y*) = (*x* = *y*)
proof
assume *Rep x* = *Rep y*
then have *Abs* (*Rep x*) = *Abs* (*Rep y*) **by** (*simp only*:)
moreover have *Abs* (*Rep x*) = *x* **by** (*rule Rep-inverse*)
moreover have *Abs* (*Rep y*) = *y* **by** (*rule Rep-inverse*)

ultimately show $x = y$ **by** *simp*
next
assume $x = y$
thus *Rep* $x =$ *Rep* y **by** (*simp only*:)
qed

lemma *Abs-inject*:
assumes x: $x \in A$ **and** y: $y \in A$
shows (*Abs* $x =$ *Abs* $y) = (x = y)$
proof
assume *Abs* $x =$ *Abs* y
then have *Rep* (*Abs* x) = *Rep* (*Abs* y) **by** (*simp only*:)
moreover from x **have** *Rep* (*Abs* x) = x **by** (*rule Abs-inverse*)
moreover from y **have** *Rep* (*Abs* y) = y **by** (*rule Abs-inverse*)
ultimately show $x = y$ **by** *simp*
next
assume $x = y$
thus *Abs* $x =$ *Abs* y **by** (*simp only*:)
qed

lemma *Rep-cases* [*cases set*]:
assumes y: $y \in A$
and *hyp*: !!x. $y =$ *Rep* x ==> P
shows P
proof (*rule hyp*)
from y **have** *Rep* (*Abs* y) = y **by** (*rule Abs-inverse*)
thus $y =$ *Rep* (*Abs* y) **..**
qed

lemma *Abs-cases* [*cases type*]:
assumes r: !!y. $x =$ *Abs* y ==> $y \in A$ ==> P
shows P
proof (*rule* r)
have *Abs* (*Rep* x) = x **by** (*rule Rep-inverse*)
thus $x =$ *Abs* (*Rep* x) **..**
show *Rep* $x \in A$ **by** (*rule Rep*)
qed

lemma *Rep-induct* [*induct set*]:
assumes y: $y \in A$
and *hyp*: !!x. P (*Rep* x)
shows P y
proof −
have P (*Rep* (*Abs* y)) **by** (*rule hyp*)
moreover from y **have** *Rep* (*Abs* y) = y **by** (*rule Abs-inverse*)
ultimately show P y **by** *simp*
qed

lemma *Abs-induct* [*induct type*]:

```
  assumes r: !!y. y ∈ A ==> P (Abs y)
  shows P x
proof −
  have Rep x ∈ A by (rule Rep)
  then have P (Abs (Rep x)) by (rule r)
  moreover have Abs (Rep x) = x by (rule Rep-inverse)
  ultimately show P x by simp
qed

lemma Rep-range: range Rep = A
proof
  show range Rep <= A using Rep by (auto simp add: image-def)
  show A <= range Rep
  proof
    fix x assume x : A
    hence x = Rep (Abs x) by (rule Abs-inverse [symmetric])
    thus x : range Rep by (rule range-eqI)
  qed
qed

lemma Abs-image: Abs ‘ A = UNIV
proof
  show Abs ‘ A <= UNIV by (rule subset-UNIV)
next
  show UNIV <= Abs ‘ A
  proof
    fix x
    have x = Abs (Rep x) by (rule Rep-inverse [symmetric])
    moreover have Rep x : A by (rule Rep)
    ultimately show x : Abs ‘ A by (rule image-eqI)
  qed
qed

end

use Tools/typedef.ML setup Typedef.setup
use Tools/typecopy.ML setup Typecopy.setup
use Tools/typedef-codegen.ML setup TypedefCodegen.setup

end
```

8 Complete-Lattice: Complete lattices, with special focus on sets

```
theory Complete-Lattice
imports Set
begin
```

notation
less-eq (**infix** $\sqsubseteq$ *50*) **and**
less (**infix** $\sqsubset$ *50*) **and**
inf (**infixl** $\sqcap$ *70*) **and**
sup (**infixl** $\sqcup$ *65*) **and**
top ($\top$) **and**
bot ($\bot$)

8.1 Syntactic infimum and supremum operations

class *Inf* =
fixes *Inf* :: *'a set* $\Rightarrow$ *'a* ($\bigsqcap$- [*900*] *900*)

class *Sup* =
fixes *Sup* :: *'a set* $\Rightarrow$ *'a* ($\bigsqcup$- [*900*] *900*)

8.2 Abstract complete lattices

class *complete-lattice* = *bounded-lattice* + *Inf* + *Sup* +
assumes *Inf-lower*: $x \in A \Longrightarrow \bigsqcap A \sqsubseteq x$
and *Inf-greatest*: $(\bigwedge x.\ x \in A \Longrightarrow z \sqsubseteq x) \Longrightarrow z \sqsubseteq \bigsqcap A$
assumes *Sup-upper*: $x \in A \Longrightarrow x \sqsubseteq \bigsqcup A$
and *Sup-least*: $(\bigwedge x.\ x \in A \Longrightarrow x \sqsubseteq z) \Longrightarrow \bigsqcup A \sqsubseteq z$
begin

lemma *dual-complete-lattice*:
class.complete-lattice Sup Inf (*op* $\geq$) (*op* $>$) (*op* $\sqcup$) (*op* $\sqcap$) $\top$ $\bot$
by (*auto intro*!: *class.complete-lattice.intro dual-bounded-lattice*)
(*unfold-locales*, (*fact bot-least top-greatest*
Sup-upper Sup-least Inf-lower Inf-greatest)+)

lemma *Inf-Sup*: $\bigsqcap A = \bigsqcup \{b.\ \forall a \in A.\ b \sqsubseteq a\}$
by (*auto intro*: *antisym Inf-lower Inf-greatest Sup-upper Sup-least*)

lemma *Sup-Inf*: $\bigsqcup A = \bigsqcap \{b.\ \forall a \in A.\ a \sqsubseteq b\}$
by (*auto intro*: *antisym Inf-lower Inf-greatest Sup-upper Sup-least*)

lemma *Inf-empty*:
$\bigsqcap \{\} = \top$
by (*auto intro*: *antisym Inf-greatest*)

lemma *Sup-empty*:
$\bigsqcup \{\} = \bot$
by (*auto intro*: *antisym Sup-least*)

lemma *Inf-insert*: $\bigsqcap$ *insert a A* $= a \sqcap \bigsqcap A$
by (*auto intro*: *le-infI le-infI1 le-infI2 antisym Inf-greatest Inf-lower*)

lemma *Sup-insert*: $\bigsqcup$ *insert a A* $= a \sqcup \bigsqcup A$

by (*auto intro*: *le-supI le-supI1 le-supI2 antisym Sup-least Sup-upper*)

lemma *Inf-singleton* [*simp*]:
$\bigsqcap\{a\} = a$
by (*auto intro*: *antisym Inf-lower Inf-greatest*)

lemma *Sup-singleton* [*simp*]:
$\bigsqcup\{a\} = a$
by (*auto intro*: *antisym Sup-upper Sup-least*)

lemma *Inf-binary*:
$\bigsqcap\{a, b\} = a \sqcap b$
by (*simp add*: *Inf-empty Inf-insert*)

lemma *Sup-binary*:
$\bigsqcup\{a, b\} = a \sqcup b$
by (*simp add*: *Sup-empty Sup-insert*)

lemma *Inf-UNIV*:
$\bigsqcap UNIV = bot$
by (*simp add*: *Sup-Inf Sup-empty* [*symmetric*])

lemma *Sup-UNIV*:
$\bigsqcup UNIV = top$
by (*simp add*: *Inf-Sup Inf-empty* [*symmetric*])

lemma *Sup-le-iff*: $Sup\ A \sqsubseteq b \longleftrightarrow (\forall\, a{\in}A.\ a \sqsubseteq b)$
by (*auto intro*: *Sup-least dest*: *Sup-upper*)

lemma *le-Inf-iff*: $b \sqsubseteq Inf\ A \longleftrightarrow (\forall\, a{\in}A.\ b \sqsubseteq a)$
by (*auto intro*: *Inf-greatest dest*: *Inf-lower*)

definition $SUPR :: 'b\ set \Rightarrow ('b \Rightarrow 'a) \Rightarrow 'a$ **where**
$SUPR\ A\ f = \bigsqcup\ (f\ `\ A)$

definition $INFI :: 'b\ set \Rightarrow ('b \Rightarrow 'a) \Rightarrow 'a$ **where**
$INFI\ A\ f = \bigsqcap\ (f\ `\ A)$

end

syntax
-SUP1 :: *pttrns* => *'b* => *'b* (*(3SUP -./ -) [0, 10] 10*)
-SUP :: *pttrn* => *'a set* => *'b* => *'b* (*(3SUP -:-./ -) [0, 0, 10] 10*)
-INF1 :: *pttrns* => *'b* => *'b* (*(3INF -./ -) [0, 10] 10*)
-INF :: *pttrn* => *'a set* => *'b* => *'b* (*(3INF -:-./ -) [0, 0, 10] 10*)

translations
$SUP\ x\ y.\ B$ == $SUP\ x.\ SUP\ y.\ B$
$SUP\ x.\ B$ == $CONST\ SUPR\ CONST\ UNIV\ (\%x.\ B)$

SUP x. B == *SUP x:CONST UNIV. B*
SUP x:A. B == *CONST SUPR A (%x. B)*
INF x y. B == *INF x. INF y. B*
INF x. B == *CONST INFI CONST UNIV (%x. B)*
INF x. B == *INF x:CONST UNIV. B*
INF x:A. B == *CONST INFI A (%x. B)*

print-translation ⟪
[*Syntax.preserve-binder-abs2-tr'* @{*const-syntax SUPR*} @{*syntax-const -SUP*},
Syntax.preserve-binder-abs2-tr' @{*const-syntax INFI*} @{*syntax-const -INF*}]
⟫ — to avoid eta-contraction of body

context *complete-lattice*
begin

lemma *le-SUPI*: $i : A \Longrightarrow M\ i \sqsubseteq (SUP\ i{:}A.\ M\ i)$
by (*auto simp add*: *SUPR-def intro*: *Sup-upper*)

lemma *SUP-leI*: $(\bigwedge i.\ i : A \Longrightarrow M\ i \sqsubseteq u) \Longrightarrow (SUP\ i{:}A.\ M\ i) \sqsubseteq u$
by (*auto simp add*: *SUPR-def intro*: *Sup-least*)

lemma *INF-leI*: $i : A \Longrightarrow (INF\ i{:}A.\ M\ i) \sqsubseteq M\ i$
by (*auto simp add*: *INFI-def intro*: *Inf-lower*)

lemma *le-INFI*: $(\bigwedge i.\ i : A \Longrightarrow u \sqsubseteq M\ i) \Longrightarrow u \sqsubseteq (INF\ i{:}A.\ M\ i)$
by (*auto simp add*: *INFI-def intro*: *Inf-greatest*)

lemma *SUP-le-iff*: $(SUP\ i{:}A.\ M\ i) \sqsubseteq u \longleftrightarrow (\forall\, i \in A.\ M\ i \sqsubseteq u)$
unfolding *SUPR-def* **by** (*auto simp add*: *Sup-le-iff*)

lemma *le-INF-iff*: $u \sqsubseteq (INF\ i{:}A.\ M\ i) \longleftrightarrow (\forall\, i \in A.\ u \sqsubseteq M\ i)$
unfolding *INFI-def* **by** (*auto simp add*: *le-Inf-iff*)

lemma *SUP-const*[*simp*]: $A \neq \{\} \Longrightarrow (SUP\ i{:}A.\ M) = M$
by (*auto intro*: *antisym SUP-leI le-SUPI*)

lemma *INF-const*[*simp*]: $A \neq \{\} \Longrightarrow (INF\ i{:}A.\ M) = M$
by (*auto intro*: *antisym INF-leI le-INFI*)

end

8.3 *bool* and - ⇒ - as complete lattice

instantiation *bool* :: *complete-lattice*
begin

definition
Inf-bool-def: $\bigsqcap A \longleftrightarrow (\forall\, x{\in}A.\ x)$

definition
Sup-bool-def: $\bigsqcup A \longleftrightarrow (\exists\, x{\in}A.\ x)$

instance proof
qed (*auto simp add*: *Inf-bool-def Sup-bool-def le-bool-def*)

end

lemma *Inf-empty-bool* [*simp*]:
$\bigsqcap\{\}$
unfolding *Inf-bool-def* **by** *auto*

lemma *not-Sup-empty-bool* [*simp*]:
$\neg \bigsqcup\{\}$
unfolding *Sup-bool-def* **by** *auto*

lemma *INFI-bool-eq*:
INFI = *Ball*
proof (*rule ext*)+
fix *A* :: *'a set*
fix *P* :: *'a* ⇒ *bool*
show (*INF x*:*A*. *P x*) ⟷ (∀ *x* ∈ *A*. *P x*)
by (*auto simp add*: *Ball-def INFI-def Inf-bool-def*)
qed

lemma *SUPR-bool-eq*:
SUPR = *Bex*
proof (*rule ext*)+
fix *A* :: *'a set*
fix *P* :: *'a* ⇒ *bool*
show (*SUP x*:*A*. *P x*) ⟷ (∃ *x* ∈ *A*. *P x*)
by (*auto simp add*: *Bex-def SUPR-def Sup-bool-def*)
qed

instantiation *fun* :: (*type*, *complete-lattice*) *complete-lattice*
begin

definition
Inf-fun-def [*code del*]: $\bigsqcap A = (\lambda x.\ \bigsqcap\{y.\ \exists f{\in}A.\ y = f\,x\})$

definition
Sup-fun-def [*code del*]: $\bigsqcup A = (\lambda x.\ \bigsqcup\{y.\ \exists f{\in}A.\ y = f\,x\})$

instance proof
qed (*auto simp add*: *Inf-fun-def Sup-fun-def le-fun-def*
intro: *Inf-lower Sup-upper Inf-greatest Sup-least*)

end

lemma *Inf-empty-fun*:
$\bigsqcap\{\} = (\lambda\text{-}.\ \bigsqcap\{\})$
by (*simp add*: *Inf-fun-def*)

lemma *Sup-empty-fun*:
$\bigsqcup\{\} = (\lambda\text{-}.\ \bigsqcup\{\})$
by (*simp add*: *Sup-fun-def*)

8.4 Union

abbreviation *Union* :: *'a set set* ⇒ *'a set* **where**
Union S ≡ $\bigsqcup S$

notation (*xsymbols*)
Union ($\bigcup$ - [*90*] *90*)

lemma *Union-eq*:
$\bigcup A = \{x.\ \exists B \in A.\ x \in B\}$
proof (*rule set-ext*)
fix *x*
have ($\exists Q{\in}\{P.\ \exists B{\in}A.\ P \longleftrightarrow x \in B\}.\ Q$) ⟷ ($\exists B{\in}A.\ x \in B$)
by *auto*
then show $x \in \bigcup A \longleftrightarrow x \in \{x.\ \exists B{\in}A.\ x \in B\}$
by (*simp add*: *Sup-fun-def Sup-bool-def*) (*simp add*: *mem-def*)
qed

lemma *Union-iff* [*simp*, *no-atp*]:
$A \in \bigcup C \longleftrightarrow (\exists X{\in}C.\ A{\in}X)$
by (*unfold Union-eq*) *blast*

lemma *UnionI* [*intro*]:
$X \in C \Longrightarrow A \in X \Longrightarrow A \in \bigcup C$
— The order of the premises presupposes that *C* is rigid; *A* may be flexible.
by *auto*

lemma *UnionE* [*elim*!]:
$A \in \bigcup C \Longrightarrow (\bigwedge X.\ A{\in}X \Longrightarrow X{\in}C \Longrightarrow R) \Longrightarrow R$
by *auto*

lemma *Union-upper*: $B \in A$ ==> $B \subseteq$ *Union A*
by (*iprover intro*: *subsetI UnionI*)

lemma *Union-least*: (!!$X.\ X \in A$ ==> $X \subseteq C$) ==> *Union A* $\subseteq C$
by (*iprover intro*: *subsetI elim*: *UnionE dest*: *subsetD*)

lemma *Un-eq-Union*: $A \cup B = \bigcup\{A, B\}$
by *blast*

lemma *Union-empty* [*simp*]: *Union*({}) = {}

by *blast*

lemma *Union-UNIV* [*simp*]: *Union UNIV* = *UNIV*
by *blast*

lemma *Union-insert* [*simp*]: *Union* (*insert a B*) = $a \cup \bigcup B$
by *blast*

lemma *Union-Un-distrib* [*simp*]: $\bigcup(A\ Un\ B) = \bigcup A \cup \bigcup B$
by *blast*

lemma *Union-Int-subset*: $\bigcup(A \cap B) \subseteq \bigcup A \cap \bigcup B$
by *blast*

lemma *Union-empty-conv* [*simp*,*no-atp*]: $(\bigcup A = \{\}) = (\forall x{\in}A.\ x = \{\})$
by *blast*

lemma *empty-Union-conv* [*simp*,*no-atp*]: $(\{\} = \bigcup A) = (\forall x{\in}A.\ x = \{\})$
by *blast*

lemma *Union-disjoint*: $(\bigcup C \cap A = \{\}) = (\forall B{\in}C.\ B \cap A = \{\})$
by *blast*

lemma *subset-Pow-Union*: $A \subseteq Pow\ (\bigcup A)$
by *blast*

lemma *Union-Pow-eq* [*simp*]: $\bigcup(Pow\ A) = A$
by *blast*

lemma *Union-mono*: $A \subseteq B \Longrightarrow \bigcup A \subseteq \bigcup B$
by *blast*

8.5 Unions of families

abbreviation *UNION* :: *'a set* ⇒ (*'a* ⇒ *'b set*) ⇒ *'b set* **where**
UNION ≡ *SUPR*

syntax
-UNION1 :: *pttrns* => *'b set* => *'b set* (*3UN -./ -*) [*0*, *10*] *10*)
-UNION :: *pttrn* => *'a set* => *'b set* => *'b set* (*3UN -:-./ -*) [*0*, *0*, *10*] *10*)

syntax (*xsymbols*)
-UNION1 :: *pttrns* => *'b set* => *'b set* (*3⋃-./ -*) [*0*, *10*] *10*)
-UNION :: *pttrn* => *'a set* => *'b set* => *'b set* (*3⋃-∈-./ -*) [*0*, *0*, *10*] *10*)

syntax (*latex* **output**)
-UNION1 :: *pttrns* => *'b set* => *'b set* (*3⋃(00-)/ -*) [*0*, *10*] *10*)
-UNION :: *pttrn* => *'a set* => *'b set* => *'b set* (*3⋃(00-∈-)/ -*) [*0*, *0*, *10*]

10)

translations
UN x y. B == *UN x. UN y. B*
UN x. B == *CONST UNION CONST UNIV* (%*x. B*)
UN x. B == *UN x*:*CONST UNIV. B*
UN x:*A. B* == *CONST UNION A* (%*x. B*)

Note the difference between ordinary xsymbol syntax of indexed unions and intersections (e.g. $\bigcup a_1{\in}A_1.\ B$) and their LaTeX rendition: $\bigcup_{a_1 \in A_1} B$. The former does not make the index expression a subscript of the union/intersection symbol because this leads to problems with nested subscripts in Proof General.

print-translation ⟪
[*Syntax.preserve-binder-abs2-tr'* @{*const-syntax UNION*} @{*syntax-const -UNION*}]
⟫ — to avoid eta-contraction of body

lemma *UNION-eq-Union-image*:
$(\bigcup x{\in}A.\ B\ x) = \bigcup (B`A)$
by (*fact SUPR-def*)

lemma *Union-def*:
$\bigcup S = (\bigcup x{\in}S.\ x)$
by (*simp add*: *UNION-eq-Union-image image-def*)

lemma *UNION-def* [*no-atp*]:
$(\bigcup x{\in}A.\ B\ x) = \{y.\ \exists x{\in}A.\ y \in B\ x\}$
by (*auto simp add*: *UNION-eq-Union-image Union-eq*)

lemma *Union-image-eq* [*simp*]:
$\bigcup (B`A) = (\bigcup x{\in}A.\ B\ x)$
by (*rule sym*) (*fact UNION-eq-Union-image*)

lemma *UN-iff* [*simp*]: (*b*: (*UN x*:*A. B x*)) = (*EX x*:*A. b*: *B x*)
by (*unfold UNION-def*) *blast*

lemma *UN-I* [*intro*]: *a*:*A* ==> *b*: *B a* ==> *b*: (*UN x*:*A. B x*)
— The order of the premises presupposes that *A* is rigid; *b* may be flexible.
by *auto*

lemma *UN-E* [*elim*!]: *b* : (*UN x*:*A. B x*) ==> (!!*x. x*:*A* ==> *b*: *B x* ==> *R*) ==> *R*
by (*unfold UNION-def*) *blast*

lemma *UN-cong* [*cong*]:
A = *B* ==> (!!*x. x*:*B* ==> *C x* = *D x*) ==> (*UN x*:*A. C x*) = (*UN x*:*B. D x*)
by (*simp add*: *UNION-def*)

lemma *strong-UN-cong*:
$A = B \Longrightarrow (!!x.\ x{:}B =simp=> C\ x = D\ x) \Longrightarrow (UN\ x{:}A.\ C\ x) = (UN\ x{:}B.\ D\ x)$
by (*simp add*: *UNION-def simp-implies-def*)

lemma *image-eq-UN*: $f`A = (UN\ x{:}A.\ \{f\ x\})$
by *blast*

lemma *UN-upper*: $a \in A \Longrightarrow B\ a \subseteq (\bigcup x{\in}A.\ B\ x)$
by (*fact le-SUPI*)

lemma *UN-least*: $(!!x.\ x \in A \Longrightarrow B\ x \subseteq C) \Longrightarrow (\bigcup x{\in}A.\ B\ x) \subseteq C$
by (*iprover intro*: *subsetI elim*: *UN-E dest*: *subsetD*)

lemma *Collect-bex-eq* [*no-atp*]: $\{x.\ \exists y{\in}A.\ P\ x\ y\} = (\bigcup y{\in}A.\ \{x.\ P\ x\ y\})$
by *blast*

lemma *UN-insert-distrib*: $u \in A \Longrightarrow (\bigcup x{\in}A.\ insert\ a\ (B\ x)) = insert\ a\ (\bigcup x{\in}A.\ B\ x)$
by *blast*

lemma *UN-empty* [*simp*,*no-atp*]: $(\bigcup x{\in}\{\}.\ B\ x) = \{\}$
by *blast*

lemma *UN-empty2* [*simp*]: $(\bigcup x{\in}A.\ \{\}) = \{\}$
by *blast*

lemma *UN-singleton* [*simp*]: $(\bigcup x{\in}A.\ \{x\}) = A$
by *blast*

lemma *UN-absorb*: $k \in I \Longrightarrow A\ k \cup (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ A\ i)$
by *auto*

lemma *UN-insert* [*simp*]: $(\bigcup x{\in}insert\ a\ A.\ B\ x) = B\ a \cup UNION\ A\ B$
by *blast*

lemma *UN-Un*[*simp*]: $(\bigcup i \in A \cup B.\ M\ i) = (\bigcup i{\in}A.\ M\ i) \cup (\bigcup i{\in}B.\ M\ i)$
by *blast*

lemma *UN-UN-flatten*: $(\bigcup x \in (\bigcup y{\in}A.\ B\ y).\ C\ x) = (\bigcup y{\in}A.\ \bigcup x{\in}B\ y.\ C\ x)$
by *blast*

lemma *UN-subset-iff*: $((\bigcup i{\in}I.\ A\ i) \subseteq B) = (\forall i{\in}I.\ A\ i \subseteq B)$
by (*fact SUP-le-iff*)

lemma *image-Union*: $f\ `\ \bigcup S = (\bigcup x{\in}S.\ f\ `\ x)$
by *blast*

lemma *UN-constant* [*simp*]: $(\bigcup y{\in}A.\ c) = (if\ A = \{\}\ then\ \{\}\ else\ c)$

by *auto*

lemma *UN-eq*: $(\bigcup x{\in}A.\ B\ x) = \bigcup(\{Y.\ \exists x{\in}A.\ Y = B\ x\})$
by *blast*

lemma *UNION-empty-conv*[*simp*]:
$(\{\} = (UN\ x{:}A.\ B\ x)) = (\forall x{\in}A.\ B\ x = \{\})$
$((UN\ x{:}A.\ B\ x) = \{\}) = (\forall x{\in}A.\ B\ x = \{\})$
by *blast*+

lemma *Collect-ex-eq* [*no-atp*]: $\{x.\ \exists y.\ P\ x\ y\} = (\bigcup y.\ \{x.\ P\ x\ y\})$
by *blast*

lemma *ball-UN*: $(\forall z \in UNION\ A\ B.\ P\ z) = (\forall x{\in}A.\ \forall z \in B\ x.\ P\ z)$
by *blast*

lemma *bex-UN*: $(\exists z \in UNION\ A\ B.\ P\ z) = (\exists x{\in}A.\ \exists z{\in}B\ x.\ P\ z)$
by *blast*

lemma *Un-eq-UN*: $A \cup B = (\bigcup b.\ if\ b\ then\ A\ else\ B)$
by (*auto simp add*: *split-if-mem2*)

lemma *UN-bool-eq*: $(\bigcup b{::}bool.\ A\ b) = (A\ True \cup A\ False)$
by (*auto intro*: *bool-contrapos*)

lemma *UN-Pow-subset*: $(\bigcup x{\in}A.\ Pow\ (B\ x)) \subseteq Pow\ (\bigcup x{\in}A.\ B\ x)$
by *blast*

lemma *UN-mono*:
$A \subseteq B \Longrightarrow (!!x.\ x \in A \Longrightarrow f\ x \subseteq g\ x) \Longrightarrow$
$(\bigcup x{\in}A.\ f\ x) \subseteq (\bigcup x{\in}B.\ g\ x)$
by (*blast dest*: *subsetD*)

lemma *vimage-Union*: $f\ -`\ (Union\ A) = (UN\ X{:}A.\ f\ -`\ X)$
by *blast*

lemma *vimage-UN*: $f-`(UN\ x{:}A.\ B\ x) = (UN\ x{:}A.\ f\ -`\ B\ x)$
by *blast*

lemma *vimage-eq-UN*: $f-`B = (UN\ y{:}\ B.\ f-`\{y\})$
— NOT suitable for rewriting
by *blast*

lemma *image-UN*: $(f\ `\ (UNION\ A\ B)) = (UN\ x{:}A.(f\ `\ (B\ x)))$
by *blast*

8.6 Inter

abbreviation *Inter* :: *'a set set* ⇒ *'a set* **where**

Inter S ≡ ⨅ *S*

notation (*xsymbols*)
Inter (⋂ - [*90*] *90*)

lemma *Inter-eq* [*code del*]:
⋂ *A* = {*x*. ∀ *B* ∈ *A*. *x* ∈ *B*}
proof (*rule set-ext*)
fix *x*
have (∀ *Q*∈{*P*. ∃ *B*∈*A*. *P* ⟷ *x* ∈ *B*}. *Q*) ⟷ (∀ *B*∈*A*. *x* ∈ *B*)
by *auto*
then show *x* ∈ ⋂ *A* ⟷ *x* ∈ {*x*. ∀ *B* ∈ *A*. *x* ∈ *B*}
by (*simp add*: *Inf-fun-def Inf-bool-def*) (*simp add*: *mem-def*)
qed

lemma *Inter-iff* [*simp*,*no-atp*]: (*A* : *Inter C*) = (*ALL X*:*C*. *A*:*X*)
by (*unfold Inter-eq*) *blast*

lemma *InterI* [*intro*!]: (!!*X*. *X*:*C* ==> *A*:*X*) ==> *A* : *Inter C*
by (*simp add*: *Inter-eq*)

A "destruct" rule – every *X* in *C* contains *A* as an element, but *A* ∈ *X* can hold when *X* ∈ *C* does not! This rule is analogous to *spec*.

lemma *InterD* [*elim*]: *A* : *Inter C* ==> *X*:*C* ==> *A*:*X*
by *auto*

lemma *InterE* [*elim*]: *A* : *Inter C* ==> (*X*~:*C* ==> *R*) ==> (*A*:*X* ==> *R*) ==> *R*
— "Classical" elimination rule – does not require proving *X* ∈ *C*.
by (*unfold Inter-eq*) *blast*

lemma *Inter-lower*: *B* ∈ *A* ==> *Inter A* ⊆ *B*
by *blast*

lemma *Inter-subset*:
[| !!*X*. *X* ∈ *A* ==> *X* ⊆ *B*; *A* ~= {} |] ==> ⋂ *A* ⊆ *B*
by *blast*

lemma *Inter-greatest*: (!!*X*. *X* ∈ *A* ==> *C* ⊆ *X*) ==> *C* ⊆ *Inter A*
by (*iprover intro*: *InterI subsetI dest*: *subsetD*)

lemma *Int-eq-Inter*: *A* ∩ *B* = ⋂ {*A*, *B*}
by *blast*

lemma *Inter-empty* [*simp*]: ⋂ {} = *UNIV*
by *blast*

lemma *Inter-UNIV* [*simp*]: ⋂ *UNIV* = {}
by *blast*

lemma *Inter-insert* [*simp*]: $\bigcap(\mathit{insert}\ a\ B) = a \cap \bigcap B$
by *blast*

lemma *Inter-Un-subset*: $\bigcap A \cup \bigcap B \subseteq \bigcap(A \cap B)$
by *blast*

lemma *Inter-Un-distrib*: $\bigcap(A \cup B) = \bigcap A \cap \bigcap B$
by *blast*

lemma *Inter-UNIV-conv* [*simp*,*no-atp*]:
$(\bigcap A = \mathit{UNIV}) = (\forall x{\in}A.\ x = \mathit{UNIV})$
$(\mathit{UNIV} = \bigcap A) = (\forall x{\in}A.\ x = \mathit{UNIV})$
by *blast+*

lemma *Inter-anti-mono*: $B \subseteq A \Longrightarrow \bigcap A \subseteq \bigcap B$
by *blast*

8.7 Intersections of families

abbreviation *INTER* :: *'a set* ⇒ (*'a* ⇒ *'b set*) ⇒ *'b set* **where**
INTER ≡ *INFI*

syntax
-INTER1 :: *pttrns* => *'b set* => *'b set* (*(3INT -./ -) [0, 10] 10*)
-INTER :: *pttrn* => *'a set* => *'b set* => *'b set* (*(3INT -:-./ -) [0, 0, 10] 10*)

syntax (*xsymbols*)
-INTER1 :: *pttrns* => *'b set* => *'b set* (*(3⋂-./ -) [0, 10] 10*)
-INTER :: *pttrn* => *'a set* => *'b set* => *'b set* (*(3⋂-∈-./ -) [0, 0, 10] 10*)

syntax (*latex* **output**)
-INTER1 :: *pttrns* => *'b set* => *'b set* (*(3⋂(00-)/ -) [0, 10] 10*)
-INTER :: *pttrn* => *'a set* => *'b set* => *'b set* (*(3⋂(00-∈-)/ -) [0, 0, 10] 10*)

translations
INT x y. B == *INT x. INT y. B*
INT x. B == *CONST INTER CONST UNIV* (%*x. B*)
INT x. B == *INT x*:*CONST UNIV. B*
INT x:*A. B* == *CONST INTER A* (%*x. B*)

print-translation ⟪
[*Syntax.preserve-binder-abs2-tr'* @{*const-syntax INTER*} @{*syntax-const -INTER*}]
⟫ — to avoid eta-contraction of body

lemma *INTER-eq-Inter-image*:
$(\bigcap x{\in}A.\ B\ x) = \bigcap(B`A)$

by (*fact INFI-def*)

lemma *Inter-def*:
$\bigcap S = (\bigcap x{\in}S.\ x)$
by (*simp add*: *INTER-eq-Inter-image image-def*)

lemma *INTER-def*:
$(\bigcap x{\in}A.\ B\ x) = \{y.\ \forall x{\in}A.\ y \in B\ x\}$
by (*auto simp add*: *INTER-eq-Inter-image Inter-eq*)

lemma *Inter-image-eq* [*simp*]:
$\bigcap (B`A) = (\bigcap x{\in}A.\ B\ x)$
by (*rule sym*) (*fact INTER-eq-Inter-image*)

lemma *INT-iff* [*simp*]: $(b{:}\ (INT\ x{:}A.\ B\ x)) = (ALL\ x{:}A.\ b{:}\ B\ x)$
by (*unfold INTER-def*) *blast*

lemma *INT-I* [*intro!*]: $(!!x.\ x{:}A \Longrightarrow b{:}\ B\ x) \Longrightarrow b : (INT\ x{:}A.\ B\ x)$
by (*unfold INTER-def*) *blast*

lemma *INT-D* [*elim*]: $b : (INT\ x{:}A.\ B\ x) \Longrightarrow a{:}A \Longrightarrow b{:}\ B\ a$
by *auto*

lemma *INT-E* [*elim*]: $b : (INT\ x{:}A.\ B\ x) \Longrightarrow (b{:}\ B\ a \Longrightarrow R) \Longrightarrow (a^{\sim}{:}A \Longrightarrow R) \Longrightarrow R$
— "Classical" elimination – by the Excluded Middle on $a \in A$.
by (*unfold INTER-def*) *blast*

lemma *INT-cong* [*cong*]:
$A = B \Longrightarrow (!!x.\ x{:}B \Longrightarrow C\ x = D\ x) \Longrightarrow (INT\ x{:}A.\ C\ x) = (INT\ x{:}B.\ D\ x)$
by (*simp add*: *INTER-def*)

lemma *Collect-ball-eq*: $\{x.\ \forall y{\in}A.\ P\ x\ y\} = (\bigcap y{\in}A.\ \{x.\ P\ x\ y\})$
by *blast*

lemma *Collect-all-eq*: $\{x.\ \forall y.\ P\ x\ y\} = (\bigcap y.\ \{x.\ P\ x\ y\})$
by *blast*

lemma *INT-lower*: $a \in A \Longrightarrow (\bigcap x{\in}A.\ B\ x) \subseteq B\ a$
by (*fact INF-leI*)

lemma *INT-greatest*: $(!!x.\ x \in A \Longrightarrow C \subseteq B\ x) \Longrightarrow C \subseteq (\bigcap x{\in}A.\ B\ x)$
by (*fact le-INFI*)

lemma *INT-empty* [*simp*]: $(\bigcap x{\in}\{\}.\ B\ x) = UNIV$
by *blast*

lemma *INT-absorb*: $k \in I \Longrightarrow A\ k \cap (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ A\ i)$

by *blast*

lemma *INT-subset-iff*: $(B \subseteq (\bigcap i{\in}I.\ A\ i)) = (\forall i{\in}I.\ B \subseteq A\ i)$
by (*fact le-INF-iff*)

lemma *INT-insert* [*simp*]: $(\bigcap x \in \mathit{insert}\ a\ A.\ B\ x) = B\ a \cap \mathit{INTER}\ A\ B$
by *blast*

lemma *INT-Un*: $(\bigcap i \in A \cup B.\ M\ i) = (\bigcap i \in A.\ M\ i) \cap (\bigcap i{\in}B.\ M\ i)$
by *blast*

lemma *INT-insert-distrib*:
$u \in A \Longrightarrow (\bigcap x{\in}A.\ \mathit{insert}\ a\ (B\ x)) = \mathit{insert}\ a\ (\bigcap x{\in}A.\ B\ x)$
by *blast*

lemma *INT-constant* [*simp*]: $(\bigcap y{\in}A.\ c) = (\mathit{if}\ A = \{\}\ \mathit{then}\ \mathit{UNIV}\ \mathit{else}\ c)$
by *auto*

lemma *INT-eq*: $(\bigcap x{\in}A.\ B\ x) = \bigcap(\{Y.\ \exists x{\in}A.\ Y = B\ x\})$
— Look: it has an *existential* quantifier
by *blast*

lemma *INTER-UNIV-conv*[*simp*]:
$(\mathit{UNIV} = (\mathit{INT}\ x{:}A.\ B\ x)) = (\forall x{\in}A.\ B\ x = \mathit{UNIV})$
$((\mathit{INT}\ x{:}A.\ B\ x) = \mathit{UNIV}) = (\forall x{\in}A.\ B\ x = \mathit{UNIV})$
by *blast*+

lemma *INT-bool-eq*: $(\bigcap b{::}\mathit{bool}.\ A\ b) = (A\ \mathit{True} \cap A\ \mathit{False})$
by (*auto intro*: *bool-induct*)

lemma *Pow-INT-eq*: $\mathit{Pow}\ (\bigcap x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ \mathit{Pow}\ (B\ x))$
by *blast*

lemma *INT-anti-mono*:
$B \subseteq A \Longrightarrow (!!x.\ x \in A \Longrightarrow f\ x \subseteq g\ x) \Longrightarrow$
$(\bigcap x{\in}A.\ f\ x) \subseteq (\bigcap x{\in}A.\ g\ x)$
— The last inclusion is POSITIVE!
by (*blast dest*: *subsetD*)

lemma *vimage-INT*: $f{-}`(\mathit{INT}\ x{:}A.\ B\ x) = (\mathit{INT}\ x{:}A.\ f\ {-}`\ B\ x)$
by *blast*

8.8 Distributive laws

lemma *Int-Union*: $A \cap \bigcup B = (\bigcup C{\in}B.\ A \cap C)$
by *blast*

lemma *Int-Union2*: $\bigcup B \cap A = (\bigcup C{\in}B.\ C \cap A)$
by *blast*

lemma *Un-Union-image*: $(\bigcup x{\in}C.\ A\ x \cup B\ x) = \bigcup(A`C) \cup \bigcup(B`C)$
— Devlin, Fundamentals of Contemporary Set Theory, page 12, exercise 5:
— Union of a family of unions
by *blast*

lemma *UN-Un-distrib*: $(\bigcup i{\in}I.\ A\ i \cup B\ i) = (\bigcup i{\in}I.\ A\ i) \cup (\bigcup i{\in}I.\ B\ i)$
— Equivalent version
by *blast*

lemma *Un-Inter*: $A \cup \bigcap B = (\bigcap C{\in}B.\ A \cup C)$
by *blast*

lemma *Int-Inter-image*: $(\bigcap x{\in}C.\ A\ x \cap B\ x) = \bigcap(A`C) \cap \bigcap(B`C)$
by *blast*

lemma *INT-Int-distrib*: $(\bigcap i{\in}I.\ A\ i \cap B\ i) = (\bigcap i{\in}I.\ A\ i) \cap (\bigcap i{\in}I.\ B\ i)$
— Equivalent version
by *blast*

lemma *Int-UN-distrib*: $B \cap (\bigcup i{\in}I.\ A\ i) = (\bigcup i{\in}I.\ B \cap A\ i)$
— Halmos, Naive Set Theory, page 35.
by *blast*

lemma *Un-INT-distrib*: $B \cup (\bigcap i{\in}I.\ A\ i) = (\bigcap i{\in}I.\ B \cup A\ i)$
by *blast*

lemma *Int-UN-distrib2*: $(\bigcup i{\in}I.\ A\ i) \cap (\bigcup j{\in}J.\ B\ j) = (\bigcup i{\in}I.\ \bigcup j{\in}J.\ A\ i \cap B\ j)$
by *blast*

lemma *Un-INT-distrib2*: $(\bigcap i{\in}I.\ A\ i) \cup (\bigcap j{\in}J.\ B\ j) = (\bigcap i{\in}I.\ \bigcap j{\in}J.\ A\ i \cup B\ j)$
by *blast*

8.9 Complement

lemma *Compl-UN* [*simp*]: $-(\bigcup x{\in}A.\ B\ x) = (\bigcap x{\in}A.\ -B\ x)$
by *blast*

lemma *Compl-INT* [*simp*]: $-(\bigcap x{\in}A.\ B\ x) = (\bigcup x{\in}A.\ -B\ x)$
by *blast*

8.10 Miniscoping and maxiscoping

Miniscoping: pushing in quantifiers and big Unions and Intersections.

lemma *UN-simps* [*simp*]:
$!!a\ B\ C.\ (UN\ x{:}C.\ insert\ a\ (B\ x)) = (if\ C{=}\{\}\ then\ \{\}\ else\ insert\ a\ (UN\ x{:}C.\ B\ x))$

!!$A\ B\ C.\ (UN\ x{:}C.\ A\ x\ Un\ B) = ((if\ C{=}\{\}\ then\ \{\}\ else\ (UN\ x{:}C.\ A\ x)\ Un\ B))$
!!$A\ B\ C.\ (UN\ x{:}C.\ A\ Un\ B\ x) = ((if\ C{=}\{\}\ then\ \{\}\ else\ A\ Un\ (UN\ x{:}C.\ B\ x)))$
!!$A\ B\ C.\ (UN\ x{:}C.\ A\ x\ Int\ B) = ((UN\ x{:}C.\ A\ x)\ Int\ B)$
!!$A\ B\ C.\ (UN\ x{:}C.\ A\ Int\ B\ x) = (A\ Int\ (UN\ x{:}C.\ B\ x))$
!!$A\ B\ C.\ (UN\ x{:}C.\ A\ x - B) = ((UN\ x{:}C.\ A\ x) - B)$
!!$A\ B\ C.\ (UN\ x{:}C.\ A - B\ x) = (A - (INT\ x{:}C.\ B\ x))$
!!$A\ B.\ (UN\ x{:}\ Union\ A.\ B\ x) = (UN\ y{:}A.\ UN\ x{:}y.\ B\ x)$
!!$A\ B\ C.\ (UN\ z{:}\ UNION\ A\ B.\ C\ z) = (UN\ \ x{:}A.\ UN\ z{:}\ B(x).\ C\ z)$
!!$A\ B\ f.\ (UN\ x{:}f`A.\ B\ x) = (UN\ a{:}A.\ B\ (f\ a))$
by *auto*

lemma *INT-simps* [*simp*]:
!!$A\ B\ C.\ (INT\ x{:}C.\ A\ x\ Int\ B) = (if\ C{=}\{\}\ then\ UNIV\ else\ (INT\ x{:}C.\ A\ x)\ Int\ B)$
!!$A\ B\ C.\ (INT\ x{:}C.\ A\ Int\ B\ x) = (if\ C{=}\{\}\ then\ UNIV\ else\ A\ Int\ (INT\ x{:}C.\ B\ x))$
!!$A\ B\ C.\ (INT\ x{:}C.\ A\ x - B) = (if\ C{=}\{\}\ then\ UNIV\ else\ (INT\ x{:}C.\ A\ x) - B)$
!!$A\ B\ C.\ (INT\ x{:}C.\ A - B\ x) = (if\ C{=}\{\}\ then\ UNIV\ else\ A - (UN\ x{:}C.\ B\ x))$
!!$a\ B\ C.\ (INT\ x{:}C.\ insert\ a\ (B\ x)) = insert\ a\ (INT\ x{:}C.\ B\ x)$
!!$A\ B\ C.\ (INT\ x{:}C.\ A\ x\ Un\ B) = ((INT\ x{:}C.\ A\ x)\ Un\ B)$
!!$A\ B\ C.\ (INT\ x{:}C.\ A\ Un\ B\ x) = (A\ Un\ (INT\ x{:}C.\ B\ x))$
!!$A\ B.\ (INT\ x{:}\ Union\ A.\ B\ x) = (INT\ y{:}A.\ INT\ x{:}y.\ B\ x)$
!!$A\ B\ C.\ (INT\ z{:}\ UNION\ A\ B.\ C\ z) = (INT\ x{:}A.\ INT\ z{:}\ B(x).\ C\ z)$
!!$A\ B\ f.\ (INT\ x{:}f`A.\ B\ x) = (INT\ a{:}A.\ B\ (f\ a))$
by *auto*

lemma *ball-simps* [*simp*,*no-atp*]:
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ x\ |\ Q) = ((ALL\ x{:}A.\ P\ x)\ |\ Q)$
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ |\ Q\ x) = (P\ |\ (ALL\ x{:}A.\ Q\ x))$
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ --{>}\ Q\ x) = (P\ --{>}\ (ALL\ x{:}A.\ Q\ x))$
!!$A\ P\ Q.\ (ALL\ x{:}A.\ P\ x\ --{>}\ Q) = ((EX\ x{:}A.\ P\ x)\ --{>}\ Q)$
!!$P.\ (ALL\ x{:}\{\}.\ P\ x) = True$
!!$P.\ (ALL\ x{:}UNIV.\ P\ x) = (ALL\ x.\ P\ x)$
!!$a\ B\ P.\ (ALL\ x{:}insert\ a\ B.\ P\ x) = (P\ a\ \&\ (ALL\ x{:}B.\ P\ x))$
!!$A\ P.\ (ALL\ x{:}Union\ A.\ P\ x) = (ALL\ y{:}A.\ ALL\ x{:}y.\ P\ x)$
!!$A\ B\ P.\ (ALL\ x{:}\ UNION\ A\ B.\ P\ x) = (ALL\ a{:}A.\ ALL\ x{:}\ B\ a.\ P\ x)$
!!$P\ Q.\ (ALL\ x{:}Collect\ Q.\ P\ x) = (ALL\ x.\ Q\ x\ --{>}\ P\ x)$
!!$A\ P\ f.\ (ALL\ x{:}f`A.\ P\ x) = (ALL\ x{:}A.\ P\ (f\ x))$
!!$A\ P.\ (\sim(ALL\ x{:}A.\ P\ x)) = (EX\ x{:}A.\ \sim P\ x)$
by *auto*

lemma *bex-simps* [*simp*,*no-atp*]:
!!$A\ P\ Q.\ (EX\ x{:}A.\ P\ x\ \&\ Q) = ((EX\ x{:}A.\ P\ x)\ \&\ Q)$
!!$A\ P\ Q.\ (EX\ x{:}A.\ P\ \&\ Q\ x) = (P\ \&\ (EX\ x{:}A.\ Q\ x))$
!!$P.\ (EX\ x{:}\{\}.\ P\ x) = False$

$!!P.\ (EX\ x{:}UNIV.\ P\ x) = (EX\ x.\ P\ x)$
$!!a\ B\ P.\ (EX\ x{:}insert\ a\ B.\ P\ x) = (P(a) \mid (EX\ x{:}B.\ P\ x))$
$!!A\ P.\ (EX\ x{:}Union\ A.\ P\ x) = (EX\ y{:}A.\ EX\ x{:}y.\ P\ x)$
$!!A\ B\ P.\ (EX\ x{:}\ UNION\ A\ B.\ P\ x) = (EX\ a{:}A.\ EX\ x{:}B\ a.\ P\ x)$
$!!P\ Q.\ (EX\ x{:}Collect\ Q.\ P\ x) = (EX\ x.\ Q\ x\ \&\ P\ x)$
$!!A\ P\ f.\ (EX\ x{:}f`A.\ P\ x) = (EX\ x{:}A.\ P\ (f\ x))$
$!!A\ P.\ (\sim(EX\ x{:}A.\ P\ x)) = (ALL\ x{:}A.\ \sim P\ x)$
by *auto*

lemma *ball-conj-distrib*:
$(ALL\ x{:}A.\ P\ x\ \&\ Q\ x) = ((ALL\ x{:}A.\ P\ x)\ \&\ (ALL\ x{:}A.\ Q\ x))$
by *blast*

lemma *bex-disj-distrib*:
$(EX\ x{:}A.\ P\ x \mid Q\ x) = ((EX\ x{:}A.\ P\ x) \mid (EX\ x{:}A.\ Q\ x))$
by *blast*

Maxiscoping: pulling out big Unions and Intersections.

lemma *UN-extend-simps*:
$!!a\ B\ C.\ insert\ a\ (UN\ x{:}C.\ B\ x) = (if\ C{=}\{\}\ then\ \{a\}\ else\ (UN\ x{:}C.\ insert\ a\ (B\ x)))$
$!!A\ B\ C.\ (UN\ x{:}C.\ A\ x)\ Un\ B = (if\ C{=}\{\}\ then\ B\ else\ (UN\ x{:}C.\ A\ x\ Un\ B))$
$!!A\ B\ C.\ A\ Un\ (UN\ x{:}C.\ B\ x) = (if\ C{=}\{\}\ then\ A\ else\ (UN\ x{:}C.\ A\ Un\ B\ x))$
$!!A\ B\ C.\ ((UN\ x{:}C.\ A\ x)\ Int\ B) = (UN\ x{:}C.\ A\ x\ Int\ B)$
$!!A\ B\ C.\ (A\ Int\ (UN\ x{:}C.\ B\ x)) = (UN\ x{:}C.\ A\ Int\ B\ x)$
$!!A\ B\ C.\ ((UN\ x{:}C.\ A\ x) - B) = (UN\ x{:}C.\ A\ x - B)$
$!!A\ B\ C.\ (A - (INT\ x{:}C.\ B\ x)) = (UN\ x{:}C.\ A - B\ x)$
$!!A\ B.\ (UN\ y{:}A.\ UN\ x{:}y.\ B\ x) = (UN\ x{:}\ Union\ A.\ B\ x)$
$!!A\ B\ C.\ (UN\ \ x{:}A.\ UN\ z{:}\ B(x).\ C\ z) = (UN\ z{:}\ UNION\ A\ B.\ C\ z)$
$!!A\ B\ f.\ (UN\ a{:}A.\ B\ (f\ a)) = (UN\ x{:}f`A.\ B\ x)$
by *auto*

lemma *INT-extend-simps*:
$!!A\ B\ C.\ (INT\ x{:}C.\ A\ x)\ Int\ B = (if\ C{=}\{\}\ then\ B\ else\ (INT\ x{:}C.\ A\ x\ Int\ B))$
$!!A\ B\ C.\ A\ Int\ (INT\ x{:}C.\ B\ x) = (if\ C{=}\{\}\ then\ A\ else\ (INT\ x{:}C.\ A\ Int\ B\ x))$
$!!A\ B\ C.\ (INT\ x{:}C.\ A\ x) - B = (if\ C{=}\{\}\ then\ UNIV{-}B\ else\ (INT\ x{:}C.\ A\ x - B))$
$!!A\ B\ C.\ A - (UN\ x{:}C.\ B\ x) = (if\ C{=}\{\}\ then\ A\ else\ (INT\ x{:}C.\ A - B\ x))$
$!!a\ B\ C.\ insert\ a\ (INT\ x{:}C.\ B\ x) = (INT\ x{:}C.\ insert\ a\ (B\ x))$
$!!A\ B\ C.\ ((INT\ x{:}C.\ A\ x)\ Un\ B) = (INT\ x{:}C.\ A\ x\ Un\ B)$
$!!A\ B\ C.\ A\ Un\ (INT\ x{:}C.\ B\ x) = (INT\ x{:}C.\ A\ Un\ B\ x)$
$!!A\ B.\ (INT\ y{:}A.\ INT\ x{:}y.\ B\ x) = (INT\ x{:}\ Union\ A.\ B\ x)$
$!!A\ B\ C.\ (INT\ x{:}A.\ INT\ z{:}\ B(x).\ C\ z) = (INT\ z{:}\ UNION\ A\ B.\ C\ z)$
$!!A\ B\ f.\ (INT\ a{:}A.\ B\ (f\ a)) = (INT\ x{:}f`A.\ B\ x)$
by *auto*

no-notation
less-eq (**infix** $\sqsubseteq$ *50*) **and**

less (**infix** $\sqsubset$ *50*) **and**
inf (**infixl** $\sqcap$ *70*) **and**
sup (**infixl** $\sqcup$ *65*) **and**
Inf ($\bigsqcap$ - [*900*] *900*) **and**
Sup ($\bigsqcup$ - [*900*] *900*) **and**
top ($\top$) **and**
bot ($\bot$)

lemmas *mem-simps* =
insert-iff empty-iff Un-iff Int-iff Compl-iff Diff-iff
mem-Collect-eq UN-iff Union-iff INT-iff Inter-iff
— Each of these has ALREADY been added [*simp*] above.

end

9 Inductive: Knaster-Tarski Fixpoint Theorem and inductive definitions

theory *Inductive*
imports *Complete-Lattice*
uses
(*Tools/inductive.ML*)
Tools/dseq.ML
(*Tools/inductive-codegen.ML*)
Tools/Datatype/datatype-aux.ML
Tools/Datatype/datatype-prop.ML
Tools/Datatype/datatype-case.ML
(*Tools/Datatype/datatype-abs-proofs.ML*)
(*Tools/Datatype/datatype-data.ML*)
(*Tools/old-primrec.ML*)
(*Tools/primrec.ML*)
(*Tools/Datatype/datatype-codegen.ML*)
begin

9.1 Least and greatest fixed points

context *complete-lattice*
begin

definition
lfp :: ('*a* $\Rightarrow$ '*a*) $\Rightarrow$ '*a* **where**
lfp f = *Inf* {*u*. *f u* $\leq$ *u*} — least fixed point

definition
gfp :: ('*a* $\Rightarrow$ '*a*) $\Rightarrow$ '*a* **where**
gfp f = *Sup* {*u*. *u* $\leq$ *f u*} — greatest fixed point

9.2 Proof of Knaster-Tarski Theorem using *lfp*

lfp f is the least upper bound of the set $\{u.\ f\ u \leq u\}$

lemma *lfp-lowerbound*: $f\ A \leq A \Longrightarrow$ *lfp f* $\leq A$
by (*auto simp add*: *lfp-def intro*: *Inf-lower*)

lemma *lfp-greatest*: $(!!u.\ f\ u \leq u \Longrightarrow A \leq u) \Longrightarrow A \leq$ *lfp f*
by (*auto simp add*: *lfp-def intro*: *Inf-greatest*)

end

lemma *lfp-lemma2*: *mono f* $\Longrightarrow$ *f* (*lfp f*) $\leq$ *lfp f*
by (*iprover intro*: *lfp-greatest order-trans monoD lfp-lowerbound*)

lemma *lfp-lemma3*: *mono f* $\Longrightarrow$ *lfp f* $\leq$ *f* (*lfp f*)
by (*iprover intro*: *lfp-lemma2 monoD lfp-lowerbound*)

lemma *lfp-unfold*: *mono f* $\Longrightarrow$ *lfp f* = *f* (*lfp f*)
by (*iprover intro*: *order-antisym lfp-lemma2 lfp-lemma3*)

lemma *lfp-const*: *lfp* ($\lambda x.\ t$) = *t*
by (*rule lfp-unfold*) (*simp add*:*mono-def*)

9.3 General induction rules for least fixed points

theorem *lfp-induct*:
assumes *mono*: *mono f* **and** *ind*: *f* (*inf* (*lfp f*) *P*) <= *P*
shows *lfp f* <= *P*
proof −
have *inf* (*lfp f*) *P* <= *lfp f* **by** (*rule inf-le1*)
with *mono* **have** *f* (*inf* (*lfp f*) *P*) <= *f* (*lfp f*) **..**
also from *mono* **have** *f* (*lfp f*) = *lfp f* **by** (*rule lfp-unfold* [*symmetric*])
finally have *f* (*inf* (*lfp f*) *P*) <= *lfp f* **.**
from *this* **and** *ind* **have** *f* (*inf* (*lfp f*) *P*) <= *inf* (*lfp f*) *P* **by** (*rule le-infI*)
hence *lfp f* <= *inf* (*lfp f*) *P* **by** (*rule lfp-lowerbound*)
also have *inf* (*lfp f*) *P* <= *P* **by** (*rule inf-le2*)
finally show *?thesis* **.**
qed

lemma *lfp-induct-set*:
assumes *lfp*: *a*: *lfp*(*f*)
and *mono*: *mono*(*f*)
and *indhyp*: !!*x*. [| *x*: *f*(*lfp*(*f*) *Int* {*x*. *P*(*x*)}) |] ==> *P*(*x*)
shows *P*(*a*)
by (*rule lfp-induct* [*THEN subsetD*, *THEN CollectD*, *OF mono - lfp*])
(*auto simp*: *intro*: *indhyp*)

lemma *lfp-ordinal-induct*:
fixes *f* :: *'a*::*complete-lattice* ⇒ *'a*

assumes *mono*: *mono f*
and *P-f*: $\bigwedge S.\ P\ S \Longrightarrow P\ (f\ S)$
and *P-Union*: $\bigwedge M.\ \forall S{\in}M.\ P\ S \Longrightarrow P\ (Sup\ M)$
shows *P* (*lfp f*)
proof −
let *?M* = $\{S.\ S \leq lfp\ f \wedge P\ S\}$
have *P* (*Sup ?M*) **using** *P-Union* **by** *simp*
also have *Sup ?M* = *lfp f*
proof (*rule antisym*)
show *Sup ?M* ≤ *lfp f* **by** (*blast intro*: *Sup-least*)
hence *f* (*Sup ?M*) ≤ *f* (*lfp f*) **by** (*rule mono* [*THEN monoD*])
hence *f* (*Sup ?M*) ≤ *lfp f* **using** *mono* [*THEN lfp-unfold*] **by** *simp*
hence *f* (*Sup ?M*) ∈ *?M* **using** *P-f P-Union* **by** *simp*
hence *f* (*Sup ?M*) ≤ *Sup ?M* **by** (*rule Sup-upper*)
thus *lfp f* ≤ *Sup ?M* **by** (*rule lfp-lowerbound*)
qed
finally show *?thesis* **.**
qed

lemma *lfp-ordinal-induct-set*:
assumes *mono*: *mono f*
and *P-f*: !!*S*. *P S* ==> *P*(*f S*)
and *P-Union*: !!*M*. !*S*:*M*. *P S* ==> *P*(*Union M*)
shows *P*(*lfp f*)
using *assms* **by** (*rule lfp-ordinal-induct* [**where** *P*=*P*])

Definition forms of *lfp-unfold* and *lfp-induct*, to control unfolding

lemma *def-lfp-unfold*: [| *h*==*lfp*(*f*); *mono*(*f*) |] ==> *h* = *f*(*h*)
by (*auto intro*!: *lfp-unfold*)

lemma *def-lfp-induct*:
[| *A* == *lfp*(*f*); *mono*(*f*);
f (*inf A P*) ≤ *P*
|] ==> *A* ≤ *P*
by (*blast intro*: *lfp-induct*)

lemma *def-lfp-induct-set*:
[| *A* == *lfp*(*f*); *mono*(*f*); *a*:*A*;
!!*x*. [| *x*: *f*(*A Int* {*x*. *P*(*x*)}) |] ==> *P*(*x*)
|] ==> *P*(*a*)
by (*blast intro*: *lfp-induct-set*)

lemma *lfp-mono*: (!!*Z*. *f Z* ≤ *g Z*) ==> *lfp f* ≤ *lfp g*
by (*rule lfp-lowerbound* [*THEN lfp-greatest*], *blast intro*: *order-trans*)

9.4 Proof of Knaster-Tarski Theorem using *gfp*

gfp f is the greatest lower bound of the set $\{u.\ u \leq f\ u\}$

lemma *gfp-upperbound*: $X \leq f\ X \Longrightarrow X \leq gfp\ f$
by (*auto simp add*: *gfp-def intro*: *Sup-upper*)

lemma *gfp-least*: $(!!u.\ u \leq f\ u \Longrightarrow u \leq X) \Longrightarrow gfp\ f \leq X$
by (*auto simp add*: *gfp-def intro*: *Sup-least*)

lemma *gfp-lemma2*: $mono\ f \Longrightarrow gfp\ f \leq f\ (gfp\ f)$
by (*iprover intro*: *gfp-least order-trans monoD gfp-upperbound*)

lemma *gfp-lemma3*: $mono\ f \Longrightarrow f\ (gfp\ f) \leq gfp\ f$
by (*iprover intro*: *gfp-lemma2 monoD gfp-upperbound*)

lemma *gfp-unfold*: $mono\ f \Longrightarrow gfp\ f = f\ (gfp\ f)$
by (*iprover intro*: *order-antisym gfp-lemma2 gfp-lemma3*)

9.5 Coinduction rules for greatest fixed points

weak version

lemma *weak-coinduct*: $[|\ a: X;\ \ X \subseteq f(X)\ |] \Longrightarrow a : gfp(f)$
by (*rule gfp-upperbound* [*THEN subsetD*], *auto*)

lemma *weak-coinduct-image*: $!!X.\ [|\ a : X;\ g`X \subseteq f\ (g`X)\ |] \Longrightarrow g\ a : gfp\ f$
apply (*erule gfp-upperbound* [*THEN subsetD*])
apply (*erule imageI*)
done

lemma *coinduct-lemma*:
$[|\ X \leq f\ (sup\ X\ (gfp\ f));\ \ mono\ f\ |] \Longrightarrow sup\ X\ (gfp\ f) \leq f\ (sup\ X\ (gfp\ f))$
apply (*frule gfp-lemma2*)
apply (*drule mono-sup*)
apply (*rule le-supI*)
apply *assumption*
apply (*rule order-trans*)
apply (*rule order-trans*)
apply *assumption*
apply (*rule sup-ge2*)
apply *assumption*
done

strong version, thanks to Coen and Frost

lemma *coinduct-set*: $[|\ mono(f);\ \ a: X;\ \ X \subseteq f(X\ Un\ gfp(f))\ |] \Longrightarrow a : gfp(f)$
by (*blast intro*: *weak-coinduct* [*OF - coinduct-lemma*])

lemma *coinduct*: $[|\ mono(f);\ X \leq f\ (sup\ X\ (gfp\ f))\ |] \Longrightarrow X \leq gfp(f)$
apply (*rule order-trans*)
apply (*rule sup-ge1*)
apply (*erule gfp-upperbound* [*OF coinduct-lemma*])
apply *assumption*
done

lemma *gfp-fun-UnI2*: [| *mono*(*f*); *a*: *gfp*(*f*) |] ==> *a*: *f*(*X Un gfp*(*f*))
by (*blast dest*: *gfp-lemma2 mono-Un*)

9.6 Even Stronger Coinduction Rule, by Martin Coen

Weakens the condition $X \subseteq f\ X$ to one expressed using both *lfp* and *gfp*

lemma *coinduct3-mono-lemma*: *mono*(*f*) ==> *mono*(%*x*. *f*(*x*) *Un X Un B*)
by (*iprover intro*: *subset-refl monoI Un-mono monoD*)

lemma *coinduct3-lemma*:
[| *X* ⊆ *f*(*lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*))); *mono*(*f*) |]
==> *lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*)) ⊆ *f*(*lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*)))
apply (*rule subset-trans*)
apply (*erule coinduct3-mono-lemma* [*THEN lfp-lemma3*])
apply (*rule Un-least* [*THEN Un-least*])
apply (*rule subset-refl*, *assumption*)
apply (*rule gfp-unfold* [*THEN equalityD1*, *THEN subset-trans*], *assumption*)
apply (*rule monoD* [**where** *f*=*f*], *assumption*)
apply (*subst coinduct3-mono-lemma* [*THEN lfp-unfold*], *auto*)
done

lemma *coinduct3*:
[| *mono*(*f*); *a*:*X*; *X* ⊆ *f*(*lfp*(%*x*. *f*(*x*) *Un X Un gfp*(*f*))) |] ==> *a* : *gfp*(*f*)
apply (*rule coinduct3-lemma* [*THEN* [*2*] *weak-coinduct*])
apply (*rule coinduct3-mono-lemma* [*THEN lfp-unfold*, *THEN ssubst*], *auto*)
done

Definition forms of *gfp-unfold* and *coinduct*, to control unfolding

lemma *def-gfp-unfold*: [| *A*==*gfp*(*f*); *mono*(*f*) |] ==> *A* = *f*(*A*)
by (*auto intro*!: *gfp-unfold*)

lemma *def-coinduct*:
[| *A*==*gfp*(*f*); *mono*(*f*); *X* ≤ *f*(*sup X A*) |] ==> *X* ≤ *A*
by (*iprover intro*!: *coinduct*)

lemma *def-coinduct-set*:
[| *A*==*gfp*(*f*); *mono*(*f*); *a*:*X*; *X* ⊆ *f*(*X Un A*) |] ==> *a*: *A*
by (*auto intro*!: *coinduct-set*)

lemma *def-Collect-coinduct*:
[| *A* == *gfp*(%*w*. *Collect*(*P*(*w*))); *mono*(%*w*. *Collect*(*P*(*w*)));
a: *X*; !!*z*. *z*: *X* ==> *P* (*X Un A*) *z* |] ==>
a : *A*
apply (*erule def-coinduct-set*, *auto*)
done

lemma *def-coinduct3*:

[| A==*gfp*(f); *mono*(f); a:X; $X \subseteq f$(*lfp*(%x. $f(x)$ *Un* X *Un* A)) |] ==> a: A
by (*auto intro*!: *coinduct3*)

Monotonicity of *gfp*!

lemma *gfp-mono*: (!!Z. f $Z \leq g$ Z) ==> *gfp* $f \leq$ *gfp* g
by (*rule gfp-upperbound* [*THEN gfp-least*], *blast intro*: *order-trans*)

9.7 Inductive predicates and sets

Package setup.

theorems *basic-monos* =
subset-refl imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj
Collect-mono in-mono vimage-mono

use *Tools/inductive.ML*
setup *Inductive.setup*

theorems [*mono*] =
imp-refl disj-mono conj-mono ex-mono all-mono if-bool-eq-conj
imp-mono not-mono
Ball-def Bex-def
induct-rulify-fallback

9.8 Inductive datatypes and primitive recursion

Package setup.

use *Tools/Datatype/datatype-abs-proofs.ML*
use *Tools/Datatype/datatype-data.ML*
setup *Datatype-Data.setup*

use *Tools/Datatype/datatype-codegen.ML*
setup *Datatype-Codegen.setup*

use *Tools/old-primrec.ML*
use *Tools/primrec.ML*

use *Tools/inductive-codegen.ML*
setup *InductiveCodegen.setup*

Lambda-abstractions with pattern matching:

syntax
-lam-pats-syntax :: *cases-syn* => *'a* => *'b* ((%-) *10*)
syntax (*xsymbols*)
-lam-pats-syntax :: *cases-syn* => *'a* => *'b* ((λ-) *10*)

parse-translation (**advanced**) ⟪
let
fun fun-tr ctxt [*cs*] =

```
    let
      val x = Free (Name.variant (Term.add-free-names cs []) x, dummyT);
      val ft = Datatype-Case.case-tr true Datatype-Data.info-of-constr ctxt [x, cs];
    in lambda x ft end
in [(@{syntax-const -lam-pats-syntax}, fun-tr)] end
⟩⟩
```

end

10 Fun: Notions about functions

theory *Fun*
imports *Complete-Lattice*
begin

As a simplification rule, it replaces all function equalities by first-order equalities.

lemma *expand-fun-eq*: $f = g \longleftrightarrow (\forall x.\ f\ x = g\ x)$
apply (*rule iffI*)
apply (*simp* (*no-asm-simp*))
apply (*rule ext*)
apply (*simp* (*no-asm-simp*))
done

lemma *apply-inverse*:
$f\ x = u \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow g\ (f\ x) = x) \Longrightarrow P\ x \Longrightarrow x = g\ u$
by *auto*

10.1 The Identity Function *id*

definition
$id :: 'a \Rightarrow 'a$
where
$id = (\lambda x.\ x)$

lemma *id-apply* [*simp*]: $id\ x = x$
by (*simp add*: *id-def*)

lemma *image-ident* [*simp*]: $(\%x.\ x)\ `\ Y = Y$
by *blast*

lemma *image-id* [*simp*]: $id\ `\ Y = Y$
by (*simp add*: *id-def*)

lemma *vimage-ident* [*simp*]: $(\%x.\ x) -` Y = Y$
by *blast*

lemma *vimage-id* [*simp*]: $id -` A = A$

by (*simp add*: *id-def*)

10.2 The Composition Operator $f \circ g$

definition
comp :: $('b \Rightarrow 'c) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'c$ (**infixl** *o* 55)
where
$f\ o\ g = (\lambda x.\ f\ (g\ x))$

notation (*xsymbols*)
comp (**infixl** $\circ$ 55)

notation (*HTML* **output**)
comp (**infixl** $\circ$ 55)

compatibility

lemmas *o-def* = *comp-def*

lemma *o-apply* [*simp*]: $(f\ o\ g)\ x = f\ (g\ x)$
by (*simp add*: *comp-def*)

lemma *o-assoc*: $f\ o\ (g\ o\ h) = f\ o\ g\ o\ h$
by (*simp add*: *comp-def*)

lemma *id-o* [*simp*]: $id\ o\ g = g$
by (*simp add*: *comp-def*)

lemma *o-id* [*simp*]: $f\ o\ id = f$
by (*simp add*: *comp-def*)

lemma *o-eq-dest*:
$a\ o\ b = c\ o\ d \Longrightarrow a\ (b\ v) = c\ (d\ v)$
by (*simp only*: *o-def*) (*fact fun-cong*)

lemma *o-eq-elim*:
$a\ o\ b = c\ o\ d \Longrightarrow ((\bigwedge v.\ a\ (b\ v) = c\ (d\ v)) \Longrightarrow R) \Longrightarrow R$
by (*erule meta-mp*) (*fact o-eq-dest*)

lemma *image-compose*: $(f\ o\ g)\ `\ r = f`(g`r)$
by (*simp add*: *comp-def*, *blast*)

lemma *vimage-compose*: $(g \circ f) -` x = f -` (g -` x)$
by *auto*

lemma *UN-o*: $UNION\ A\ (g\ o\ f) = UNION\ (f`A)\ g$
by (*unfold comp-def*, *blast*)

10.3 The Forward Composition Operator *fcomp*

definition

fcomp :: $('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'c) \Rightarrow 'a \Rightarrow 'c$ (**infixl** *o>* *60*)
where
f *o>* $g = (\lambda x.\ g\ (f\ x))$

lemma *fcomp-apply*: $(f$ *o>* $g)\ x = g\ (f\ x)$
by (*simp add*: *fcomp-def*)

lemma *fcomp-assoc*: $(f$ *o>* $g)$ *o>* $h = f$ *o>* $(g$ *o>* $h)$
by (*simp add*: *fcomp-def*)

lemma *id-fcomp* [*simp*]: *id o>* $g = g$
by (*simp add*: *fcomp-def*)

lemma *fcomp-id* [*simp*]: f *o>* *id* $= f$
by (*simp add*: *fcomp-def*)

code-const *fcomp*
(*Eval* **infixl** *1 #>*)

no-notation *fcomp* (**infixl** *o>* *60*)

10.4 Injectivity and Surjectivity

definition
inj-on :: [$'a$ => $'b$, $'a$ *set*] => *bool* **where**
— injective
inj-on f A == ! *x*:*A*. ! *y*:*A*. *f*(*x*)=*f*(*y*) −−> *x*=*y*

A common special case: functions injective over the entire domain type.

abbreviation
inj f == *inj-on f UNIV*

definition
bij-betw :: ($'a$ => $'b$) => $'a$ *set* => $'b$ *set* => *bool* **where** — bijective
[*code del*]: *bij-betw f A B* ⟷ *inj-on f A* & *f* ‘ *A* = *B*

definition
surj :: ($'a$ => $'b$) => *bool* **where**
— surjective
surj f == ! *y*. ? *x*. *y*=*f*(*x*)

definition
bij :: ($'a$ => $'b$) => *bool* **where**
— bijective
bij f == *inj f* & *surj f*

lemma *injI*:
assumes $\bigwedge x\ y.\ f\ x = f\ y \Longrightarrow x = y$
shows *inj f*

using *assms* **unfolding** *inj-on-def* **by** *auto*

For Proofs in *Tools/Datatype/datatype-rep-proofs*

lemma *datatype-injI*:
(!! *x*. *ALL y*. *f*(*x*) = *f*(*y*) −−> *x*=*y*) ==> *inj*(*f*)
by (*simp add*: *inj-on-def*)

theorem *range-ex1-eq*: *inj f* ⟹ *b* : *range f* = (*EX*! *x*. *b* = *f x*)
by (*unfold inj-on-def*, *blast*)

lemma *injD*: [| *inj*(*f*); *f*(*x*) = *f*(*y*) |] ==> *x*=*y*
by (*simp add*: *inj-on-def*)

lemma *inj-on-eq-iff*: *inj-on f A* ==> *x*:*A* ==> *y*:*A* ==> (*f*(*x*) = *f*(*y*)) = (*x*=*y*)
by (*force simp add*: *inj-on-def*)

lemma *inj-eq*: *inj f* ==> (*f*(*x*) = *f*(*y*)) = (*x*=*y*)
by (*simp add*: *inj-on-eq-iff*)

lemma *inj-on-id*[*simp*]: *inj-on id A*
by (*simp add*: *inj-on-def*)

lemma *inj-on-id2*[*simp*]: *inj-on* (%*x*. *x*) *A*
by (*simp add*: *inj-on-def*)

lemma *surj-id*[*simp*]: *surj id*
by (*simp add*: *surj-def*)

lemma *bij-id*[*simp*]: *bij id*
by (*simp add*: *bij-def*)

lemma *inj-onI*:
(!! *x y*. [| *x*:*A*; *y*:*A*; *f*(*x*) = *f*(*y*) |] ==> *x*=*y*) ==> *inj-on f A*
by (*simp add*: *inj-on-def*)

lemma *inj-on-inverseI*: (!!*x*. *x*:*A* ==> *g*(*f*(*x*)) = *x*) ==> *inj-on f A*
by (*auto dest*: *arg-cong* [*of concl*: *g*] *simp add*: *inj-on-def*)

lemma *inj-onD*: [| *inj-on f A*; *f*(*x*)=*f*(*y*); *x*:*A*; *y*:*A* |] ==> *x*=*y*
by (*unfold inj-on-def*, *blast*)

lemma *inj-on-iff*: [| *inj-on f A*; *x*:*A*; *y*:*A* |] ==> (*f*(*x*)=*f*(*y*)) = (*x*=*y*)
by (*blast dest*!: *inj-onD*)

lemma *comp-inj-on*:
[| *inj-on f A*; *inj-on g* (*f*‘*A*) |] ==> *inj-on* (*g o f*) *A*
by (*simp add*: *comp-def inj-on-def*)

lemma *inj-on-imageI*: *inj-on* (*g o f*) *A* ⟹ *inj-on g* (*f* ‘ *A*)

apply(*simp add*:*inj-on-def image-def*)
apply *blast*
done

lemma *inj-on-image-iff*: ⟦ *ALL x*:*A*. *ALL y*:*A*. (*g*(*f x*) = *g*(*f y*)) = (*g x* = *g y*);
inj-on f A ⟧ ⟹ *inj-on g* (*f* ‘ *A*) = *inj-on g A*
apply(*unfold inj-on-def*)
apply *blast*
done

lemma *inj-on-contraD*: [| *inj-on f A*; ~*x*=*y*; *x*:*A*; *y*:*A* |] ==> ~ *f*(*x*)=*f*(*y*)
by (*unfold inj-on-def*, *blast*)

lemma *inj-singleton*: *inj* (%*s*. {*s*})
by (*simp add*: *inj-on-def*)

lemma *inj-on-empty*[*iff*]: *inj-on f* {}
by(*simp add*: *inj-on-def*)

lemma *subset-inj-on*: [| *inj-on f B*; *A* <= *B* |] ==> *inj-on f A*
by (*unfold inj-on-def*, *blast*)

lemma *inj-on-Un*:
inj-on f (*A Un B*) =
(*inj-on f A* & *inj-on f B* & *f*‘(*A*−*B*) *Int f*‘(*B*−*A*) = {})
apply(*unfold inj-on-def*)
apply (*blast intro*:*sym*)
done

lemma *inj-on-insert*[*iff*]:
inj-on f (*insert a A*) = (*inj-on f A* & *f a* ~: *f*‘(*A*−{*a*}))
apply(*unfold inj-on-def*)
apply (*blast intro*:*sym*)
done

lemma *inj-on-diff*: *inj-on f A* ==> *inj-on f* (*A*−*B*)
apply(*unfold inj-on-def*)
apply (*blast*)
done

lemma *surjI*: (!! *x*. *g*(*f x*) = *x*) ==> *surj g*
apply (*simp add*: *surj-def*)
apply (*blast intro*: *sym*)
done

lemma *surj-range*: *surj f* ==> *range f* = *UNIV*
by (*auto simp add*: *surj-def*)

lemma *surjD*: *surj f* ==> *EX x*. *y* = *f x*

by (*simp add*: *surj-def*)

lemma *surjE*: *surj f* ==> (!!*x*. *y* = *f x* ==> *C*) ==> *C*
by (*simp add*: *surj-def*, *blast*)

lemma *comp-surj*: [| *surj f*; *surj g* |] ==> *surj* (*g o f*)
apply (*simp add*: *comp-def surj-def*, *clarify*)
apply (*drule-tac x* = *y* **in** *spec*, *clarify*)
apply (*drule-tac x* = *x* **in** *spec*, *blast*)
done

lemma *bijI*: [| *inj f*; *surj f* |] ==> *bij f*
by (*simp add*: *bij-def*)

lemma *bij-is-inj*: *bij f* ==> *inj f*
by (*simp add*: *bij-def*)

lemma *bij-is-surj*: *bij f* ==> *surj f*
by (*simp add*: *bij-def*)

lemma *bij-betw-imp-inj-on*: *bij-betw f A B* ⟹ *inj-on f A*
by (*simp add*: *bij-betw-def*)

lemma *bij-comp*: *bij f* ⟹ *bij g* ⟹ *bij* (*g o f*)
by(*fastsimp intro*: *comp-inj-on comp-surj simp*: *bij-def surj-range*)

lemma *bij-betw-trans*:
bij-betw f A B ⟹ *bij-betw g B C* ⟹ *bij-betw* (*g o f*) *A C*
by(*auto simp add*:*bij-betw-def comp-inj-on*)

lemma *bij-betw-inv*: **assumes** *bij-betw f A B* **shows** *EX g*. *bij-betw g B A*
proof −
have *i*: *inj-on f A* **and** *s*: *f* ‘ *A* = *B*
using *assms* **by**(*auto simp*:*bij-betw-def*)
let *?P* = %*b a*. *a*:*A* ∧ *f a* = *b* **let** *?g* = %*b*. *The* (*?P b*)
{ fix *a b* **assume** *P*: *?P b a*
hence *ex1*: ∃ *a*. *?P b a* **using** *s* **unfolding** *image-def* **by** *blast*
hence *uex1*: ∃!*a*. *?P b a* **by**(*blast dest*:*inj-onD*[*OF i*])
hence *?g b* = *a* **using** *the1-equality*[*OF uex1*, *OF P*] *P* **by** *simp*
} note *g* = *this*
have *inj-on ?g B*
proof(*rule inj-onI*)
fix *x y* **assume** *x*:*B y*:*B ?g x* = *?g y*
from *s* ‹*x*:*B*› **obtain** *a1* **where** *a1*: *?P x a1* **unfolding** *image-def* **by** *blast*
from *s* ‹*y*:*B*› **obtain** *a2* **where** *a2*: *?P y a2* **unfolding** *image-def* **by** *blast*
from *g*[*OF a1*] *a1 g*[*OF a2*] *a2* ‹*?g x* = *?g y*› **show** *x*=*y* **by** *simp*
qed
moreover have *?g* ‘ *B* = *A*
proof(*auto simp*:*image-def*)

fix *b* **assume** *b*:*B*
with *s* **obtain** *a* **where** *P*: *?P b a* **unfolding** *image-def* **by** *blast*
thus *?g b* ∈ *A* **using** *g*[*OF P*] **by** *auto*
next
fix *a* **assume** *a*:*A*
then obtain *b* **where** *P*: *?P b a* **using** *s* **unfolding** *image-def* **by** *blast*
then have *b*:*B* **using** *s* **unfolding** *image-def* **by** *blast*
with *g*[*OF P*] **show** ∃ *b*∈*B*. *a* = *?g b* **by** *blast*
qed
ultimately show *?thesis* **by**(*auto simp*:*bij-betw-def*)
qed

lemma *surj-image-vimage-eq*: *surj f* ==> *f* ‘ (*f* −‘ *A*) = *A*
by (*simp add*: *surj-range*)

lemma *inj-vimage-image-eq*: *inj f* ==> *f* −‘ (*f* ‘ *A*) = *A*
by (*simp add*: *inj-on-def*, *blast*)

lemma *vimage-subsetD*: *surj f* ==> *f* −‘ *B* <= *A* ==> *B* <= *f* ‘ *A*
apply (*unfold surj-def*)
apply (*blast intro*: *sym*)
done

lemma *vimage-subsetI*: *inj f* ==> *B* <= *f* ‘ *A* ==> *f* −‘ *B* <= *A*
by (*unfold inj-on-def*, *blast*)

lemma *vimage-subset-eq*: *bij f* ==> (*f* −‘ *B* <= *A*) = (*B* <= *f* ‘ *A*)
apply (*unfold bij-def*)
apply (*blast del*: *subsetI intro*: *vimage-subsetI vimage-subsetD*)
done

lemma *inj-on-Un-image-eq-iff*: *inj-on f* (*A* ∪ *B*) ⟹ *f* ‘ *A* = *f* ‘ *B* ⟷ *A* = *B*
by(*blast dest*: *inj-onD*)

lemma *inj-on-image-Int*:
[| *inj-on f C*; *A*<=*C*; *B*<=*C* |] ==> *f*‘(*A Int B*) = *f*‘*A Int f*‘*B*
apply (*simp add*: *inj-on-def*, *blast*)
done

lemma *inj-on-image-set-diff*:
[| *inj-on f C*; *A*<=*C*; *B*<=*C* |] ==> *f*‘(*A*−*B*) = *f*‘*A* − *f*‘*B*
apply (*simp add*: *inj-on-def*, *blast*)
done

lemma *image-Int*: *inj f* ==> *f*‘(*A Int B*) = *f*‘*A Int f*‘*B*
by (*simp add*: *inj-on-def*, *blast*)

lemma *image-set-diff*: *inj f* ==> *f*‘(*A*−*B*) = *f*‘*A* − *f*‘*B*
by (*simp add*: *inj-on-def*, *blast*)

lemma *inj-image-mem-iff*: *inj f ==> (f a : f`A) = (a : A)*
by (*blast dest*: *injD*)

lemma *inj-image-subset-iff*: *inj f ==> (f`A <= f`B) = (A<=B)*
by (*simp add*: *inj-on-def*, *blast*)

lemma *inj-image-eq-iff*: *inj f ==> (f`A = f`B) = (A = B)*
by (*blast dest*: *injD*)

lemma *image-INT*:
[| *inj-on f C*; *ALL x:A. B x <= C*; *j:A* |]
==> *f ` (INTER A B) = (INT x:A. f ` B x)*
apply (*simp add*: *inj-on-def*, *blast*)
done

lemma *bij-image-INT*: *bij f ==> f ` (INTER A B) = (INT x:A. f ` B x)*
apply (*simp add*: *bij-def*)
apply (*simp add*: *inj-on-def surj-def*, *blast*)
done

lemma *surj-Compl-image-subset*: *surj f ==> -(f`A) <= f`(-A)*
by (*auto simp add*: *surj-def*)

lemma *inj-image-Compl-subset*: *inj f ==> f`(-A) <= -(f`A)*
by (*auto simp add*: *inj-on-def*)

lemma *bij-image-Compl-eq*: *bij f ==> f`(-A) = -(f`A)*
apply (*simp add*: *bij-def*)
apply (*rule equalityI*)
apply (*simp-all* (*no-asm-simp*) *add*: *inj-image-Compl-subset surj-Compl-image-subset*)
done

lemma (**in** *ordered-ab-group-add*) *inj-uminus*[*simp*, *intro*]: *inj-on uminus A*
by (*auto intro*!: *inj-onI*)

lemma (**in** *linorder*) *strict-mono-imp-inj-on*: *strict-mono f ⟹ inj-on f A*
by (*auto intro*!: *inj-onI dest*: *strict-mono-eq*)

10.5 Function Updating

definition
fun-upd :: *('a => 'b) => 'a => 'b => ('a => 'b)* **where**
fun-upd f a b == % x. if x=a then b else f x

nonterminals
updbinds updbind

syntax
-updbind :: [*'a*, *'a*] => *updbind* ((*2*- :=/ -))
:: *updbind* => *updbinds* (-)
-updbinds:: [*updbind*, *updbinds*] => *updbinds* (-,/ -)
-Update :: [*'a*, *updbinds*] => *'a* (-/'((-)') [*1000*, *0*] *900*)

translations
-Update f (*-updbinds b bs*) == *-Update* (*-Update f b*) *bs*
f(*x*:=*y*) == *CONST fun-upd f x y*

lemma *fun-upd-idem-iff*: (*f*(*x*:=*y*) = *f*) = (*f x* = *y*)
apply (*simp add*: *fun-upd-def*, *safe*)
apply (*erule subst*)
apply (*rule-tac* [*2*] *ext*, *auto*)
done

lemmas *fun-upd-idem* = *fun-upd-idem-iff* [*THEN iffD2*, *standard*]

lemmas *fun-upd-triv* = *refl* [*THEN fun-upd-idem*]
declare *fun-upd-triv* [*iff*]

lemma *fun-upd-apply* [*simp*]: (*f*(*x*:=*y*))*z* = (*if z*=*x then y else f z*)
by (*simp add*: *fun-upd-def*)

lemma *fun-upd-same*: (*f*(*x*:=*y*)) *x* = *y*
by *simp*

lemma *fun-upd-other*: *z*~=*x* ==> (*f*(*x*:=*y*)) *z* = *f z*
by *simp*

lemma *fun-upd-upd* [*simp*]: *f*(*x*:=*y*,*x*:=*z*) = *f*(*x*:=*z*)
by (*simp add*: *expand-fun-eq*)

lemma *fun-upd-twist*: *a* ~= *c* ==> (*m*(*a*:=*b*))(*c*:=*d*) = (*m*(*c*:=*d*))(*a*:=*b*)
by (*rule ext*, *auto*)

lemma *inj-on-fun-updI*: ⟦ *inj-on f A*; *y* ∉ *f*`*A* ⟧ ⟹ *inj-on* (*f*(*x*:=*y*)) *A*
by (*fastsimp simp*:*inj-on-def image-def*)

lemma *fun-upd-image*:
f(*x*:=*y*) ` *A* = (*if x* ∈ *A then insert y* (*f* ` (*A*−{*x*})) *else f* ` *A*)
by *auto*

lemma *fun-upd-comp*: *f* ∘ (*g*(*x* := *y*)) = (*f* ∘ *g*)(*x* := *f y*)

by (*auto intro*: *ext*)

10.6 *override-on*

definition
override-on :: $('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'a \Rightarrow 'b$
where
override-on f g A = $(\lambda a.$ *if a* $\in$ *A then g a else f a*)

lemma *override-on-emptyset*[*simp*]: *override-on f g* {} = *f*
by(*simp add*:*override-on-def*)

lemma *override-on-apply-notin*[*simp*]: *a* $\sim$: *A* ==> (*override-on f g A*) *a* = *f a*
by(*simp add*:*override-on-def*)

lemma *override-on-apply-in*[*simp*]: *a* : *A* ==> (*override-on f g A*) *a* = *g a*
by(*simp add*:*override-on-def*)

10.7 *swap*

definition
swap :: $'a \Rightarrow 'a \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b)$
where
swap a b f = *f* (*a* := *f b*, *b*:= *f a*)

lemma *swap-self* [*simp*]: *swap a a f* = *f*
by (*simp add*: *swap-def*)

lemma *swap-commute*: *swap a b f* = *swap b a f*
by (*rule ext*, *simp add*: *fun-upd-def swap-def*)

lemma *swap-nilpotent* [*simp*]: *swap a b* (*swap a b f*) = *f*
by (*rule ext*, *simp add*: *fun-upd-def swap-def*)

lemma *swap-triple*:
assumes *a* $\neq$ *c* **and** *b* $\neq$ *c*
shows *swap a b* (*swap b c* (*swap a b f*)) = *swap a c f*
using *assms* **by** (*simp add*: *expand-fun-eq swap-def*)

lemma *comp-swap*: *f* $\circ$ *swap a b g* = *swap a b* (*f* $\circ$ *g*)
by (*rule ext*, *simp add*: *fun-upd-def swap-def*)

lemma *inj-on-imp-inj-on-swap*:
[|*inj-on f A*; *a* $\in$ *A*; *b* $\in$ *A*|] ==> *inj-on* (*swap a b f*) *A*
by (*simp add*: *inj-on-def swap-def*, *blast*)

lemma *inj-on-swap-iff* [*simp*]:
assumes *A*: *a* $\in$ *A b* $\in$ *A* **shows** *inj-on* (*swap a b f*) *A* = *inj-on f A*
proof
assume *inj-on* (*swap a b f*) *A*

with *A* **have** *inj-on* (*swap a b* (*swap a b f*)) *A*
by (*iprover intro*: *inj-on-imp-inj-on-swap*)
thus *inj-on f A* **by** *simp*
next
assume *inj-on f A*
with *A* **show** *inj-on* (*swap a b f*) *A* **by** (*iprover intro*: *inj-on-imp-inj-on-swap*)
qed

lemma *surj-imp-surj-swap*: *surj f* ==> *surj* (*swap a b f*)
apply (*simp add*: *surj-def swap-def*, *clarify*)
apply (*case-tac y* = *f b*, *blast*)
apply (*case-tac y* = *f a*, *auto*)
done

lemma *surj-swap-iff* [*simp*]: *surj* (*swap a b f*) = *surj f*
proof
assume *surj* (*swap a b f*)
hence *surj* (*swap a b* (*swap a b f*)) **by** (*rule surj-imp-surj-swap*)
thus *surj f* **by** *simp*
next
assume *surj f*
thus *surj* (*swap a b f*) **by** (*rule surj-imp-surj-swap*)
qed

lemma *bij-swap-iff*: *bij* (*swap a b f*) = *bij f*
by (*simp add*: *bij-def*)

hide-const (**open**) *swap*

10.8 Inversion of injective functions

definition *the-inv-into* :: *'a set* => (*'a* => *'b*) => (*'b* => *'a*) **where**
the-inv-into A f == %*x*. *THE y*. *y* : *A* & *f y* = *x*

lemma *the-inv-into-f-f*:
[| *inj-on f A*; *x* : *A* |] ==> *the-inv-into A f* (*f x*) = *x*
apply (*simp add*: *the-inv-into-def inj-on-def*)
apply *blast*
done

lemma *f-the-inv-into-f*:
inj-on f A ==> *y* : *f*'*A* ==> *f* (*the-inv-into A f y*) = *y*
apply (*simp add*: *the-inv-into-def*)
apply (*rule the1I2*)
apply(*blast dest*: *inj-onD*)
apply *blast*
done

lemma *the-inv-into-into*:

```
  [| inj-on f A; x : f ` A; A <= B |] ==> the-inv-into A f x : B
apply (simp add: the-inv-into-def)
apply (rule the1I2)
 apply(blast dest: inj-onD)
apply blast
done

lemma the-inv-into-onto[simp]:
  inj-on f A ==> the-inv-into A f ` (f ` A) = A
by (fast intro:the-inv-into-into the-inv-into-f-f[symmetric])

lemma the-inv-into-f-eq:
  [| inj-on f A; f x = y; x : A |] ==> the-inv-into A f y = x
  apply (erule subst)
  apply (erule the-inv-into-f-f, assumption)
  done

lemma the-inv-into-comp:
  [| inj-on f (g ` A); inj-on g A; x : f ` g ` A |] ==>
  the-inv-into A (f o g) x = (the-inv-into A g o the-inv-into (g ` A) f) x
apply (rule the-inv-into-f-eq)
  apply (fast intro: comp-inj-on)
 apply (simp add: f-the-inv-into-f the-inv-into-into)
apply (simp add: the-inv-into-into)
done

lemma inj-on-the-inv-into:
  inj-on f A ⟹ inj-on (the-inv-into A f) (f ` A)
by (auto intro: inj-onI simp: image-def the-inv-into-f-f)

lemma bij-betw-the-inv-into:
  bij-betw f A B ⟹ bij-betw (the-inv-into A f) B A
by (auto simp add: bij-betw-def inj-on-the-inv-into the-inv-into-into)

abbreviation the-inv :: ('a ⇒ 'b) ⇒ ('b ⇒ 'a) where
  the-inv f ≡ the-inv-into UNIV f

lemma the-inv-f-f:
  assumes inj f
  shows the-inv f (f x) = x using assms UNIV-I
  by (rule the-inv-into-f-f)
```

10.9 Proof tool setup

simplifies terms of the form f(...,x:=y,...,x:=z,...) to f(...,x:=z,...)

```
simproc-setup fun-upd2 (f(v := w, x := y)) = ⟪ fn - =>
let
  fun gen-fun-upd NONE T - - = NONE
    | gen-fun-upd (SOME f) T x y = SOME (Const (@{const-name fun-upd}, T)
```

```
$ f $ x $ y)
  fun dest-fun-T1 (Type (-, T :: Ts)) = T
  fun find-double (t as Const (@{const-name fun-upd},T) $ f $ x $ y) =
    let
      fun find (Const (@{const-name fun-upd},T) $ g $ v $ w) =
            if v aconv x then SOME g else gen-fun-upd (find g) T v w
        | find t = NONE
    in (dest-fun-T1 T, gen-fun-upd (find f) T x y) end

  fun proc ss ct =
    let
      val ctxt = Simplifier.the-context ss
      val t = Thm.term-of ct
    in
      case find-double t of
        (T, NONE) => NONE
      | (T, SOME rhs) =>
          SOME (Goal.prove ctxt [] [] (Logic.mk-equals (t, rhs))
            (fn - =>
              rtac eq-reflection 1 THEN
              rtac ext 1 THEN
              simp-tac (Simplifier.inherit-context ss @{simpset}) 1))
    end
in proc end
⟩⟩
```

10.10 Code generator setup

```
types-code
  fun  ((- ->/ -))
attach (term-of) ⟨⟨
fun term-of-fun-type - aT - bT - = Free (<function>, aT --> bT);
⟩⟩
attach (test) ⟨⟨
fun gen-fun-type aF aT bG bT i =
  let
    val tab = Unsynchronized.ref [];
    fun mk-upd (x, (-, y)) t = Const (Fun.fun-upd,
      (aT --> bT) --> aT --> bT --> aT --> bT) $ t $ aF x $ y ()
  in
    (fn x =>
       case AList.lookup op = (!tab) x of
         NONE =>
           let val p as (y, -) = bG i
           in (tab := (x, p) :: !tab; y) end
       | SOME (y, -) => y,
     fn () => Basics.fold mk-upd (!tab) (Const (HOL.undefined, aT --> bT)))
  end;
⟩⟩
```

code-const *op* ∘
(*SML* **infixl** *5 o*)
(*Haskell* **infixr** *9* .)

code-const *id*
(*Haskell id*)

end

11 Product-Type: Cartesian products

theory *Product-Type*
imports *Typedef Inductive Fun*
uses
(*Tools/split-rule.ML*)
(*Tools/inductive-set.ML*)
begin

11.1 *bool* is a datatype

rep-datatype *True False* **by** (*auto intro*: *bool-induct*)

declare *case-split* [*cases type*: *bool*]
— prefer plain propositional version

lemma
shows [*code*]: *eq-class.eq False P* ⟷ ¬ *P*
and [*code*]: *eq-class.eq True P* ⟷ *P*
and [*code*]: *eq-class.eq P False* ⟷ ¬ *P*
and [*code*]: *eq-class.eq P True* ⟷ *P*
and [*code nbe*]: *eq-class.eq P P* ⟷ *True*
by (*simp-all add*: *eq*)

code-const *eq-class.eq* :: *bool* ⇒ *bool* ⇒ *bool*
(*Haskell* **infixl** *4* ==)

code-instance *bool* :: *eq*
(*Haskell* −)

11.2 The *unit* type

typedef *unit* = {*True*}
proof
show *True* : *?unit* **..**
qed

definition

Unity :: *unit* (′(′))
where
() = *Abs-unit True*

lemma *unit-eq* [*no-atp*]: *u* = ()
by (*induct u*) (*simp add*: *unit-def Unity-def*)

Simplification procedure for *unit-eq*. Cannot use this rule directly — it loops!

ML ⟪
val unit-eq-proc =
let val unit-meta-eq = *mk-meta-eq* @{*thm unit-eq*} *in*
Simplifier.simproc @{*theory*} *unit-eq* [*x*::*unit*]
(*fn - => fn - => fn t => if HOLogic.is-unit t then NONE else SOME unit-meta-eq*)
end;

Addsimprocs [*unit-eq-proc*];
⟫

rep-datatype () **by** *simp*

lemma *unit-all-eq1*: (!!*x*::*unit*. *PROP P x*) == *PROP P* ()
by *simp*

lemma *unit-all-eq2*: (!!*x*::*unit*. *PROP P*) == *PROP P*
by (*rule triv-forall-equality*)

This rewrite counters the effect of *unit-eq-proc* on %*u*::*unit*. *f u*, replacing it by *f* rather than by %*u*. *f* ().

lemma *unit-abs-eta-conv* [*simp*,*no-atp*]: (%*u*::*unit*. *f* ()) = *f*
by (*rule ext*) *simp*

instantiation *unit* :: *default*
begin

definition *default* = ()

instance ..

end

lemma [*code*]:
eq-class.eq (*u*::*unit*) *v* ⟷ *True* **unfolding** *eq unit-eq* [*of u*] *unit-eq* [*of v*] **by** *rule*+

code-type *unit*
(*SML unit*)
(*OCaml unit*)

(*Haskell* ())
(*Scala Unit*)

code-const *Unity*
(*SML* ())
(*OCaml* ())
(*Haskell* ())
(*Scala* ())

code-instance *unit* :: *eq*
(*Haskell* −)

code-const *eq-class.eq* :: *unit* ⇒ *unit* ⇒ *bool*
(*Haskell* **infixl** *4* ==)

code-reserved *SML*
unit

code-reserved *OCaml*
unit

code-reserved *Scala*
Unit

11.3 The product type

11.3.1 Type definition

definition *Pair-Rep* :: *'a* ⇒ *'b* ⇒ *'a* ⇒ *'b* ⇒ *bool* **where**
Pair-Rep a b = (λ*x y*. *x* = *a* ∧ *y* = *b*)

global

typedef (*Prod*)
(*'a*, *'b*) ∗ (**infixr** ∗ *20*)
= {*f*. ∃ *a b*. *f* = *Pair-Rep* (*a*::*'a*) (*b*::*'b*)}
proof
fix *a b* **show** *Pair-Rep a b* ∈ *?Prod*
by *rule+*
qed

type-notation (*xsymbols*)
∗ ((- ×/ -) [*21*, *20*] *20*)
type-notation (*HTML* **output**)
∗ ((- ×/ -) [*21*, *20*] *20*)

consts
Pair :: *'a* ⇒ *'b* ⇒ *'a* × *'b*

local

defs
Pair-def: *Pair a b* == *Abs-Prod* (*Pair-Rep a b*)

rep-datatype (*prod*) *Pair* **proof** −
fix *P* :: *'a* × *'b* ⇒ *bool* **and** *p*
assume ⋀*a b*. *P* (*Pair a b*)
then show *P p* **by** (*cases p*) (*auto simp add*: *Prod-def Pair-def Pair-Rep-def*)
next
fix *a c* :: *'a* **and** *b d* :: *'b*
have *Pair-Rep a b* = *Pair-Rep c d* ⟷ *a* = *c* ∧ *b* = *d*
by (*auto simp add*: *Pair-Rep-def expand-fun-eq*)
moreover have *Pair-Rep a b* ∈ *Prod* **and** *Pair-Rep c d* ∈ *Prod*
by (*auto simp add*: *Prod-def*)
ultimately show *Pair a b* = *Pair c d* ⟷ *a* = *c* ∧ *b* = *d*
by (*simp add*: *Pair-def Abs-Prod-inject*)
qed

11.3.2 Tuple syntax

global consts
split :: (*'a* ⇒ *'b* ⇒ *'c*) ⇒ *'a* × *'b* ⇒ *'c*

local defs
split-prod-case: *split* == *prod-case*

Patterns – extends pre-defined type *pttrn* used in abstractions.

nonterminals
tuple-args patterns

syntax
-tuple :: *'a* => *tuple-args* => *'a* * *'b* ((*1'*(-,/ -*'*)))
-tuple-arg :: *'a* => *tuple-args* (-)
-tuple-args :: *'a* => *tuple-args* => *tuple-args* (-,/ -)
-pattern :: [*pttrn*, *patterns*] => *pttrn* (*'*(-,/ -*'*))
:: *pttrn* => *patterns* (-)
-patterns :: [*pttrn*, *patterns*] => *patterns* (-,/ -)

translations
(*x*, *y*) == *CONST Pair x y*
-tuple x (*-tuple-args y z*) == *-tuple x* (*-tuple-arg* (*-tuple y z*))
%(*x*, *y*, *zs*). *b* == *CONST split* (%*x* (*y*, *zs*). *b*)
%(*x*, *y*). *b* == *CONST split* (%*x y*. *b*)
-abs (*CONST Pair x y*) *t* => %(*x*, *y*). *t*
— The last rule accommodates tuples in 'case C ... (x,y) ... =¿ ...' The (x,y) is parsed as 'Pair x y' because it is logic, not pttrn

print-translation ⟪

```
let
  fun split-tr' [Abs (x, T, t as (Abs abs))] =
        (* split (%x y. t) => %(x,y) t *)
        let
          val (y, t') = atomic-abs-tr' abs;
          val (x', t'') = atomic-abs-tr' (x, T, t');
        in
          Syntax.const @{syntax-const -abs} $
            (Syntax.const @{syntax-const -pattern} $ x' $ y) $ t''
        end
    | split-tr' [Abs (x, T, (s as Const (@{const-syntax split}, -) $ t))] =
        (* split (%x. (split (%y z. t))) => %(x,y,z). t *)
        let
          val Const (@{syntax-const -abs}, -) $
            (Const (@{syntax-const -pattern}, -) $ y $ z) $ t' = split-tr' [t];
          val (x', t'') = atomic-abs-tr' (x, T, t');
        in
          Syntax.const @{syntax-const -abs} $
            (Syntax.const @{syntax-const -pattern} $ x' $
              (Syntax.const @{syntax-const -patterns} $ y $ z)) $ t''
        end
    | split-tr' [Const (@{const-syntax split}, -) $ t] =
        (* split (split (%x y z. t)) => %((x, y), z). t *)
        split-tr' [(split-tr' [t])] (* inner split-tr' creates next pattern *)
    | split-tr' [Const (@{syntax-const -abs}, -) $ x-y $ Abs abs] =
        (* split (%pttrn z. t) => %(pttrn,z). t *)
        let val (z, t) = atomic-abs-tr' abs in
          Syntax.const @{syntax-const -abs} $
            (Syntax.const @{syntax-const -pattern} $ x-y $ z) $ t
        end
    | split-tr' - = raise Match;
in [(@{const-syntax split}, split-tr')] end
⟫
```

typed-print-translation ⟪

```
let
  fun split-guess-names-tr' - T [Abs (x, -, Abs -)] = raise Match
    | split-guess-names-tr' - T [Abs (x, xT, t)] =
        (case (head-of t) of
          Const (@{const-syntax split}, -) => raise Match
        | - =>
          let
            val (- :: yT :: -) = binder-types (domain-type T) handle Bind => raise
Match;
            val (y, t') = atomic-abs-tr' (y, yT, incr-boundvars 1 t $ Bound 0);
            val (x', t'') = atomic-abs-tr' (x, xT, t');
          in
            Syntax.const @{syntax-const -abs} $
```

```
          (Syntax.const @{syntax-const -pattern} $ x' $ y) $ t''
        end)
  | split-guess-names-tr' - T [t] =
      (case head-of t of
        Const (@{const-syntax split}, -) => raise Match
      | - =>
        let
         val (xT :: yT :: -) = binder-types (domain-type T) handle Bind => raise
Match;
           val (y, t') = atomic-abs-tr' (y, yT, incr-boundvars 2 t $ Bound 1 $
Bound 0);
          val (x', t'') = atomic-abs-tr' (x, xT, t');
        in
          Syntax.const @{syntax-const -abs} $
            (Syntax.const @{syntax-const -pattern} $ x' $ y) $ t''
        end)
  | split-guess-names-tr' - - - = raise Match;
in [(@{const-syntax split}, split-guess-names-tr')] end
⟫
```

11.3.3 Code generator setup

```
lemma split-case-cert:
  assumes CASE ≡ split f
  shows CASE (a, b) ≡ f a b
  using assms by (simp add: split-prod-case)

setup ⟪
  Code.add-case @{thm split-case-cert}
⟫

code-type *
  (SML infix 2 *)
  (OCaml infix 2 *)
  (Haskell !((-),/ (-)))
  (Scala ((-),/ (-)))

code-const Pair
  (SML !((-),/ (-)))
  (OCaml !((-),/ (-)))
  (Haskell !((-),/ (-)))
  (Scala !((-),/ (-)))

code-instance * :: eq
  (Haskell -)

code-const eq-class.eq :: 'a::eq × 'b::eq ⇒ 'a × 'b ⇒ bool
  (Haskell infixl 4 ==)
```

```
types-code
  *     ((- */ -))
attach (term-of) ⟪
fun term-of-id-42 aF aT bF bT (x, y) = HOLogic.pair-const aT bT $ aF x $ bF
y;
⟫
attach (test) ⟪
fun gen-id-42 aG aT bG bT i =
  let
    val (x, t) = aG i;
    val (y, u) = bG i
  in ((x, y), fn () => HOLogic.pair-const aT bT $ t () $ u ()) end;
⟫

consts-code
  Pair    ((-,/ -))

setup ⟪
let

fun strip-abs-split 0 t = ([], t)
  | strip-abs-split i (Abs (s, T, t)) =
      let
        val s' = Codegen.new-name t s;
        val v = Free (s', T)
      in apfst (cons v) (strip-abs-split (i-1) (subst-bound (v, t))) end
  | strip-abs-split i (u as Const (@{const-name split}, -) $ t) =
      (case strip-abs-split (i+1) t of
        (v :: v' :: vs, u) => (HOLogic.mk-prod (v, v') :: vs, u)
      | - => ([], u))
  | strip-abs-split i t =
      strip-abs-split i (Abs (x, hd (binder-types (fastype-of t)), t $ Bound 0));

fun let-codegen thy defs dep thyname brack t gr =
  (case strip-comb t of
    (t1 as Const (@{const-name Let}, -), t2 :: t3 :: ts) =>
    let
      fun dest-let (l as Const (@{const-name Let}, -) $ t $ u) =
          (case strip-abs-split 1 u of
            ([p], u') => apfst (cons (p, t)) (dest-let u')
          | - => ([], l))
        | dest-let t = ([], t);
      fun mk-code (l, r) gr =
        let
          val (pl, gr1) = Codegen.invoke-codegen thy defs dep thyname false l gr;
          val (pr, gr2) = Codegen.invoke-codegen thy defs dep thyname false r gr1;
        in ((pl, pr), gr2) end
    in case dest-let (t1 $ t2 $ t3) of
        ([], -) => NONE
```

```
    | (ps, u) =>
        let
          val (qs, gr1) = fold-map mk-code ps gr;
         val (pu, gr2) = Codegen.invoke-codegen thy defs dep thyname false u gr1;
          val (pargs, gr3) = fold-map
            (Codegen.invoke-codegen thy defs dep thyname true) ts gr2
        in
          SOME (Codegen.mk-app brack
            (Pretty.blk (0, [Codegen.str let , Pretty.blk (0, flat
                (separate [Codegen.str ;, Pretty.brk 1] (map (fn (pl, pr) =>
                  [Pretty.block [Codegen.str val , pl, Codegen.str =,
                    Pretty.brk 1, pr]]) qs))),
              Pretty.brk 1, Codegen.str in , pu,
              Pretty.brk 1, Codegen.str end])) pargs, gr3)
        end
    end
  | - => NONE);

fun split-codegen thy defs dep thyname brack t gr = (case strip-comb t of
    (t1 as Const (@{const-name split}, -), t2 :: ts) =>
      let
        val ([p], u) = strip-abs-split 1 (t1 $ t2);
        val (q, gr1) = Codegen.invoke-codegen thy defs dep thyname false p gr;
        val (pu, gr2) = Codegen.invoke-codegen thy defs dep thyname false u gr1;
        val (pargs, gr3) = fold-map
          (Codegen.invoke-codegen thy defs dep thyname true) ts gr2
      in
        SOME (Codegen.mk-app brack
          (Pretty.block [Codegen.str (fn , q, Codegen.str =>,
            Pretty.brk 1, pu, Codegen.str )]) pargs, gr2)
      end
  | - => NONE);

in

  Codegen.add-codegen let-codegen let-codegen
  #> Codegen.add-codegen split-codegen split-codegen

end
⟩⟩
```

11.3.4 Fundamental operations and properties

lemma *surj-pair* [*simp*]: *EX x y. p = (x, y)*
  by (*cases p*) *simp*

global consts
  fst :: *'a × 'b ⇒ 'a*
  snd :: *'a × 'b ⇒ 'b*

local defs
fst-def: *fst p == case p of (a, b) ⇒ a*
snd-def: *snd p == case p of (a, b) ⇒ b*

lemma *fst-conv* [*simp*, *code*]: *fst (a, b) = a*
unfolding *fst-def* **by** *simp*

lemma *snd-conv* [*simp*, *code*]: *snd (a, b) = b*
unfolding *snd-def* **by** *simp*

code-const *fst* **and** *snd*
(*Haskell fst* **and** *snd*)

lemma *prod-case-unfold*: *prod-case = (%c p. c (fst p) (snd p))*
by (*simp add*: *expand-fun-eq split*: *prod.split*)

lemma *fst-eqD*: *fst (x, y) = a ==> x = a*
by *simp*

lemma *snd-eqD*: *snd (x, y) = a ==> y = a*
by *simp*

lemma *pair-collapse* [*simp*]: *(fst p, snd p) = p*
by (*cases p*) *simp*

lemmas *surjective-pairing = pair-collapse* [*symmetric*]

lemma *Pair-fst-snd-eq*: *s = t ⟷ fst s = fst t ∧ snd s = snd t*
by (*cases s*, *cases t*) *simp*

lemma *prod-eqI* [*intro?*]: *fst p = fst q ⟹ snd p = snd q ⟹ p = q*
by (*simp add*: *Pair-fst-snd-eq*)

lemma *split-conv* [*simp*, *code*]: *split f (a, b) = f a b*
by (*simp add*: *split-prod-case*)

lemma *splitI*: *f a b ⟹ split f (a, b)*
by (*rule split-conv* [*THEN iffD2*])

lemma *splitD*: *split f (a, b) ⟹ f a b*
by (*rule split-conv* [*THEN iffD1*])

lemma *split-Pair* [*simp*]: *(λ(x, y). (x, y)) = id*
by (*simp add*: *split-prod-case expand-fun-eq split*: *prod.split*)

lemma *split-eta*: *(λ(x, y). f (x, y)) = f*
— Subsumes the old *split-Pair* when *f* is the identity function.
by (*simp add*: *split-prod-case expand-fun-eq split*: *prod.split*)

lemma *split-comp*: *split* $(f \circ g)$ $x = f$ $(g$ $(fst\ x))$ $(snd\ x)$
by (*cases x*) *simp*

lemma *split-twice*: *split f* (*split g p*) = *split* $(\lambda x\ y.$ *split f* $(g\ x\ y))$ *p*
by (*cases p*) *simp*

lemma *The-split*: *The* (*split P*) = (*THE xy. P* (*fst xy*) (*snd xy*))
by (*simp add*: *split-prod-case prod-case-unfold*)

lemma *split-weak-cong*: $p = q \Longrightarrow$ *split c p* = *split c q*
— Prevents simplification of *c*: much faster
by (*erule arg-cong*)

lemma *cond-split-eta*: $(!!x\ y.\ f\ x\ y = g\ (x, y)) ==> (\%(x, y).\ f\ x\ y) = g$
by (*simp add*: *split-eta*)

lemma *split-paired-all*: $(!!x.$ *PROP P x*$) ==$ $(!!a\ b.$ *PROP P* $(a, b))$
proof
fix *a b*
assume $!!x.$ *PROP P x*
then show *PROP P* (a, b) **.**
next
fix *x*
assume $!!a\ b.$ *PROP P* (a, b)
from ‹*PROP P* (*fst x*, *snd x*)› **show** *PROP P x* **by** *simp*
qed

The rule *split-paired-all* does not work with the Simplifier because it also affects premises in congrence rules, where this can lead to premises of the form $!!a\ b.$ ... = *?P*(a, b) which cannot be solved by reflexivity.

lemmas *split-tupled-all* = *split-paired-all unit-all-eq2*

ML ⟪
(∗ *replace parameters of product type by individual component parameters* ∗)
val safe-full-simp-tac = *generic-simp-tac true* (*true*, *false*, *false*);
local (∗ *filtering with exists-paired-all is an essential optimization* ∗)
fun exists-paired-all (*Const* (*all*, -) $ *Abs* (-, *T*, *t*)) =
can HOLogic.dest-prodT T orelse exists-paired-all t
| *exists-paired-all* (*t* $ *u*) = *exists-paired-all t orelse exists-paired-all u*
| *exists-paired-all* (*Abs* (-, -, *t*)) = *exists-paired-all t*
| *exists-paired-all* - = *false*;
val ss = *HOL-basic-ss*
addsimps [@{*thm split-paired-all*}, @{*thm unit-all-eq2*}, @{*thm unit-abs-eta-conv*}]
addsimprocs [*unit-eq-proc*];
in
val split-all-tac = *SUBGOAL* (*fn* (*t*, *i*) =>
if exists-paired-all t then safe-full-simp-tac ss i else no-tac);
val unsafe-split-all-tac = *SUBGOAL* (*fn* (*t*, *i*) =>

```
    if exists-paired-all t then full-simp-tac ss i else no-tac);
  fun split-all th =
  if exists-paired-all (Thm.prop-of th) then full-simplify ss th else th;
 end;
⟩⟩
```

declaration ⟨⟨ *fn - =>*
Classical.map-cs (*fn cs => cs addSbefore* (*split-all-tac*, *split-all-tac*))
⟩⟩

lemma *split-paired-All* [*simp*]: (*ALL x. P x*) = (*ALL a b. P* (*a*, *b*))
— [*iff*] is not a good idea because it makes *blast* loop
by *fast*

lemma *split-paired-Ex* [*simp*]: (*EX x. P x*) = (*EX a b. P* (*a*, *b*))
by *fast*

lemma *split-paired-The*: (*THE x. P x*) = (*THE* (*a*, *b*). *P* (*a*, *b*))
— Can't be added to simpset: loops!
by (*simp add*: *split-eta*)

Simplification procedure for *cond-split-eta*. Using *split-eta* as a rewrite rule is not general enough, and using *cond-split-eta* directly would render some existing proofs very inefficient; similarly for *split-beta*.

ML ⟨⟨

```
local
 val cond-split-eta-ss = HOL-basic-ss addsimps @{thms cond-split-eta};
 fun Pair-pat k 0 (Bound m) = (m = k)
   | Pair-pat k i (Const (@{const-name Pair}, -) $ Bound m $ t) =
       i > 0 andalso m = k + i andalso Pair-pat k (i - 1) t
   | Pair-pat - - - = false;
 fun no-args k i (Abs (-, -, t)) = no-args (k + 1) i t
   | no-args k i (t $ u) = no-args k i t andalso no-args k i u
   | no-args k i (Bound m) = m < k orelse m > k + i
   | no-args - - - = true;
 fun split-pat tp i (Abs (-, -, t)) = if tp 0 i t then SOME (i, t) else NONE
   | split-pat tp i (Const (@{const-name split}, -) $ Abs (-, -, t)) = split-pat tp (i
+ 1) t
   | split-pat tp i - = NONE;
 fun metaeq ss lhs rhs = mk-meta-eq (Goal.prove (Simplifier.the-context ss) [] []
       (HOLogic.mk-Trueprop (HOLogic.mk-eq (lhs, rhs)))
       (K (simp-tac (Simplifier.inherit-context ss cond-split-eta-ss) 1)));

 fun beta-term-pat k i (Abs (-, -, t)) = beta-term-pat (k + 1) i t
   | beta-term-pat k i (t $ u) =
       Pair-pat k i (t $ u) orelse (beta-term-pat k i t andalso beta-term-pat k i u)
   | beta-term-pat k i t = no-args k i t;
 fun eta-term-pat k i (f $ arg) = no-args k i f andalso Pair-pat k i arg
   | eta-term-pat - - - = false;
```

```
  fun subst_arg k i (Abs (x, T, t)) = Abs (x, T, subst_arg (k+1) i t)
    | subst_arg k i (t $ u) =
        if Pair_pat k i (t $ u) then incr_boundvars k arg
        else (subst_arg k i t $ subst_arg k i u)
    | subst_arg k i t = t;
  fun beta_proc ss (s as Const (@{const_name split}, _) $ Abs (_, _, t) $ arg) =
        (case split_pat beta_term_pat 1 t of
          SOME (i, f) => SOME (metaeq ss s (subst_arg 0 i f))
        | NONE => NONE)
    | beta_proc _ _ = NONE;
  fun eta_proc ss (s as Const (@{const_name split}, _) $ Abs (_, _, t)) =
        (case split_pat eta_term_pat 1 t of
          SOME (_, ft) => SOME (metaeq ss s (let val (f $ arg) = ft in f end))
        | NONE => NONE)
    | eta_proc _ _ = NONE;
in
  val split_beta_proc = Simplifier.simproc @{theory} split_beta [split f z] (K beta_proc);
  val split_eta_proc = Simplifier.simproc @{theory} split_eta [split f] (K eta_proc);
end;

Addsimprocs [split_beta_proc, split_eta_proc];
⟫
```

lemma *split_beta* [*mono*]: (%(*x*, *y*). *P x y*) *z* = *P* (*fst z*) (*snd z*)
by (*subst surjective_pairing*, *rule split_conv*)

lemma *split_split* [*no_atp*]: *R*(*split c p*) = (*ALL x y*. *p* = (*x*, *y*) −−> *R*(*c x y*))
— For use with *split* and the Simplifier.
by (*insert surj_pair* [*of p*], *clarify*, *simp*)

split_split could be declared as [*split*] done after the Splitter has been speeded up significantly; precompute the constants involved and don't do anything unless the current goal contains one of those constants.

lemma *split_split_asm* [*no_atp*]: *R* (*split c p*) = (~(*EX x y*. *p* = (*x*, *y*) & (~*R* (*c x y*))))
by (*subst split_split*, *simp*)

split used as a logical connective or set former.

These rules are for use with *blast*; could instead call *simp* using *split* as rewrite.

lemma *splitI2*: !!*p*. [| !!*a b*. *p* = (*a*, *b*) ==> *c a b* |] ==> *split c p*
apply (*simp only*: *split_tupled_all*)
apply (*simp* (*no_asm_simp*))
done

lemma *splitI2'*: !!*p*. [| !!*a b*. (*a*, *b*) = *p* ==> *c a b x* |] ==> *split c p x*
apply (*simp only*: *split_tupled_all*)

apply (*simp* (*no-asm-simp*))
done

lemma *splitE*: *split c p* ==> (!!*x y*. *p* = (*x*, *y*) ==> *c x y* ==> *Q*) ==> *Q*
by (*induct p*) (*auto simp add*: *split-prod-case*)

lemma *splitE'*: *split c p z* ==> (!!*x y*. *p* = (*x*, *y*) ==> *c x y z* ==> *Q*) ==> *Q*
by (*induct p*) (*auto simp add*: *split-prod-case*)

lemma *splitE2*:
[| *Q* (*split P z*); !!*x y*. [|*z* = (*x*, *y*); *Q* (*P x y*)|] ==> *R* |] ==> *R*
proof −
assume *q*: *Q* (*split P z*)
assume *r*: !!*x y*. [|*z* = (*x*, *y*); *Q* (*P x y*)|] ==> *R*
show *R*
apply (*rule r surjective-pairing*)+
apply (*rule split-beta* [*THEN subst*], *rule q*)
done
qed

lemma *splitD'*: *split R* (*a*,*b*) *c* ==> *R a b c*
by *simp*

lemma *mem-splitI*: *z*: *c a b* ==> *z*: *split c* (*a*, *b*)
by *simp*

lemma *mem-splitI2*: !!*p*. [| !!*a b*. *p* = (*a*, *b*) ==> *z*: *c a b* |] ==> *z*: *split c p*
by (*simp only*: *split-tupled-all*, *simp*)

lemma *mem-splitE*:
assumes *major*: *z* ∈ *split c p*
and *cases*: ⋀*x y*. *p* = (*x*, *y*) ⟹ *z* ∈ *c x y* ⟹ *Q*
shows *Q*
by (*rule major* [*unfolded split-prod-case prod-case-unfold*] *cases surjective-pairing*)+

declare *mem-splitI2* [*intro*!] *mem-splitI* [*intro*!] *splitI2'* [*intro*!] *splitI2* [*intro*!] *splitI*
[*intro*!]
declare *mem-splitE* [*elim*!] *splitE'* [*elim*!] *splitE* [*elim*!]

ML ⟪
local (∗ *filtering with exists-p-split is an essential optimization* ∗)
fun exists-p-split (*Const* (@{*const-name split*},-) $ - $ (*Const* (@{*const-name*
Pair},-)$-$-)) = *true*
| *exists-p-split* (*t* $ *u*) = *exists-p-split t orelse exists-p-split u*
| *exists-p-split* (*Abs* (-, -, *t*)) = *exists-p-split t*
| *exists-p-split* - = *false*;
val ss = *HOL-basic-ss addsimps* @{*thms split-conv*};
in
val split-conv-tac = *SUBGOAL* (*fn* (*t*, *i*) =>

if exists-p-split t then safe-full-simp-tac ss i else no-tac);
end;
⟫

declaration ⟪ *fn - =>*
Classical.map-cs (*fn cs => cs addSbefore* (*split-conv-tac*, *split-conv-tac*))
⟫

lemma *split-eta-SetCompr* [*simp*,*no-atp*]: (%*u*. *EX x y*. *u* = (*x*, *y*) & *P* (*x*, *y*)) = *P*
by (*rule ext*) *fast*

lemma *split-eta-SetCompr2* [*simp*,*no-atp*]: (%*u*. *EX x y*. *u* = (*x*, *y*) & *P x y*) = *split P*
by (*rule ext*) *fast*

lemma *split-part* [*simp*]: (%(*a*,*b*). *P* & *Q a b*) = (%*ab*. *P* & *split Q ab*)
— Allows simplifications of nested splits in case of independent predicates.
by (*rule ext*) *blast*

lemma *split-comp-eq*:
fixes *f* :: *'a => 'b => 'c* **and** *g* :: *'d => 'a*
shows (%*u*. *f* (*g* (*fst u*)) (*snd u*)) = (*split* (%*x*. *f* (*g x*)))
by (*rule ext*) *auto*

lemma *pair-imageI* [*intro*]: (*a*, *b*) : *A ==> f a b* : (%(*a*, *b*). *f a b*) ' *A*
apply (*rule-tac x* = (*a*, *b*) **in** *image-eqI*)
apply *auto*
done

lemma *The-split-eq* [*simp*]: (*THE* (*x'*,*y'*). *x* = *x'* & *y* = *y'*) = (*x*, *y*)
by *blast*

Setup of internal *split-rule*.

lemmas *prod-caseI* = *prod.cases* [*THEN iffD2*, *standard*]

lemma *prod-caseI2*: !!*p*. [| !!*a b*. *p* = (*a*, *b*) ==> *c a b* |] ==> *prod-case c p*
by *auto*

lemma *prod-caseI2'*: !!*p*. [| !!*a b*. (*a*, *b*) = *p* ==> *c a b x* |] ==> *prod-case c p x*
by (*auto simp*: *split-tupled-all*)

lemma *prod-caseE*: *prod-case c p* ==> (!!*x y*. *p* = (*x*, *y*) ==> *c x y* ==> *Q*) ==> *Q*
by (*induct p*) *auto*

lemma *prod-caseE'*: *prod-case c p z* ==> (!!*x y*. *p* = (*x*, *y*) ==> *c x y z* ==>

Q) ==> *Q*
by (*induct p*) *auto*

declare *prod-caseI2'* [*intro*!] *prod-caseI2* [*intro*!] *prod-caseI* [*intro*!]
declare *prod-caseE'* [*elim*!] *prod-caseE* [*elim*!]

lemma *prod-case-split*:
prod-case = *split*
by (*auto simp add*: *expand-fun-eq*)

lemma *prod-case-beta*:
prod-case f p = *f* (*fst p*) (*snd p*)
unfolding *prod-case-split split-beta* **..**

lemma *prod-cases3* [*cases type*]:
obtains (*fields*) *a b c* **where** *y* = (*a*, *b*, *c*)
by (*cases y*, *case-tac b*) *blast*

lemma *prod-induct3* [*case-names fields*, *induct type*]:
(!!*a b c*. *P* (*a*, *b*, *c*)) ==> *P x*
by (*cases x*) *blast*

lemma *prod-cases4* [*cases type*]:
obtains (*fields*) *a b c d* **where** *y* = (*a*, *b*, *c*, *d*)
by (*cases y*, *case-tac c*) *blast*

lemma *prod-induct4* [*case-names fields*, *induct type*]:
(!!*a b c d*. *P* (*a*, *b*, *c*, *d*)) ==> *P x*
by (*cases x*) *blast*

lemma *prod-cases5* [*cases type*]:
obtains (*fields*) *a b c d e* **where** *y* = (*a*, *b*, *c*, *d*, *e*)
by (*cases y*, *case-tac d*) *blast*

lemma *prod-induct5* [*case-names fields*, *induct type*]:
(!!*a b c d e*. *P* (*a*, *b*, *c*, *d*, *e*)) ==> *P x*
by (*cases x*) *blast*

lemma *prod-cases6* [*cases type*]:
obtains (*fields*) *a b c d e f* **where** *y* = (*a*, *b*, *c*, *d*, *e*, *f*)
by (*cases y*, *case-tac e*) *blast*

lemma *prod-induct6* [*case-names fields*, *induct type*]:
(!!*a b c d e f*. *P* (*a*, *b*, *c*, *d*, *e*, *f*)) ==> *P x*
by (*cases x*) *blast*

lemma *prod-cases7* [*cases type*]:
obtains (*fields*) *a b c d e f g* **where** *y* = (*a*, *b*, *c*, *d*, *e*, *f*, *g*)
by (*cases y*, *case-tac f*) *blast*

lemma *prod-induct7* [*case-names fields*, *induct type*]:
(!!*a b c d e f g*. *P* (*a*, *b*, *c*, *d*, *e*, *f*, *g*)) ==> *P x*
by (*cases x*) *blast*

lemma *split-def*:
split = (λ*c p*. *c* (*fst p*) (*snd p*))
unfolding *split-prod-case prod-case-unfold* **..**

definition *internal-split* :: ('*a* ⇒ '*b* ⇒ '*c*) ⇒ '*a* × '*b* ⇒ '*c* **where**
internal-split == *split*

lemma *internal-split-conv*: *internal-split c* (*a*, *b*) = *c a b*
by (*simp only*: *internal-split-def split-conv*)

use *Tools/split-rule.ML*
setup *Split-Rule.setup*

hide-const *internal-split*

11.3.5 Derived operations

global consts
curry :: ('*a* × '*b* ⇒ '*c*) ⇒ '*a* ⇒ '*b* ⇒ '*c*

local defs
curry-def: *curry* == (%*c x y*. *c* (*Pair x y*))

lemma *curry-conv* [*simp*, *code*]: *curry f a b* = *f* (*a*, *b*)
by (*simp add*: *curry-def*)

lemma *curryI* [*intro*!]: *f* (*a*, *b*) ⟹ *curry f a b*
by (*simp add*: *curry-def*)

lemma *curryD* [*dest*!]: *curry f a b* ⟹ *f* (*a*, *b*)
by (*simp add*: *curry-def*)

lemma *curryE*: *curry f a b* ⟹ (*f* (*a*, *b*) ⟹ *Q*) ⟹ *Q*
by (*simp add*: *curry-def*)

lemma *curry-split* [*simp*]: *curry* (*split f*) = *f*
by (*simp add*: *curry-def split-def*)

lemma *split-curry* [*simp*]: *split* (*curry f*) = *f*
by (*simp add*: *curry-def split-def*)

The composition-uncurry combinator.

notation *fcomp* (**infixl** *o*> *60*)

definition *scomp* :: $('a \Rightarrow 'b \times 'c) \Rightarrow ('b \Rightarrow 'c \Rightarrow 'd) \Rightarrow 'a \Rightarrow 'd$ (**infixl** $o\!\rightarrow$ *60*)
where
$f\ o\!\rightarrow\ g = (\lambda x.\ split\ g\ (f\ x))$

lemma *scomp-apply*: $(f\ o\!\rightarrow\ g)\ x = split\ g\ (f\ x)$
by (*simp add*: *scomp-def*)

lemma *Pair-scomp*: $Pair\ x\ o\!\rightarrow\ f = f\ x$
by (*simp add*: *expand-fun-eq scomp-apply*)

lemma *scomp-Pair*: $x\ o\!\rightarrow\ Pair = x$
by (*simp add*: *expand-fun-eq scomp-apply*)

lemma *scomp-scomp*: $(f\ o\!\rightarrow\ g)\ o\!\rightarrow\ h = f\ o\!\rightarrow\ (\lambda x.\ g\ x\ o\!\rightarrow\ h)$
by (*simp add*: *expand-fun-eq split-twice scomp-def*)

lemma *scomp-fcomp*: $(f\ o\!\rightarrow\ g)\ o\!>\ h = f\ o\!\rightarrow\ (\lambda x.\ g\ x\ o\!>\ h)$
by (*simp add*: *expand-fun-eq scomp-apply fcomp-def split-def*)

lemma *fcomp-scomp*: $(f\ o\!>\ g)\ o\!\rightarrow\ h = f\ o\!>\ (g\ o\!\rightarrow\ h)$
by (*simp add*: *expand-fun-eq scomp-apply fcomp-apply*)

code-const *scomp*
(*Eval* **infixl** *3* #−>)

no-notation *fcomp* (**infixl** $o\!>$ *60*)
no-notation *scomp* (**infixl** $o\!\rightarrow$ *60*)

prod-fun — action of the product functor upon functions.

definition *prod-fun* :: $('a \Rightarrow 'c) \Rightarrow ('b \Rightarrow 'd) \Rightarrow 'a \times 'b \Rightarrow 'c \times 'd$ **where**
[*code del*]: $prod\text{-}fun\ f\ g = (\lambda(x,\ y).\ (f\ x,\ g\ y))$

lemma *prod-fun* [*simp*, *code*]: $prod\text{-}fun\ f\ g\ (a,\ b) = (f\ a,\ g\ b)$
by (*simp add*: *prod-fun-def*)

lemma *fst-prod-fun*[*simp*]: $fst\ (prod\text{-}fun\ f\ g\ x) = f\ (fst\ x)$
by (*cases x*, *auto*)

lemma *snd-prod-fun*[*simp*]: $snd\ (prod\text{-}fun\ f\ g\ x) = g\ (snd\ x)$
by (*cases x*, *auto*)

lemma *fst-comp-prod-fun*[*simp*]: $fst \circ prod\text{-}fun\ f\ g = f \circ fst$
by (*rule ext*) *auto*

lemma *snd-comp-prod-fun*[*simp*]: $snd \circ prod\text{-}fun\ f\ g = g \circ snd$
by (*rule ext*) *auto*

lemma *prod-fun-compose*:

prod-fun (*f1 o f2*) (*g1 o g2*) = (*prod-fun f1 g1 o prod-fun f2 g2*)
by (*rule ext*) *auto*

lemma *prod-fun-ident* [*simp*]: *prod-fun* (%*x*. *x*) (%*y*. *y*) = (%*z*. *z*)
by (*rule ext*) *auto*

lemma *prod-fun-imageI* [*intro*]: (*a*, *b*) : *r* ==> (*f a*, *g b*) : *prod-fun f g* ‘ *r*
apply (*rule image-eqI*)
apply (*rule prod-fun* [*symmetric*], *assumption*)
done

lemma *prod-fun-imageE* [*elim*!]:
assumes *major*: *c*: (*prod-fun f g*)‘*r*
and *cases*: !!*x y*. [| *c*=(*f*(*x*),*g*(*y*)); (*x*,*y*):*r* |] ==> *P*
shows *P*
apply (*rule major* [*THEN imageE*])
apply (*case-tac x*)
apply (*rule cases*)
apply (*blast intro*: *prod-fun*)
apply *blast*
done

definition *apfst* :: (′*a* ⇒ ′*c*) ⇒ ′*a* × ′*b* ⇒ ′*c* × ′*b* **where**
apfst f = *prod-fun f id*

definition *apsnd* :: (′*b* ⇒ ′*c*) ⇒ ′*a* × ′*b* ⇒ ′*a* × ′*c* **where**
apsnd f = *prod-fun id f*

lemma *apfst-conv* [*simp*, *code*]:
apfst f (*x*, *y*) = (*f x*, *y*)
by (*simp add*: *apfst-def*)

lemma *apsnd-conv* [*simp*, *code*]:
apsnd f (*x*, *y*) = (*x*, *f y*)
by (*simp add*: *apsnd-def*)

lemma *fst-apfst* [*simp*]:
fst (*apfst f x*) = *f* (*fst x*)
by (*cases x*) *simp*

lemma *fst-apsnd* [*simp*]:
fst (*apsnd f x*) = *fst x*
by (*cases x*) *simp*

lemma *snd-apfst* [*simp*]:
snd (*apfst f x*) = *snd x*
by (*cases x*) *simp*

lemma *snd-apsnd* [*simp*]:
snd (*apsnd f x*) = *f* (*snd x*)
by (*cases x*) *simp*

lemma *apfst-compose*:
apfst f (*apfst g x*) = *apfst* (*f* ∘ *g*) *x*
by (*cases x*) *simp*

lemma *apsnd-compose*:
apsnd f (*apsnd g x*) = *apsnd* (*f* ∘ *g*) *x*
by (*cases x*) *simp*

lemma *apfst-apsnd* [*simp*]:
apfst f (*apsnd g x*) = (*f* (*fst x*), *g* (*snd x*))
by (*cases x*) *simp*

lemma *apsnd-apfst* [*simp*]:
apsnd f (*apfst g x*) = (*g* (*fst x*), *f* (*snd x*))
by (*cases x*) *simp*

lemma *apfst-id* [*simp*] :
apfst id = *id*
by (*simp add*: *expand-fun-eq*)

lemma *apsnd-id* [*simp*] :
apsnd id = *id*
by (*simp add*: *expand-fun-eq*)

lemma *apfst-eq-conv* [*simp*]:
apfst f x = *apfst g x* ⟷ *f* (*fst x*) = *g* (*fst x*)
by (*cases x*) *simp*

lemma *apsnd-eq-conv* [*simp*]:
apsnd f x = *apsnd g x* ⟷ *f* (*snd x*) = *g* (*snd x*)
by (*cases x*) *simp*

lemma *apsnd-apfst-commute*:
apsnd f (*apfst g p*) = *apfst g* (*apsnd f p*)
by *simp*

Disjoint union of a family of sets – Sigma.

definition *Sigma* :: [*'a set*, *'a* => *'b set*] => (*'a* × *'b*) *set* **where**
Sigma-def: *Sigma A B* == *UN x*:*A*. *UN y*:*B x*. {*Pair x y*}

abbreviation
Times :: [*'a set*, *'b set*] => (*'a* ∗ *'b*) *set*
(**infixr** <∗> *80*) **where**
A <∗> *B* == *Sigma A* (%-. *B*)

notation (*xsymbols*)
Times (**infixr** × *80*)

notation (*HTML* **output**)
Times (**infixr** × *80*)

syntax
-Sigma :: [*pttrn*, *'a set*, *'b set*] => (*'a* ∗ *'b*) *set* ((*3SIGMA -:-./ -*) [*0*, *0*, *10*] *10*)
translations
SIGMA x:*A*. *B* == *CONST Sigma A* (%*x*. *B*)

lemma *SigmaI* [*intro*!]: [| *a*:*A*; *b*:*B*(*a*) |] ==> (*a*,*b*) : *Sigma A B*
by (*unfold Sigma-def*) *blast*

lemma *SigmaE* [*elim*!]:
[| *c*: *Sigma A B*;
!!*x y*.[| *x*:*A*; *y*:*B*(*x*); *c*=(*x*,*y*) |] ==> *P*
|] ==> *P*
— The general elimination rule.
by (*unfold Sigma-def*) *blast*

Elimination of (*a*, *b*) ∈ *A* × *B* – introduces no eigenvariables.

lemma *SigmaD1*: (*a*, *b*) : *Sigma A B* ==> *a* : *A*
by *blast*

lemma *SigmaD2*: (*a*, *b*) : *Sigma A B* ==> *b* : *B a*
by *blast*

lemma *SigmaE2*:
[| (*a*, *b*) : *Sigma A B*;
[| *a*:*A*; *b*:*B*(*a*) |] ==> *P*
|] ==> *P*
by *blast*

lemma *Sigma-cong*:
⟦*A* = *B*; !!*x*. *x* ∈ *B* ⟹ *C x* = *D x*⟧
⟹ (*SIGMA x*: *A*. *C x*) = (*SIGMA x*: *B*. *D x*)
by *auto*

lemma *Sigma-mono*: [| *A* <= *C*; !!*x*. *x*:*A* ==> *B x* <= *D x* |] ==> *Sigma A B* <= *Sigma C D*
by *blast*

lemma *Sigma-empty1* [*simp*]: *Sigma* {} *B* = {}
by *blast*

lemma *Sigma-empty2* [*simp*]: *A* <*> {} = {}
by *blast*

lemma *UNIV-Times-UNIV* [*simp*]: $UNIV <*> UNIV = UNIV$
by *auto*

lemma *Compl-Times-UNIV1* [*simp*]: $- (UNIV <*> A) = UNIV <*> (-A)$
by *auto*

lemma *Compl-Times-UNIV2* [*simp*]: $- (A <*> UNIV) = (-A) <*> UNIV$
by *auto*

lemma *mem-Sigma-iff* [*iff*]: $((a,b): Sigma\ A\ B) = (a{:}A\ \&\ b{:}B(a))$
by *blast*

lemma *Times-subset-cancel2*: $x{:}C ==> (A <*> C <= B <*> C) = (A <= B)$
by *blast*

lemma *Times-eq-cancel2*: $x{:}C ==> (A <*> C = B <*> C) = (A = B)$
by (*blast elim*: *equalityE*)

lemma *SetCompr-Sigma-eq*:
$Collect\ (split\ (\%x\ y.\ P\ x\ \&\ Q\ x\ y)) = (SIGMA\ x{:}Collect\ P.\ Collect\ (Q\ x))$
by *blast*

lemma *Collect-split* [*simp*]: $\{(a,b).\ P\ a\ \&\ Q\ b\} = Collect\ P <*> Collect\ Q$
by *blast*

lemma *UN-Times-distrib*:
$(UN\ (a,b){:}(A <*> B).\ E\ a <*> F\ b) = (UNION\ A\ E) <*> (UNION\ B\ F)$
— Suggested by Pierre Chartier
by *blast*

lemma *split-paired-Ball-Sigma* [*simp*,*no-atp*]:
$(ALL\ z{:}\ Sigma\ A\ B.\ P\ z) = (ALL\ x{:}A.\ ALL\ y{:}\ B\ x.\ P(x,y))$
by *blast*

lemma *split-paired-Bex-Sigma* [*simp*,*no-atp*]:
$(EX\ z{:}\ Sigma\ A\ B.\ P\ z) = (EX\ x{:}A.\ EX\ y{:}\ B\ x.\ P(x,y))$
by *blast*

lemma *Sigma-Un-distrib1*: $(SIGMA\ i{:}I\ Un\ J.\ C(i)) = (SIGMA\ i{:}I.\ C(i))\ Un\ (SIGMA\ j{:}J.\ C(j))$
by *blast*

lemma *Sigma-Un-distrib2*: $(SIGMA\ i{:}I.\ A(i)\ Un\ B(i)) = (SIGMA\ i{:}I.\ A(i))\ Un\ (SIGMA\ i{:}I.\ B(i))$
by *blast*

lemma *Sigma-Int-distrib1*: $(SIGMA\ i{:}I\ Int\ J.\ C(i)) = (SIGMA\ i{:}I.\ C(i))\ Int\ (SIGMA\ j{:}J.\ C(j))$
by *blast*

lemma *Sigma-Int-distrib2*: $(SIGMA\ i{:}I.\ A(i)\ Int\ B(i)) = (SIGMA\ i{:}I.\ A(i))\ Int\ (SIGMA\ i{:}I.\ B(i))$
by *blast*

lemma *Sigma-Diff-distrib1*: $(SIGMA\ i{:}I - J.\ C(i)) = (SIGMA\ i{:}I.\ C(i)) - (SIGMA\ j{:}J.\ C(j))$
by *blast*

lemma *Sigma-Diff-distrib2*: $(SIGMA\ i{:}I.\ A(i) - B(i)) = (SIGMA\ i{:}I.\ A(i)) - (SIGMA\ i{:}I.\ B(i))$
by *blast*

lemma *Sigma-Union*: $Sigma\ (Union\ X)\ B = (UN\ A{:}X.\ Sigma\ A\ B)$
by *blast*

Non-dependent versions are needed to avoid the need for higher-order matching, especially when the rules are re-oriented.

lemma *Times-Un-distrib1*: $(A\ Un\ B) <*> C = (A <*> C)\ Un\ (B <*> C)$
by *blast*

lemma *Times-Int-distrib1*: $(A\ Int\ B) <*> C = (A <*> C)\ Int\ (B <*> C)$
by *blast*

lemma *Times-Diff-distrib1*: $(A - B) <*> C = (A <*> C) - (B <*> C)$
by *blast*

lemma *Times-empty*[*simp*]: $A \times B = \{\} \longleftrightarrow A = \{\} \vee B = \{\}$
by *auto*

lemma *fst-image-times*[*simp*]: $fst\ `\ (A \times B) = (if\ B = \{\}\ then\ \{\}\ else\ A)$
by (*auto intro*!: *image-eqI*)

lemma *snd-image-times*[*simp*]: $snd\ `\ (A \times B) = (if\ A = \{\}\ then\ \{\}\ else\ B)$
by (*auto intro*!: *image-eqI*)

lemma *insert-times-insert*[*simp*]:
$insert\ a\ A \times insert\ b\ B =$
$insert\ (a,b)\ (A \times insert\ b\ B \cup insert\ a\ A \times B)$
by *blast*

lemma *vimage-Times*: $f\ -`\ (A \times B) = ((fst \circ f)\ -`\ A) \cap ((snd \circ f)\ -`\ B)$
by (*auto*, *case-tac f x*, *auto*)

The following *prod-fun* lemmas are due to Joachim Breitner:

lemma *prod-fun-inj-on*:
assumes *inj-on f A* **and** *inj-on g B*
shows *inj-on* (*prod-fun f g*) $(A \times B)$
proof (*rule inj-onI*)

fix $x :: 'a \times 'c$ **and** $y :: 'a \times 'c$
assume $x \in A \times B$ **hence** *fst* $x \in A$ **and** *snd* $x \in B$ **by** *auto*
assume $y \in A \times B$ **hence** *fst* $y \in A$ **and** *snd* $y \in B$ **by** *auto*
assume *prod-fun f g x* = *prod-fun f g y*
hence *fst* (*prod-fun f g x*) = *fst* (*prod-fun f g y*) **by** (*auto*)
hence *f* (*fst x*) = *f* (*fst y*) **by** (*cases x*,*cases y*,*auto*)
with ‹*inj-on f A*› **and** ‹*fst x* ∈ *A*› **and** ‹*fst y* ∈ *A*›
have *fst x* = *fst y* **by** (*auto dest*:*dest*:*inj-onD*)
moreover from ‹*prod-fun f g x* = *prod-fun f g y*›
have *snd* (*prod-fun f g x*) = *snd* (*prod-fun f g y*) **by** (*auto*)
hence *g* (*snd x*) = *g* (*snd y*) **by** (*cases x*,*cases y*,*auto*)
with ‹*inj-on g B*› **and** ‹*snd x* ∈ *B*› **and** ‹*snd y* ∈ *B*›
have *snd x* = *snd y* **by** (*auto dest*:*dest*:*inj-onD*)
ultimately show *x* = *y* **by**(*rule prod-eqI*)
qed

lemma *prod-fun-surj*:
assumes *surj f* **and** *surj g*
shows *surj* (*prod-fun f g*)
unfolding *surj-def*
proof
fix $y :: 'b \times 'd$
from ‹*surj f*› **obtain** *a* **where** *fst y* = *f a* **by** (*auto elim*:*surjE*)
moreover
from ‹*surj g*› **obtain** *b* **where** *snd y* = *g b* **by** (*auto elim*:*surjE*)
ultimately have (*fst y*, *snd y*) = *prod-fun f g* (*a*,*b*) **by** *auto*
thus ∃ *x*. *y* = *prod-fun f g x* **by** *auto*
qed

lemma *prod-fun-surj-on*:
assumes *f* ‘ *A* = *A'* **and** *g* ‘ *B* = *B'*
shows *prod-fun f g* ‘ (*A* × *B*) = *A'* × *B'*
unfolding *image-def*
proof(*rule set-ext*,*rule iffI*)
fix $x :: 'a \times 'c$
assume $x \in \{y::'a \times 'c.\ \exists x::'b \times 'd \in A \times B.\ y = prod\text{-}fun\ f\ g\ x\}$
then obtain *y* **where** *y* ∈ *A* × *B* **and** *x* = *prod-fun f g y* **by** *blast*
from ‹*image f A* = *A'*› **and** ‹*y* ∈ *A* × *B*› **have** *f* (*fst y*) ∈ *A'* **by** *auto*
moreover from ‹*image g B* = *B'*› **and** ‹*y* ∈ *A* × *B*› **have** *g* (*snd y*) ∈ *B'* **by** *auto*
ultimately have (*f* (*fst y*), *g* (*snd y*)) ∈ (*A'* × *B'*) **by** *auto*
with ‹*x* = *prod-fun f g y*› **show** *x* ∈ *A'* × *B'* **by** (*cases y*, *auto*)
next
fix $x :: 'a \times 'c$
assume *x* ∈ *A'* × *B'* **hence** *fst x* ∈ *A'* **and** *snd x* ∈ *B'* **by** *auto*
from ‹*image f A* = *A'*› **and** ‹*fst x* ∈ *A'*› **have** *fst x* ∈ *image f A* **by** *auto*
then obtain *a* **where** *a* ∈ *A* **and** *fst x* = *f a* **by** (*rule imageE*)
moreover from ‹*image g B* = *B'*› **and** ‹*snd x* ∈ *B'*›
obtain *b* **where** *b* ∈ *B* **and** *snd x* = *g b* **by** *auto*

ultimately have (*fst x*, *snd x*) = *prod-fun f g* (*a*,*b*) **by** *auto*
moreover from ‹*a* ∈ *A*› **and** ‹*b* ∈ *B*› **have** (*a* , *b*) ∈ *A* × *B* **by** *auto*
ultimately have ∃ *y* ∈ *A* × *B*. *x* = *prod-fun f g y* **by** *auto*
thus *x* ∈ {*x*. ∃ *y* ∈ *A* × *B*. *x* = *prod-fun f g y*} **by** *auto*
qed

lemma *swap-inj-on*:
inj-on (λ(*i*, *j*). (*j*, *i*)) *A*
by (*auto intro*!: *inj-onI*)

lemma *swap-product*:
(%(*i*, *j*). (*j*, *i*)) ‘ (*A* × *B*) = *B* × *A*
by (*simp add*: *split-def image-def*) *blast*

lemma *image-split-eq-Sigma*:
(λ*x*. (*f x*, *g x*)) ‘ *A* = *Sigma* (*f* ‘ *A*) (λ*x*. *g* ‘ (*f* −‘ {*x*} ∩ *A*))
proof (*safe intro*!: *imageI vimageI*)
fix *a b* **assume** ∗: *a* ∈ *A b* ∈ *A* **and** *eq*: *f a* = *f b*
show (*f b*, *g a*) ∈ (λ*x*. (*f x*, *g x*)) ‘ *A*
using ∗ *eq*[*symmetric*] **by** *auto*
qed *simp-all*

11.4 Inductively defined sets

use *Tools/inductive-set.ML*
setup *Inductive-Set.setup*

11.5 Legacy theorem bindings and duplicates

lemma *PairE*:
obtains *x y* **where** *p* = (*x*, *y*)
by (*fact prod.exhaust*)

lemma *Pair-inject*:
assumes (*a*, *b*) = (*a*′, *b*′)
and *a* = *a*′ ==> *b* = *b*′ ==> *R*
shows *R*
using *assms* **by** *simp*

lemmas *Pair-eq* = *prod.inject*

lemmas *split* = *split-conv* — for backwards compatibility

end

12 Sum-Type: The Disjoint Sum of Two Types

theory *Sum-Type*

imports *Typedef Inductive Fun*
begin

12.1 Construction of the sum type and its basic abstract operations

definition *Inl-Rep* :: $'a \Rightarrow 'a \Rightarrow 'b \Rightarrow bool \Rightarrow bool$ **where**
Inl-Rep a x y p $\longleftrightarrow x = a \wedge p$

definition *Inr-Rep* :: $'b \Rightarrow 'a \Rightarrow 'b \Rightarrow bool \Rightarrow bool$ **where**
Inr-Rep b x y p $\longleftrightarrow y = b \wedge \neg p$

global

typedef (*Sum*) $('a, 'b)$ + (**infixr** + *10*) = $\{f.\ (\exists\, a.\ f = \textit{Inl-Rep}\ (a::'a)) \vee (\exists\, b.\ f = \textit{Inr-Rep}\ (b::'b))\}$
by *auto*

local

lemma *Inl-RepI*: *Inl-Rep a* $\in$ *Sum*
by (*auto simp add*: *Sum-def*)

lemma *Inr-RepI*: *Inr-Rep b* $\in$ *Sum*
by (*auto simp add*: *Sum-def*)

lemma *inj-on-Abs-Sum*: *A* $\subseteq$ *Sum* $\Longrightarrow$ *inj-on Abs-Sum A*
by (*rule inj-on-inverseI*, *rule Abs-Sum-inverse*) *auto*

lemma *Inl-Rep-inject*: *inj-on Inl-Rep A*
proof (*rule inj-onI*)
show $\bigwedge a\ c.$ *Inl-Rep a* = *Inl-Rep c* $\Longrightarrow$ *a* = *c*
by (*auto simp add*: *Inl-Rep-def expand-fun-eq*)
qed

lemma *Inr-Rep-inject*: *inj-on Inr-Rep A*
proof (*rule inj-onI*)
show $\bigwedge b\ d.$ *Inr-Rep b* = *Inr-Rep d* $\Longrightarrow$ *b* = *d*
by (*auto simp add*: *Inr-Rep-def expand-fun-eq*)
qed

lemma *Inl-Rep-not-Inr-Rep*: *Inl-Rep a* $\neq$ *Inr-Rep b*
by (*auto simp add*: *Inl-Rep-def Inr-Rep-def expand-fun-eq*)

definition *Inl* :: $'a \Rightarrow 'a + 'b$ **where**
Inl = *Abs-Sum* $\circ$ *Inl-Rep*

definition *Inr* :: $'b \Rightarrow 'a + 'b$ **where**
Inr = *Abs-Sum* $\circ$ *Inr-Rep*

lemma *inj-Inl* [*simp*]: *inj-on Inl A*
by (*auto simp add*: *Inl-def intro*!: *comp-inj-on Inl-Rep-inject inj-on-Abs-Sum Inl-RepI*)

lemma *Inl-inject*: *Inl x = Inl y ⟹ x = y*
using *inj-Inl* **by** (*rule injD*)

lemma *inj-Inr* [*simp*]: *inj-on Inr A*
by (*auto simp add*: *Inr-def intro*!: *comp-inj-on Inr-Rep-inject inj-on-Abs-Sum Inr-RepI*)

lemma *Inr-inject*: *Inr x = Inr y ⟹ x = y*
using *inj-Inr* **by** (*rule injD*)

lemma *Inl-not-Inr*: *Inl a ≠ Inr b*
proof −
  from *Inl-RepI* [*of a*] *Inr-RepI* [*of b*] **have** {*Inl-Rep a*, *Inr-Rep b*} ⊆ *Sum* **by** *auto*
  with *inj-on-Abs-Sum* **have** *inj-on Abs-Sum* {*Inl-Rep a*, *Inr-Rep b*} **.**
  with *Inl-Rep-not-Inr-Rep* [*of a b*] *inj-on-contraD* **have** *Abs-Sum* (*Inl-Rep a*) ≠ *Abs-Sum* (*Inr-Rep b*) **by** *auto*
  then show *?thesis* **by** (*simp add*: *Inl-def Inr-def*)
qed

lemma *Inr-not-Inl*: *Inr b ≠ Inl a*
  using *Inl-not-Inr* **by** (*rule not-sym*)

lemma *sumE*:
  assumes ⋀*x*::′*a*. *s = Inl x ⟹ P*
    and ⋀*y*::′*b*. *s = Inr y ⟹ P*
  shows *P*
proof (*rule Abs-Sum-cases* [*of s*])
  fix *f*
  assume *s = Abs-Sum f* **and** *f ∈ Sum*
  with *assms* **show** *P* **by** (*auto simp add*: *Sum-def Inl-def Inr-def*)
qed

rep-datatype (*sum*) *Inl Inr*
proof −
  fix *P*
  fix *s* :: ′*a* + ′*b*
  assume *x*: ⋀*x*::′*a*. *P* (*Inl x*) **and** *y*: ⋀*y*::′*b*. *P* (*Inr y*)
  then show *P s* **by** (*auto intro*: *sumE* [*of s*])
qed (*auto dest*: *Inl-inject Inr-inject simp add*: *Inl-not-Inr*)

12.2 Projections

lemma *sum-case-KK* [*simp*]: *sum-case* (λ*x*. *a*) (λ*x*. *a*) = (λ*x*. *a*)
  by (*rule ext*) (*simp split*: *sum.split*)

lemma *surjective-sum*: *sum-case* ($\lambda x{::}'a.\ f\ (Inl\ x)$) ($\lambda y{::}'b.\ f\ (Inr\ y)$) = *f*
proof
  fix *s* :: $'a + 'b$
  show (*case s of Inl* ($x{::}'a$) $\Rightarrow$ *f* (*Inl x*) | *Inr* ($y{::}'b$) $\Rightarrow$ *f* (*Inr y*)) = *f s*
    by (*cases s*) *simp-all*
qed

lemma *sum-case-inject*:
  assumes *a*: *sum-case f1 f2* = *sum-case g1 g2*
  assumes *r*: *f1* = *g1* $\Longrightarrow$ *f2* = *g2* $\Longrightarrow$ *P*
  shows *P*
proof (*rule r*)
  show *f1* = *g1* **proof**
    fix *x* :: $'a$
    from *a* **have** *sum-case f1 f2* (*Inl x*) = *sum-case g1 g2* (*Inl x*) **by** *simp*
    then show *f1 x* = *g1 x* **by** *simp*
  qed
  show *f2* = *g2* **proof**
    fix *y* :: $'b$
    from *a* **have** *sum-case f1 f2* (*Inr y*) = *sum-case g1 g2* (*Inr y*) **by** *simp*
    then show *f2 y* = *g2 y* **by** *simp*
  qed
qed

lemma *sum-case-weak-cong*:
  s = *t* $\Longrightarrow$ *sum-case f g s* = *sum-case f g t*
  — Prevents simplification of *f* and *g*: much faster.
  by *simp*

primrec *Projl* :: $'a + 'b \Rightarrow 'a$ **where**
  Projl-Inl: *Projl* (*Inl x*) = *x*

primrec *Projr* :: $'a + 'b \Rightarrow 'b$ **where**
  Projr-Inr: *Projr* (*Inr x*) = *x*

primrec *Suml* :: $('a \Rightarrow 'c) \Rightarrow 'a + 'b \Rightarrow 'c$ **where**
  Suml f (*Inl x*) = *f x*

primrec *Sumr* :: $('b \Rightarrow 'c) \Rightarrow 'a + 'b \Rightarrow 'c$ **where**
  Sumr f (*Inr x*) = *f x*

lemma *Suml-inject*:
  assumes *Suml f* = *Suml g* **shows** *f* = *g*
proof
  fix *x* :: $'a$
  let *?s* = *Inl x* :: $'a + 'b$
  from *assms* **have** *Suml f ?s* = *Suml g ?s* **by** *simp*
  then show *f x* = *g x* **by** *simp*
qed

lemma *Sumr-inject*:
assumes *Sumr f* = *Sumr g* **shows** *f* = *g*
proof
fix *x* :: *'b*
let *?s* = *Inr x* :: *'a* + *'b*
from *assms* **have** *Sumr f ?s* = *Sumr g ?s* **by** *simp*
then show *f x* = *g x* **by** *simp*
qed

12.3 The Disjoint Sum of Sets

definition *Plus* :: *'a set* ⇒ *'b set* ⇒ (*'a* + *'b*) *set* (**infixr** <+> *65*) **where**
A <+> *B* = *Inl* ' *A* ∪ *Inr* ' *B*

lemma *InlI* [*intro*!]: *a* ∈ *A* ⟹ *Inl a* ∈ *A* <+> *B*
by (*simp add*: *Plus-def*)

lemma *InrI* [*intro*!]: *b* ∈ *B* ⟹ *Inr b* ∈ *A* <+> *B*
by (*simp add*: *Plus-def*)

Exhaustion rule for sums, a degenerate form of induction

lemma *PlusE* [*elim*!]:
u ∈ *A* <+> *B* ⟹ (⋀*x*. *x* ∈ *A* ⟹ *u* = *Inl x* ⟹ *P*) ⟹ (⋀*y*. *y* ∈ *B* ⟹ *u* = *Inr y* ⟹ *P*) ⟹ *P*
by (*auto simp add*: *Plus-def*)

lemma *Plus-eq-empty-conv* [*simp*]: *A* <+> *B* = {} ⟷ *A* = {} ∧ *B* = {}
by *auto*

lemma *UNIV-Plus-UNIV* [*simp*]: *UNIV* <+> *UNIV* = *UNIV*
proof (*rule set-ext*)
fix *u* :: *'a* + *'b*
show *u* ∈ *UNIV* <+> *UNIV* ⟷ *u* ∈ *UNIV* **by** (*cases u*) *auto*
qed

hide-const (**open**) *Suml Sumr Projl Projr*

end

13 Rings: Rings

theory *Rings*
imports *Groups*
begin

class *semiring* = *ab-semigroup-add* + *semigroup-mult* +
assumes *left-distrib*[*algebra-simps*, *field-simps*]: (*a* + *b*) ∗ *c* = *a* ∗ *c* + *b* ∗ *c*

assumes *right-distrib*[*algebra-simps*, *field-simps*]: $a * (b + c) = a * b + a * c$
begin

For the *combine-numerals* simproc

lemma *combine-common-factor*:
$a * e + (b * e + c) = (a + b) * e + c$
by (*simp add*: *left-distrib add-ac*)

end

class *mult-zero* = *times* + *zero* +
assumes *mult-zero-left* [*simp*]: $0 * a = 0$
assumes *mult-zero-right* [*simp*]: $a * 0 = 0$

class *semiring-0* = *semiring* + *comm-monoid-add* + *mult-zero*

class *semiring-0-cancel* = *semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0*
proof
fix $a :: 'a$
have $0 * a + 0 * a = 0 * a + 0$ **by** (*simp add*: *left-distrib* [*symmetric*])
thus $0 * a = 0$ **by** (*simp only*: *add-left-cancel*)
next
fix $a :: 'a$
have $a * 0 + a * 0 = a * 0 + 0$ **by** (*simp add*: *right-distrib* [*symmetric*])
thus $a * 0 = 0$ **by** (*simp only*: *add-left-cancel*)
qed

end

class *comm-semiring* = *ab-semigroup-add* + *ab-semigroup-mult* +
assumes *distrib*: $(a + b) * c = a * c + b * c$
begin

subclass *semiring*
proof
fix $a\ b\ c :: 'a$
show $(a + b) * c = a * c + b * c$ **by** (*simp add*: *distrib*)
have $a * (b + c) = (b + c) * a$ **by** (*simp add*: *mult-ac*)
also have $... = b * a + c * a$ **by** (*simp only*: *distrib*)
also have $... = a * b + a * c$ **by** (*simp add*: *mult-ac*)
finally show $a * (b + c) = a * b + a * c$ **by** *blast*
qed

end

class *comm-semiring-0* = *comm-semiring* + *comm-monoid-add* + *mult-zero*

begin

subclass *semiring-0* **..**

end

class *comm-semiring-0-cancel* = *comm-semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0-cancel* **..**

subclass *comm-semiring-0* **..**

end

class *zero-neq-one* = *zero* + *one* +
  assumes *zero-neq-one* [*simp*]: $0 \neq 1$
begin

lemma *one-neq-zero* [*simp*]: $1 \neq 0$
by (*rule not-sym*) (*rule zero-neq-one*)

end

class *semiring-1* = *zero-neq-one* + *semiring-0* + *monoid-mult*

Abstract divisibility

class *dvd* = *times*
begin

definition *dvd* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**infixl** *dvd* 50) **where**
  [*code del*]: b *dvd* $a \longleftrightarrow (\exists k.\ a = b * k)$

lemma *dvdI* [*intro?*]: $a = b * k \Longrightarrow b$ *dvd* a
  unfolding *dvd-def* **..**

lemma *dvdE* [*elim?*]: b *dvd* $a \Longrightarrow (\bigwedge k.\ a = b * k \Longrightarrow P) \Longrightarrow P$
  unfolding *dvd-def* **by** *blast*

end

class *comm-semiring-1* = *zero-neq-one* + *comm-semiring-0* + *comm-monoid-mult* + *dvd*

begin

subclass *semiring-1* **..**

lemma *dvd-refl*[*simp*]: a *dvd* a

proof
  show $a = a * 1$ **by** *simp*
qed

lemma *dvd-trans*:
  assumes *a dvd b* **and** *b dvd c*
  shows *a dvd c*
proof −
  from *assms* **obtain** *v* **where** $b = a * v$ **by** (*auto elim*!: *dvdE*)
  moreover from *assms* **obtain** *w* **where** $c = b * w$ **by** (*auto elim*!: *dvdE*)
  ultimately have $c = a * (v * w)$ **by** (*simp add*: *mult-assoc*)
  then show *?thesis* **..**
qed

lemma *dvd-0-left-iff* [*no-atp*, *simp*]: *0 dvd a* ⟷ $a = 0$
by (*auto intro*: *dvd-refl elim*!: *dvdE*)

lemma *dvd-0-right* [*iff*]: *a dvd 0*
proof
  show $0 = a * 0$ **by** *simp*
qed

lemma *one-dvd* [*simp*]: *1 dvd a*
by (*auto intro*!: *dvdI*)

lemma *dvd-mult*[*simp*]: *a dvd c* ⟹ *a dvd* $(b * c)$
by (*auto intro*!: *mult-left-commute dvdI elim*!: *dvdE*)

lemma *dvd-mult2*[*simp*]: *a dvd b* ⟹ *a dvd* $(b * c)$
  apply (*subst mult-commute*)
  apply (*erule dvd-mult*)
  done

lemma *dvd-triv-right* [*simp*]: *a dvd* $b * a$
by (*rule dvd-mult*) (*rule dvd-refl*)

lemma *dvd-triv-left* [*simp*]: *a dvd* $a * b$
by (*rule dvd-mult2*) (*rule dvd-refl*)

lemma *mult-dvd-mono*:
  assumes *a dvd b*
    and *c dvd d*
  shows $a * c$ *dvd* $b * d$
proof −
  from ‹*a dvd b*› **obtain** b' **where** $b = a * b'$ **..**
  moreover from ‹*c dvd d*› **obtain** d' **where** $d = c * d'$ **..**
  ultimately have $b * d = (a * c) * (b' * d')$ **by** (*simp add*: *mult-ac*)
  then show *?thesis* **..**
qed

lemma *dvd-mult-left*: $a * b$ *dvd* $c \Longrightarrow a$ *dvd* c
by (*simp add*: *dvd-def mult-assoc*, *blast*)

lemma *dvd-mult-right*: $a * b$ *dvd* $c \Longrightarrow b$ *dvd* c
unfolding *mult-ac* [*of* a] **by** (*rule dvd-mult-left*)

lemma *dvd-0-left*: *0 dvd* $a \Longrightarrow a = 0$
by *simp*

lemma *dvd-add*[*simp*]:
assumes a *dvd* b **and** a *dvd* c **shows** a *dvd* $(b + c)$
proof −
from ‹a *dvd* b› **obtain** b' **where** $b = a * b'$ **..**
moreover from ‹a *dvd* c› **obtain** c' **where** $c = a * c'$ **..**
ultimately have $b + c = a * (b' + c')$ **by** (*simp add*: *right-distrib*)
then show *?thesis* **..**
qed

end

class *no-zero-divisors* = *zero* + *times* +
assumes *no-zero-divisors*: $a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow a * b \neq 0$
begin

lemma *divisors-zero*:
assumes $a * b = 0$
shows $a = 0 \vee b = 0$
proof (*rule classical*)
assume $\neg (a = 0 \vee b = 0)$
then have $a \neq 0$ **and** $b \neq 0$ **by** *auto*
with *no-zero-divisors* **have** $a * b \neq 0$ **by** *blast*
with *assms* **show** *?thesis* **by** *simp*
qed

end

class *semiring-1-cancel* = *semiring* + *cancel-comm-monoid-add*
+ *zero-neq-one* + *monoid-mult*
begin

subclass *semiring-0-cancel* **..**

subclass *semiring-1* **..**

end

class *comm-semiring-1-cancel* = *comm-semiring* + *cancel-comm-monoid-add*
+ *zero-neq-one* + *comm-monoid-mult*

begin

subclass *semiring-1-cancel* **..**
subclass *comm-semiring-0-cancel* **..**
subclass *comm-semiring-1* **..**

end

class *ring* = *semiring* + *ab-group-add*
begin

subclass *semiring-0-cancel* **..**

Distribution rules

lemma *minus-mult-left*: $-(a * b) = -a * b$
by (*rule minus-unique*) (*simp add*: *left-distrib* [*symmetric*])

lemma *minus-mult-right*: $-(a * b) = a * -b$
by (*rule minus-unique*) (*simp add*: *right-distrib* [*symmetric*])

Extract signs from products

lemmas *mult-minus-left* [*simp*, *no-atp*] = *minus-mult-left* [*symmetric*]
lemmas *mult-minus-right* [*simp*,*no-atp*] = *minus-mult-right* [*symmetric*]

lemma *minus-mult-minus* [*simp*]: $-a * -b = a * b$
by *simp*

lemma *minus-mult-commute*: $-a * b = a * -b$
by *simp*

lemma *right-diff-distrib*[*algebra-simps*, *field-simps*]: $a * (b - c) = a * b - a * c$
by (*simp add*: *right-distrib diff-minus*)

lemma *left-diff-distrib*[*algebra-simps*, *field-simps*]: $(a - b) * c = a * c - b * c$
by (*simp add*: *left-distrib diff-minus*)

lemmas *ring-distribs*[*no-atp*] =
right-distrib left-distrib left-diff-distrib right-diff-distrib

lemma *eq-add-iff1*:
$a * e + c = b * e + d \longleftrightarrow (a - b) * e + c = d$
by (*simp add*: *algebra-simps*)

lemma *eq-add-iff2*:
$a * e + c = b * e + d \longleftrightarrow c = (b - a) * e + d$
by (*simp add*: *algebra-simps*)

end

lemmas *ring-distribs*[*no-atp*] =
right-distrib left-distrib left-diff-distrib right-diff-distrib

class *comm-ring* = *comm-semiring* + *ab-group-add*
begin

subclass *ring* **..**
subclass *comm-semiring-0-cancel* **..**

end

class *ring-1* = *ring* + *zero-neq-one* + *monoid-mult*
begin

subclass *semiring-1-cancel* **..**

end

class *comm-ring-1* = *comm-ring* + *zero-neq-one* + *comm-monoid-mult*

begin

subclass *ring-1* **..**
subclass *comm-semiring-1-cancel* **..**

lemma *dvd-minus-iff* [*simp*]: *x dvd* − *y* ⟷ *x dvd y*
proof
assume *x dvd* − *y*
then have *x dvd* − *1* ∗ − *y* **by** (*rule dvd-mult*)
then show *x dvd y* **by** *simp*
next
assume *x dvd y*
then have *x dvd* − *1* ∗ *y* **by** (*rule dvd-mult*)
then show *x dvd* − *y* **by** *simp*
qed

lemma *minus-dvd-iff* [*simp*]: − *x dvd y* ⟷ *x dvd y*
proof
assume − *x dvd y*
then obtain *k* **where** *y* = − *x* ∗ *k* **..**
then have *y* = *x* ∗ − *k* **by** *simp*
then show *x dvd y* **..**
next
assume *x dvd y*
then obtain *k* **where** *y* = *x* ∗ *k* **..**
then have *y* = − *x* ∗ − *k* **by** *simp*
then show − *x dvd y* **..**
qed

lemma *dvd-diff*[*simp*]: *x dvd y* $\Longrightarrow$ *x dvd z* $\Longrightarrow$ *x dvd* (*y* − *z*)
by (*simp only*: *diff-minus dvd-add dvd-minus-iff*)

end

class *ring-no-zero-divisors* = *ring* + *no-zero-divisors*
begin

lemma *mult-eq-0-iff* [*simp*]:
shows *a* ∗ *b* = *0* ⟷ (*a* = *0* ∨ *b* = *0*)
proof (*cases a* = *0* ∨ *b* = *0*)
case *False* **then have** *a* ≠ *0* **and** *b* ≠ *0* **by** *auto*
then show *?thesis* **using** *no-zero-divisors* **by** *simp*
next
case *True* **then show** *?thesis* **by** *auto*
qed

Cancellation of equalities with a common factor

lemma *mult-cancel-right* [*simp*, *no-atp*]:
a ∗ *c* = *b* ∗ *c* ⟷ *c* = *0* ∨ *a* = *b*
proof −
have (*a* ∗ *c* = *b* ∗ *c*) = ((*a* − *b*) ∗ *c* = *0*)
by (*simp add*: *algebra-simps*)
thus *?thesis* **by** (*simp add*: *disj-commute*)
qed

lemma *mult-cancel-left* [*simp*, *no-atp*]:
c ∗ *a* = *c* ∗ *b* ⟷ *c* = *0* ∨ *a* = *b*
proof −
have (*c* ∗ *a* = *c* ∗ *b*) = (*c* ∗ (*a* − *b*) = *0*)
by (*simp add*: *algebra-simps*)
thus *?thesis* **by** *simp*
qed

end

class *ring-1-no-zero-divisors* = *ring-1* + *ring-no-zero-divisors*
begin

lemma *square-eq-1-iff*:
x ∗ *x* = *1* ⟷ *x* = *1* ∨ *x* = − *1*
proof −
have (*x* − *1*) ∗ (*x* + *1*) = *x* ∗ *x* − *1*
by (*simp add*: *algebra-simps*)
hence *x* ∗ *x* = *1* ⟷ (*x* − *1*) ∗ (*x* + *1*) = *0*
by *simp*
thus *?thesis*
by (*simp add*: *eq-neg-iff-add-eq-0*)
qed

lemma *mult-cancel-right1* [*simp*]:
  $c = b * c \longleftrightarrow c = 0 \vee b = 1$
by (*insert mult-cancel-right* [*of 1 c b*], *force*)

lemma *mult-cancel-right2* [*simp*]:
  $a * c = c \longleftrightarrow c = 0 \vee a = 1$
by (*insert mult-cancel-right* [*of a c 1*], *simp*)

lemma *mult-cancel-left1* [*simp*]:
  $c = c * b \longleftrightarrow c = 0 \vee b = 1$
by (*insert mult-cancel-left* [*of c 1 b*], *force*)

lemma *mult-cancel-left2* [*simp*]:
  $c * a = c \longleftrightarrow c = 0 \vee a = 1$
by (*insert mult-cancel-left* [*of c a 1*], *simp*)

end

class *idom* = *comm-ring-1* + *no-zero-divisors*
begin

subclass *ring-1-no-zero-divisors* **..**

lemma *square-eq-iff*: $a * a = b * b \longleftrightarrow (a = b \vee a = - b)$
proof
  assume $a * a = b * b$
  then have $(a - b) * (a + b) = 0$
    by (*simp add*: *algebra-simps*)
  then show $a = b \vee a = - b$
    by (*simp add*: *eq-neg-iff-add-eq-0*)
next
  assume $a = b \vee a = - b$
  then show $a * a = b * b$ **by** *auto*
qed

lemma *dvd-mult-cancel-right* [*simp*]:
  $a * c \ dvd \ b * c \longleftrightarrow c = 0 \vee a \ dvd \ b$
proof −
  have $a * c \ dvd \ b * c \longleftrightarrow (\exists k.\ b * c = (a * k) * c)$
    unfolding *dvd-def* **by** (*simp add*: *mult-ac*)
  also have $(\exists k.\ b * c = (a * k) * c) \longleftrightarrow c = 0 \vee a \ dvd \ b$
    unfolding *dvd-def* **by** *simp*
  finally show *?thesis* **.**
qed

lemma *dvd-mult-cancel-left* [*simp*]:
  $c * a \ dvd \ c * b \longleftrightarrow c = 0 \vee a \ dvd \ b$
proof −

```
  have c * a dvd c * b ⟷ (∃k. b * c = (a * k) * c)
    unfolding dvd-def by (simp add: mult-ac)
  also have (∃k. b * c = (a * k) * c) ⟷ c = 0 ∨ a dvd b
    unfolding dvd-def by simp
  finally show ?thesis .
qed

end

class inverse =
  fixes inverse :: 'a ⇒ 'a
    and divide :: 'a ⇒ 'a ⇒ 'a  (infixl '/ 70)

class division-ring = ring-1 + inverse +
  assumes left-inverse [simp]:  a ≠ 0 ⟹ inverse a * a = 1
  assumes right-inverse [simp]: a ≠ 0 ⟹ a * inverse a = 1
  assumes divide-inverse: a / b = a * inverse b
begin

subclass ring-1-no-zero-divisors
proof
  fix a b :: 'a
  assume a: a ≠ 0 and b: b ≠ 0
  show a * b ≠ 0
  proof
    assume ab: a * b = 0
    hence 0 = inverse a * (a * b) * inverse b by simp
    also have ... = (inverse a * a) * (b * inverse b)
      by (simp only: mult-assoc)
    also have ... = 1 using a b by simp
    finally show False by simp
  qed
qed

lemma nonzero-imp-inverse-nonzero:
  a ≠ 0 ⟹ inverse a ≠ 0
proof
  assume ianz: inverse a = 0
  assume a ≠ 0
  hence 1 = a * inverse a by simp
  also have ... = 0 by (simp add: ianz)
  finally have 1 = 0 .
  thus False by (simp add: eq-commute)
qed

lemma inverse-zero-imp-zero:
  inverse a = 0 ⟹ a = 0
apply (rule classical)
apply (drule nonzero-imp-inverse-nonzero)
```

apply *auto*
done

lemma *inverse-unique*:
assumes *ab*: $a * b = 1$
shows $inverse\ a = b$
proof −
have $a \neq 0$ **using** *ab* **by** (*cases* $a = 0$) *simp-all*
moreover have $inverse\ a * (a * b) = inverse\ a$ **by** (*simp add*: *ab*)
ultimately show *?thesis* **by** (*simp add*: *mult-assoc* [*symmetric*])
qed

lemma *nonzero-inverse-minus-eq*:
$a \neq 0 \Longrightarrow inverse\ (-a) = -\ inverse\ a$
by (*rule inverse-unique*) *simp*

lemma *nonzero-inverse-inverse-eq*:
$a \neq 0 \Longrightarrow inverse\ (inverse\ a) = a$
by (*rule inverse-unique*) *simp*

lemma *nonzero-inverse-eq-imp-eq*:
assumes $inverse\ a = inverse\ b$ **and** $a \neq 0$ **and** $b \neq 0$
shows $a = b$
proof −
from ‹$inverse\ a = inverse\ b$›
have $inverse\ (inverse\ a) = inverse\ (inverse\ b)$ **by** (*rule arg-cong*)
with ‹$a \neq 0$› **and** ‹$b \neq 0$› **show** $a = b$
by (*simp add*: *nonzero-inverse-inverse-eq*)
qed

lemma *inverse-1* [*simp*]: $inverse\ 1 = 1$
by (*rule inverse-unique*) *simp*

lemma *nonzero-inverse-mult-distrib*:
assumes $a \neq 0$ **and** $b \neq 0$
shows $inverse\ (a * b) = inverse\ b * inverse\ a$
proof −
have $a * (b * inverse\ b) * inverse\ a = 1$ **using** *assms* **by** *simp*
hence $a * b * (inverse\ b * inverse\ a) = 1$ **by** (*simp only*: *mult-assoc*)
thus *?thesis* **by** (*rule inverse-unique*)
qed

lemma *division-ring-inverse-add*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow inverse\ a + inverse\ b = inverse\ a * (a + b) * inverse\ b$
by (*simp add*: *algebra-simps*)

lemma *division-ring-inverse-diff*:
$a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow inverse\ a - inverse\ b = inverse\ a * (b - a) * inverse\ b$
by (*simp add*: *algebra-simps*)

lemma *right-inverse-eq*: $b \neq 0 \Longrightarrow a \mathbin{/} b = 1 \longleftrightarrow a = b$
proof
  assume *neq*: $b \neq 0$
  {
    hence $a = (a \mathbin{/} b) * b$ **by** (*simp add*: *divide-inverse mult-assoc*)
    also assume $a \mathbin{/} b = 1$
    finally show $a = b$ **by** *simp*
  next
    assume $a = b$
    with *neq* **show** $a \mathbin{/} b = 1$ **by** (*simp add*: *divide-inverse*)
  }
qed

lemma *nonzero-inverse-eq-divide*: $a \neq 0 \Longrightarrow \mathit{inverse}\ a = 1 \mathbin{/} a$
by (*simp add*: *divide-inverse*)

lemma *divide-self* [*simp*]: $a \neq 0 \Longrightarrow a \mathbin{/} a = 1$
by (*simp add*: *divide-inverse*)

lemma *divide-zero-left* [*simp*]: $0 \mathbin{/} a = 0$
by (*simp add*: *divide-inverse*)

lemma *inverse-eq-divide*: $\mathit{inverse}\ a = 1 \mathbin{/} a$
by (*simp add*: *divide-inverse*)

lemma *add-divide-distrib*: $(a+b) \mathbin{/} c = a/c + b/c$
by (*simp add*: *divide-inverse algebra-simps*)

lemma *divide-1* [*simp*]: $a \mathbin{/} 1 = a$
  by (*simp add*: *divide-inverse*)

lemma *times-divide-eq-right* [*simp*]: $a * (b \mathbin{/} c) = (a * b) \mathbin{/} c$
  by (*simp add*: *divide-inverse mult-assoc*)

lemma *minus-divide-left*: $-\ (a \mathbin{/} b) = (-a) \mathbin{/} b$
  by (*simp add*: *divide-inverse*)

lemma *nonzero-minus-divide-right*: $b \neq 0 \Longrightarrow -\ (a \mathbin{/} b) = a \mathbin{/} (-\ b)$
  by (*simp add*: *divide-inverse nonzero-inverse-minus-eq*)

lemma *nonzero-minus-divide-divide*: $b \neq 0 \Longrightarrow (-a) \mathbin{/} (-b) = a \mathbin{/} b$
  by (*simp add*: *divide-inverse nonzero-inverse-minus-eq*)

lemma *divide-minus-left* [*simp*, *no-atp*]: $(-a) \mathbin{/} b = -\ (a \mathbin{/} b)$
  by (*simp add*: *divide-inverse*)

lemma *diff-divide-distrib*: $(a - b) \mathbin{/} c = a \mathbin{/} c - b \mathbin{/} c$
  by (*simp add*: *diff-minus add-divide-distrib*)

lemma *nonzero-eq-divide-eq* [*field-simps*]: $c \neq 0 \Longrightarrow a = b \,/\, c \longleftrightarrow a * c = b$
proof −
  assume [*simp*]: $c \neq 0$
  have $a = b \,/\, c \longleftrightarrow a * c = (b \,/\, c) * c$ **by** *simp*
  also have ... $\longleftrightarrow a * c = b$ **by** (*simp add*: *divide-inverse mult-assoc*)
  finally show *?thesis* **.**
qed

lemma *nonzero-divide-eq-eq* [*field-simps*]: $c \neq 0 \Longrightarrow b \,/\, c = a \longleftrightarrow b = a * c$
proof −
  assume [*simp*]: $c \neq 0$
  have $b \,/\, c = a \longleftrightarrow (b \,/\, c) * c = a * c$ **by** *simp*
  also have ... $\longleftrightarrow b = a * c$ **by** (*simp add*: *divide-inverse mult-assoc*)
  finally show *?thesis* **.**
qed

lemma *divide-eq-imp*: $c \neq 0 \Longrightarrow b = a * c \Longrightarrow b \,/\, c = a$
  by (*simp add*: *divide-inverse mult-assoc*)

lemma *eq-divide-imp*: $c \neq 0 \Longrightarrow a * c = b \Longrightarrow a = b \,/\, c$
  by (*drule sym*) (*simp add*: *divide-inverse mult-assoc*)

end

class *division-ring-inverse-zero* = *division-ring* +
  assumes *inverse-zero* [*simp*]: *inverse 0* = *0*
begin

lemma *divide-zero* [*simp*]:
  $a \,/\, 0 = 0$
  by (*simp add*: *divide-inverse*)

lemma *divide-self-if* [*simp*]:
  $a \,/\, a = (\textit{if } a = 0 \textit{ then } 0 \textit{ else } 1)$
  by *simp*

lemma *inverse-nonzero-iff-nonzero* [*simp*]:
  inverse a = *0* $\longleftrightarrow$ *a* = *0*
  by *rule* (*fact inverse-zero-imp-zero*, *simp*)

lemma *inverse-minus-eq* [*simp*]:
  $\textit{inverse } (-a) = -\textit{ inverse } a$
proof *cases*
  assume *a=0* **thus** *?thesis* **by** *simp*
next
  assume *a≠0*
  thus *?thesis* **by** (*simp add*: *nonzero-inverse-minus-eq*)
qed

lemma *inverse-eq-imp-eq*:
  inverse a = inverse b ⟹ a = b
apply (*cases a=0 | b=0*)
 apply (*force dest*!: *inverse-zero-imp-zero*
            simp add: *eq-commute* [*of 0::'a*])
apply (*force dest*!: *nonzero-inverse-eq-imp-eq*)
done

lemma *inverse-eq-iff-eq* [*simp*]:
  inverse a = inverse b ⟷ a = b
  by (*force dest*!: *inverse-eq-imp-eq*)

lemma *inverse-inverse-eq* [*simp*]:
  inverse (inverse a) = a
proof *cases*
  assume *a=0* **thus** *?thesis* **by** *simp*
next
  assume *a≠0*
  thus *?thesis* **by** (*simp add*: *nonzero-inverse-inverse-eq*)
qed

end

class *mult-mono = times + zero + ord +*
  assumes *mult-left-mono*: *a ≤ b ⟹ 0 ≤ c ⟹ c * a ≤ c * b*
  assumes *mult-right-mono*: *a ≤ b ⟹ 0 ≤ c ⟹ a * c ≤ b * c*

The theory of partially ordered rings is taken from the books:

- *Lattice Theory* by Garret Birkhoff, American Mathematical Society 1979
- *Partially Ordered Algebraic Systems*, Pergamon Press 1963

Most of the used notions can also be looked up in

- http://www.mathworld.com by Eric Weisstein et. al.
- *Algebra I* by van der Waerden, Springer.

class *ordered-semiring = mult-mono + semiring-0 + ordered-ab-semigroup-add*
begin

lemma *mult-mono*:
  a ≤ b ⟹ c ≤ d ⟹ 0 ≤ b ⟹ 0 ≤ c
    *⟹ a * c ≤ b * d*
apply (*erule mult-right-mono* [*THEN order-trans*], *assumption*)
apply (*erule mult-left-mono*, *assumption*)
done

lemma *mult-mono'*:
$a \leq b \Longrightarrow c \leq d \Longrightarrow 0 \leq a \Longrightarrow 0 \leq c$
$\Longrightarrow a * c \leq b * d$
apply (*rule mult-mono*)
apply (*fast intro*: *order-trans*)+
done

end

class *ordered-cancel-semiring* = *mult-mono* + *ordered-ab-semigroup-add*
+ *semiring* + *cancel-comm-monoid-add*
begin

subclass *semiring-0-cancel* ..
subclass *ordered-semiring* ..

lemma *mult-nonneg-nonneg*: $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 \leq a * b$
using *mult-left-mono* [*of 0 b a*] **by** *simp*

lemma *mult-nonneg-nonpos*: $0 \leq a \Longrightarrow b \leq 0 \Longrightarrow a * b \leq 0$
using *mult-left-mono* [*of b 0 a*] **by** *simp*

lemma *mult-nonpos-nonneg*: $a \leq 0 \Longrightarrow 0 \leq b \Longrightarrow a * b \leq 0$
using *mult-right-mono* [*of a 0 b*] **by** *simp*

Legacy - use *mult-nonpos-nonneg*

lemma *mult-nonneg-nonpos2*: $0 \leq a \Longrightarrow b \leq 0 \Longrightarrow b * a \leq 0$
by (*drule mult-right-mono* [*of b 0*], *auto*)

lemma *split-mult-neg-le*: $(0 \leq a \;\&\; b \leq 0) \mid (a \leq 0 \;\&\; 0 \leq b) \Longrightarrow a * b \leq 0$
by (*auto simp add*: *mult-nonneg-nonpos mult-nonneg-nonpos2*)

end

class *linordered-semiring* = *semiring* + *comm-monoid-add* + *linordered-cancel-ab-semigroup-add*
+ *mult-mono*
begin

subclass *ordered-cancel-semiring* ..

subclass *ordered-comm-monoid-add* ..

lemma *mult-left-less-imp-less*:
$c * a < c * b \Longrightarrow 0 \leq c \Longrightarrow a < b$
by (*force simp add*: *mult-left-mono not-le* [*symmetric*])

lemma *mult-right-less-imp-less*:
$a * c < b * c \Longrightarrow 0 \leq c \Longrightarrow a < b$

by (*force simp add*: *mult-right-mono not-le* [*symmetric*])

end

class *linordered-semiring-1* = *linordered-semiring* + *semiring-1*
begin

lemma *convex-bound-le*:
  assumes $x \leq a$ $y \leq a$ $0 \leq u$ $0 \leq v$ $u + v = 1$
  shows $u * x + v * y \leq a$
proof−
  from *assms* **have** $u * x + v * y \leq u * a + v * a$
    by (*simp add*: *add-mono mult-left-mono*)
  thus *?thesis* **using** *assms* **unfolding** *left-distrib*[*symmetric*] **by** *simp*
qed

end

class *linordered-semiring-strict* = *semiring* + *comm-monoid-add* + *linordered-cancel-ab-semigroup-add* +
  assumes *mult-strict-left-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
  assumes *mult-strict-right-mono*: $a < b \Longrightarrow 0 < c \Longrightarrow a * c < b * c$
begin

subclass *semiring-0-cancel* **..**

subclass *linordered-semiring*
proof
  fix a b c :: $'a$
  assume *A*: $a \leq b$ $0 \leq c$
  from *A* **show** $c * a \leq c * b$
    unfolding *le-less*
    using *mult-strict-left-mono* **by** (*cases* $c = 0$) *auto*
  from *A* **show** $a * c \leq b * c$
    unfolding *le-less*
    using *mult-strict-right-mono* **by** (*cases* $c = 0$) *auto*
qed

lemma *mult-left-le-imp-le*:
  $c * a \leq c * b \Longrightarrow 0 < c \Longrightarrow a \leq b$
by (*force simp add*: *mult-strict-left-mono -not-less* [*symmetric*])

lemma *mult-right-le-imp-le*:
  $a * c \leq b * c \Longrightarrow 0 < c \Longrightarrow a \leq b$
by (*force simp add*: *mult-strict-right-mono not-less* [*symmetric*])

lemma *mult-pos-pos*: $0 < a \Longrightarrow 0 < b \Longrightarrow 0 < a * b$
using *mult-strict-left-mono* [*of 0 b a*] **by** *simp*

lemma *mult-pos-neg*: $0 < a \Longrightarrow b < 0 \Longrightarrow a * b < 0$
using *mult-strict-left-mono* [*of b 0 a*] **by** *simp*

lemma *mult-neg-pos*: $a < 0 \Longrightarrow 0 < b \Longrightarrow a * b < 0$
using *mult-strict-right-mono* [*of a 0 b*] **by** *simp*

Legacy - use *mult-neg-pos*

lemma *mult-pos-neg2*: $0 < a \Longrightarrow b < 0 \Longrightarrow b * a < 0$
by (*drule mult-strict-right-mono* [*of b 0*], *auto*)

lemma *zero-less-mult-pos*:
$0 < a * b \Longrightarrow 0 < a \Longrightarrow 0 < b$
apply (*cases* $b \leq 0$)
apply (*auto simp add*: *le-less not-less*)
apply (*drule-tac mult-pos-neg* [*of a b*])
apply (*auto dest*: *less-not-sym*)
done

lemma *zero-less-mult-pos2*:
$0 < b * a \Longrightarrow 0 < a \Longrightarrow 0 < b$
apply (*cases* $b \leq 0$)
apply (*auto simp add*: *le-less not-less*)
apply (*drule-tac mult-pos-neg2* [*of a b*])
apply (*auto dest*: *less-not-sym*)
done

Strict monotonicity in both arguments

lemma *mult-strict-mono*:
assumes $a < b$ **and** $c < d$ **and** $0 < b$ **and** $0 \leq c$
shows $a * c < b * d$
using *assms* **apply** (*cases c=0*)
apply (*simp add*: *mult-pos-pos*)
apply (*erule mult-strict-right-mono* [*THEN less-trans*])
apply (*force simp add*: *le-less*)
apply (*erule mult-strict-left-mono*, *assumption*)
done

This weaker variant has more natural premises

lemma *mult-strict-mono'*:
assumes $a < b$ **and** $c < d$ **and** $0 \leq a$ **and** $0 \leq c$
shows $a * c < b * d$
by (*rule mult-strict-mono*) (*insert assms*, *auto*)

lemma *mult-less-le-imp-less*:
assumes $a < b$ **and** $c \leq d$ **and** $0 \leq a$ **and** $0 < c$
shows $a * c < b * d$
using *assms* **apply** (*subgoal-tac* $a * c < b * c$)
apply (*erule less-le-trans*)
apply (*erule mult-left-mono*)

apply *simp*
apply (*erule mult-strict-right-mono*)
apply *assumption*
done

lemma *mult-le-less-imp-less*:
assumes $a \leq b$ **and** $c < d$ **and** $0 < a$ **and** $0 \leq c$
shows $a * c < b * d$
using *assms* **apply** (*subgoal-tac* $a * c \leq b * c$)
apply (*erule le-less-trans*)
apply (*erule mult-strict-left-mono*)
apply *simp*
apply (*erule mult-right-mono*)
apply *simp*
done

lemma *mult-less-imp-less-left*:
assumes *less*: $c * a < c * b$ **and** *nonneg*: $0 \leq c$
shows $a < b$
proof (*rule ccontr*)
assume $\neg\ a < b$
hence $b \leq a$ **by** (*simp add*: *linorder-not-less*)
hence $c * b \leq c * a$ **using** *nonneg* **by** (*rule mult-left-mono*)
with *this* **and** *less* **show** *False* **by** (*simp add*: *not-less* [*symmetric*])
qed

lemma *mult-less-imp-less-right*:
assumes *less*: $a * c < b * c$ **and** *nonneg*: $0 \leq c$
shows $a < b$
proof (*rule ccontr*)
assume $\neg\ a < b$
hence $b \leq a$ **by** (*simp add*: *linorder-not-less*)
hence $b * c \leq a * c$ **using** *nonneg* **by** (*rule mult-right-mono*)
with *this* **and** *less* **show** *False* **by** (*simp add*: *not-less* [*symmetric*])
qed

end

class *linordered-semiring-1-strict* = *linordered-semiring-strict* + *semiring-1*
begin

subclass *linordered-semiring-1* **..**

lemma *convex-bound-lt*:
assumes $x < a$ $y < a$ $0 \leq u$ $0 \leq v$ $u + v = 1$
shows $u * x + v * y < a$
proof −
from *assms* **have** $u * x + v * y < u * a + v * a$
by (*cases* $u = 0$)

(*auto intro*!: *add-less-le-mono mult-strict-left-mono mult-left-mono*)
thus *?thesis* **using** *assms* **unfolding** *left-distrib*[*symmetric*] **by** *simp*
qed

end

class *mult-mono1* = *times* + *zero* + *ord* +
assumes *mult-mono1*: $a \leq b \Longrightarrow 0 \leq c \Longrightarrow c * a \leq c * b$

class *ordered-comm-semiring* = *comm-semiring-0*
+ *ordered-ab-semigroup-add* + *mult-mono1*
begin

subclass *ordered-semiring*
proof
fix *a b c* :: *'a*
assume $a \leq b$ $0 \leq c$
thus $c * a \leq c * b$ **by** (*rule mult-mono1*)
thus $a * c \leq b * c$ **by** (*simp only*: *mult-commute*)
qed

end

class *ordered-cancel-comm-semiring* = *comm-semiring-0-cancel*
+ *ordered-ab-semigroup-add* + *mult-mono1*
begin

subclass *ordered-comm-semiring* **..**
subclass *ordered-cancel-semiring* **..**

end

class *linordered-comm-semiring-strict* = *comm-semiring-0* + *linordered-cancel-ab-semigroup-add*
+
assumes *mult-strict-left-mono-comm*: $a < b \Longrightarrow 0 < c \Longrightarrow c * a < c * b$
begin

subclass *linordered-semiring-strict*
proof
fix *a b c* :: *'a*
assume $a < b$ $0 < c$
thus $c * a < c * b$ **by** (*rule mult-strict-left-mono-comm*)
thus $a * c < b * c$ **by** (*simp only*: *mult-commute*)
qed

subclass *ordered-cancel-comm-semiring*
proof
fix *a b c* :: *'a*
assume $a \leq b$ $0 \leq c$

thus $c * a \leq c * b$
unfolding *le-less*
using *mult-strict-left-mono* **by** (*cases* $c = 0$) *auto*
qed

end

class *ordered-ring* = *ring* + *ordered-cancel-semiring*
begin

subclass *ordered-ab-group-add* **..**

lemma *less-add-iff1*:
$a * e + c < b * e + d \longleftrightarrow (a - b) * e + c < d$
by (*simp add*: *algebra-simps*)

lemma *less-add-iff2*:
$a * e + c < b * e + d \longleftrightarrow c < (b - a) * e + d$
by (*simp add*: *algebra-simps*)

lemma *le-add-iff1*:
$a * e + c \leq b * e + d \longleftrightarrow (a - b) * e + c \leq d$
by (*simp add*: *algebra-simps*)

lemma *le-add-iff2*:
$a * e + c \leq b * e + d \longleftrightarrow c \leq (b - a) * e + d$
by (*simp add*: *algebra-simps*)

lemma *mult-left-mono-neg*:
$b \leq a \Longrightarrow c \leq 0 \Longrightarrow c * a \leq c * b$
apply (*drule mult-left-mono* [*of* - - − *c*])
apply *simp-all*
done

lemma *mult-right-mono-neg*:
$b \leq a \Longrightarrow c \leq 0 \Longrightarrow a * c \leq b * c$
apply (*drule mult-right-mono* [*of* - - − *c*])
apply *simp-all*
done

lemma *mult-nonpos-nonpos*: $a \leq 0 \Longrightarrow b \leq 0 \Longrightarrow 0 \leq a * b$
using *mult-right-mono-neg* [*of a 0 b*] **by** *simp*

lemma *split-mult-pos-le*:
$(0 \leq a \wedge 0 \leq b) \vee (a \leq 0 \wedge b \leq 0) \Longrightarrow 0 \leq a * b$
by (*auto simp add*: *mult-nonneg-nonneg mult-nonpos-nonpos*)

end

class *linordered-ring* = *ring* + *linordered-semiring* + *linordered-ab-group-add* + *abs-if*
begin

subclass *ordered-ring* **..**

subclass *ordered-ab-group-add-abs*
proof
fix *a b*
show $|a + b| \leq |a| + |b|$
by (*auto simp add*: *abs-if not-less*)
(*auto simp del*: *minus-add-distrib simp add*: *minus-add-distrib* [*symmetric*],
auto intro!: *less-imp-le add-neg-neg*)
qed (*auto simp add*: *abs-if*)

lemma *zero-le-square* [*simp*]: $0 \leq a * a$
using *linear* [*of 0 a*]
by (*auto simp add*: *mult-nonneg-nonneg mult-nonpos-nonpos*)

lemma *not-square-less-zero* [*simp*]: $\neg (a * a < 0)$
by (*simp add*: *not-less*)

end

class *linordered-ring-strict* = *ring* + *linordered-semiring-strict*
+ *ordered-ab-group-add* + *abs-if*
begin

subclass *linordered-ring* **..**

lemma *mult-strict-left-mono-neg*: $b < a \Longrightarrow c < 0 \Longrightarrow c * a < c * b$
using *mult-strict-left-mono* [*of b a* $-$ *c*] **by** *simp*

lemma *mult-strict-right-mono-neg*: $b < a \Longrightarrow c < 0 \Longrightarrow a * c < b * c$
using *mult-strict-right-mono* [*of b a* $-$ *c*] **by** *simp*

lemma *mult-neg-neg*: $a < 0 \Longrightarrow b < 0 \Longrightarrow 0 < a * b$
using *mult-strict-right-mono-neg* [*of a 0 b*] **by** *simp*

subclass *ring-no-zero-divisors*
proof
fix *a b*
assume $a \neq 0$ **then have** *A*: $a < 0 \vee 0 < a$ **by** (*simp add*: *neq-iff*)
assume $b \neq 0$ **then have** *B*: $b < 0 \vee 0 < b$ **by** (*simp add*: *neq-iff*)
have $a * b < 0 \vee 0 < a * b$
proof (*cases* $a < 0$)
case *True* **note** $A' = this$
show *?thesis* **proof** (*cases* $b < 0$)

```
    case True with A'
    show ?thesis by (auto dest: mult-neg-neg)
  next
    case False with B have 0 < b by auto
    with A' show ?thesis by (auto dest: mult-strict-right-mono)
  qed
next
  case False with A have A': 0 < a by auto
  show ?thesis proof (cases b < 0)
    case True with A'
    show ?thesis by (auto dest: mult-strict-right-mono-neg)
  next
    case False with B have 0 < b by auto
    with A' show ?thesis by (auto dest: mult-pos-pos)
  qed
qed
then show a * b ≠ 0 by (simp add: neq-iff)
qed

lemma zero-less-mult-iff:
  0 < a * b ⟷ 0 < a ∧ 0 < b ∨ a < 0 ∧ b < 0
  apply (auto simp add: mult-pos-pos mult-neg-neg)
  apply (simp-all add: not-less le-less)
  apply (erule disjE) apply assumption defer
  apply (erule disjE) defer apply (drule sym) apply simp
  apply (erule disjE) defer apply (drule sym) apply simp
  apply (erule disjE) apply assumption apply (drule sym) apply simp
  apply (drule sym) apply simp
  apply (blast dest: zero-less-mult-pos)
  apply (blast dest: zero-less-mult-pos2)
  done

lemma zero-le-mult-iff:
  0 ≤ a * b ⟷ 0 ≤ a ∧ 0 ≤ b ∨ a ≤ 0 ∧ b ≤ 0
by (auto simp add: eq-commute [of 0] le-less not-less zero-less-mult-iff)

lemma mult-less-0-iff:
  a * b < 0 ⟷ 0 < a ∧ b < 0 ∨ a < 0 ∧ 0 < b
  apply (insert zero-less-mult-iff [of -a b])
  apply force
  done

lemma mult-le-0-iff:
  a * b ≤ 0 ⟷ 0 ≤ a ∧ b ≤ 0 ∨ a ≤ 0 ∧ 0 ≤ b
  apply (insert zero-le-mult-iff [of -a b])
  apply force
  done
```

Cancellation laws for $c * a < c * b$ and $a * c < b * c$, also with the relations

$\leq$ and equality.

These "disjunction" versions produce two cases when the comparison is an assumption, but effectively four when the comparison is a goal.

lemma *mult-less-cancel-right-disj*:
$a * c < b * c \longleftrightarrow 0 < c \wedge a < b \vee c < 0 \wedge b < a$
apply (*cases* $c = 0$)
apply (*auto simp add*: *neq-iff mult-strict-right-mono*
mult-strict-right-mono-neg)
apply (*auto simp add*: *not-less*
not-le [*symmetric*, *of* $a*c$]
not-le [*symmetric*, *of* a])
apply (*erule-tac* [!] *notE*)
apply (*auto simp add*: *less-imp-le mult-right-mono*
mult-right-mono-neg)
done

lemma *mult-less-cancel-left-disj*:
$c * a < c * b \longleftrightarrow 0 < c \wedge a < b \vee c < 0 \wedge b < a$
apply (*cases* $c = 0$)
apply (*auto simp add*: *neq-iff mult-strict-left-mono*
mult-strict-left-mono-neg)
apply (*auto simp add*: *not-less*
not-le [*symmetric*, *of* $c*a$]
not-le [*symmetric*, *of* a])
apply (*erule-tac* [!] *notE*)
apply (*auto simp add*: *less-imp-le mult-left-mono*
mult-left-mono-neg)
done

The "conjunction of implication" lemmas produce two cases when the comparison is a goal, but give four when the comparison is an assumption.

lemma *mult-less-cancel-right*:
$a * c < b * c \longleftrightarrow (0 \leq c \longrightarrow a < b) \wedge (c \leq 0 \longrightarrow b < a)$
using *mult-less-cancel-right-disj* [*of* a c b] **by** *auto*

lemma *mult-less-cancel-left*:
$c * a < c * b \longleftrightarrow (0 \leq c \longrightarrow a < b) \wedge (c \leq 0 \longrightarrow b < a)$
using *mult-less-cancel-left-disj* [*of* c a b] **by** *auto*

lemma *mult-le-cancel-right*:
$a * c \leq b * c \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
by (*simp add*: *not-less* [*symmetric*] *mult-less-cancel-right-disj*)

lemma *mult-le-cancel-left*:
$c * a \leq c * b \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
by (*simp add*: *not-less* [*symmetric*] *mult-less-cancel-left-disj*)

lemma *mult-le-cancel-left-pos*:

$0 < c \Longrightarrow c * a \leq c * b \longleftrightarrow a \leq b$
by (*auto simp*: *mult-le-cancel-left*)

lemma *mult-le-cancel-left-neg*:
$c < 0 \Longrightarrow c * a \leq c * b \longleftrightarrow b \leq a$
by (*auto simp*: *mult-le-cancel-left*)

lemma *mult-less-cancel-left-pos*:
$0 < c \Longrightarrow c * a < c * b \longleftrightarrow a < b$
by (*auto simp*: *mult-less-cancel-left*)

lemma *mult-less-cancel-left-neg*:
$c < 0 \Longrightarrow c * a < c * b \longleftrightarrow b < a$
by (*auto simp*: *mult-less-cancel-left*)

end

lemmas *mult-sign-intros* =
mult-nonneg-nonneg mult-nonneg-nonpos
mult-nonpos-nonneg mult-nonpos-nonpos
mult-pos-pos mult-pos-neg
mult-neg-pos mult-neg-neg

class *ordered-comm-ring* = *comm-ring* + *ordered-comm-semiring*
begin

subclass *ordered-ring* **..**
subclass *ordered-cancel-comm-semiring* **..**

end

class *linordered-semidom* = *comm-semiring-1-cancel* + *linordered-comm-semiring-strict*
+

assumes *zero-less-one* [*simp*]: $0 < 1$
begin

lemma *pos-add-strict*:
shows $0 < a \Longrightarrow b < c \Longrightarrow b < a + c$
using *add-strict-mono* [*of 0 a b c*] **by** *simp*

lemma *zero-le-one* [*simp*]: $0 \leq 1$
by (*rule zero-less-one* [*THEN less-imp-le*])

lemma *not-one-le-zero* [*simp*]: $\neg\ 1 \leq 0$
by (*simp add*: *not-le*)

lemma *not-one-less-zero* [*simp*]: $\neg\ 1 < 0$
by (*simp add*: *not-less*)

lemma *less-1-mult*:
assumes $1 < m$ **and** $1 < n$
shows $1 < m * n$
using *assms mult-strict-mono* [*of 1 m 1 n*]
by (*simp add*: *less-trans* [*OF zero-less-one*])

end

class *linordered-idom* = *comm-ring-1* +
linordered-comm-semiring-strict + *ordered-ab-group-add* +
abs-if + *sgn-if*

begin

subclass *linordered-semiring-1-strict* **..**
subclass *linordered-ring-strict* **..**
subclass *ordered-comm-ring* **..**
subclass *idom* **..**

subclass *linordered-semidom*
proof
have $0 \leq 1 * 1$ **by** (*rule zero-le-square*)
thus $0 < 1$ **by** (*simp add*: *le-less*)
qed

lemma *linorder-neqE-linordered-idom*:
assumes $x \neq y$ **obtains** $x < y \mid y < x$
using *assms* **by** (*rule neqE*)

These cancellation simprules also produce two cases when the comparison is a goal.

lemma *mult-le-cancel-right1*:
$c \leq b * c \longleftrightarrow (0 < c \longrightarrow 1 \leq b) \wedge (c < 0 \longrightarrow b \leq 1)$
by (*insert mult-le-cancel-right* [*of 1 c b*], *simp*)

lemma *mult-le-cancel-right2*:
$a * c \leq c \longleftrightarrow (0 < c \longrightarrow a \leq 1) \wedge (c < 0 \longrightarrow 1 \leq a)$
by (*insert mult-le-cancel-right* [*of a c 1*], *simp*)

lemma *mult-le-cancel-left1*:
$c \leq c * b \longleftrightarrow (0 < c \longrightarrow 1 \leq b) \wedge (c < 0 \longrightarrow b \leq 1)$
by (*insert mult-le-cancel-left* [*of c 1 b*], *simp*)

lemma *mult-le-cancel-left2*:
$c * a \leq c \longleftrightarrow (0 < c \longrightarrow a \leq 1) \wedge (c < 0 \longrightarrow 1 \leq a)$
by (*insert mult-le-cancel-left* [*of c a 1*], *simp*)

lemma *mult-less-cancel-right1*:

$c < b * c \longleftrightarrow (0 \leq c \longrightarrow 1 < b) \wedge (c \leq 0 \longrightarrow b < 1)$
by (*insert mult-less-cancel-right* [*of 1 c b*], *simp*)

lemma *mult-less-cancel-right2*:
$a * c < c \longleftrightarrow (0 \leq c \longrightarrow a < 1) \wedge (c \leq 0 \longrightarrow 1 < a)$
by (*insert mult-less-cancel-right* [*of a c 1*], *simp*)

lemma *mult-less-cancel-left1*:
$c < c * b \longleftrightarrow (0 \leq c \longrightarrow 1 < b) \wedge (c \leq 0 \longrightarrow b < 1)$
by (*insert mult-less-cancel-left* [*of c 1 b*], *simp*)

lemma *mult-less-cancel-left2*:
$c * a < c \longleftrightarrow (0 \leq c \longrightarrow a < 1) \wedge (c \leq 0 \longrightarrow 1 < a)$
by (*insert mult-less-cancel-left* [*of c a 1*], *simp*)

lemma *sgn-sgn* [*simp*]:
$sgn\ (sgn\ a) = sgn\ a$
unfolding *sgn-if* **by** *simp*

lemma *sgn-0-0*:
$sgn\ a = 0 \longleftrightarrow a = 0$
unfolding *sgn-if* **by** *simp*

lemma *sgn-1-pos*:
$sgn\ a = 1 \longleftrightarrow a > 0$
unfolding *sgn-if* **by** *simp*

lemma *sgn-1-neg*:
$sgn\ a = -\ 1 \longleftrightarrow a < 0$
unfolding *sgn-if* **by** *auto*

lemma *sgn-pos* [*simp*]:
$0 < a \Longrightarrow sgn\ a = 1$
unfolding *sgn-1-pos* **.**

lemma *sgn-neg* [*simp*]:
$a < 0 \Longrightarrow sgn\ a = -\ 1$
unfolding *sgn-1-neg* **.**

lemma *sgn-times*:
$sgn\ (a * b) = sgn\ a * sgn\ b$
by (*auto simp add*: *sgn-if zero-less-mult-iff*)

lemma *abs-sgn*: $|k| = k * sgn\ k$
unfolding *sgn-if abs-if* **by** *auto*

lemma *sgn-greater* [*simp*]:
$0 < sgn\ a \longleftrightarrow 0 < a$
unfolding *sgn-if* **by** *auto*

lemma *sgn-less* [*simp*]:
  sgn a < 0 ⟷ a < 0
  unfolding *sgn-if* **by** *auto*

lemma *abs-dvd-iff* [*simp*]: *|m| dvd k ⟷ m dvd k*
  by (*simp add*: *abs-if*)

lemma *dvd-abs-iff* [*simp*]: *m dvd |k| ⟷ m dvd k*
  by (*simp add*: *abs-if*)

lemma *dvd-if-abs-eq*:
  |l| = |k| ⟹ l dvd k
by(*subst abs-dvd-iff*[*symmetric*]) *simp*

end

Simprules for comparisons where common factors can be cancelled.

lemmas *mult-compare-simps*[*no-atp*] =
    mult-le-cancel-right mult-le-cancel-left
    mult-le-cancel-right1 mult-le-cancel-right2
    mult-le-cancel-left1 mult-le-cancel-left2
    mult-less-cancel-right mult-less-cancel-left
    mult-less-cancel-right1 mult-less-cancel-right2
    mult-less-cancel-left1 mult-less-cancel-left2
    mult-cancel-right mult-cancel-left
    mult-cancel-right1 mult-cancel-right2
    mult-cancel-left1 mult-cancel-left2

Reasoning about inequalities with division

context *linordered-semidom*
begin

lemma *less-add-one*: *a < a + 1*
proof −
  have *a + 0 < a + 1*
    by (*blast intro*: *zero-less-one add-strict-left-mono*)
  thus *?thesis* **by** *simp*
qed

lemma *zero-less-two*: *0 < 1 + 1*
by (*blast intro*: *less-trans zero-less-one less-add-one*)

end

context *linordered-idom*
begin

lemma *mult-right-le-one-le*:

$0 \leq x \Longrightarrow 0 \leq y \Longrightarrow y \leq 1 \Longrightarrow x * y \leq x$
by (*auto simp add*: *mult-le-cancel-left2*)

lemma *mult-left-le-one-le*:
$0 \leq x \Longrightarrow 0 \leq y \Longrightarrow y \leq 1 \Longrightarrow y * x \leq x$
by (*auto simp add*: *mult-le-cancel-right2*)

end

Absolute Value

context *linordered-idom*
begin

lemma *mult-sgn-abs*:
sgn $x * |x| = x$
unfolding *abs-if sgn-if* **by** *auto*

lemma *abs-one* [*simp*]:
$|1| = 1$
by (*simp add*: *abs-if zero-less-one* [*THEN less-not-sym*])

end

class *ordered-ring-abs* = *ordered-ring* + *ordered-ab-group-add-abs* +
assumes *abs-eq-mult*:
$(0 \leq a \vee a \leq 0) \wedge (0 \leq b \vee b \leq 0) \Longrightarrow |a * b| = |a| * |b|$

context *linordered-idom*
begin

subclass *ordered-ring-abs* **proof**
qed (*auto simp add*: *abs-if not-less mult-less-0-iff*)

lemma *abs-mult*:
$|a * b| = |a| * |b|$
by (*rule abs-eq-mult*) *auto*

lemma *abs-mult-self*:
$|a| * |a| = a * a$
by (*simp add*: *abs-if*)

lemma *abs-mult-less*:
$|a| < c \Longrightarrow |b| < d \Longrightarrow |a| * |b| < c * d$
proof −
assume *ac*: $|a| < c$
hence *cpos*: $0<c$ **by** (*blast intro*: *le-less-trans abs-ge-zero*)
assume $|b| < d$
thus *?thesis* **by** (*simp add*: *ac cpos mult-strict-mono*)
qed

lemma *less-minus-self-iff*:
  $a < -\ a \longleftrightarrow a < 0$
  by (*simp only*: *less-le less-eq-neg-nonpos equal-neg-zero*)

lemma *abs-less-iff*:
  $|a| < b \longleftrightarrow a < b \wedge -\ a < b$
  by (*simp add*: *less-le abs-le-iff*) (*auto simp add*: *abs-if*)

lemma *abs-mult-pos*:
  $0 \leq x \Longrightarrow |y| * x = |y * x|$
  by (*simp add*: *abs-mult*)

end

code-modulename *SML*
  Rings Arith

code-modulename *OCaml*
  Rings Arith

code-modulename *Haskell*
  Rings Arith

end

14 Fields: Fields

theory *Fields*
imports *Rings*
begin

class *field* = *comm-ring-1* + *inverse* +
  assumes *field-inverse*: $a \neq 0 \Longrightarrow inverse\ a * a = 1$
  assumes *field-divide-inverse*: $a\ /\ b = a * inverse\ b$
begin

subclass *division-ring*
proof
  fix $a :: 'a$
  assume $a \neq 0$
  thus $inverse\ a * a = 1$ **by** (*rule field-inverse*)
  thus $a * inverse\ a = 1$ **by** (*simp only*: *mult-commute*)
next
  fix $a\ b :: 'a$
  show $a\ /\ b = a * inverse\ b$ **by** (*rule field-divide-inverse*)
qed

subclass *idom* **..**

There is no slick version using division by zero.

lemma *inverse-add*:
[| $a \neq 0$; $b \neq 0$ |]
==> $inverse\ a + inverse\ b = (a + b) * inverse\ a * inverse\ b$
by (*simp add*: *division-ring-inverse-add mult-ac*)

lemma *nonzero-mult-divide-mult-cancel-left* [*simp*, *no-atp*]:
assumes [*simp*]: $b \neq 0$ **and** [*simp*]: $c \neq 0$ **shows** $(c*a)/(c*b) = a/b$
proof −
have $(c*a)/(c*b) = c * a * (inverse\ b * inverse\ c)$
by (*simp add*: *divide-inverse nonzero-inverse-mult-distrib*)
also have ... = $a * inverse\ b * (inverse\ c * c)$
by (*simp only*: *mult-ac*)
also have ... = $a * inverse\ b$ **by** *simp*
finally show *?thesis* **by** (*simp add*: *divide-inverse*)
qed

lemma *nonzero-mult-divide-mult-cancel-right* [*simp*, *no-atp*]:
$[\![b \neq 0; c \neq 0]\!] \Longrightarrow (a * c) / (b * c) = a / b$
by (*simp add*: *mult-commute* [*of* - *c*])

lemma *times-divide-eq-left* [*simp*]: $(b / c) * a = (b * a) / c$
by (*simp add*: *divide-inverse mult-ac*)

These are later declared as simp rules.

lemmas *times-divide-eq* [*no-atp*] = *times-divide-eq-right times-divide-eq-left*

lemma *add-frac-eq*:
assumes $y \neq 0$ **and** $z \neq 0$
shows $x / y + w / z = (x * z + w * y) / (y * z)$
proof −
have $x / y + w / z = (x * z) / (y * z) + (y * w) / (y * z)$
using *assms* **by** *simp*
also have ... = $(x * z + y * w) / (y * z)$
by (*simp only*: *add-divide-distrib*)
finally show *?thesis*
by (*simp only*: *mult-commute*)
qed

Special Cancellation Simprules for Division

lemma *nonzero-mult-divide-cancel-right* [*simp*, *no-atp*]:
$b \neq 0 \Longrightarrow a * b / b = a$
using *nonzero-mult-divide-mult-cancel-right* [*of 1 b a*] **by** *simp*

lemma *nonzero-mult-divide-cancel-left* [*simp*, *no-atp*]:
$a \neq 0 \Longrightarrow a * b / a = b$
using *nonzero-mult-divide-mult-cancel-left* [*of 1 a b*] **by** *simp*

lemma *nonzero-divide-mult-cancel-right* [*simp*, *no-atp*]:
$[\![a \neq 0;\ b \neq 0]\!] \Longrightarrow b\ /\ (a * b) = 1\ /\ a$
using *nonzero-mult-divide-mult-cancel-right* [*of a b 1*] **by** *simp*

lemma *nonzero-divide-mult-cancel-left* [*simp*, *no-atp*]:
$[\![a \neq 0;\ b \neq 0]\!] \Longrightarrow a\ /\ (a * b) = 1\ /\ b$
using *nonzero-mult-divide-mult-cancel-left* [*of b a 1*] **by** *simp*

lemma *nonzero-mult-divide-mult-cancel-left2* [*simp*, *no-atp*]:
$[\![b \neq 0;\ c \neq 0]\!] \Longrightarrow (c * a)\ /\ (b * c) = a\ /\ b$
using *nonzero-mult-divide-mult-cancel-left* [*of b c a*] **by** (*simp add*: *mult-ac*)

lemma *nonzero-mult-divide-mult-cancel-right2* [*simp*, *no-atp*]:
$[\![b \neq 0;\ c \neq 0]\!] \Longrightarrow (a * c)\ /\ (c * b) = a\ /\ b$
using *nonzero-mult-divide-mult-cancel-right* [*of b c a*] **by** (*simp add*: *mult-ac*)

lemma *add-divide-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x + y\ /\ z = (z * x + y)\ /\ z$
by (*simp add*: *add-divide-distrib*)

lemma *divide-add-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x\ /\ z + y = (x + z * y)\ /\ z$
by (*simp add*: *add-divide-distrib*)

lemma *diff-divide-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x - y\ /\ z = (z * x - y)\ /\ z$
by (*simp add*: *diff-divide-distrib*)

lemma *divide-diff-eq-iff* [*field-simps*]:
$z \neq 0 \Longrightarrow x\ /\ z - y = (x - z * y)\ /\ z$
by (*simp add*: *diff-divide-distrib*)

lemma *diff-frac-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow x\ /\ y - w\ /\ z = (x * z - w * y)\ /\ (y * z)$
by (*simp add*: *field-simps*)

lemma *frac-eq-eq*:
$y \neq 0 \Longrightarrow z \neq 0 \Longrightarrow (x\ /\ y = w\ /\ z) = (x * z = w * y)$
by (*simp add*: *field-simps*)

end

class *field-inverse-zero* = *field* +
assumes *field-inverse-zero*: *inverse 0 = 0*
begin

subclass *division-ring-inverse-zero* **proof**
qed (*fact field-inverse-zero*)

This version builds in division by zero while also re-orienting the right-hand side.

lemma *inverse-mult-distrib* [*simp*]:
  inverse (*a* ∗ *b*) = *inverse a* ∗ *inverse b*
proof *cases*
  assume *a* ≠ *0* & *b* ≠ *0*
  thus *?thesis* **by** (*simp add*: *nonzero-inverse-mult-distrib mult-ac*)
next
  assume ~ (*a* ≠ *0* & *b* ≠ *0*)
  thus *?thesis* **by** *force*
qed

lemma *inverse-divide* [*simp*]:
  inverse (*a* / *b*) = *b* / *a*
  by (*simp add*: *divide-inverse mult-commute*)

Calculations with fractions

There is a whole bunch of simp-rules just for class *field* but none for class *field* and *nonzero-divides* because the latter are covered by a simproc.

lemma *mult-divide-mult-cancel-left*:
  c ≠ *0* ⟹ (*c* ∗ *a*) / (*c* ∗ *b*) = *a* / *b*
apply (*cases b* = *0*)
apply *simp-all*
done

lemma *mult-divide-mult-cancel-right*:
  c ≠ *0* ⟹ (*a* ∗ *c*) / (*b* ∗ *c*) = *a* / *b*
apply (*cases b* = *0*)
apply *simp-all*
done

lemma *divide-divide-eq-right* [*simp*, *no-atp*]:
  a / (*b* / *c*) = (*a* ∗ *c*) / *b*
  by (*simp add*: *divide-inverse mult-ac*)

lemma *divide-divide-eq-left* [*simp*, *no-atp*]:
  (*a* / *b*) / *c* = *a* / (*b* ∗ *c*)
  by (*simp add*: *divide-inverse mult-assoc*)

Special Cancellation Simprules for Division

lemma *mult-divide-mult-cancel-left-if* [*simp*,*no-atp*]:
  shows (*c* ∗ *a*) / (*c* ∗ *b*) = (*if c* = *0 then 0 else a* / *b*)
  by (*simp add*: *mult-divide-mult-cancel-left*)

Division and Unary Minus

lemma *minus-divide-right*:
  − (*a* / *b*) = *a* / − *b*

by (*simp add*: *divide-inverse*)

lemma *divide-minus-right* [*simp*, *no-atp*]:
a / − *b* = − (*a* / *b*)
by (*simp add*: *divide-inverse*)

lemma *minus-divide-divide*:
(− *a*) / (− *b*) = *a* / *b*
apply (*cases b=0*, *simp*)
apply (*simp add*: *nonzero-minus-divide-divide*)
done

lemma *eq-divide-eq*:
a = *b* / *c* ⟷ (*if c* ≠ *0 then a* ∗ *c* = *b else a* = *0*)
by (*simp add*: *nonzero-eq-divide-eq*)

lemma *divide-eq-eq*:
b / *c* = *a* ⟷ (*if c* ≠ *0 then b* = *a* ∗ *c else a* = *0*)
by (*force simp add*: *nonzero-divide-eq-eq*)

lemma *inverse-eq-1-iff* [*simp*]:
inverse x = *1* ⟷ *x* = *1*
by (*insert inverse-eq-iff-eq* [*of x 1*], *simp*)

lemma *divide-eq-0-iff* [*simp*, *no-atp*]:
a / *b* = *0* ⟷ *a* = *0* ∨ *b* = *0*
by (*simp add*: *divide-inverse*)

lemma *divide-cancel-right* [*simp*, *no-atp*]:
a / *c* = *b* / *c* ⟷ *c* = *0* ∨ *a* = *b*
apply (*cases c=0*, *simp*)
apply (*simp add*: *divide-inverse*)
done

lemma *divide-cancel-left* [*simp*, *no-atp*]:
c / *a* = *c* / *b* ⟷ *c* = *0* ∨ *a* = *b*
apply (*cases c=0*, *simp*)
apply (*simp add*: *divide-inverse*)
done

lemma *divide-eq-1-iff* [*simp*, *no-atp*]:
a / *b* = *1* ⟷ *b* ≠ *0* ∧ *a* = *b*
apply (*cases b=0*, *simp*)
apply (*simp add*: *right-inverse-eq*)
done

lemma *one-eq-divide-iff* [*simp*, *no-atp*]:
1 = *a* / *b* ⟷ *b* ≠ *0* ∧ *a* = *b*
by (*simp add*: *eq-commute* [*of 1*])

lemma *times-divide-times-eq*:
$(x\ /\ y) * (z\ /\ w) = (x * z)\ /\ (y * w)$
by *simp*

lemma *add-frac-num*:
$y \neq 0 \Longrightarrow x\ /\ y + z = (x + z * y)\ /\ y$
by (*simp add*: *add-divide-distrib*)

lemma *add-num-frac*:
$y \neq 0 \Longrightarrow z + x\ /\ y = (x + z * y)\ /\ y$
by (*simp add*: *add-divide-distrib add.commute*)

end

Ordered Fields

class *linordered-field* = *field* + *linordered-idom*
begin

lemma *positive-imp-inverse-positive*:
assumes *a-gt-0*: $0 < a$
shows $0 <$ *inverse a*
proof −
have $0 < a *$ *inverse a*
by (*simp add*: *a-gt-0* [*THEN less-imp-not-eq2*])
thus $0 <$ *inverse a*
by (*simp add*: *a-gt-0* [*THEN less-not-sym*] *zero-less-mult-iff*)
qed

lemma *negative-imp-inverse-negative*:
$a < 0 \Longrightarrow$ *inverse a* < 0
by (*insert positive-imp-inverse-positive* [*of* $-a$],
simp add: *nonzero-inverse-minus-eq less-imp-not-eq*)

lemma *inverse-le-imp-le*:
assumes *invle*: *inverse a* $\leq$ *inverse b* **and** *apos*: $0 < a$
shows $b \leq a$
proof (*rule classical*)
assume $^{\sim}$ $b \leq a$
hence $a < b$ **by** (*simp add*: *linorder-not-le*)
hence *bpos*: $0 < b$ **by** (*blast intro*: *apos less-trans*)
hence $a *$ *inverse a* $\leq a *$ *inverse b*
by (*simp add*: *apos invle less-imp-le mult-left-mono*)
hence (*a* $*$ *inverse a*) $* b \leq$ (*a* $*$ *inverse b*) $* b$
by (*simp add*: *bpos less-imp-le mult-right-mono*)
thus $b \leq a$ **by** (*simp add*: *mult-assoc apos bpos less-imp-not-eq2*)
qed

lemma *inverse-positive-imp-positive*:

assumes *inv-gt-0*: $0 < inverse\ a$ **and** *nz*: $a \neq 0$
shows $0 < a$
proof −
have $0 < inverse\ (inverse\ a)$
using *inv-gt-0* **by** (*rule positive-imp-inverse-positive*)
thus $0 < a$
using *nz* **by** (*simp add*: *nonzero-inverse-inverse-eq*)
qed

lemma *inverse-negative-imp-negative*:
assumes *inv-less-0*: $inverse\ a < 0$ **and** *nz*: $a \neq 0$
shows $a < 0$
proof −
have $inverse\ (inverse\ a) < 0$
using *inv-less-0* **by** (*rule negative-imp-inverse-negative*)
thus $a < 0$ **using** *nz* **by** (*simp add*: *nonzero-inverse-inverse-eq*)
qed

lemma *linordered-field-no-lb*:
$\forall x.\ \exists y.\ y < x$
proof
fix $x::'a$
have *m1*: $-\ (1::'a) < 0$ **by** *simp*
from *add-strict-right-mono*[*OF m1*, **where** $c{=}x$]
have $(-\ 1) + x < x$ **by** *simp*
thus $\exists y.\ y < x$ **by** *blast*
qed

lemma *linordered-field-no-ub*:
$\forall\ x.\ \exists y.\ y > x$
proof
fix $x::'a$
have *m1*: $(1::'a) > 0$ **by** *simp*
from *add-strict-right-mono*[*OF m1*, **where** $c{=}x$]
have $1 + x > x$ **by** *simp*
thus $\exists y.\ y > x$ **by** *blast*
qed

lemma *less-imp-inverse-less*:
assumes *less*: $a < b$ **and** *apos*: $0 < a$
shows $inverse\ b < inverse\ a$
proof (*rule ccontr*)
assume $\sim\ inverse\ b < inverse\ a$
hence $inverse\ a \leq inverse\ b$ **by** *simp*
hence $\sim\ (a < b)$
by (*simp add*: *not-less inverse-le-imp-le* [*OF - apos*])
thus *False* **by** (*rule notE* [*OF - less*])
qed

lemma *inverse-less-imp-less*:
inverse a < inverse b ⟹ 0 < a ⟹ b < a
apply (*simp add*: *less-le* [*of inverse a*] *less-le* [*of b*])
apply (*force dest*!: *inverse-le-imp-le nonzero-inverse-eq-imp-eq*)
done

Both premises are essential. Consider -1 and 1.

lemma *inverse-less-iff-less* [*simp*,*no-atp*]:
0 < a ⟹ 0 < b ⟹ inverse a < inverse b ⟷ b < a
by (*blast intro*: *less-imp-inverse-less dest*: *inverse-less-imp-less*)

lemma *le-imp-inverse-le*:
a ≤ b ⟹ 0 < a ⟹ inverse b ≤ inverse a
by (*force simp add*: *le-less less-imp-inverse-less*)

lemma *inverse-le-iff-le* [*simp*,*no-atp*]:
0 < a ⟹ 0 < b ⟹ inverse a ≤ inverse b ⟷ b ≤ a
by (*blast intro*: *le-imp-inverse-le dest*: *inverse-le-imp-le*)

These results refer to both operands being negative. The opposite-sign case is trivial, since inverse preserves signs.

lemma *inverse-le-imp-le-neg*:
inverse a ≤ inverse b ⟹ b < 0 ⟹ b ≤ a
apply (*rule classical*)
apply (*subgoal-tac a < 0*)
prefer *2* **apply** *force*
apply (*insert inverse-le-imp-le* [*of* −*b* −*a*])
apply (*simp add*: *nonzero-inverse-minus-eq*)
done

lemma *less-imp-inverse-less-neg*:
a < b ⟹ b < 0 ⟹ inverse b < inverse a
apply (*subgoal-tac a < 0*)
prefer *2* **apply** (*blast intro*: *less-trans*)
apply (*insert less-imp-inverse-less* [*of* −*b* −*a*])
apply (*simp add*: *nonzero-inverse-minus-eq*)
done

lemma *inverse-less-imp-less-neg*:
inverse a < inverse b ⟹ b < 0 ⟹ b < a
apply (*rule classical*)
apply (*subgoal-tac a < 0*)
prefer *2*
apply *force*
apply (*insert inverse-less-imp-less* [*of* −*b* −*a*])
apply (*simp add*: *nonzero-inverse-minus-eq*)
done

lemma *inverse-less-iff-less-neg* [*simp*,*no-atp*]:

a < 0 ⟹ b < 0 ⟹ inverse a < inverse b ⟷ b < a
apply (*insert inverse-less-iff-less* [*of* −*b* −*a*])
apply (*simp del*: *inverse-less-iff-less*
add: *nonzero-inverse-minus-eq*)
done

lemma *le-imp-inverse-le-neg*:
a ≤ b ⟹ b < 0 ==> inverse b ≤ inverse a
by (*force simp add*: *le-less less-imp-inverse-less-neg*)

lemma *inverse-le-iff-le-neg* [*simp*,*no-atp*]:
a < 0 ⟹ b < 0 ⟹ inverse a ≤ inverse b ⟷ b ≤ a
by (*blast intro*: *le-imp-inverse-le-neg dest*: *inverse-le-imp-le-neg*)

lemma *one-less-inverse*:
0 < a ⟹ a < 1 ⟹ 1 < inverse a
using *less-imp-inverse-less* [*of a 1*, *unfolded inverse-1*] **.**

lemma *one-le-inverse*:
0 < a ⟹ a ≤ 1 ⟹ 1 ≤ inverse a
using *le-imp-inverse-le* [*of a 1*, *unfolded inverse-1*] **.**

lemma *pos-le-divide-eq* [*field-simps*]: *0 < c ==> (a ≤ b/c) = (a∗c ≤ b)*
proof −
assume *less*: *0<c*
hence (*a ≤ b/c*) = (*a∗c ≤ (b/c)∗c*)
by (*simp add*: *mult-le-cancel-right less-not-sym* [*OF less*] *del*: *times-divide-eq*)
also have ... = (*a∗c ≤ b*)
by (*simp add*: *less-imp-not-eq2* [*OF less*] *divide-inverse mult-assoc*)
finally show *?thesis* **.**
qed

lemma *neg-le-divide-eq* [*field-simps*]: *c < 0 ==> (a ≤ b/c) = (b ≤ a∗c)*
proof −
assume *less*: *c<0*
hence (*a ≤ b/c*) = ((*b/c*)∗*c ≤ a∗c*)
by (*simp add*: *mult-le-cancel-right less-not-sym* [*OF less*] *del*: *times-divide-eq*)
also have ... = (*b ≤ a∗c*)
by (*simp add*: *less-imp-not-eq* [*OF less*] *divide-inverse mult-assoc*)
finally show *?thesis* **.**
qed

lemma *pos-less-divide-eq* [*field-simps*]:
0 < c ==> (a < b/c) = (a∗c < b)
proof −
assume *less*: *0<c*
hence (*a < b/c*) = (*a∗c < (b/c)∗c*)
by (*simp add*: *mult-less-cancel-right-disj less-not-sym* [*OF less*] *del*: *times-divide-eq*)
also have ... = (*a∗c < b*)

by (*simp add*: *less-imp-not-eq2* [*OF less*] *divide-inverse mult-assoc*)
finally show *?thesis* **.**
qed

lemma *neg-less-divide-eq* [*field-simps*]:
$c < 0 \Longrightarrow (a < b/c) = (b < a*c)$
proof −
assume *less*: *c<0*
hence $(a < b/c) = ((b/c)*c < a*c)$
by (*simp add*: *mult-less-cancel-right-disj less-not-sym* [*OF less*] *del*: *times-divide-eq*)
also have ... $= (b < a*c)$
by (*simp add*: *less-imp-not-eq* [*OF less*] *divide-inverse mult-assoc*)
finally show *?thesis* **.**
qed

lemma *pos-divide-less-eq* [*field-simps*]:
$0 < c \Longrightarrow (b/c < a) = (b < a*c)$
proof −
assume *less*: *0<c*
hence $(b/c < a) = ((b/c)*c < a*c)$
by (*simp add*: *mult-less-cancel-right-disj less-not-sym* [*OF less*] *del*: *times-divide-eq*)
also have ... $= (b < a*c)$
by (*simp add*: *less-imp-not-eq2* [*OF less*] *divide-inverse mult-assoc*)
finally show *?thesis* **.**
qed

lemma *neg-divide-less-eq* [*field-simps*]:
$c < 0 \Longrightarrow (b/c < a) = (a*c < b)$
proof −
assume *less*: *c<0*
hence $(b/c < a) = (a*c < (b/c)*c)$
by (*simp add*: *mult-less-cancel-right-disj less-not-sym* [*OF less*] *del*: *times-divide-eq*)
also have ... $= (a*c < b)$
by (*simp add*: *less-imp-not-eq* [*OF less*] *divide-inverse mult-assoc*)
finally show *?thesis* **.**
qed

lemma *pos-divide-le-eq* [*field-simps*]: $0 < c \Longrightarrow (b/c \leq a) = (b \leq a*c)$
proof −
assume *less*: *0<c*
hence $(b/c \leq a) = ((b/c)*c \leq a*c)$
by (*simp add*: *mult-le-cancel-right less-not-sym* [*OF less*] *del*: *times-divide-eq*)
also have ... $= (b \leq a*c)$
by (*simp add*: *less-imp-not-eq2* [*OF less*] *divide-inverse mult-assoc*)
finally show *?thesis* **.**
qed

lemma *neg-divide-le-eq* [*field-simps*]: $c < 0 \Longrightarrow (b/c \leq a) = (a*c \leq b)$
proof −

assume *less*: $c<0$
hence $(b/c \leq a) = (a*c \leq (b/c)*c)$
by (*simp add*: *mult-le-cancel-right less-not-sym* [*OF less*] *del*: *times-divide-eq*)
also have $... = (a*c \leq b)$
by (*simp add*: *less-imp-not-eq* [*OF less*] *divide-inverse mult-assoc*)
finally show *?thesis* **.**
qed

Lemmas *sign-simps* is a first attempt to automate proofs of positivity/negativity needed for *field-simps*. Have not added *sign-simps* to *field-simps* because the former can lead to case explosions.

lemmas *sign-simps* [*no-atp*] = *algebra-simps*
zero-less-mult-iff mult-less-0-iff

lemmas (**in** −) *sign-simps* [*no-atp*] = *algebra-simps*
zero-less-mult-iff mult-less-0-iff

lemma *divide-pos-pos*:
$0 < x ==> 0 < y ==> 0 < x / y$
by(*simp add*:*field-simps*)

lemma *divide-nonneg-pos*:
$0 <= x ==> 0 < y ==> 0 <= x / y$
by(*simp add*:*field-simps*)

lemma *divide-neg-pos*:
$x < 0 ==> 0 < y ==> x / y < 0$
by(*simp add*:*field-simps*)

lemma *divide-nonpos-pos*:
$x <= 0 ==> 0 < y ==> x / y <= 0$
by(*simp add*:*field-simps*)

lemma *divide-pos-neg*:
$0 < x ==> y < 0 ==> x / y < 0$
by(*simp add*:*field-simps*)

lemma *divide-nonneg-neg*:
$0 <= x ==> y < 0 ==> x / y <= 0$
by(*simp add*:*field-simps*)

lemma *divide-neg-neg*:
$x < 0 ==> y < 0 ==> 0 < x / y$
by(*simp add*:*field-simps*)

lemma *divide-nonpos-neg*:
$x <= 0 ==> y < 0 ==> 0 <= x / y$

by(*simp add*:*field-simps*)

lemma *divide-strict-right-mono*:
[|$a < b$; $0 < c$|] ==> a / $c < b$ / c
by (*simp add*: *less-imp-not-eq2 divide-inverse mult-strict-right-mono*
positive-imp-inverse-positive)

lemma *divide-strict-right-mono-neg*:
[|$b < a$; $c < 0$|] ==> a / $c < b$ / c
apply (*drule divide-strict-right-mono* [*of* - - $-c$], *simp*)
apply (*simp add*: *less-imp-not-eq nonzero-minus-divide-right* [*symmetric*])
done

The last premise ensures that a and b have the same sign

lemma *divide-strict-left-mono*:
[|$b < a$; $0 < c$; $0 < a*b$|] ==> c / $a < c$ / b
by(*auto simp*: *field-simps times-divide-eq zero-less-mult-iff mult-strict-right-mono*)

lemma *divide-left-mono*:
[|$b \leq a$; $0 \leq c$; $0 < a*b$|] ==> c / $a \leq c$ / b
by(*auto simp*: *field-simps times-divide-eq zero-less-mult-iff mult-right-mono*)

lemma *divide-strict-left-mono-neg*:
[|$a < b$; $c < 0$; $0 < a*b$|] ==> c / $a < c$ / b
by(*auto simp*: *field-simps times-divide-eq zero-less-mult-iff mult-strict-right-mono-neg*)

lemma *mult-imp-div-pos-le*: $0 < y$ ==> $x <= z * y$ ==>
x / $y <= z$
by (*subst pos-divide-le-eq*, *assumption*+)

lemma *mult-imp-le-div-pos*: $0 < y$ ==> $z * y <= x$ ==>
$z <= x$ / y
by(*simp add*:*field-simps*)

lemma *mult-imp-div-pos-less*: $0 < y$ ==> $x < z * y$ ==>
x / $y < z$
by(*simp add*:*field-simps*)

lemma *mult-imp-less-div-pos*: $0 < y$ ==> $z * y < x$ ==>
$z < x$ / y
by(*simp add*:*field-simps*)

lemma *frac-le*: $0 <= x$ ==>
$x <= y$ ==> $0 < w$ ==> $w <= z$ ==> x / $z <= y$ / w
apply (*rule mult-imp-div-pos-le*)
apply *simp*
apply (*subst times-divide-eq-left*)
apply (*rule mult-imp-le-div-pos*, *assumption*)

apply (*rule mult-mono*)
apply *simp-all*
done

lemma *frac-less*: *0 <= x ==>*
x < y ==> 0 < w ==> w <= z ==> x / z < y / w
apply (*rule mult-imp-div-pos-less*)
apply *simp*
apply (*subst times-divide-eq-left*)
apply (*rule mult-imp-less-div-pos*, *assumption*)
apply (*erule mult-less-le-imp-less*)
apply *simp-all*
done

lemma *frac-less2*: *0 < x ==>*
x <= y ==> 0 < w ==> w < z ==> x / z < y / w
apply (*rule mult-imp-div-pos-less*)
apply *simp-all*
apply (*rule mult-imp-less-div-pos*, *assumption*)
apply (*erule mult-le-less-imp-less*)
apply *simp-all*
done

It's not obvious whether these should be simprules or not. Their effect is to gather terms into one big fraction, like a*b*c / x*y*z. The rationale for that is unclear, but many proofs seem to need them.

lemma *less-half-sum*: *a < b ==> a < (a+b) / (1+1)*
by (*simp add*: *field-simps zero-less-two*)

lemma *gt-half-sum*: *a < b ==> (a+b)/(1+1) < b*
by (*simp add*: *field-simps zero-less-two*)

subclass *dense-linorder*
proof
fix *x y* :: *'a*
from *less-add-one* **show** ∃ *y*. *x < y* **..**
from *less-add-one* **have** *x + (− 1) < (x + 1) + (− 1)* **by** (*rule add-strict-right-mono*)
then have *x − 1 < x + 1 − 1* **by** (*simp only*: *diff-minus* [*symmetric*])
then have *x − 1 < x* **by** (*simp add*: *algebra-simps*)
then show ∃ *y*. *y < x* **..**
show *x < y* ⟹ ∃ *z*>*x*. *z < y* **by** (*blast intro*!: *less-half-sum gt-half-sum*)
qed

lemma *nonzero-abs-inverse*:
a ≠ 0 ==> |inverse a| = inverse |a|
apply (*auto simp add*: *neq-iff abs-if nonzero-inverse-minus-eq*
negative-imp-inverse-negative)
apply (*blast intro*: *positive-imp-inverse-positive elim*: *less-asym*)
done

lemma *nonzero-abs-divide*:
  $b \neq 0 \Longrightarrow |a\ /\ b| = |a|\ /\ |b|$
by (*simp add*: *divide-inverse abs-mult nonzero-abs-inverse*)

lemma *field-le-epsilon*:
assumes *e*: $\bigwedge e.\ 0 < e \Longrightarrow x \leq y + e$
shows $x \leq y$
proof (*rule dense-le*)
fix *t* **assume** $t < x$
hence $0 < x - t$ **by** (*simp add*: *less-diff-eq*)
from *e* [*OF this*] **have** $x + 0 \leq x + (y - t)$ **by** (*simp add*: *algebra-simps*)
then have $0 \leq y - t$ **by** (*simp only*: *add-le-cancel-left*)
then show $t \leq y$ **by** (*simp add*: *algebra-simps*)
qed

end

class *linordered-field-inverse-zero* = *linordered-field* + *field-inverse-zero*
begin

lemma *le-divide-eq*:
$(a \leq b/c) =$
(*if* $0 < c$ *then* $a*c \leq b$
else if $c < 0$ *then* $b \leq a*c$
else $a \leq 0$)
apply (*cases* $c=0$, *simp*)
apply (*force simp add*: *pos-le-divide-eq neg-le-divide-eq linorder-neq-iff*)
done

lemma *inverse-positive-iff-positive* [*simp*]:
$(0 < inverse\ a) = (0 < a)$
apply (*cases* $a = 0$, *simp*)
apply (*blast intro*: *inverse-positive-imp-positive positive-imp-inverse-positive*)
done

lemma *inverse-negative-iff-negative* [*simp*]:
$(inverse\ a < 0) = (a < 0)$
apply (*cases* $a = 0$, *simp*)
apply (*blast intro*: *inverse-negative-imp-negative negative-imp-inverse-negative*)
done

lemma *inverse-nonnegative-iff-nonnegative* [*simp*]:
$0 \leq inverse\ a \longleftrightarrow 0 \leq a$
by (*simp add*: *not-less* [*symmetric*])

lemma *inverse-nonpositive-iff-nonpositive* [*simp*]:
$inverse\ a \leq 0 \longleftrightarrow a \leq 0$
by (*simp add*: *not-less* [*symmetric*])

lemma *one-less-inverse-iff*:
  $1 < inverse\ x \longleftrightarrow 0 < x \wedge x < 1$
proof *cases*
  assume $0 < x$
    with *inverse-less-iff-less* [*OF zero-less-one*, *of x*]
    show *?thesis* **by** *simp*
next
  assume *notless*: $\sim (0 < x)$
  have $\sim (1 < inverse\ x)$
  proof
    assume $1 < inverse\ x$
    also with *notless* **have** $... \leq 0$ **by** *simp*
    also have $... < 1$ **by** (*rule zero-less-one*)
    finally show *False* **by** *auto*
  qed
  with *notless* **show** *?thesis* **by** *simp*
qed

lemma *one-le-inverse-iff*:
  $1 \leq inverse\ x \longleftrightarrow 0 < x \wedge x \leq 1$
proof (*cases* $x = 1$)
  case *True* **then show** *?thesis* **by** *simp*
next
  case *False* **then have** $inverse\ x \neq 1$ **by** *simp*
  then have $1 \neq inverse\ x$ **by** *blast*
  then have $1 \leq inverse\ x \longleftrightarrow 1 < inverse\ x$ **by** (*simp add*: *le-less*)
  with *False* **show** *?thesis* **by** (*auto simp add*: *one-less-inverse-iff*)
qed

lemma *inverse-less-1-iff*:
  $inverse\ x < 1 \longleftrightarrow x \leq 0 \vee 1 < x$
  by (*simp add*: *not-le* [*symmetric*] *one-le-inverse-iff*)

lemma *inverse-le-1-iff*:
  $inverse\ x \leq 1 \longleftrightarrow x \leq 0 \vee 1 \leq x$
  by (*simp add*: *not-less* [*symmetric*] *one-less-inverse-iff*)

lemma *divide-le-eq*:
  $(b/c \leq a) =$
   (*if* $0 < c$ *then* $b \leq a*c$
         else if $c < 0$ *then* $a*c \leq b$
         else $0 \leq a$)
apply (*cases* $c=0$, *simp*)
apply (*force simp add*: *pos-divide-le-eq neg-divide-le-eq*)
done

lemma *less-divide-eq*:
  $(a < b/c) =$

(*if* $0 < c$ *then* $a*c < b$
else if $c < 0$ *then* $b < a*c$
else $a < 0$)
apply (*cases* $c=0$, *simp*)
apply (*force simp add*: *pos-less-divide-eq neg-less-divide-eq*)
done

lemma *divide-less-eq*:
$(b/c < a) =$
(*if* $0 < c$ *then* $b < a*c$
else if $c < 0$ *then* $a*c < b$
else $0 < a$)
apply (*cases* $c=0$, *simp*)
apply (*force simp add*: *pos-divide-less-eq neg-divide-less-eq*)
done

Division and Signs

lemma *zero-less-divide-iff*:
$(0 < a/b) = (0 < a \ \& \ 0 < b \mid a < 0 \ \& \ b < 0)$
by (*simp add*: *divide-inverse zero-less-mult-iff*)

lemma *divide-less-0-iff*:
$(a/b < 0) =$
$(0 < a \ \& \ b < 0 \mid a < 0 \ \& \ 0 < b)$
by (*simp add*: *divide-inverse mult-less-0-iff*)

lemma *zero-le-divide-iff*:
$(0 \leq a/b) =$
$(0 \leq a \ \& \ 0 \leq b \mid a \leq 0 \ \& \ b \leq 0)$
by (*simp add*: *divide-inverse zero-le-mult-iff*)

lemma *divide-le-0-iff*:
$(a/b \leq 0) =$
$(0 \leq a \ \& \ b \leq 0 \mid a \leq 0 \ \& \ 0 \leq b)$
by (*simp add*: *divide-inverse mult-le-0-iff*)

Division and the Number One

Simplify expressions equated with 1

lemma *zero-eq-1-divide-iff* [*simp*,*no-atp*]:
$(0 = 1/a) = (a = 0)$
apply (*cases* $a=0$, *simp*)
apply (*auto simp add*: *nonzero-eq-divide-eq*)
done

lemma *one-divide-eq-0-iff* [*simp*,*no-atp*]:
$(1/a = 0) = (a = 0)$
apply (*cases* $a=0$, *simp*)
apply (*insert zero-neq-one* [*THEN not-sym*])

apply (*auto simp add*: *nonzero-divide-eq-eq*)
done

Simplify expressions such as $0 < 1/x$ to $0 < x$

lemma *zero-le-divide-1-iff* [*simp*, *no-atp*]:
$0 \leq 1 \,/\, a \longleftrightarrow 0 \leq a$
by (*simp add*: *zero-le-divide-iff*)

lemma *zero-less-divide-1-iff* [*simp*, *no-atp*]:
$0 < 1 \,/\, a \longleftrightarrow 0 < a$
by (*simp add*: *zero-less-divide-iff*)

lemma *divide-le-0-1-iff* [*simp*, *no-atp*]:
$1 \,/\, a \leq 0 \longleftrightarrow a \leq 0$
by (*simp add*: *divide-le-0-iff*)

lemma *divide-less-0-1-iff* [*simp*, *no-atp*]:
$1 \,/\, a < 0 \longleftrightarrow a < 0$
by (*simp add*: *divide-less-0-iff*)

lemma *divide-right-mono*:
$[|a \leq b; 0 \leq c|] \Longrightarrow a/c \leq b/c$
by (*force simp add*: *divide-strict-right-mono le-less*)

lemma *divide-right-mono-neg*: $a <= b$
$\Longrightarrow c <= 0 \Longrightarrow b \,/\, c <= a \,/\, c$
apply (*drule divide-right-mono* [*of* - - − *c*])
apply *auto*
done

lemma *divide-left-mono-neg*: $a <= b$
$\Longrightarrow c <= 0 \Longrightarrow 0 < a * b \Longrightarrow c \,/\, a <= c \,/\, b$
apply (*drule divide-left-mono* [*of* - - − *c*])
apply (*auto simp add*: *mult-commute*)
done

Simplify quotients that are compared with the value 1.

lemma *le-divide-eq-1* [*no-atp*]:
$(1 \leq b \,/\, a) = ((0 < a \;\&\; a \leq b) \mid (a < 0 \;\&\; b \leq a))$
by (*auto simp add*: *le-divide-eq*)

lemma *divide-le-eq-1* [*no-atp*]:
$(b \,/\, a \leq 1) = ((0 < a \;\&\; b \leq a) \mid (a < 0 \;\&\; a \leq b) \mid a=0)$
by (*auto simp add*: *divide-le-eq*)

lemma *less-divide-eq-1* [*no-atp*]:
$(1 < b \,/\, a) = ((0 < a \;\&\; a < b) \mid (a < 0 \;\&\; b < a))$
by (*auto simp add*: *less-divide-eq*)

lemma *divide-less-eq-1* [*no-atp*]:
$(b \ / \ a < 1) = ((0 < a \ \& \ b < a) \mid (a < 0 \ \& \ a < b) \mid a=0)$
by (*auto simp add*: *divide-less-eq*)

Conditional Simplification Rules: No Case Splits

lemma *le-divide-eq-1-pos* [*simp*,*no-atp*]:
$0 < a \Longrightarrow (1 \leq b/a) = (a \leq b)$
by (*auto simp add*: *le-divide-eq*)

lemma *le-divide-eq-1-neg* [*simp*,*no-atp*]:
$a < 0 \Longrightarrow (1 \leq b/a) = (b \leq a)$
by (*auto simp add*: *le-divide-eq*)

lemma *divide-le-eq-1-pos* [*simp*,*no-atp*]:
$0 < a \Longrightarrow (b/a \leq 1) = (b \leq a)$
by (*auto simp add*: *divide-le-eq*)

lemma *divide-le-eq-1-neg* [*simp*,*no-atp*]:
$a < 0 \Longrightarrow (b/a \leq 1) = (a \leq b)$
by (*auto simp add*: *divide-le-eq*)

lemma *less-divide-eq-1-pos* [*simp*,*no-atp*]:
$0 < a \Longrightarrow (1 < b/a) = (a < b)$
by (*auto simp add*: *less-divide-eq*)

lemma *less-divide-eq-1-neg* [*simp*,*no-atp*]:
$a < 0 \Longrightarrow (1 < b/a) = (b < a)$
by (*auto simp add*: *less-divide-eq*)

lemma *divide-less-eq-1-pos* [*simp*,*no-atp*]:
$0 < a \Longrightarrow (b/a < 1) = (b < a)$
by (*auto simp add*: *divide-less-eq*)

lemma *divide-less-eq-1-neg* [*simp*,*no-atp*]:
$a < 0 \Longrightarrow b/a < 1 <-> a < b$
by (*auto simp add*: *divide-less-eq*)

lemma *eq-divide-eq-1* [*simp*,*no-atp*]:
$(1 = b/a) = ((a \neq 0 \ \& \ a = b))$
by (*auto simp add*: *eq-divide-eq*)

lemma *divide-eq-eq-1* [*simp*,*no-atp*]:
$(b/a = 1) = ((a \neq 0 \ \& \ a = b))$
by (*auto simp add*: *divide-eq-eq*)

lemma *abs-inverse* [*simp*]:
$|inverse\ a| =$
$inverse\ |a|$
apply (*cases a=0*, *simp*)

apply (*simp add*: *nonzero-abs-inverse*)
done

lemma *abs-divide* [*simp*]:
  $|a\ /\ b| = |a|\ /\ |b|$
apply (*cases b=0*, *simp*)
apply (*simp add*: *nonzero-abs-divide*)
done

lemma *abs-div-pos*: $0 < y \Longrightarrow$
  $|x|\ /\ y = |x\ /\ y|$
 apply (*subst abs-divide*)
 apply (*simp add*: *order-less-imp-le*)
done

lemma *field-le-mult-one-interval*:
 assumes $*$: $\bigwedge z.\ [\![\ 0 < z\ ;\ z < 1\ ]\!] \Longrightarrow z * x \leq y$
 shows $x \leq y$
proof (*cases* $0 < x$)
 assume $0 < x$
 thus *?thesis*
  using *dense-le-bounded*[*of 0 1 y*/*x*] $*$
  unfolding *le-divide-eq if-P*[*OF* ‹$0 < x$›] **by** *simp*
next
 assume $\neg 0 < x$ **hence** $x \leq 0$ **by** *simp*
 obtain s::$'a$ **where** s: $0 < s\ s < 1$ **using** *dense*[*of 0 1*::$'a$] **by** *auto*
 hence $x \leq s * x$ **using** *mult-le-cancel-right*[*of 1 x s*] ‹$x \leq 0$› **by** *auto*
 also note $*$[*OF s*]
 finally show *?thesis* **.**
qed

end

code-modulename *SML*
 Fields Arith

code-modulename *OCaml*
 Fields Arith

code-modulename *Haskell*
 Fields Arith

end

15 Nat: Natural numbers

theory *Nat*
imports *Inductive Typedef Fun Fields*

uses
~~/*src*/*Tools*/*rat.ML*
~~/*src*/*Provers*/*Arith*/*cancel-sums.ML*
Tools/*arith-data.ML*
(*Tools*/*nat-arith.ML*)
~~/*src*/*Provers*/*Arith*/*fast-lin-arith.ML*
(*Tools*/*lin-arith.ML*)
begin

15.1 Type *ind*

typedecl *ind*

axiomatization
Zero-Rep :: *ind* **and**
Suc-Rep :: *ind* => *ind*
where
— the axiom of infinity in 2 parts
Suc-Rep-inject: *Suc-Rep x* = *Suc-Rep y* ==> *x* = *y* **and**
Suc-Rep-not-Zero-Rep: *Suc-Rep x* ≠ *Zero-Rep*

15.2 Type nat

Type definition

inductive *Nat* :: *ind* ⇒ *bool*
where
Zero-RepI: *Nat Zero-Rep*
| *Suc-RepI*: *Nat i* ⟹ *Nat* (*Suc-Rep i*)

global

typedef (**open** *Nat*)
nat = *Nat*
by (*rule exI*, *unfold mem-def*, *rule Nat.Zero-RepI*)

definition *Suc* :: *nat* => *nat* **where**
Suc-def: *Suc* == (%*n*. *Abs-Nat* (*Suc-Rep* (*Rep-Nat n*)))

local

instantiation *nat* :: *zero*
begin

definition *Zero-nat-def* [*code del*]:
0 = *Abs-Nat Zero-Rep*

instance ..

end

lemma *Suc-not-Zero*: *Suc m ≠ 0*
by (*simp add*: *Zero-nat-def Suc-def Abs-Nat-inject* [*unfolded mem-def*]
Rep-Nat [*unfolded mem-def*] *Suc-RepI Zero-RepI Suc-Rep-not-Zero-Rep* [*unfolded mem-def*])

lemma *Zero-not-Suc*: *0 ≠ Suc m*
by (*rule not-sym*, *rule Suc-not-Zero not-sym*)

lemma *Suc-Rep-inject'*: *Suc-Rep x = Suc-Rep y ⟷ x = y*
by (*rule iffI*, *rule Suc-Rep-inject*) *simp-all*

rep-datatype *0* :: *nat Suc*
apply (*unfold Zero-nat-def Suc-def*)
apply (*rule Rep-Nat-inverse* [*THEN subst*]) — types force good instantiation
apply (*erule Rep-Nat* [*unfolded mem-def*, *THEN Nat.induct*])
apply (*iprover elim*: *Abs-Nat-inverse* [*unfolded mem-def*, *THEN subst*])
apply (*simp-all add*: *Abs-Nat-inject* [*unfolded mem-def*] *Rep-Nat* [*unfolded mem-def*]
Suc-RepI Zero-RepI Suc-Rep-not-Zero-Rep [*unfolded mem-def*]
Suc-Rep-not-Zero-Rep [*unfolded mem-def*, *symmetric*]
Suc-Rep-inject' Rep-Nat-inject)
done

lemma *nat-induct* [*case-names 0 Suc*, *induct type*: *nat*]:
— for backward compatibility – names of variables differ
fixes *n*
assumes *P 0*
and ⋀*n*. *P n* ⟹ *P* (*Suc n*)
shows *P n*
using *assms* **by** (*rule nat.induct*)

declare *nat.exhaust* [*case-names 0 Suc*, *cases type*: *nat*]

lemmas *nat-rec-0* = *nat.recs*(*1*)
and *nat-rec-Suc* = *nat.recs*(*2*)

lemmas *nat-case-0* = *nat.cases*(*1*)
and *nat-case-Suc* = *nat.cases*(*2*)

Injectiveness and distinctness lemmas

lemma *inj-Suc*[*simp*]: *inj-on Suc N*
by (*simp add*: *inj-on-def*)

lemma *Suc-neq-Zero*: *Suc m = 0* ⟹ *R*
by (*rule notE*, *rule Suc-not-Zero*)

lemma *Zero-neq-Suc*: *0 = Suc m* ⟹ *R*
by (*rule Suc-neq-Zero*, *erule sym*)

lemma *Suc-inject*: $Suc\ x = Suc\ y \Longrightarrow x = y$
by (*rule inj-Suc* [*THEN injD*])

lemma *n-not-Suc-n*: $n \neq Suc\ n$
by (*induct n*) *simp-all*

lemma *Suc-n-not-n*: $Suc\ n \neq n$
by (*rule not-sym*, *rule n-not-Suc-n*)

A special form of induction for reasoning about $m < n$ and $m - n$

lemma *diff-induct*: $(!!x.\ P\ x\ 0) ==> (!!y.\ P\ 0\ (Suc\ y)) ==>$
$(!!x\ y.\ P\ x\ y ==> P\ (Suc\ x)\ (Suc\ y)) ==> P\ m\ n$
apply (*rule-tac x* = *m* **in** *spec*)
apply (*induct n*)
prefer *2*
apply (*rule allI*)
apply (*induct-tac x*, *iprover*+)
done

15.3 Arithmetic operators

instantiation *nat* :: {*minus*, *comm-monoid-add*}
begin

primrec *plus-nat*
where
add-0: $0 + n = (n::nat)$
| *add-Suc*: $Suc\ m + n = Suc\ (m + n)$

lemma *add-0-right* [*simp*]: $m + 0 = (m::nat)$
by (*induct m*) *simp-all*

lemma *add-Suc-right* [*simp*]: $m + Suc\ n = Suc\ (m + n)$
by (*induct m*) *simp-all*

declare *add-0* [*code*]

lemma *add-Suc-shift* [*code*]: $Suc\ m + n = m + Suc\ n$
by *simp*

primrec *minus-nat*
where
diff-0: $m - 0 = (m::nat)$
| *diff-Suc*: $m - Suc\ n = (case\ m - n\ of\ 0 => 0 \mid Suc\ k => k)$

declare *diff-Suc* [*simp del*]
declare *diff-0* [*code*]

lemma *diff-0-eq-0* [*simp*, *code*]: *0 − n = (0::nat)*
by (*induct n*) (*simp-all add*: *diff-Suc*)

lemma *diff-Suc-Suc* [*simp*, *code*]: *Suc m − Suc n = m − n*
by (*induct n*) (*simp-all add*: *diff-Suc*)

instance proof
fix *n m q* :: *nat*
show *(n + m) + q = n + (m + q)* **by** (*induct n*) *simp-all*
show *n + m = m + n* **by** (*induct n*) *simp-all*
show *0 + n = n* **by** *simp*
qed

end

hide-fact (**open**) *add-0 add-0-right diff-0*

instantiation *nat* :: *comm-semiring-1-cancel*
begin

definition
One-nat-def [*simp*]: *1 = Suc 0*

primrec *times-nat*
where
mult-0: *0 * n = (0::nat)*
| *mult-Suc*: *Suc m * n = n + (m * n)*

lemma *mult-0-right* [*simp*]: *(m::nat) * 0 = 0*
by (*induct m*) *simp-all*

lemma *mult-Suc-right* [*simp*]: *m * Suc n = m + (m * n)*
by (*induct m*) (*simp-all add*: *add-left-commute*)

lemma *add-mult-distrib*: *(m + n) * k = (m * k) + ((n * k)::nat)*
by (*induct m*) (*simp-all add*: *add-assoc*)

instance proof
fix *n m q* :: *nat*
show *0 ≠ (1::nat)* **unfolding** *One-nat-def* **by** *simp*
show *1 * n = n* **unfolding** *One-nat-def* **by** *simp*
show *n * m = m * n* **by** (*induct n*) *simp-all*
show *(n * m) * q = n * (m * q)* **by** (*induct n*) (*simp-all add*: *add-mult-distrib*)
show *(n + m) * q = n * q + m * q* **by** (*rule add-mult-distrib*)
assume *n + m = n + q* **thus** *m = q* **by** (*induct n*) *simp-all*
qed

end

15.3.1 Addition

lemma *nat-add-assoc*: $(m + n) + k = m + ((n + k)::nat)$
by (*rule add-assoc*)

lemma *nat-add-commute*: $m + n = n + (m::nat)$
by (*rule add-commute*)

lemma *nat-add-left-commute*: $x + (y + z) = y + ((x + z)::nat)$
by (*rule add-left-commute*)

lemma *nat-add-left-cancel* [*simp*]: $(k + m = k + n) = (m = (n::nat))$
by (*rule add-left-cancel*)

lemma *nat-add-right-cancel* [*simp*]: $(m + k = n + k) = (m=(n::nat))$
by (*rule add-right-cancel*)

Reasoning about $m + 0 = 0$, etc.

lemma *add-is-0* [*iff*]:
fixes *m n* :: *nat*
shows $(m + n = 0) = (m = 0 \ \& \ n = 0)$
by (*cases m*) *simp-all*

lemma *add-is-1*:
$(m+n= Suc\ 0) = (m= Suc\ 0 \ \& \ n=0 \mid m=0 \ \& \ n= Suc\ 0)$
by (*cases m*) *simp-all*

lemma *one-is-add*:
$(Suc\ 0 = m + n) = (m = Suc\ 0 \ \& \ n = 0 \mid m = 0 \ \& \ n = Suc\ 0)$
by (*rule trans*, *rule eq-commute*, *rule add-is-1*)

lemma *add-eq-self-zero*:
fixes *m n* :: *nat*
shows $m + n = m \Longrightarrow n = 0$
by (*induct m*) *simp-all*

lemma *inj-on-add-nat*[*simp*]: *inj-on* (%*n*::*nat*. *n*+*k*) *N*
apply (*induct k*)
apply *simp*
apply(*drule comp-inj-on*[*OF - inj-Suc*])
apply (*simp add*:*o-def*)
done

15.3.2 Difference

lemma *diff-self-eq-0* [*simp*]: $(m::nat) - m = 0$
by (*induct m*) *simp-all*

lemma *diff-diff-left*: $(i::nat) - j - k = i - (j + k)$
by (*induct i j rule*: *diff-induct*) *simp-all*

lemma *Suc-diff-diff* [*simp*]: $(Suc\ m - n) - Suc\ k = m - n - k$
by (*simp add*: *diff-diff-left*)

lemma *diff-commute*: $(i::nat) - j - k = i - k - j$
by (*simp add*: *diff-diff-left add-commute*)

lemma *diff-add-inverse*: $(n + m) - n = (m::nat)$
by (*induct n*) *simp-all*

lemma *diff-add-inverse2*: $(m + n) - n = (m::nat)$
by (*simp add*: *diff-add-inverse add-commute* [*of m n*])

lemma *diff-cancel*: $(k + m) - (k + n) = m - (n::nat)$
by (*induct k*) *simp-all*

lemma *diff-cancel2*: $(m + k) - (n + k) = m - (n::nat)$
by (*simp add*: *diff-cancel add-commute*)

lemma *diff-add-0*: $n - (n + m) = (0::nat)$
by (*induct n*) *simp-all*

lemma *diff-Suc-1* [*simp*]: $Suc\ n - 1 = n$
unfolding *One-nat-def* **by** *simp*

Difference distributes over multiplication

lemma *diff-mult-distrib*: $((m::nat) - n) * k = (m * k) - (n * k)$
by (*induct m n rule*: *diff-induct*) (*simp-all add*: *diff-cancel*)

lemma *diff-mult-distrib2*: $k * ((m::nat) - n) = (k * m) - (k * n)$
by (*simp add*: *diff-mult-distrib mult-commute* [*of k*])
— NOT added as rewrites, since sometimes they are used from right-to-left

15.3.3 Multiplication

lemma *nat-mult-assoc*: $(m * n) * k = m * ((n * k)::nat)$
by (*rule mult-assoc*)

lemma *nat-mult-commute*: $m * n = n * (m::nat)$
by (*rule mult-commute*)

lemma *add-mult-distrib2*: $k * (m + n) = (k * m) + ((k * n)::nat)$
by (*rule right-distrib*)

lemma *mult-is-0* [*simp*]: $((m::nat) * n = 0) = (m=0 \mid n=0)$
by (*induct m*) *auto*

lemmas *nat-distrib* =
add-mult-distrib add-mult-distrib2 diff-mult-distrib diff-mult-distrib2

lemma *mult-eq-1-iff* [*simp*]: $(m * n = Suc\ 0) = (m = Suc\ 0 \ \&\ n = Suc\ 0)$
apply (*induct m*)
apply *simp*
apply (*induct n*)
apply *auto*
done

lemma *one-eq-mult-iff* [*simp*,*no-atp*]: $(Suc\ 0 = m * n) = (m = Suc\ 0 \ \&\ n = Suc\ 0)$
apply (*rule trans*)
apply (*rule-tac* [*2*] *mult-eq-1-iff*, *fastsimp*)
done

lemma *nat-mult-eq-1-iff* [*simp*]: $m * n = (1::nat) \longleftrightarrow m = 1 \wedge n = 1$
unfolding *One-nat-def* **by** (*rule mult-eq-1-iff*)

lemma *nat-1-eq-mult-iff* [*simp*]: $(1::nat) = m * n \longleftrightarrow m = 1 \wedge n = 1$
unfolding *One-nat-def* **by** (*rule one-eq-mult-iff*)

lemma *mult-cancel1* [*simp*]: $(k * m = k * n) = (m = n \mid (k = (0::nat)))$
proof −
have $k \neq 0 \Longrightarrow k * m = k * n \Longrightarrow m = n$
proof (*induct n arbitrary*: *m*)
case *0* **then show** $m = 0$ **by** *simp*
next
case (*Suc n*) **then show** $m = Suc\ n$
by (*cases m*) (*simp-all add*: *eq-commute* [*of 0*])
qed
then show *?thesis* **by** *auto*
qed

lemma *mult-cancel2* [*simp*]: $(m * k = n * k) = (m = n \mid (k = (0::nat)))$
by (*simp add*: *mult-commute*)

lemma *Suc-mult-cancel1*: $(Suc\ k * m = Suc\ k * n) = (m = n)$
by (*subst mult-cancel1*) *simp*

15.4 Orders on *nat*

15.4.1 Operation definition

instantiation *nat* :: *linorder*
begin

primrec *less-eq-nat* **where**
$(0::nat) \leq n \longleftrightarrow True$
| $Suc\ m \leq n \longleftrightarrow (case\ n\ of\ 0 \Rightarrow False \mid Suc\ n \Rightarrow m \leq n)$

declare *less-eq-nat.simps* [*simp del*]

lemma [*code*]: (*0*::*nat*) ≤ *n* ⟷ *True* **by** (*simp add*: *less-eq-nat.simps*)
lemma *le0* [*iff*]: *0* ≤ (*n*::*nat*) **by** (*simp add*: *less-eq-nat.simps*)

definition *less-nat* **where**
less-eq-Suc-le: *n* < *m* ⟷ *Suc n* ≤ *m*

lemma *Suc-le-mono* [*iff*]: *Suc n* ≤ *Suc m* ⟷ *n* ≤ *m*
by (*simp add*: *less-eq-nat.simps*(*2*))

lemma *Suc-le-eq* [*code*]: *Suc m* ≤ *n* ⟷ *m* < *n*
unfolding *less-eq-Suc-le* **..**

lemma *le-0-eq* [*iff*]: (*n*::*nat*) ≤ *0* ⟷ *n* = *0*
by (*induct n*) (*simp-all add*: *less-eq-nat.simps*(*2*))

lemma *not-less0* [*iff*]: ¬ *n* < (*0*::*nat*)
by (*simp add*: *less-eq-Suc-le*)

lemma *less-nat-zero-code* [*code*]: *n* < (*0*::*nat*) ⟷ *False*
by *simp*

lemma *Suc-less-eq* [*iff*]: *Suc m* < *Suc n* ⟷ *m* < *n*
by (*simp add*: *less-eq-Suc-le*)

lemma *less-Suc-eq-le* [*code*]: *m* < *Suc n* ⟷ *m* ≤ *n*
by (*simp add*: *less-eq-Suc-le*)

lemma *le-SucI*: *m* ≤ *n* ⟹ *m* ≤ *Suc n*
by (*induct m arbitrary*: *n*)
(*simp-all add*: *less-eq-nat.simps*(*2*) *split*: *nat.splits*)

lemma *Suc-leD*: *Suc m* ≤ *n* ⟹ *m* ≤ *n*
by (*cases n*) (*auto intro*: *le-SucI*)

lemma *less-SucI*: *m* < *n* ⟹ *m* < *Suc n*
by (*simp add*: *less-eq-Suc-le*) (*erule Suc-leD*)

lemma *Suc-lessD*: *Suc m* < *n* ⟹ *m* < *n*
by (*simp add*: *less-eq-Suc-le*) (*erule Suc-leD*)

instance
proof
fix *n m* :: *nat*
show *n* < *m* ⟷ *n* ≤ *m* ∧ ¬ *m* ≤ *n*
proof (*induct n arbitrary*: *m*)
case *0* **then show** *?case* **by** (*cases m*) (*simp-all add*: *less-eq-Suc-le*)
next
case (*Suc n*) **then show** *?case* **by** (*cases m*) (*simp-all add*: *less-eq-Suc-le*)
qed

next
fix n :: *nat* **show** $n \leq n$ **by** (*induct n*) *simp-all*
next
fix n m :: *nat* **assume** $n \leq m$ **and** $m \leq n$
then show $n = m$
by (*induct n arbitrary*: m)
(*simp-all add*: *less-eq-nat.simps*(*2*) *split*: *nat.splits*)
next
fix n m q :: *nat* **assume** $n \leq m$ **and** $m \leq q$
then show $n \leq q$
proof (*induct n arbitrary*: m q)
case *0* **show** *?case* **by** *simp*
next
case (*Suc n*) **then show** *?case*
by (*simp-all* (*no-asm-use*) *add*: *less-eq-nat.simps*(*2*) *split*: *nat.splits*, *clarify*,
simp-all (*no-asm-use*) *add*: *less-eq-nat.simps*(*2*) *split*: *nat.splits*, *clarify*,
simp-all (*no-asm-use*) *add*: *less-eq-nat.simps*(*2*) *split*: *nat.splits*)
qed
next
fix n m :: *nat* **show** $n \leq m \vee m \leq n$
by (*induct n arbitrary*: m)
(*simp-all add*: *less-eq-nat.simps*(*2*) *split*: *nat.splits*)
qed

end

instantiation *nat* :: *bot*
begin

definition *bot-nat* :: *nat* **where**
bot-nat = *0*

instance proof
qed (*simp add*: *bot-nat-def*)

end

15.4.2 Introduction properties

lemma *lessI* [*iff*]: $n <$ *Suc n*
by (*simp add*: *less-Suc-eq-le*)

lemma *zero-less-Suc* [*iff*]: *0* $<$ *Suc n*
by (*simp add*: *less-Suc-eq-le*)

15.4.3 Elimination properties

lemma *less-not-refl*: $\sim n <$ (n::*nat*)
by (*rule order-less-irrefl*)

lemma *less-not-refl2*: $n < m \Longrightarrow m \neq (n::nat)$
by (*rule not-sym*) (*rule less-imp-neq*)

lemma *less-not-refl3*: $(s::nat) < t \Longrightarrow s \neq t$
by (*rule less-imp-neq*)

lemma *less-irrefl-nat*: $(n::nat) < n \Longrightarrow R$
by (*rule notE*, *rule less-not-refl*)

lemma *less-zeroE*: $(n::nat) < 0 \Longrightarrow R$
by (*rule notE*) (*rule not-less0*)

lemma *less-Suc-eq*: $(m < Suc\ n) = (m < n \mid m = n)$
unfolding *less-Suc-eq-le le-less* **..**

lemma *less-Suc0* [*iff*]: $(n < Suc\ 0) = (n = 0)$
by (*simp add*: *less-Suc-eq*)

lemma *less-one* [*iff*, *no-atp*]: $(n < (1::nat)) = (n = 0)$
unfolding *One-nat-def* **by** (*rule less-Suc0*)

lemma *Suc-mono*: $m < n \Longrightarrow Suc\ m < Suc\ n$
by *simp*

"Less than" is antisymmetric, sort of

lemma *less-antisym*: $[\![\neg\ n < m;\ n < Suc\ m\ ]\!] \Longrightarrow m = n$
unfolding *not-less less-Suc-eq-le* **by** (*rule antisym*)

lemma *nat-neq-iff*: $((m::nat) \neq n) = (m < n \mid n < m)$
by (*rule linorder-neq-iff*)

lemma *nat-less-cases*: **assumes** *major*: $(m::nat) < n \Longrightarrow P\ n\ m$
and *eqCase*: $m = n \Longrightarrow P\ n\ m$ **and** *lessCase*: $n<m \Longrightarrow P\ n\ m$
shows $P\ n\ m$
apply (*rule less-linear* [*THEN disjE*])
apply (*erule-tac* [*2*] *disjE*)
apply (*erule lessCase*)
apply (*erule sym* [*THEN eqCase*])
apply (*erule major*)
done

15.4.4 Inductive (?) properties

lemma *Suc-lessI*: $m < n \Longrightarrow Suc\ m \neq n \Longrightarrow Suc\ m < n$
unfolding *less-eq-Suc-le* [*of m*] *le-less* **by** *simp*

lemma *lessE*:
assumes *major*: $i < k$
and *p1*: $k = Suc\ i \Longrightarrow P$ **and** *p2*: $!!j.\ i < j \Longrightarrow k = Suc\ j \Longrightarrow P$

shows P
proof −
from *major* **have** $\exists j.\ i \leq j \wedge k = Suc\ j$
unfolding *less-eq-Suc-le* **by** (*induct k*) *simp-all*
then have $(\exists j.\ i < j \wedge k = Suc\ j) \vee k = Suc\ i$
by (*clarsimp simp add*: *less-le*)
with *p1 p2* **show** P **by** *auto*
qed

lemma *less-SucE*: **assumes** *major*: $m < Suc\ n$
and *less*: $m < n \Longrightarrow P$ **and** *eq*: $m = n \Longrightarrow P$ **shows** P
apply (*rule major* [*THEN lessE*])
apply (*rule eq*, *blast*)
apply (*rule less*, *blast*)
done

lemma *Suc-lessE*: **assumes** *major*: $Suc\ i < k$
and *minor*: $!!j.\ i < j \Longrightarrow k = Suc\ j \Longrightarrow P$ **shows** P
apply (*rule major* [*THEN lessE*])
apply (*erule lessI* [*THEN minor*])
apply (*erule Suc-lessD* [*THEN minor*], *assumption*)
done

lemma *Suc-less-SucD*: $Suc\ m < Suc\ n \Longrightarrow m < n$
by *simp*

lemma *less-trans-Suc*:
assumes *le*: $i < j$ **shows** $j < k \Longrightarrow Suc\ i < k$
apply (*induct k*, *simp-all*)
apply (*insert le*)
apply (*simp add*: *less-Suc-eq*)
apply (*blast dest*: *Suc-lessD*)
done

Can be used with *less-Suc-eq* to get $n = m \vee n < m$

lemma *not-less-eq*: $\neg\ m < n \longleftrightarrow n < Suc\ m$
unfolding *not-less less-Suc-eq-le* **..**

lemma *not-less-eq-eq*: $\neg\ m \leq n \longleftrightarrow Suc\ n \leq m$
unfolding *not-le Suc-le-eq* **..**

Properties of "less than or equal"

lemma *le-imp-less-Suc*: $m \leq n \Longrightarrow m < Suc\ n$
unfolding *less-Suc-eq-le* **.**

lemma *Suc-n-not-le-n*: $\sim\ Suc\ n \leq n$
unfolding *not-le less-Suc-eq-le* **..**

lemma *le-Suc-eq*: $(m \leq Suc\ n) = (m \leq n \mid m = Suc\ n)$

by (*simp add*: *less-Suc-eq-le* [*symmetric*] *less-Suc-eq*)

lemma *le-SucE*: $m \leq Suc\ n \Longrightarrow (m \leq n \Longrightarrow R) \Longrightarrow (m = Suc\ n \Longrightarrow R) \Longrightarrow R$
by (*drule le-Suc-eq* [*THEN iffD1*], *iprover*+)

lemma *Suc-leI*: $m < n \Longrightarrow Suc(m) \leq n$
unfolding *Suc-le-eq* **.**

Stronger version of *Suc-leD*

lemma *Suc-le-lessD*: $Suc\ m \leq n \Longrightarrow m < n$
unfolding *Suc-le-eq* **.**

lemma *less-imp-le-nat*: $m < n \Longrightarrow m \leq (n::nat)$
unfolding *less-eq-Suc-le* **by** (*rule Suc-leD*)

For instance, $(Suc\ m < Suc\ n) = (Suc\ m \leq n) = (m < n)$

lemmas *le-simps* = *less-imp-le-nat less-Suc-eq-le Suc-le-eq*

Equivalence of $m \leq n$ and $m < n \vee m = n$

lemma *less-or-eq-imp-le*: $m < n \mid m = n \Longrightarrow m \leq (n::nat)$
unfolding *le-less* **.**

lemma *le-eq-less-or-eq*: $(m \leq (n::nat)) = (m < n \mid m=n)$
by (*rule le-less*)

Useful with *blast*.

lemma *eq-imp-le*: $(m::nat) = n \Longrightarrow m \leq n$
by *auto*

lemma *le-refl*: $n \leq (n::nat)$
by *simp*

lemma *le-trans*: $[\!|\ i \leq j; j \leq k\ |\!] \Longrightarrow i \leq (k::nat)$
by (*rule order-trans*)

lemma *le-antisym*: $[\!|\ m \leq n; n \leq m\ |\!] \Longrightarrow m = (n::nat)$
by (*rule antisym*)

lemma *nat-less-le*: $((m::nat) < n) = (m \leq n\ \&\ m \neq n)$
by (*rule less-le*)

lemma *le-neq-implies-less*: $(m::nat) \leq n \Longrightarrow m \neq n \Longrightarrow m < n$
unfolding *less-le* **..**

lemma *nat-le-linear*: $(m::nat) \leq n \mid n \leq m$
by (*rule linear*)

lemmas *linorder-neqE-nat* = *linorder-neqE* [**where** *'a* = *nat*]

lemma *le-less-Suc-eq*: $m \leq n \Longrightarrow (n < Suc\ m) = (n = m)$
unfolding *less-Suc-eq-le* **by** *auto*

lemma *not-less-less-Suc-eq*: $\sim n < m \Longrightarrow (n < Suc\ m) = (n = m)$
unfolding *not-less* **by** (*rule le-less-Suc-eq*)

lemmas *not-less-simps* = *not-less-less-Suc-eq le-less-Suc-eq*

These two rules ease the use of primitive recursion. NOTE USE OF ==

lemma *def-nat-rec-0*: $(!!n.\ f\ n == nat\text{-}rec\ c\ h\ n) \Longrightarrow f\ 0 = c$
by *simp*

lemma *def-nat-rec-Suc*: $(!!n.\ f\ n == nat\text{-}rec\ c\ h\ n) \Longrightarrow f\ (Suc\ n) = h\ n\ (f\ n)$
by *simp*

lemma *not0-implies-Suc*: $n \neq 0 \Longrightarrow \exists\, m.\ n = Suc\ m$
by (*cases n*) *simp-all*

lemma *gr0-implies-Suc*: $n > 0 \Longrightarrow \exists\, m.\ n = Suc\ m$
by (*cases n*) *simp-all*

lemma *gr-implies-not0*: **fixes** *n* :: *nat* **shows** $m<n \Longrightarrow n \neq 0$
by (*cases n*) *simp-all*

lemma *neq0-conv*[*iff*]: **fixes** *n* :: *nat* **shows** $(n \neq 0) = (0 < n)$
by (*cases n*) *simp-all*

This theorem is useful with *blast*

lemma *gr0I*: $((n::nat) = 0 \Longrightarrow False) \Longrightarrow 0 < n$
by (*rule neq0-conv*[*THEN iffD1*], *iprover*)

lemma *gr0-conv-Suc*: $(0 < n) = (\exists\, m.\ n = Suc\ m)$
by (*fast intro*: *not0-implies-Suc*)

lemma *not-gr0* [*iff*,*no-atp*]: $!!n::nat.\ (\sim (0 < n)) = (n = 0)$
using *neq0-conv* **by** *blast*

lemma *Suc-le-D*: $(Suc\ n \leq m') \Longrightarrow (?\ m.\ m' = Suc\ m)$
by (*induct m'*) *simp-all*

Useful in certain inductive arguments

lemma *less-Suc-eq-0-disj*: $(m < Suc\ n) = (m = 0 \mid (\exists j.\ m = Suc\ j\ \&\ j < n))$
by (*cases m*) *simp-all*

15.4.5 *min* and *max*

lemma *mono-Suc*: *mono Suc*

by (*rule monoI*) *simp*

lemma *min-0L* [*simp*]: *min 0 n* = (*0*::*nat*)
by (*rule min-leastL*) *simp*

lemma *min-0R* [*simp*]: *min n 0* = (*0*::*nat*)
by (*rule min-leastR*) *simp*

lemma *min-Suc-Suc* [*simp*]: *min* (*Suc m*) (*Suc n*) = *Suc* (*min m n*)
by (*simp add*: *mono-Suc min-of-mono*)

lemma *min-Suc1*:
min (*Suc n*) *m* = (*case m of 0* => *0* | *Suc m'* => *Suc*(*min n m'*))
by (*simp split*: *nat.split*)

lemma *min-Suc2*:
min m (*Suc n*) = (*case m of 0* => *0* | *Suc m'* => *Suc*(*min m' n*))
by (*simp split*: *nat.split*)

lemma *max-0L* [*simp*]: *max 0 n* = (*n*::*nat*)
by (*rule max-leastL*) *simp*

lemma *max-0R* [*simp*]: *max n 0* = (*n*::*nat*)
by (*rule max-leastR*) *simp*

lemma *max-Suc-Suc* [*simp*]: *max* (*Suc m*) (*Suc n*) = *Suc*(*max m n*)
by (*simp add*: *mono-Suc max-of-mono*)

lemma *max-Suc1*:
max (*Suc n*) *m* = (*case m of 0* => *Suc n* | *Suc m'* => *Suc*(*max n m'*))
by (*simp split*: *nat.split*)

lemma *max-Suc2*:
max m (*Suc n*) = (*case m of 0* => *Suc n* | *Suc m'* => *Suc*(*max m' n*))
by (*simp split*: *nat.split*)

15.4.6 Monotonicity of Addition

lemma *Suc-pred* [*simp*]: *n*>*0* ==> *Suc* (*n* − *Suc 0*) = *n*
by (*simp add*: *diff-Suc split*: *nat.split*)

lemma *Suc-diff-1* [*simp*]: *0* < *n* ==> *Suc* (*n* − *1*) = *n*
unfolding *One-nat-def* **by** (*rule Suc-pred*)

lemma *nat-add-left-cancel-le* [*simp*]: (*k* + *m* ≤ *k* + *n*) = (*m*≤(*n*::*nat*))
by (*induct k*) *simp-all*

lemma *nat-add-left-cancel-less* [*simp*]: (*k* + *m* < *k* + *n*) = (*m*<(*n*::*nat*))
by (*induct k*) *simp-all*

lemma *add-gr-0* [*iff*]: !!*m*::*nat*. (*m* + *n* > *0*) = (*m*>*0* | *n*>*0*)
by(*auto dest*:*gr0-implies-Suc*)

strict, in 1st argument

lemma *add-less-mono1*: *i* < *j* ==> *i* + *k* < *j* + (*k*::*nat*)
by (*induct k*) *simp-all*

strict, in both arguments

lemma *add-less-mono*: [|*i* < *j*; *k* < *l*|] ==> *i* + *k* < *j* + (*l*::*nat*)
apply (*rule add-less-mono1* [*THEN less-trans*], *assumption*+)
apply (*induct j*, *simp-all*)
done

Deleted *less-natE*; use *less-imp-Suc-add RS exE*

lemma *less-imp-Suc-add*: *m* < *n* ==> (∃ *k*. *n* = *Suc* (*m* + *k*))
apply (*induct n*)
apply (*simp-all add*: *order-le-less*)
apply (*blast elim*!: *less-SucE*
intro!: *Nat.add-0-right* [*symmetric*] *add-Suc-right* [*symmetric*])
done

strict, in 1st argument; proof is by induction on *k* > *0*

lemma *mult-less-mono2*: (*i*::*nat*) < *j* ==> *0*<*k* ==> *k* * *i* < *k* * *j*
apply(*auto simp*: *gr0-conv-Suc*)
apply (*induct-tac m*)
apply (*simp-all add*: *add-less-mono*)
done

The naturals form an ordered *comm-semiring-1-cancel*

instance *nat* :: *linordered-semidom*
proof
fix *i j k* :: *nat*
show *0* < (*1*::*nat*) **by** *simp*
show *i* ≤ *j* ==> *k* + *i* ≤ *k* + *j* **by** *simp*
show *i* < *j* ==> *0* < *k* ==> *k* * *i* < *k* * *j* **by** (*simp add*: *mult-less-mono2*)
qed

instance *nat* :: *no-zero-divisors*
proof
fix *a*::*nat* **and** *b*::*nat* **show** *a* ~= *0* ⟹ *b* ~= *0* ⟹ *a* * *b* ~= *0* **by** *auto*
qed

lemma *nat-mult-1*: (*1*::*nat*) * *n* = *n*
by *simp*

lemma *nat-mult-1-right*: *n* * (*1*::*nat*) = *n*
by *simp*

15.4.7 Additional theorems about *op* ≤

Complete induction, aka course-of-values induction

instance *nat* :: *wellorder* **proof**
  fix *P* **and** *n* :: *nat*
  assume *step*: ⋀*n*::*nat*. (⋀*m*. *m* < *n* ⟹ *P m*) ⟹ *P n*
  have ⋀*q*. *q* ≤ *n* ⟹ *P q*
  proof (*induct n*)
    case (*0 n*)
    have *P 0* **by** (*rule step*) *auto*
    thus *?case* **using** *0* **by** *auto*
  next
    case (*Suc m n*)
    then have *n* ≤ *m* ∨ *n* = *Suc m* **by** (*simp add*: *le-Suc-eq*)
    thus *?case*
    proof
      assume *n* ≤ *m* **thus** *P n* **by** (*rule Suc*(*1*))
    next
      assume *n*: *n* = *Suc m*
      show *P n*
        by (*rule step*) (*rule Suc*(*1*), *simp add*: *n le-simps*)
    qed
  qed
  then show *P n* **by** *auto*
qed

lemma *Least-Suc*:
    [| *P n*; ~ *P 0* |] ==> (*LEAST n*. *P n*) = *Suc* (*LEAST m*. *P*(*Suc m*))
  apply (*case-tac n*, *auto*)
  apply (*frule LeastI*)
  apply (*drule-tac P* = %*x*. *P* (*Suc x*) **in** *LeastI*)
  apply (*subgoal-tac* (*LEAST x*. *P x*) ≤ *Suc* (*LEAST x*. *P* (*Suc x*)))
  apply (*erule-tac* [*2*] *Least-le*)
  apply (*case-tac LEAST x*. *P x*, *auto*)
  apply (*drule-tac P* = %*x*. *P* (*Suc x*) **in** *Least-le*)
  apply (*blast intro*: *order-antisym*)
  done

lemma *Least-Suc2*:
   [|*P n*; *Q m*; ~*P 0*; !*k*. *P* (*Suc k*) = *Q k*|] ==> *Least P* = *Suc* (*Least Q*)
  apply (*erule* (*1*) *Least-Suc* [*THEN ssubst*])
  apply *simp*
  done

lemma *ex-least-nat-le*: ¬*P*(*0*) ⟹ *P*(*n*::*nat*) ⟹ ∃*k*≤*n*. (∀*i*<*k*. ¬*P i*) & *P*(*k*)
  apply (*cases n*)
   apply *blast*
  apply (*rule-tac x*=*LEAST k*. *P*(*k*) **in** *exI*)
  apply (*blast intro*: *Least-le dest*: *not-less-Least intro*: *LeastI-ex*)

done

lemma *ex-least-nat-less*: $\neg P(0) \Longrightarrow P(n::nat) \Longrightarrow \exists k{<}n.\ (\forall i{\leq}k.\ \neg P\ i)\ \&\ P(k{+}1)$
unfolding *One-nat-def*
apply (*cases n*)
apply *blast*
apply (*frule* (*1*) *ex-least-nat-le*)
apply (*erule exE*)
apply (*case-tac k*)
apply *simp*
apply (*rename-tac k1*)
apply (*rule-tac x=k1* **in** *exI*)
apply (*auto simp add*: *less-eq-Suc-le*)
done

lemma *nat-less-induct*:
assumes !!*n*. $\forall m::nat.\ m < n \longrightarrow P\ m \Longrightarrow P\ n$ **shows** *P n*
using *assms less-induct* **by** *blast*

lemma *measure-induct-rule* [*case-names less*]:
fixes *f* :: $'a \Rightarrow nat$
assumes *step*: $\bigwedge x.\ (\bigwedge y.\ f\ y < f\ x \Longrightarrow P\ y) \Longrightarrow P\ x$
shows *P a*
by (*induct m*≡*f a arbitrary*: *a rule*: *less-induct*) (*auto intro*: *step*)

old style induction rules:

lemma *measure-induct*:
fixes *f* :: $'a \Rightarrow nat$
shows $(\bigwedge x.\ \forall y.\ f\ y < f\ x \longrightarrow P\ y \Longrightarrow P\ x) \Longrightarrow P\ a$
by (*rule measure-induct-rule* [*of f P a*]) *iprover*

lemma *full-nat-induct*:
assumes *step*: (!!*n*. (*ALL m*. *Suc m* <= *n* --> *P m*) ==> *P n*)
shows *P n*
by (*rule less-induct*) (*auto intro*: *step simp*:*le-simps*)

An induction rule for estabilishing binary relations

lemma *less-Suc-induct*:
assumes *less*: *i* < *j*
and *step*: !!*i*. *P i* (*Suc i*)
and *trans*: !!*i j k*. *i* < *j* ==> *j* < *k* ==> *P i j* ==> *P j k* ==> *P i k*
shows *P i j*
proof −
from *less* **obtain** *k* **where** *j*: *j* = *Suc* (*i* + *k*) **by** (*auto dest*: *less-imp-Suc-add*)
have *P i* (*Suc* (*i* + *k*))
proof (*induct k*)
case *0*
show *?case* **by** (*simp add*: *step*)
next

```
    case (Suc k)
    have 0 + i < Suc k + i by (rule add-less-mono1) simp
    hence i < Suc (i + k) by (simp add: add-commute)
    from trans[OF this lessI Suc step]
    show ?case by simp
  qed
  thus P i j by (simp add: j)
qed
```

The method of infinite descent, frequently used in number theory. Provided by Roelof Oosterhuis. $P(n)$ is true for all $n \in \mathbb{N}$ if

- case "0": given $n = 0$ prove $P(n)$,
- case "smaller": given $n > 0$ and $\neg P(n)$ prove there exists a smaller integer m such that $\neg P(m)$.

A compact version without explicit base case:

```
lemma infinite-descent:
  ⟦ !!n::nat. ¬ P n ⟹ ∃ m<n. ¬ P m ⟧ ⟹ P n
by (induct n rule: less-induct, auto)
```

```
lemma infinite-descent0[case-names 0 smaller]:
  ⟦ P 0; !!n. n>0 ⟹ ¬ P n ⟹ (∃ m::nat. m < n ∧ ¬P m) ⟧ ⟹ P n
by (rule infinite-descent) (case-tac n>0, auto)
```

Infinite descent using a mapping to $\mathbb{N}$: $P(x)$ is true for all $x \in D$ if there exists a $V : D \to \mathbb{N}$ and

- case "0": given $V(x) = 0$ prove $P(x)$,
- case "smaller": given $V(x) > 0$ and $\neg P(x)$ prove there exists a $y \in D$ such that $V(y) < V(x)$ and $\neg P(y)$.

NB: the proof also shows how to use the previous lemma.

```
corollary infinite-descent0-measure [case-names 0 smaller]:
  assumes A0: !!x. V x = (0::nat) ⟹ P x
    and   A1: !!x. V x > 0 ⟹ ¬P x ⟹ (∃ y. V y < V x ∧ ¬P y)
  shows P x
proof −
  obtain n where n = V x by auto
  moreover have ⋀x. V x = n ⟹ P x
  proof (induct n rule: infinite-descent0)
    case 0 — i.e. V(x) = 0
    with A0 show P x by auto
  next — now n > 0 and P(x) does not hold for some x with V(x) = n
    case (smaller n)
    then obtain x where vxn: V x = n  and V x > 0 ∧ ¬ P x by auto
```

with *A1* **obtain** y **where** $V\ y < V\ x \wedge \neg\ P\ y$ **by** *auto*
with *vxn* **obtain** m **where** $m = V\ y \wedge m<n \wedge \neg\ P\ y$ **by** *auto*
then show *?case* **by** *auto*
qed
ultimately show $P\ x$ **by** *auto*
qed

Again, without explicit base case:

lemma *infinite-descent-measure*:
assumes $!!x.\ \neg\ P\ x \Longrightarrow \exists\, y.\ (V::'a\Rightarrow nat)\ y < V\ x \wedge \neg\ P\ y$ **shows** $P\ x$
proof −
from *assms* **obtain** n **where** $n = V\ x$ **by** *auto*
moreover have $!!x.\ V\ x = n \Longrightarrow P\ x$
proof (*induct n rule*: *infinite-descent*, *auto*)
fix x **assume** $\neg\ P\ x$
with *assms* **show** $\exists\, m < V\ x.\ \exists\, y.\ V\ y = m \wedge \neg\ P\ y$ **by** *auto*
qed
ultimately show $P\ x$ **by** *auto*
qed

A [clumsy] way of lifting $<$ monotonicity to $\leq$ monotonicity

lemma *less-mono-imp-le-mono*:
$[\![\ !!i\ j::nat.\ i < j \Longrightarrow f\ i < f\ j;\ i \leq j\ ]\!] \Longrightarrow f\ i \leq ((f\ j)::nat)$
by (*simp add*: *order-le-less*) (*blast*)

non-strict, in 1st argument

lemma *add-le-mono1*: $i \leq j \Longrightarrow i + k \leq j + (k::nat)$
by (*rule add-right-mono*)

non-strict, in both arguments

lemma *add-le-mono*: $[|\ i \leq j;\ \ k \leq l\ |] \Longrightarrow i + k \leq j + (l::nat)$
by (*rule add-mono*)

lemma *le-add2*: $n \leq ((m + n)::nat)$
by (*insert add-right-mono* [*of 0 m n*], *simp*)

lemma *le-add1*: $n \leq ((n + m)::nat)$
by (*simp add*: *add-commute*, *rule le-add2*)

lemma *less-add-Suc1*: $i < Suc\ (i + m)$
by (*rule le-less-trans*, *rule le-add1*, *rule lessI*)

lemma *less-add-Suc2*: $i < Suc\ (m + i)$
by (*rule le-less-trans*, *rule le-add2*, *rule lessI*)

lemma *less-iff-Suc-add*: $(m < n) = (\exists\, k.\ n = Suc\ (m + k))$
by (*iprover intro*!: *less-add-Suc1 less-imp-Suc-add*)

lemma *trans-le-add1*: $(i::nat) \leq j \Longrightarrow i \leq j + m$
by (*rule le-trans*, *assumption*, *rule le-add1*)

lemma *trans-le-add2*: $(i::nat) \leq j \Longrightarrow i \leq m + j$
by (*rule le-trans*, *assumption*, *rule le-add2*)

lemma *trans-less-add1*: $(i::nat) < j \Longrightarrow i < j + m$
by (*rule less-le-trans*, *assumption*, *rule le-add1*)

lemma *trans-less-add2*: $(i::nat) < j \Longrightarrow i < m + j$
by (*rule less-le-trans*, *assumption*, *rule le-add2*)

lemma *add-lessD1*: $i + j < (k::nat) \Longrightarrow i < k$
apply (*rule le-less-trans* [*of* - *i+j*])
apply (*simp-all add*: *le-add1*)
done

lemma *not-add-less1* [*iff*]: $\sim (i + j < (i::nat))$
apply (*rule notI*)
apply (*drule add-lessD1*)
apply (*erule less-irrefl* [*THEN notE*])
done

lemma *not-add-less2* [*iff*]: $\sim (j + i < (i::nat))$
by (*simp add*: *add-commute*)

lemma *add-leD1*: $m + k \leq n \Longrightarrow m \leq (n::nat)$
apply (*rule order-trans* [*of* - *m+k*])
apply (*simp-all add*: *le-add1*)
done

lemma *add-leD2*: $m + k \leq n \Longrightarrow k \leq (n::nat)$
apply (*simp add*: *add-commute*)
apply (*erule add-leD1*)
done

lemma *add-leE*: $(m::nat) + k \leq n \Longrightarrow (m \leq n \Longrightarrow k \leq n \Longrightarrow R) \Longrightarrow R$
by (*blast dest*: *add-leD1 add-leD2*)

needs $!!k$ for *add-ac* to work

lemma *less-add-eq-less*: $!!k::nat.\ k < l \Longrightarrow m + l = k + n \Longrightarrow m < n$
by (*force simp del*: *add-Suc-right*
simp add: *less-iff-Suc-add add-Suc-right* [*symmetric*] *add-ac*)

15.4.8 More results about difference

Addition is the inverse of subtraction: if $n \leq m$ then $n + (m - n) = m$.

lemma *add-diff-inverse*: $\sim\ m < n \Longrightarrow n + (m - n) = (m::nat)$
by (*induct m n rule*: *diff-induct*) *simp-all*

lemma *le-add-diff-inverse* [*simp*]: $n \leq m \Longrightarrow n + (m - n) = (m::nat)$
by (*simp add*: *add-diff-inverse linorder-not-less*)

lemma *le-add-diff-inverse2* [*simp*]: $n \leq m \Longrightarrow (m - n) + n = (m::nat)$
by (*simp add*: *add-commute*)

lemma *Suc-diff-le*: $n \leq m \Longrightarrow Suc\ m - n = Suc\ (m - n)$
by (*induct m n rule*: *diff-induct*) *simp-all*

lemma *diff-less-Suc*: $m - n < Suc\ m$
apply (*induct m n rule*: *diff-induct*)
apply (*erule-tac* [*3*] *less-SucE*)
apply (*simp-all add*: *less-Suc-eq*)
done

lemma *diff-le-self* [*simp*]: $m - n \leq (m::nat)$
by (*induct m n rule*: *diff-induct*) (*simp-all add*: *le-SucI*)

lemma *le-iff-add*: $(m::nat) \leq n = (\exists k.\ n = m + k)$
by (*auto simp*: *le-add1 dest*!: *le-add-diff-inverse sym* [*of* - *n*])

lemma *less-imp-diff-less*: $(j::nat) < k \Longrightarrow j - n < k$
by (*rule le-less-trans*, *rule diff-le-self*)

lemma *diff-Suc-less* [*simp*]: $0<n \Longrightarrow n - Suc\ i < n$
by (*cases n*) (*auto simp add*: *le-simps*)

lemma *diff-add-assoc*: $k \leq (j::nat) \Longrightarrow (i + j) - k = i + (j - k)$
by (*induct j k rule*: *diff-induct*) *simp-all*

lemma *diff-add-assoc2*: $k \leq (j::nat) \Longrightarrow (j + i) - k = (j - k) + i$
by (*simp add*: *add-commute diff-add-assoc*)

lemma *le-imp-diff-is-add*: $i \leq (j::nat) \Longrightarrow (j - i = k) = (j = k + i)$
by (*auto simp add*: *diff-add-inverse2*)

lemma *diff-is-0-eq* [*simp*]: $((m::nat) - n = 0) = (m \leq n)$
by (*induct m n rule*: *diff-induct*) *simp-all*

lemma *diff-is-0-eq'* [*simp*]: $m \leq n \Longrightarrow (m::nat) - n = 0$
by (*rule iffD2*, *rule diff-is-0-eq*)

lemma *zero-less-diff* [*simp*]: $(0 < n - (m::nat)) = (m < n)$
by (*induct m n rule*: *diff-induct*) *simp-all*

lemma *less-imp-add-positive*:
assumes $i < j$
shows $\exists k::nat.\ 0 < k\ \&\ i + k = j$

proof
from *assms* **show** $0 < j - i \;\&\; i + (j - i) = j$
by (*simp add*: *order-less-imp-le*)
qed

a nice rewrite for bounded subtraction

lemma *nat-minus-add-max*:
fixes $n\ m :: nat$
shows $n - m + m = max\ n\ m$
by (*simp add*: *max-def not-le order-less-imp-le*)

lemma *nat-diff-split*:
$P(a - b::nat) = ((a<b \longrightarrow P\ 0) \;\&\; (ALL\ d.\ a = b + d \longrightarrow P\ d))$
— elimination of $-$ on *nat*
by (*cases* $a < b$)
(*auto simp add*: *diff-is-0-eq* [*THEN iffD2*] *diff-add-inverse*
not-less le-less dest!: *sym* [*of a*] *sym* [*of b*] *add-eq-self-zero*)

lemma *nat-diff-split-asm*:
$P(a - b::nat) = (\sim\ (a < b \;\&\; \sim\ P\ 0 \mid (EX\ d.\ a = b + d \;\&\; \sim\ P\ d)))$
— elimination of $-$ on *nat* in assumptions
by (*auto split*: *nat-diff-split*)

15.4.9 Monotonicity of Multiplication

lemma *mult-le-mono1*: $i \le (j::nat) \Longrightarrow i * k \le j * k$
by (*simp add*: *mult-right-mono*)

lemma *mult-le-mono2*: $i \le (j::nat) \Longrightarrow k * i \le k * j$
by (*simp add*: *mult-left-mono*)

$\le$ monotonicity, BOTH arguments

lemma *mult-le-mono*: $i \le (j::nat) \Longrightarrow k \le l \Longrightarrow i * k \le j * l$
by (*simp add*: *mult-mono*)

lemma *mult-less-mono1*: $(i::nat) < j \Longrightarrow 0 < k \Longrightarrow i * k < j * k$
by (*simp add*: *mult-strict-right-mono*)

Differs from the standard *zero-less-mult-iff* in that there are no negative numbers.

lemma *nat-0-less-mult-iff* [*simp*]: $(0 < (m::nat) * n) = (0 < m \;\&\; 0 < n)$
apply (*induct m*)
apply *simp*
apply (*case-tac n*)
apply *simp-all*
done

lemma *one-le-mult-iff* [*simp*]: $(Suc\ 0 \le m * n) = (Suc\ 0 \le m \;\&\; Suc\ 0 \le n)$
apply (*induct m*)

apply *simp*
apply (*case-tac n*)
apply *simp-all*
done

lemma *mult-less-cancel2* [*simp*]: $((m::nat) * k < n * k) = (0 < k \ \& \ m < n)$
apply (*safe intro*!: *mult-less-mono1*)
apply (*case-tac k*, *auto*)
apply (*simp del*: *le-0-eq add*: *linorder-not-le* [*symmetric*])
apply (*blast intro*: *mult-le-mono1*)
done

lemma *mult-less-cancel1* [*simp*]: $(k * (m::nat) < k * n) = (0 < k \ \& \ m < n)$
by (*simp add*: *mult-commute* [*of k*])

lemma *mult-le-cancel1* [*simp*]: $(k * (m::nat) \leq k * n) = (0 < k \longrightarrow m \leq n)$
by (*simp add*: *linorder-not-less* [*symmetric*], *auto*)

lemma *mult-le-cancel2* [*simp*]: $((m::nat) * k \leq n * k) = (0 < k \longrightarrow m \leq n)$
by (*simp add*: *linorder-not-less* [*symmetric*], *auto*)

lemma *Suc-mult-less-cancel1*: $(Suc\ k * m < Suc\ k * n) = (m < n)$
by (*subst mult-less-cancel1*) *simp*

lemma *Suc-mult-le-cancel1*: $(Suc\ k * m \leq Suc\ k * n) = (m \leq n)$
by (*subst mult-le-cancel1*) *simp*

lemma *le-square*: $m \leq m * (m::nat)$
by (*cases m*) (*auto intro*: *le-add1*)

lemma *le-cube*: $(m::nat) \leq m * (m * m)$
by (*cases m*) (*auto intro*: *le-add1*)

Lemma for *gcd*

lemma *mult-eq-self-implies-10*: $(m::nat) = m * n \Longrightarrow n = 1 \mid m = 0$
apply (*drule sym*)
apply (*rule disjCI*)
apply (*rule nat-less-cases*, *erule-tac* [*2*] -)
apply (*drule-tac* [*2*] *mult-less-mono2*)
apply (*auto*)
done

the lattice order on *nat*

instantiation *nat* :: *distrib-lattice*
begin

definition
$(inf :: nat \Rightarrow nat \Rightarrow nat) = min$

definition
(*sup* :: *nat* ⇒ *nat* ⇒ *nat*) = *max*

instance by *intro-classes*
(*auto simp add*: *inf-nat-def sup-nat-def max-def not-le min-def*
intro: *order-less-imp-le antisym elim*!: *order-trans order-less-trans*)

end

15.5 Natural operation of natural numbers on functions

We use the same logical constant for the power operations on functions and relations, in order to share the same syntax.

consts *compow* :: *nat* ⇒ ('*a* ⇒ '*b*) ⇒ ('*a* ⇒ '*b*)

abbreviation *compower* :: ('*a* ⇒ '*b*) ⇒ *nat* ⇒ '*a* ⇒ '*b* (**infixr** ^^ *80*) **where**
f ^^ *n* ≡ *compow n f*

notation (*latex* **output**)
compower ((-⁻) [*1000*] *1000*)

notation (*HTML* **output**)
compower ((-⁻) [*1000*] *1000*)

f ^^ *n* = *f o* ... *o f*, the n-fold composition of *f*

overloading
funpow == *compow* :: *nat* ⇒ ('*a* ⇒ '*a*) ⇒ ('*a* ⇒ '*a*)
begin

primrec *funpow* :: *nat* ⇒ ('*a* ⇒ '*a*) ⇒ '*a* ⇒ '*a* **where**
funpow 0 f = *id*
| *funpow* (*Suc n*) *f* = *f o funpow n f*

end

for code generation

definition *funpow* :: *nat* ⇒ ('*a* ⇒ '*a*) ⇒ '*a* ⇒ '*a* **where**
funpow-code-def [*code-post*]: *funpow* = *compow*

lemmas [*code-unfold*] = *funpow-code-def* [*symmetric*]

lemma [*code*]:
funpow 0 f = *id*
funpow (*Suc n*) *f* = *f o funpow n f*
unfolding *funpow-code-def* **by** *simp-all*

hide-const (**open**) *funpow*

lemma *funpow-add*:
f ^^ (m + n) = f ^^ m ∘ f ^^ n
by (*induct m*) *simp-all*

lemma *funpow-swap1*:
f ((f ^^ n) x) = (f ^^ n) (f x)
proof −
have *f ((f ^^ n) x) = (f ^^ (n + 1)) x* **by** *simp*
also have ... *= (f ^^ n o f ^^ 1) x* **by** (*simp only*: *funpow-add*)
also have ... *= (f ^^ n) (f x)* **by** *simp*
finally show *?thesis* **.**
qed

15.6 Embedding of the Naturals into any *semiring-1*: *of-nat*

context *semiring-1*
begin

primrec
of-nat :: *nat* ⇒ *'a*
where
of-nat-0: *of-nat 0 = 0*
| *of-nat-Suc*: *of-nat (Suc m) = 1 + of-nat m*

lemma *of-nat-1* [*simp*]: *of-nat 1 = 1*
unfolding *One-nat-def* **by** *simp*

lemma *of-nat-add* [*simp*]: *of-nat (m + n) = of-nat m + of-nat n*
by (*induct m*) (*simp-all add*: *add-ac*)

lemma *of-nat-mult*: *of-nat (m * n) = of-nat m * of-nat n*
by (*induct m*) (*simp-all add*: *add-ac left-distrib*)

primrec *of-nat-aux* :: (*'a* ⇒ *'a*) ⇒ *nat* ⇒ *'a* ⇒ *'a* **where**
of-nat-aux inc 0 i = i
| *of-nat-aux inc (Suc n) i = of-nat-aux inc n (inc i)* — tail recursive

lemma *of-nat-code*:
of-nat n = of-nat-aux (λi. i + 1) n 0
proof (*induct n*)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc n*)
have ⋀*i*. *of-nat-aux (λi. i + 1) n (i + 1) = of-nat-aux (λi. i + 1) n i + 1*
by (*induct n*) *simp-all*
from *this* [*of 0*] **have** *of-nat-aux (λi. i + 1) n 1 = of-nat-aux (λi. i + 1) n 0 + 1*
by *simp*
with *Suc* **show** *?case* **by** (*simp add*: *add-commute*)

qed

end

declare *of-nat-code* [*code*, *code-unfold*, *code-inline del*]

Class for unital semirings with characteristic zero. Includes non-ordered rings like the complex numbers.

class *semiring-char-0* = *semiring-1* +
  assumes *of-nat-eq-iff* [*simp*]: *of-nat m* = *of-nat n* ⟷ *m* = *n*
begin

Special cases where either operand is zero

lemma *of-nat-0-eq-iff* [*simp*, *no-atp*]: *0* = *of-nat n* ⟷ *0* = *n*
  by (*rule of-nat-eq-iff* [*of 0 n*, *unfolded of-nat-0*])

lemma *of-nat-eq-0-iff* [*simp*, *no-atp*]: *of-nat m* = *0* ⟷ *m* = *0*
  by (*rule of-nat-eq-iff* [*of m 0*, *unfolded of-nat-0*])

lemma *inj-of-nat*: *inj of-nat*
  by (*simp add*: *inj-on-def*)

end

context *linordered-semidom*
begin

lemma *zero-le-imp-of-nat*: *0* ≤ *of-nat m*
  by (*induct m*) *simp-all*

lemma *less-imp-of-nat-less*: *m* < *n* ⟹ *of-nat m* < *of-nat n*
  apply (*induct m n rule*: *diff-induct*, *simp-all*)
  apply (*rule add-pos-nonneg* [*OF zero-less-one zero-le-imp-of-nat*])
  done

lemma *of-nat-less-imp-less*: *of-nat m* < *of-nat n* ⟹ *m* < *n*
  apply (*induct m n rule*: *diff-induct*, *simp-all*)
  apply (*insert zero-le-imp-of-nat*)
  apply (*force simp add*: *not-less* [*symmetric*])
  done

lemma *of-nat-less-iff* [*simp*]: *of-nat m* < *of-nat n* ⟷ *m* < *n*
  by (*blast intro*: *of-nat-less-imp-less less-imp-of-nat-less*)

lemma *of-nat-le-iff* [*simp*]: *of-nat m* ≤ *of-nat n* ⟷ *m* ≤ *n*
  by (*simp add*: *not-less* [*symmetric*] *linorder-not-less* [*symmetric*])

Every *linordered-semidom* has characteristic zero.

subclass *semiring-char-0*

proof qed (*simp add*: *eq-iff order-eq-iff*)

Special cases where either operand is zero

lemma *of-nat-0-le-iff* [*simp*]: $0 \leq$ *of-nat n*
by (*rule of-nat-le-iff* [*of 0*, *simplified*])

lemma *of-nat-le-0-iff* [*simp*, *no-atp*]: *of-nat m* $\leq 0 \longleftrightarrow m = 0$
by (*rule of-nat-le-iff* [*of - 0*, *simplified*])

lemma *of-nat-0-less-iff* [*simp*]: $0 <$ *of-nat n* $\longleftrightarrow 0 < n$
by (*rule of-nat-less-iff* [*of 0*, *simplified*])

lemma *of-nat-less-0-iff* [*simp*]: $\neg$ *of-nat m* < 0
by (*rule of-nat-less-iff* [*of - 0*, *simplified*])

end

context *ring-1*
begin

lemma *of-nat-diff*: $n \leq m \Longrightarrow$ *of-nat* $(m - n) =$ *of-nat m* $-$ *of-nat n*
by (*simp add*: *algebra-simps of-nat-add* [*symmetric*])

end

context *linordered-idom*
begin

lemma *abs-of-nat* [*simp*]: $|$*of-nat n*$| =$ *of-nat n*
unfolding *abs-if* **by** *auto*

end

lemma *of-nat-id* [*simp*]: *of-nat n* $= n$
by (*induct n*) *simp-all*

lemma *of-nat-eq-id* [*simp*]: *of-nat* $=$ *id*
by (*auto simp add*: *expand-fun-eq*)

15.7 The Set of Natural Numbers

context *semiring-1*
begin

definition
Nats :: $'a$ *set* **where**
[*code del*]: *Nats* $=$ *range of-nat*

notation (*xsymbols*)

Nats (ℕ)

lemma *of-nat-in-Nats* [*simp*]: *of-nat n* ∈ ℕ
by (*simp add*: *Nats-def*)

lemma *Nats-0* [*simp*]: *0* ∈ ℕ
apply (*simp add*: *Nats-def*)
apply (*rule range-eqI*)
apply (*rule of-nat-0* [*symmetric*])
done

lemma *Nats-1* [*simp*]: *1* ∈ ℕ
apply (*simp add*: *Nats-def*)
apply (*rule range-eqI*)
apply (*rule of-nat-1* [*symmetric*])
done

lemma *Nats-add* [*simp*]: *a* ∈ ℕ ⟹ *b* ∈ ℕ ⟹ *a* + *b* ∈ ℕ
apply (*auto simp add*: *Nats-def*)
apply (*rule range-eqI*)
apply (*rule of-nat-add* [*symmetric*])
done

lemma *Nats-mult* [*simp*]: *a* ∈ ℕ ⟹ *b* ∈ ℕ ⟹ *a* * *b* ∈ ℕ
apply (*auto simp add*: *Nats-def*)
apply (*rule range-eqI*)
apply (*rule of-nat-mult* [*symmetric*])
done

lemma *Nats-cases* [*cases set*: *Nats*]:
assumes *x* ∈ ℕ
obtains (*of-nat*) *n* **where** *x* = *of-nat n*
unfolding *Nats-def*
proof −
from ‹*x* ∈ ℕ› **have** *x* ∈ *range of-nat* **unfolding** *Nats-def* **.**
then obtain *n* **where** *x* = *of-nat n* **..**
then show *thesis* **..**
qed

lemma *Nats-induct* [*case-names of-nat*, *induct set*: *Nats*]:
x ∈ ℕ ⟹ (⋀*n*. *P* (*of-nat n*)) ⟹ *P x*
by (*rule Nats-cases*) *auto*

end

15.8 Further Arithmetic Facts Concerning the Natural Numbers

lemma *subst-equals*:

assumes *1*: $t = s$ **and** *2*: $u = t$
shows $u = s$
using *2 1* **by** (*rule trans*)

setup *Arith-Data.setup*

use *Tools/nat-arith.ML*
declaration ⟪ *K Nat-Arith.setup* ⟫

use *Tools/lin-arith.ML*
setup ⟪ *Lin-Arith.global-setup* ⟫
declaration ⟪ *K Lin-Arith.setup* ⟫

lemmas [*arith-split*] = *nat-diff-split split-min split-max*

context *order*
begin

lemma *lift-Suc-mono-le*:
assumes *mono*: !!n. $f\ n \leq f(Suc\ n)$ **and** $n \leq n'$
shows $f\ n \leq f\ n'$
proof (*cases* $n < n'$)
case *True*
thus *?thesis*
by (*induct n n' rule*: *less-Suc-induct*[*consumes 1*]) (*auto intro*: *mono*)
qed (*insert* ‹$n \leq n'$›, *auto*) — trivial for $n = n'$

lemma *lift-Suc-mono-less*:
assumes *mono*: !!n. $f\ n < f(Suc\ n)$ **and** $n < n'$
shows $f\ n < f\ n'$
using ‹$n < n'$›
by (*induct n n' rule*: *less-Suc-induct*[*consumes 1*]) (*auto intro*: *mono*)

lemma *lift-Suc-mono-less-iff*:
(!!n. $f\ n < f(Suc\ n)$) $\Longrightarrow f(n) < f(m) \longleftrightarrow n<m$
by(*blast intro*: *less-asym'* *lift-Suc-mono-less*[*of f*]
dest: *linorder-not-less*[*THEN iffD1*] *le-eq-less-or-eq*[*THEN iffD1*])

end

lemma *mono-iff-le-Suc*: *mono f* = ($\forall\, n.\ f\ n \leq f\ (Suc\ n)$)
unfolding *mono-def*
by (*auto intro*:*lift-Suc-mono-le*[*of f*])

lemma *mono-nat-linear-lb*:
(!!$m\ n$::*nat*. $m<n \Longrightarrow f\ m < f\ n) \Longrightarrow f(m)+k \leq f(m+k)$
apply(*induct-tac k*)
apply *simp*
apply(*erule-tac* $x=m+n$ **in** *meta-allE*)

apply(*erule-tac x=Suc(m+n)* **in** *meta-allE*)
apply *simp*
done

Subtraction laws, mostly by Clemens Ballarin

lemma *diff-less-mono*: [| $a < (b::nat)$; $c \leq a$ |] ==> $a-c < b-c$
by *arith*

lemma *less-diff-conv*: $(i < j-k) = (i+k < (j::nat))$
by *arith*

lemma *le-diff-conv*: $(j-k \leq (i::nat)) = (j \leq i+k)$
by *arith*

lemma *le-diff-conv2*: $k \leq j$ ==> $(i \leq j-k) = (i+k \leq (j::nat))$
by *arith*

lemma *diff-diff-cancel* [*simp*]: $i \leq (n::nat)$ ==> $n - (n - i) = i$
by *arith*

lemma *le-add-diff*: $k \leq (n::nat)$ ==> $m \leq n + m - k$
by *arith*

lemma *diff-less*[*simp*]: !!$m::nat$. [| $0<n$; $0<m$ |] ==> $m - n < m$
by *arith*

Simplification of relational expressions involving subtraction

lemma *diff-diff-eq*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $((m-k) - (n-k)) = (m-n)$
by (*simp split add*: *nat-diff-split*)

hide-fact (**open**) *diff-diff-eq*

lemma *eq-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $(m-k = n-k) = (m=n)$
by (*auto split add*: *nat-diff-split*)

lemma *less-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $(m-k < n-k) = (m<n)$
by (*auto split add*: *nat-diff-split*)

lemma *le-diff-iff*: [| $k \leq m$; $k \leq (n::nat)$ |] ==> $(m-k \leq n-k) = (m \leq n)$
by (*auto split add*: *nat-diff-split*)

(Anti)Monotonicity of subtraction – by Stephan Merz

lemma *diff-le-mono*: $m \leq (n::nat)$ ==> $(m-l) \leq (n-l)$
by (*simp split add*: *nat-diff-split*)

lemma *diff-le-mono2*: $m \leq (n::nat)$ ==> $(l-n) \leq (l-m)$
by (*simp split add*: *nat-diff-split*)

lemma *diff-less-mono2*: [| $m < (n::nat)$; $m<l$ |] ==> $(l-n) < (l-m)$
by (*simp split add*: *nat-diff-split*)

lemma *diffs0-imp-equal*: !!$m::nat$. [| $m-n = 0$; $n-m = 0$ |] ==> $m=n$
by (*simp split add*: *nat-diff-split*)

lemma *min-diff*: *min* $(m - (i::nat))$ $(n - i) = min\ m\ n - i$
by *auto*

lemma *inj-on-diff-nat*:
  assumes *k-le-n*: $\forall n \in N.\ k \leq (n::nat)$
  shows *inj-on* $(\lambda n.\ n - k)$ N
proof (*rule inj-onI*)
  fix $x\ y$
  assume *a*: $x \in N\ y \in N\ x - k = y - k$
  with *k-le-n* **have** $x - k + k = y - k + k$ **by** *auto*
  with *a k-le-n* **show** $x = y$ **by** *auto*
qed

Rewriting to pull differences out

lemma *diff-diff-right* [*simp*]: $k \leq j$ --> $i - (j - k) = i + (k::nat) - j$
by *arith*

lemma *diff-Suc-diff-eq1* [*simp*]: $k \leq j$ ==> $m - Suc\ (j - k) = m + k - Suc\ j$
by *arith*

lemma *diff-Suc-diff-eq2* [*simp*]: $k \leq j$ ==> $Suc\ (j - k) - m = Suc\ j - (k + m)$
by *arith*

Lemmas for ex/Factorization

lemma *one-less-mult*: [| $Suc\ 0 < n$; $Suc\ 0 < m$ |] ==> $Suc\ 0 < m*n$
by (*cases m*) *auto*

lemma *n-less-m-mult-n*: [| $Suc\ 0 < n$; $Suc\ 0 < m$ |] ==> $n<m*n$
by (*cases m*) *auto*

lemma *n-less-n-mult-m*: [| $Suc\ 0 < n$; $Suc\ 0 < m$ |] ==> $n<n*m$
by (*cases m*) *auto*

Specialized induction principles that work "backwards":

lemma *inc-induct*[*consumes 1*, *case-names base step*]:
  assumes *less*: $i <= j$
  assumes *base*: $P\ j$
  assumes *step*: !!i. [| $i < j$; $P\ (Suc\ i)$ |] ==> $P\ i$
  shows $P\ i$
  using *less*
proof (*induct d*==$j - i$ *arbitrary*: i)
  case (*0 i*)
  hence $i = j$ **by** *simp*

with *base* **show** *?case* **by** *simp*
next
case (*Suc d i*)
hence $i < j$ *P* (*Suc i*)
by *simp-all*
thus *P i* **by** (*rule step*)
qed

lemma *strict-inc-induct*[*consumes 1*, *case-names base step*]:
assumes *less*: $i < j$
assumes *base*: !!*i*. *j* = *Suc i* ==> *P i*
assumes *step*: !!*i*. [| $i < j$; *P* (*Suc i*) |] ==> *P i*
shows *P i*
using *less*
proof (*induct d*==$j - i - 1$ *arbitrary*: *i*)
case (*0 i*)
with ‹$i < j$› **have** *j* = *Suc i* **by** *simp*
with *base* **show** *?case* **by** *simp*
next
case (*Suc d i*)
hence $i < j$ *P* (*Suc i*)
by *simp-all*
thus *P i* **by** (*rule step*)
qed

lemma *zero-induct-lemma*: *P k* ==> (!!*n*. *P* (*Suc n*) ==> *P n*) ==> *P* ($k - i$)
using *inc-induct*[*of* $k - i$ *k P*, *simplified*] **by** *blast*

lemma *zero-induct*: *P k* ==> (!!*n*. *P* (*Suc n*) ==> *P n*) ==> *P 0*
using *inc-induct*[*of 0 k P*] **by** *blast*

lemmas *add-diff-assoc* = *diff-add-assoc* [*symmetric*]
lemmas *add-diff-assoc2* = *diff-add-assoc2*[*symmetric*]
declare *diff-diff-left* [*simp*] *add-diff-assoc* [*simp*] *add-diff-assoc2*[*simp*]

At present we prove no analogue of *not-less-Least* or *Least-Suc*, since there appears to be no need.

15.9 The divides relation on *nat*

lemma *dvd-1-left* [*iff*]: *Suc 0 dvd k*
unfolding *dvd-def* **by** *simp*

lemma *dvd-1-iff-1* [*simp*]: (*m dvd Suc 0*) = (*m* = *Suc 0*)
by (*simp add*: *dvd-def*)

lemma *nat-dvd-1-iff-1* [*simp*]: *m dvd* (*1*::*nat*) ⟷ *m* = *1*
by (*simp add*: *dvd-def*)

lemma *dvd-antisym*: [| *m dvd n*; *n dvd m* |] ==> *m* = (*n*::*nat*)
unfolding *dvd-def*
by (*force dest*: *mult-eq-self-implies-10 simp add*: *mult-assoc*)

op dvd is a partial order

interpretation *dvd*: *order op dvd* λ*n m* :: *nat*. *n dvd m* ∧ ¬ *m dvd n*
proof qed (*auto intro*: *dvd-refl dvd-trans dvd-antisym*)

lemma *dvd-diff-nat*[*simp*]: [| *k dvd m*; *k dvd n* |] ==> *k dvd* (*m*−*n* :: *nat*)
unfolding *dvd-def*
by (*blast intro*: *diff-mult-distrib2* [*symmetric*])

lemma *dvd-diffD*: [| *k dvd m*−*n*; *k dvd n*; *n*≤*m* |] ==> *k dvd* (*m*::*nat*)
apply (*erule linorder-not-less* [*THEN iffD2*, *THEN add-diff-inverse*, *THEN subst*])
apply (*blast intro*: *dvd-add*)
done

lemma *dvd-diffD1*: [| *k dvd m*−*n*; *k dvd m*; *n*≤*m* |] ==> *k dvd* (*n*::*nat*)
by (*drule-tac m* = *m* **in** *dvd-diff-nat*, *auto*)

lemma *dvd-reduce*: (*k dvd n* + *k*) = (*k dvd* (*n*::*nat*))
apply (*rule iffI*)
apply (*erule-tac* [*2*] *dvd-add*)
apply (*rule-tac* [*2*] *dvd-refl*)
apply (*subgoal-tac n* = (*n*+*k*) −*k*)
prefer *2* **apply** *simp*
apply (*erule ssubst*)
apply (*erule dvd-diff-nat*)
apply (*rule dvd-refl*)
done

lemma *dvd-mult-cancel*: !!*k*::*nat*. [| *k*∗*m dvd k*∗*n*; *0*<*k* |] ==> *m dvd n*
unfolding *dvd-def*
apply (*erule exE*)
apply (*simp add*: *mult-ac*)
done

lemma *dvd-mult-cancel1*: *0*<*m* ==> (*m*∗*n dvd m*) = (*n* = (*1*::*nat*))
apply *auto*
apply (*subgoal-tac m*∗*n dvd m*∗*1*)
apply (*drule dvd-mult-cancel*, *auto*)
done

lemma *dvd-mult-cancel2*: *0*<*m* ==> (*n*∗*m dvd m*) = (*n* = (*1*::*nat*))
apply (*subst mult-commute*)
apply (*erule dvd-mult-cancel1*)
done

lemma *dvd-imp-le*: [| *k dvd n*; *0 < n* |] ==> *k* ≤ (*n*::*nat*)
by (*auto elim*!: *dvdE*) (*auto simp add*: *gr0-conv-Suc*)

lemma *nat-dvd-not-less*:
fixes *m n* :: *nat*
shows *0 < m* ⟹ *m < n* ⟹ ¬ *n dvd m*
by (*auto elim*!: *dvdE*) (*auto simp add*: *gr0-conv-Suc*)

15.10 size of a datatype value

class *size* =
fixes *size* :: *'a* ⇒ *nat* — see further theory *Wellfounded*

15.11 code module namespace

code-modulename *SML*
Nat Arith

code-modulename *OCaml*
Nat Arith

code-modulename *Haskell*
Nat Arith

end

16 Datatype: Datatype package: constructing datatypes from Cartesian Products and Disjoint Sums

theory *Datatype*
imports *Product-Type Sum-Type Nat*
uses
(*Tools/Datatype/datatype.ML*)
(*Tools/inductive-realizer.ML*)
(*Tools/Datatype/datatype-realizer.ML*)
begin

16.1 The datatype universe

typedef (*Node*)
('*a*,'*b*) *node* = {*p*. *EX f x k*. *p* = (*f*::*nat*=>'*b*+*nat*, *x*::'*a*+*nat*) & *f k* = *Inr 0*}
— it is a subtype of (*nat*=>'*b*+*nat*) ∗ ('*a*+*nat*)
by *auto*

Datatypes will be represented by sets of type *node*

types *'a item* = ('*a*, *unit*) *node set*

$('a, 'b)\ dtree = ('a, 'b)\ node\ set$

consts

Push :: $[('b + nat), nat => ('b + nat)] => (nat => ('b + nat))$

Push-Node :: $[('b + nat), ('a, 'b)\ node] => ('a, 'b)\ node$
ndepth :: $('a, 'b)\ node => nat$

Atom :: $('a + nat) => ('a, 'b)\ dtree$
Leaf :: $'a => ('a, 'b)\ dtree$
Numb :: $nat => ('a, 'b)\ dtree$
Scons :: $[('a, 'b)\ dtree, ('a, 'b)\ dtree] => ('a, 'b)\ dtree$
In0 :: $('a, 'b)\ dtree => ('a, 'b)\ dtree$
In1 :: $('a, 'b)\ dtree => ('a, 'b)\ dtree$
Lim :: $('b => ('a, 'b)\ dtree) => ('a, 'b)\ dtree$

ntrunc :: $[nat, ('a, 'b)\ dtree] => ('a, 'b)\ dtree$

uprod :: $[('a, 'b)\ dtree\ set, ('a, 'b)\ dtree\ set] => ('a, 'b)\ dtree\ set$
usum :: $[('a, 'b)\ dtree\ set, ('a, 'b)\ dtree\ set] => ('a, 'b)\ dtree\ set$

Split :: $[[('a, 'b)\ dtree, ('a, 'b)\ dtree] => 'c, ('a, 'b)\ dtree] => 'c$
Case :: $[[('a, 'b)\ dtree] => 'c, [('a, 'b)\ dtree] => 'c, ('a, 'b)\ dtree] => 'c$

dprod :: $[(('a, 'b)\ dtree * ('a, 'b)\ dtree)set, (('a, 'b)\ dtree * ('a, 'b)\ dtree)set] => (('a, 'b)\ dtree * ('a, 'b)\ dtree)set$
dsum :: $[(('a, 'b)\ dtree * ('a, 'b)\ dtree)set, (('a, 'b)\ dtree * ('a, 'b)\ dtree)set] => (('a, 'b)\ dtree * ('a, 'b)\ dtree)set$

defs

Push-Node-def: $Push\text{-}Node == (\%n\ x.\ Abs\text{-}Node\ (apfst\ (Push\ n)\ (Rep\text{-}Node\ x)))$

Push-def: $Push == (\%b\ h.\ nat\text{-}case\ b\ h)$

Atom-def: $Atom == (\%x.\ \{Abs\text{-}Node((\%k.\ Inr\ 0, x))\})$
Scons-def: $Scons\ M\ N == (Push\text{-}Node\ (Inr\ 1)\ `\ M)\ Un\ (Push\text{-}Node\ (Inr\ (Suc\ 1))\ `\ N)$

Leaf-def: $Leaf == Atom\ o\ Inl$
Numb-def: $Numb == Atom\ o\ Inr$

In0-def: $In0(M) == Scons\ (Numb\ 0)\ M$
In1-def: $In1(M) == Scons\ (Numb\ 1)\ M$

Lim-def: $Lim\ f == Union\ \{z.\ ?\ x.\ z = Push\text{-}Node\ (Inl\ x)\ `\ (f\ x)\}$

ndepth-def: $ndepth(n) == (\%(f,x).\ LEAST\ k.\ f\ k = Inr\ 0)\ (Rep\text{-}Node\ n)$
ntrunc-def: $ntrunc\ k\ N == \{n.\ n{:}N\ \&\ ndepth(n){<}k\}$

uprod-def: $uprod\ A\ B == UN\ x{:}A.\ UN\ y{:}B.\ \{\ Scons\ x\ y\ \}$
usum-def: $usum\ A\ B == In0`A\ Un\ In1`B$

Split-def: $Split\ c\ M == THE\ u.\ EX\ x\ y.\ M = Scons\ x\ y\ \&\ u = c\ x\ y$

Case-def: $Case\ c\ d\ M == THE\ u.\ (EX\ x\ .\ M = In0(x)\ \&\ u = c(x))$
$\quad | (EX\ y\ .\ M = In1(y)\ \&\ u = d(y))$

dprod-def: $dprod\ r\ s == UN\ (x,x'){:}r.\ UN\ (y,y'){:}s.\ \{(Scons\ x\ y,\ Scons\ x'\ y')\}$

dsum-def: $dsum\ r\ s == (UN\ (x,x'){:}r.\ \{(In0(x),In0(x'))\})\ Un$
$\quad (UN\ (y,y'){:}s.\ \{(In1(y),In1(y'))\})$

lemma *apfst-convE*:
$[|\ q = apfst\ f\ p;\ !!x\ y.\ [|\ p = (x,y);\ q = (f(x),y)\ |] ==> R\ |] ==> R$
by (*force simp add*: *apfst-def*)

lemma *Push-inject1*: $Push\ i\ f = Push\ j\ g\ ==> i{=}j$
apply (*simp add*: *Push-def expand-fun-eq*)
apply (*drule-tac x=0* **in** *spec*, *simp*)
done

lemma *Push-inject2*: $Push\ i\ f = Push\ j\ g\ ==> f{=}g$
apply (*auto simp add*: *Push-def expand-fun-eq*)
apply (*drule-tac x=Suc x* **in** *spec*, *simp*)
done

lemma *Push-inject*:

[| *Push i f* =*Push j g*; [| *i*=*j*; *f*=*g* |] ==> *P* |] ==> *P*
by (*blast dest*: *Push-inject1 Push-inject2*)

lemma *Push-neq-K0*: *Push* (*Inr* (*Suc k*)) *f* = (%*z*. *Inr 0*) ==> *P*
by (*auto simp add*: *Push-def expand-fun-eq split*: *nat.split-asm*)

lemmas *Abs-Node-inj* = *Abs-Node-inject* [*THEN* [*2*] *rev-iffD1*, *standard*]

lemma *Node-K0-I*: (%*k*. *Inr 0*, *a*) : *Node*
by (*simp add*: *Node-def*)

lemma *Node-Push-I*: *p*: *Node* ==> *apfst* (*Push i*) *p* : *Node*
apply (*simp add*: *Node-def Push-def*)
apply (*fast intro*!: *apfst-conv nat-case-Suc* [*THEN trans*])
done

16.2 Freeness: Distinctness of Constructors

lemma *Scons-not-Atom* [*iff*]: *Scons M N* ≠ *Atom*(*a*)
unfolding *Atom-def Scons-def Push-Node-def One-nat-def*
by (*blast intro*: *Node-K0-I Rep-Node* [*THEN Node-Push-I*]
dest!: *Abs-Node-inj*
elim!: *apfst-convE sym* [*THEN Push-neq-K0*])

lemmas *Atom-not-Scons* [*iff*] = *Scons-not-Atom* [*THEN not-sym*, *standard*]

lemma *inj-Atom*: *inj*(*Atom*)
apply (*simp add*: *Atom-def*)
apply (*blast intro*!: *inj-onI Node-K0-I dest*!: *Abs-Node-inj*)
done
lemmas *Atom-inject* = *inj-Atom* [*THEN injD*, *standard*]

lemma *Atom-Atom-eq* [*iff*]: (*Atom*(*a*)=*Atom*(*b*)) = (*a*=*b*)
by (*blast dest*!: *Atom-inject*)

lemma *inj-Leaf*: *inj*(*Leaf*)
apply (*simp add*: *Leaf-def o-def*)
apply (*rule inj-onI*)
apply (*erule Atom-inject* [*THEN Inl-inject*])
done

lemmas *Leaf-inject* [*dest*!] = *inj-Leaf* [*THEN injD*, *standard*]

lemma *inj-Numb*: *inj*(*Numb*)
apply (*simp add*: *Numb-def o-def*)
apply (*rule inj-onI*)
apply (*erule Atom-inject* [*THEN Inr-inject*])
done

lemmas *Numb-inject* [*dest*!] = *inj-Numb* [*THEN injD*, *standard*]

lemma *Push-Node-inject*:
[| *Push-Node i m* =*Push-Node j n*; [| *i*=*j*; *m*=*n* |] ==> *P*
|] ==> *P*
apply (*simp add*: *Push-Node-def*)
apply (*erule Abs-Node-inj* [*THEN apfst-convE*])
apply (*rule Rep-Node* [*THEN Node-Push-I*])+
apply (*erule sym* [*THEN apfst-convE*])
apply (*blast intro*: *Rep-Node-inject* [*THEN iffD1*] *trans sym elim*!: *Push-inject*)
done

lemma *Scons-inject-lemma1*: *Scons M N* <= *Scons M' N'* ==> *M*<=*M'*
unfolding *Scons-def One-nat-def*
by (*blast dest*!: *Push-Node-inject*)

lemma *Scons-inject-lemma2*: *Scons M N* <= *Scons M' N'* ==> *N*<=*N'*
unfolding *Scons-def One-nat-def*
by (*blast dest*!: *Push-Node-inject*)

lemma *Scons-inject1*: *Scons M N* = *Scons M' N'* ==> *M*=*M'*
apply (*erule equalityE*)
apply (*iprover intro*: *equalityI Scons-inject-lemma1*)
done

lemma *Scons-inject2*: *Scons M N* = *Scons M' N'* ==> *N*=*N'*
apply (*erule equalityE*)
apply (*iprover intro*: *equalityI Scons-inject-lemma2*)
done

lemma *Scons-inject*:
[| *Scons M N* = *Scons M' N'*; [| *M*=*M'*; *N*=*N'* |] ==> *P* |] ==> *P*
by (*iprover dest*: *Scons-inject1 Scons-inject2*)

lemma *Scons-Scons-eq* [*iff*]: (*Scons M N* = *Scons M' N'*) = (*M*=*M'* & *N*=*N'*)

by (*blast elim*!: *Scons-inject*)

lemma *Scons-not-Leaf* [*iff*]: *Scons M N* ≠ *Leaf*(*a*)
unfolding *Leaf-def o-def* **by** (*rule Scons-not-Atom*)

lemmas *Leaf-not-Scons* [*iff*] = *Scons-not-Leaf* [*THEN not-sym*, *standard*]

lemma *Scons-not-Numb* [*iff*]: *Scons M N* ≠ *Numb*(*k*)
unfolding *Numb-def o-def* **by** (*rule Scons-not-Atom*)

lemmas *Numb-not-Scons* [*iff*] = *Scons-not-Numb* [*THEN not-sym*, *standard*]

lemma *Leaf-not-Numb* [*iff*]: *Leaf*(*a*) ≠ *Numb*(*k*)
by (*simp add*: *Leaf-def Numb-def*)

lemmas *Numb-not-Leaf* [*iff*] = *Leaf-not-Numb* [*THEN not-sym*, *standard*]

lemma *ndepth-K0*: *ndepth* (*Abs-Node*(%*k*. *Inr 0*, *x*)) = *0*
by (*simp add*: *ndepth-def Node-K0-I* [*THEN Abs-Node-inverse*] *Least-equality*)

lemma *ndepth-Push-Node-aux*:
nat-case (*Inr* (*Suc i*)) *f k* = *Inr 0* −−> *Suc*(*LEAST x*. *f x* = *Inr 0*) <= *k*
apply (*induct-tac k*, *auto*)
apply (*erule Least-le*)
done

lemma *ndepth-Push-Node*:
ndepth (*Push-Node* (*Inr* (*Suc i*)) *n*) = *Suc*(*ndepth*(*n*))
apply (*insert Rep-Node* [*of n*, *unfolded Node-def*])
apply (*auto simp add*: *ndepth-def Push-Node-def*
Rep-Node [*THEN Node-Push-I*, *THEN Abs-Node-inverse*])
apply (*rule Least-equality*)
apply (*auto simp add*: *Push-def ndepth-Push-Node-aux*)
apply (*erule LeastI*)
done

lemma *ntrunc-0* [*simp*]: *ntrunc 0 M* = {}
by (*simp add*: *ntrunc-def*)

lemma *ntrunc-Atom* [*simp*]: *ntrunc* (*Suc k*) (*Atom a*) = *Atom*(*a*)
by (*auto simp add*: *Atom-def ntrunc-def ndepth-K0*)

lemma *ntrunc-Leaf* [*simp*]: *ntrunc* (*Suc k*) (*Leaf a*) = *Leaf*(*a*)
unfolding *Leaf-def o-def* **by** (*rule ntrunc-Atom*)

lemma *ntrunc-Numb* [*simp*]: *ntrunc* (*Suc k*) (*Numb i*) = *Numb*(*i*)
unfolding *Numb-def o-def* **by** (*rule ntrunc-Atom*)

lemma *ntrunc-Scons* [*simp*]:
ntrunc (*Suc k*) (*Scons M N*) = *Scons* (*ntrunc k M*) (*ntrunc k N*)
unfolding *Scons-def ntrunc-def One-nat-def*
by (*auto simp add*: *ndepth-Push-Node*)

lemma *ntrunc-one-In0* [*simp*]: *ntrunc* (*Suc 0*) (*In0 M*) = {}
apply (*simp add*: *In0-def*)
apply (*simp add*: *Scons-def*)
done

lemma *ntrunc-In0* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In0 M*) = *In0* (*ntrunc* (*Suc k*) *M*)
by (*simp add*: *In0-def*)

lemma *ntrunc-one-In1* [*simp*]: *ntrunc* (*Suc 0*) (*In1 M*) = {}
apply (*simp add*: *In1-def*)
apply (*simp add*: *Scons-def*)
done

lemma *ntrunc-In1* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In1 M*) = *In1* (*ntrunc* (*Suc k*) *M*)
by (*simp add*: *In1-def*)

16.3 Set Constructions

lemma *uprodI* [*intro*!]: [| *M*:*A*; *N*:*B* |] ==> *Scons M N* : *uprod A B*
by (*simp add*: *uprod-def*)

lemma *uprodE* [*elim*!]:
[| *c* : *uprod A B*;

!!$x\ y$. [| x:A; y:B; $c = Scons\ x\ y$ |] ==> P
|] ==> P
by (*auto simp add*: *uprod-def*)

lemma *uprodE2*: [| *Scons* $M\ N$: *uprod* $A\ B$; [| M:A; N:B |] ==> P |] ==> P
by (*auto simp add*: *uprod-def*)

lemma *usum-In0I* [*intro*]: M:A ==> *In0*(M) : *usum* $A\ B$
by (*simp add*: *usum-def*)

lemma *usum-In1I* [*intro*]: N:B ==> *In1*(N) : *usum* $A\ B$
by (*simp add*: *usum-def*)

lemma *usumE* [*elim*!]:
[| u : *usum* $A\ B$;
!!x. [| x:A; u=*In0*(x) |] ==> P;
!!y. [| y:B; u=*In1*(y) |] ==> P
|] ==> P
by (*auto simp add*: *usum-def*)

lemma *In0-not-In1* [*iff*]: *In0*(M) ≠ *In1*(N)
unfolding *In0-def In1-def One-nat-def* **by** *auto*

lemmas *In1-not-In0* [*iff*] = *In0-not-In1* [*THEN not-sym*, *standard*]

lemma *In0-inject*: *In0*(M) = *In0*(N) ==> M=N
by (*simp add*: *In0-def*)

lemma *In1-inject*: *In1*(M) = *In1*(N) ==> M=N
by (*simp add*: *In1-def*)

lemma *In0-eq* [*iff*]: (*In0* M = *In0* N) = (M=N)
by (*blast dest*!: *In0-inject*)

lemma *In1-eq* [*iff*]: (*In1* M = *In1* N) = (M=N)
by (*blast dest*!: *In1-inject*)

lemma *inj-In0*: *inj In0*
by (*blast intro*!: *inj-onI*)

lemma *inj-In1*: *inj In1*

by (*blast intro*!: *inj-onI*)

lemma *Lim-inject*: *Lim f = Lim g ==> f = g*
apply (*simp add*: *Lim-def*)
apply (*rule ext*)
apply (*blast elim*!: *Push-Node-inject*)
done

lemma *ntrunc-subsetI*: *ntrunc k M <= M*
by (*auto simp add*: *ntrunc-def*)

lemma *ntrunc-subsetD*: (!!*k*. *ntrunc k M <= N*) *==> M<=N*
by (*auto simp add*: *ntrunc-def*)

lemma *ntrunc-equality*: (!!*k*. *ntrunc k M = ntrunc k N*) *==> M=N*
apply (*rule equalityI*)
apply (*rule-tac* [!] *ntrunc-subsetD*)
apply (*rule-tac* [!] *ntrunc-subsetI* [*THEN* [*2*] *subset-trans*], *auto*)
done

lemma *ntrunc-o-equality*:
[| !!*k*. (*ntrunc*(*k*) *o h1*) = (*ntrunc*(*k*) *o h2*) |] *==> h1=h2*
apply (*rule ntrunc-equality* [*THEN ext*])
apply (*simp add*: *expand-fun-eq*)
done

lemma *uprod-mono*: [| *A<=A'*; *B<=B'* |] *==> uprod A B <= uprod A' B'*
by (*simp add*: *uprod-def*, *blast*)

lemma *usum-mono*: [| *A<=A'*; *B<=B'* |] *==> usum A B <= usum A' B'*
by (*simp add*: *usum-def*, *blast*)

lemma *Scons-mono*: [| *M<=M'*; *N<=N'* |] *==> Scons M N <= Scons M' N'*
by (*simp add*: *Scons-def*, *blast*)

lemma *In0-mono*: *M<=N ==> In0*(*M*) *<= In0*(*N*)
by (*simp add*: *In0-def Scons-mono*)

lemma *In1-mono*: *M<=N ==> In1*(*M*) *<= In1*(*N*)

by (*simp add*: *In1-def Scons-mono*)

lemma *Split* [*simp*]: *Split c* (*Scons M N*) = *c M N*
by (*simp add*: *Split-def*)

lemma *Case-In0* [*simp*]: *Case c d* (*In0 M*) = *c*(*M*)
by (*simp add*: *Case-def*)

lemma *Case-In1* [*simp*]: *Case c d* (*In1 N*) = *d*(*N*)
by (*simp add*: *Case-def*)

lemma *ntrunc-UN1*: *ntrunc k* (*UN x. f*(*x*)) = (*UN x. ntrunc k* (*f x*))
by (*simp add*: *ntrunc-def*, *blast*)

lemma *Scons-UN1-x*: *Scons* (*UN x. f x*) *M* = (*UN x. Scons* (*f x*) *M*)
by (*simp add*: *Scons-def*, *blast*)

lemma *Scons-UN1-y*: *Scons M* (*UN x. f x*) = (*UN x. Scons M* (*f x*))
by (*simp add*: *Scons-def*, *blast*)

lemma *In0-UN1*: *In0*(*UN x. f*(*x*)) = (*UN x. In0*(*f*(*x*)))
by (*simp add*: *In0-def Scons-UN1-y*)

lemma *In1-UN1*: *In1*(*UN x. f*(*x*)) = (*UN x. In1*(*f*(*x*)))
by (*simp add*: *In1-def Scons-UN1-y*)

lemma *dprodI* [*intro*!]:
[| (*M*,*M'*):*r*; (*N*,*N'*):*s* |] ==> (*Scons M N*, *Scons M' N'*) : *dprod r s*
by (*auto simp add*: *dprod-def*)

lemma *dprodE* [*elim*!]:
[| *c* : *dprod r s*;
!!*x y x' y'*. [| (*x*,*x'*) : *r*; (*y*,*y'*) : *s*;
c = (*Scons x y*, *Scons x' y'*) |] ==> *P*
|] ==> *P*
by (*auto simp add*: *dprod-def*)

lemma *dsum-In0I* [*intro*]: (*M*,*M'*):*r* ==> (*In0*(*M*), *In0*(*M'*)) : *dsum r s*
by (*auto simp add*: *dsum-def*)

lemma *dsum-In1I* [*intro*]: (*N*,*N'*):*s* ==> (*In1*(*N*), *In1*(*N'*)) : *dsum r s*
by (*auto simp add*: *dsum-def*)

lemma *dsumE* [*elim*!]:
[| *w* : *dsum r s*;
!!*x x'*. [| (*x*,*x'*) : *r*; *w* = (*In0*(*x*), *In0*(*x'*)) |] ==> *P*;
!!*y y'*. [| (*y*,*y'*) : *s*; *w* = (*In1*(*y*), *In1*(*y'*)) |] ==> *P*
|] ==> *P*
by (*auto simp add*: *dsum-def*)

lemma *dprod-mono*: [| *r*<=*r'*; *s*<=*s'* |] ==> *dprod r s* <= *dprod r' s'*
by *blast*

lemma *dsum-mono*: [| *r*<=*r'*; *s*<=*s'* |] ==> *dsum r s* <= *dsum r' s'*
by *blast*

lemma *dprod-Sigma*: (*dprod* (*A* <*>* *B*) (*C* <*>* *D*)) <= (*uprod A C*) <*> (*uprod B D*)
by *blast*

lemmas *dprod-subset-Sigma* = *subset-trans* [*OF dprod-mono dprod-Sigma*, *standard*]

lemma *dprod-subset-Sigma2*:
(*dprod* (*Sigma A B*) (*Sigma C D*)) <=
Sigma (*uprod A C*) (*Split* (%*x y*. *uprod* (*B x*) (*D y*)))
by *auto*

lemma *dsum-Sigma*: (*dsum* (*A* <*>* *B*) (*C* <*>* *D*)) <= (*usum A C*) <*> (*usum B D*)
by *blast*

lemmas *dsum-subset-Sigma* = *subset-trans* [*OF dsum-mono dsum-Sigma*, *standard*]

hides popular names

hide-type (**open**) *node item*

hide-const (**open**) *Push Node Atom Leaf Numb Lim Split Case*

use *Tools/Datatype/datatype.ML*

use *Tools/inductive-realizer.ML*
setup *InductiveRealizer.setup*

use *Tools/Datatype/datatype-realizer.ML*
setup *Datatype-Realizer.setup*

end

17 Record: Extensible records with structural subtyping

theory *Record*
imports *Datatype*
uses (*Tools/record.ML*)
begin

17.1 Introduction

Records are isomorphic to compound tuple types. To implement efficient records, we make this isomorphism explicit. Consider the record access/update simplification *alpha* (*beta-update f rec*) = *alpha rec* for distinct fields alpha and beta of some record rec with n fields. There are n ^ *2* such theorems, which prohibits storage of all of them for large n. The rules can be proved on the fly by case decomposition and simplification in O(n) time. By creating O(n) isomorphic-tuple types while defining the record, however, we can prove the access/update simplification in $O(log(n)$ ^*2*$)$ time.

The O(n) cost of case decomposition is not because O(n) steps are taken, but rather because the resulting rule must contain O(n) new variables and an O(n) size concrete record construction. To sidestep this cost, we would like to avoid case decomposition in proving access/update theorems.

Record types are defined as isomorphic to tuple types. For instance, a record type with fields $'a$, $'b$, $'c$ and $'d$ might be introduced as isomorphic to $'a \times ('b \times ('c \times 'd))$. If we balance the tuple tree to $('a \times 'b) \times ('c \times 'd)$ then accessors can be defined by converting to the underlying type then using O(log(n)) fst or snd operations. Updators can be defined similarly, if we introduce a *fst-update* and *snd-update* function. Furthermore, we can prove the access/update theorem in O(log(n)) steps by using simple rewrites on fst, snd, *fst-update* and *snd-update*.

The catch is that, although O(log(n)) steps were taken, the underlying type we converted to is a tuple tree of size O(n). Processing this term type wastes

performance. We avoid this for large n by taking each subtree of size K and defining a new type isomorphic to that tuple subtree. A record can now be defined as isomorphic to a tuple tree of these O(n/K) new types, or, if $n > K*K$, we can repeat the process, until the record can be defined in terms of a tuple tree of complexity less than the constant K.

If we prove the access/update theorem on this type with the analagous steps to the tuple tree, we consume $O(log(n)\hat{}2)$ time as the intermediate terms are $O(log(n))$ in size and the types needed have size bounded by K. To enable this analagous traversal, we define the functions seen below: *iso-tuple-fst*, *iso-tuple-snd*, *iso-tuple-fst-update* and *iso-tuple-snd-update*. These functions generalise tuple operations by taking a parameter that encapsulates a tuple isomorphism. The rewrites needed on these functions now need an additional assumption which is that the isomorphism works.

These rewrites are typically used in a structured way. They are here presented as the introduction rule *isomorphic-tuple.intros* rather than as a rewrite rule set. The introduction form is an optimisation, as net matching can be performed at one term location for each step rather than the simplifier searching the term for possible pattern matches. The rule set is used as it is viewed outside the locale, with the locale assumption (that the isomorphism is valid) left as a rule assumption. All rules are structured to aid net matching, using either a point-free form or an encapsulating predicate.

17.2 Operators and lemmas for types isomorphic to tuples

datatype $('a, 'b, 'c)$ *tuple-isomorphism* =
  Tuple-Isomorphism $'a \Rightarrow 'b \times 'c$ $'b \times 'c \Rightarrow 'a$

primrec
  repr :: $('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow 'a \Rightarrow 'b \times 'c$ **where**
  repr (*Tuple-Isomorphism r a*) = *r*

primrec
  abst :: $('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow 'b \times 'c \Rightarrow 'a$ **where**
  abst (*Tuple-Isomorphism r a*) = *a*

definition
  iso-tuple-fst :: $('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow 'a \Rightarrow 'b$ **where**
  iso-tuple-fst isom = *fst* ∘ *repr isom*

definition
  iso-tuple-snd :: $('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow 'a \Rightarrow 'c$ **where**
  iso-tuple-snd isom = *snd* ∘ *repr isom*

definition
  iso-tuple-fst-update ::
    $('a, 'b, 'c)$ *tuple-isomorphism* $\Rightarrow ('b \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'a)$ **where**

iso-tuple-fst-update isom f = abst isom ∘ apfst f ∘ repr isom

definition
iso-tuple-snd-update ::
('a, 'b, 'c) *tuple-isomorphism* ⇒ ('c ⇒ 'c) ⇒ ('a ⇒ 'a) **where**
iso-tuple-snd-update isom f = abst isom ∘ apsnd f ∘ repr isom

definition
iso-tuple-cons ::
('a, 'b, 'c) *tuple-isomorphism* ⇒ 'b ⇒ 'c ⇒ 'a **where**
iso-tuple-cons isom = curry (abst isom)

17.3 Logical infrastructure for records

definition
iso-tuple-surjective-proof-assist :: 'a ⇒ 'b ⇒ ('a ⇒ 'b) ⇒ *bool* **where**
iso-tuple-surjective-proof-assist x y f ⟷ f x = y

definition
iso-tuple-update-accessor-cong-assist ::
(('b ⇒ 'b) ⇒ ('a ⇒ 'a)) ⇒ ('a ⇒ 'b) ⇒ *bool* **where**
iso-tuple-update-accessor-cong-assist upd acc ⟷
(∀ f v. *upd* (λx. f (*acc* v)) v = *upd* f v) ∧ (∀ v. *upd id* v = v)

definition
iso-tuple-update-accessor-eq-assist ::
(('b ⇒ 'b) ⇒ ('a ⇒ 'a)) ⇒ ('a ⇒ 'b) ⇒ 'a ⇒ ('b ⇒ 'b) ⇒ 'a ⇒ 'b ⇒ *bool*
where
iso-tuple-update-accessor-eq-assist upd acc v f v' x ⟷
upd f v = v' ∧ acc v = x ∧ iso-tuple-update-accessor-cong-assist upd acc

lemma *update-accessor-congruence-foldE*:
assumes *uac*: *iso-tuple-update-accessor-cong-assist upd acc*
and *r*: r = r' **and** *v*: *acc r' = v'*
and *f*: ⋀v. v' = v ⟹ f v = f' v
shows *upd f r = upd f' r'*
using *uac r v* [*symmetric*]
apply (*subgoal-tac upd* (λx. f (*acc r'*)) r' = *upd* (λx. f' (*acc r'*)) r')
apply (*simp add*: *iso-tuple-update-accessor-cong-assist-def*)
apply (*simp add*: *f*)
done

lemma *update-accessor-congruence-unfoldE*:
iso-tuple-update-accessor-cong-assist upd acc ⟹
r = r' ⟹ *acc r' = v'* ⟹ (⋀v. v = v' ⟹ f v = f' v) ⟹
upd f r = upd f' r'
apply (*erule*(2) *update-accessor-congruence-foldE*)
apply *simp*
done

lemma *iso-tuple-update-accessor-cong-assist-id*:
iso-tuple-update-accessor-cong-assist upd acc $\Longrightarrow$ *upd id* = *id*
by *rule* (*simp add*: *iso-tuple-update-accessor-cong-assist-def*)

lemma *update-accessor-noopE*:
assumes *uac*: *iso-tuple-update-accessor-cong-assist upd acc*
and *acc*: *f* (*acc x*) = *acc x*
shows *upd f x* = *x*
using *uac*
by (*simp add*: *acc iso-tuple-update-accessor-cong-assist-id* [*OF uac*, *unfolded id-def*]
cong: *update-accessor-congruence-unfoldE* [*OF uac*])

lemma *update-accessor-noop-compE*:
assumes *uac*: *iso-tuple-update-accessor-cong-assist upd acc*
and *acc*: *f* (*acc x*) = *acc x*
shows *upd* (*g* ∘ *f*) *x* = *upd g x*
by (*simp add*: *acc cong*: *update-accessor-congruence-unfoldE*[*OF uac*])

lemma *update-accessor-cong-assist-idI*:
iso-tuple-update-accessor-cong-assist id id
by (*simp add*: *iso-tuple-update-accessor-cong-assist-def*)

lemma *update-accessor-cong-assist-triv*:
iso-tuple-update-accessor-cong-assist upd acc $\Longrightarrow$
iso-tuple-update-accessor-cong-assist upd acc
by *assumption*

lemma *update-accessor-accessor-eqE*:
iso-tuple-update-accessor-eq-assist upd acc v f v' x $\Longrightarrow$ *acc v* = *x*
by (*simp add*: *iso-tuple-update-accessor-eq-assist-def*)

lemma *update-accessor-updator-eqE*:
iso-tuple-update-accessor-eq-assist upd acc v f v' x $\Longrightarrow$ *upd f v* = *v'*
by (*simp add*: *iso-tuple-update-accessor-eq-assist-def*)

lemma *iso-tuple-update-accessor-eq-assist-idI*:
v' = *f v* $\Longrightarrow$ *iso-tuple-update-accessor-eq-assist id id v f v' v*
by (*simp add*: *iso-tuple-update-accessor-eq-assist-def update-accessor-cong-assist-idI*)

lemma *iso-tuple-update-accessor-eq-assist-triv*:
iso-tuple-update-accessor-eq-assist upd acc v f v' x $\Longrightarrow$
iso-tuple-update-accessor-eq-assist upd acc v f v' x
by *assumption*

lemma *iso-tuple-update-accessor-cong-from-eq*:
iso-tuple-update-accessor-eq-assist upd acc v f v' x $\Longrightarrow$
iso-tuple-update-accessor-cong-assist upd acc

by (*simp add*: *iso-tuple-update-accessor-eq-assist-def*)

lemma *iso-tuple-surjective-proof-assistI*:
f x = *y* ⟹ *iso-tuple-surjective-proof-assist x y f*
by (*simp add*: *iso-tuple-surjective-proof-assist-def*)

lemma *iso-tuple-surjective-proof-assist-idE*:
iso-tuple-surjective-proof-assist x y id ⟹ *x* = *y*
by (*simp add*: *iso-tuple-surjective-proof-assist-def*)

locale *isomorphic-tuple* =
fixes *isom* :: ('*a*, '*b*, '*c*) *tuple-isomorphism*
assumes *repr-inv*: ⋀*x*. *abst isom* (*repr isom x*) = *x*
and *abst-inv*: ⋀*y*. *repr isom* (*abst isom y*) = *y*
begin

lemma *repr-inj*: *repr isom x* = *repr isom y* ⟷ *x* = *y*
by (*auto dest*: *arg-cong* [*of repr isom x repr isom y abst isom*]
simp add: *repr-inv*)

lemma *abst-inj*: *abst isom x* = *abst isom y* ⟷ *x* = *y*
by (*auto dest*: *arg-cong* [*of abst isom x abst isom y repr isom*]
simp add: *abst-inv*)

lemmas *simps* = *Let-def repr-inv abst-inv repr-inj abst-inj*

lemma *iso-tuple-access-update-fst-fst*:
f o h g = *j o f* ⟹
(*f o iso-tuple-fst isom*) *o* (*iso-tuple-fst-update isom o h*) *g* =
j o (*f o iso-tuple-fst isom*)
by (*clarsimp simp*: *iso-tuple-fst-update-def iso-tuple-fst-def simps*
intro!: *ext elim*!: *o-eq-elim*)

lemma *iso-tuple-access-update-snd-snd*:
f o h g = *j o f* ⟹
(*f o iso-tuple-snd isom*) *o* (*iso-tuple-snd-update isom o h*) *g* =
j o (*f o iso-tuple-snd isom*)
by (*clarsimp simp*: *iso-tuple-snd-update-def iso-tuple-snd-def simps*
intro!: *ext elim*!: *o-eq-elim*)

lemma *iso-tuple-access-update-fst-snd*:
(*f o iso-tuple-fst isom*) *o* (*iso-tuple-snd-update isom o h*) *g* =
id o (*f o iso-tuple-fst isom*)
by (*clarsimp simp*: *iso-tuple-snd-update-def iso-tuple-fst-def simps*
intro!: *ext elim*!: *o-eq-elim*)

lemma *iso-tuple-access-update-snd-fst*:
(*f o iso-tuple-snd isom*) *o* (*iso-tuple-fst-update isom o h*) *g* =
id o (*f o iso-tuple-snd isom*)

by (*clarsimp simp*: *iso-tuple-fst-update-def iso-tuple-snd-def simps intro*!: *ext elim*!: *o-eq-elim*)

lemma *iso-tuple-update-swap-fst-fst*:
h f o j g = j g o h f ⟹
(*iso-tuple-fst-update isom o h*) *f o* (*iso-tuple-fst-update isom o j*) *g* =
(*iso-tuple-fst-update isom o j*) *g o* (*iso-tuple-fst-update isom o h*) *f*
by (*clarsimp simp*: *iso-tuple-fst-update-def simps apfst-compose intro*!: *ext*)

lemma *iso-tuple-update-swap-snd-snd*:
h f o j g = j g o h f ⟹
(*iso-tuple-snd-update isom o h*) *f o* (*iso-tuple-snd-update isom o j*) *g* =
(*iso-tuple-snd-update isom o j*) *g o* (*iso-tuple-snd-update isom o h*) *f*
by (*clarsimp simp*: *iso-tuple-snd-update-def simps apsnd-compose intro*!: *ext*)

lemma *iso-tuple-update-swap-fst-snd*:
(*iso-tuple-snd-update isom o h*) *f o* (*iso-tuple-fst-update isom o j*) *g* =
(*iso-tuple-fst-update isom o j*) *g o* (*iso-tuple-snd-update isom o h*) *f*
by (*clarsimp simp*: *iso-tuple-fst-update-def iso-tuple-snd-update-def simps intro*!: *ext*)

lemma *iso-tuple-update-swap-snd-fst*:
(*iso-tuple-fst-update isom o h*) *f o* (*iso-tuple-snd-update isom o j*) *g* =
(*iso-tuple-snd-update isom o j*) *g o* (*iso-tuple-fst-update isom o h*) *f*
by (*clarsimp simp*: *iso-tuple-fst-update-def iso-tuple-snd-update-def simps intro*!: *ext*)

lemma *iso-tuple-update-compose-fst-fst*:
h f o j g = k (*f o g*) ⟹
(*iso-tuple-fst-update isom o h*) *f o* (*iso-tuple-fst-update isom o j*) *g* =
(*iso-tuple-fst-update isom o k*) (*f o g*)
by (*clarsimp simp*: *iso-tuple-fst-update-def simps apfst-compose intro*!: *ext*)

lemma *iso-tuple-update-compose-snd-snd*:
h f o j g = k (*f o g*) ⟹
(*iso-tuple-snd-update isom o h*) *f o* (*iso-tuple-snd-update isom o j*) *g* =
(*iso-tuple-snd-update isom o k*) (*f o g*)
by (*clarsimp simp*: *iso-tuple-snd-update-def simps apsnd-compose intro*!: *ext*)

lemma *iso-tuple-surjective-proof-assist-step*:
iso-tuple-surjective-proof-assist v a (*iso-tuple-fst isom o f*) ⟹
iso-tuple-surjective-proof-assist v b (*iso-tuple-snd isom o f*) ⟹
iso-tuple-surjective-proof-assist v (*iso-tuple-cons isom a b*) *f*
by (*clarsimp simp*: *iso-tuple-surjective-proof-assist-def simps iso-tuple-fst-def iso-tuple-snd-def iso-tuple-cons-def*)

lemma *iso-tuple-fst-update-accessor-cong-assist*:
assumes *iso-tuple-update-accessor-cong-assist f g*
shows *iso-tuple-update-accessor-cong-assist*

```
    (iso-tuple-fst-update isom o f) (g o iso-tuple-fst isom)
proof −
  from assms have f id = id
    by (rule iso-tuple-update-accessor-cong-assist-id)
  with assms show ?thesis
    by (clarsimp simp: iso-tuple-update-accessor-cong-assist-def simps
      iso-tuple-fst-update-def iso-tuple-fst-def)
qed

lemma iso-tuple-snd-update-accessor-cong-assist:
  assumes iso-tuple-update-accessor-cong-assist f g
  shows iso-tuple-update-accessor-cong-assist
    (iso-tuple-snd-update isom o f) (g o iso-tuple-snd isom)
proof −
  from assms have f id = id
    by (rule iso-tuple-update-accessor-cong-assist-id)
  with assms show ?thesis
    by (clarsimp simp: iso-tuple-update-accessor-cong-assist-def simps
      iso-tuple-snd-update-def iso-tuple-snd-def)
qed

lemma iso-tuple-fst-update-accessor-eq-assist:
  assumes iso-tuple-update-accessor-eq-assist f g a u a' v
  shows iso-tuple-update-accessor-eq-assist
    (iso-tuple-fst-update isom o f) (g o iso-tuple-fst isom)
    (iso-tuple-cons isom a b) u (iso-tuple-cons isom a' b) v
proof −
  from assms have f id = id
    by (auto simp add: iso-tuple-update-accessor-eq-assist-def
      intro: iso-tuple-update-accessor-cong-assist-id)
  with assms show ?thesis
    by (clarsimp simp: iso-tuple-update-accessor-eq-assist-def
      iso-tuple-fst-update-def iso-tuple-fst-def
      iso-tuple-update-accessor-cong-assist-def iso-tuple-cons-def simps)
qed

lemma iso-tuple-snd-update-accessor-eq-assist:
  assumes iso-tuple-update-accessor-eq-assist f g b u b' v
  shows iso-tuple-update-accessor-eq-assist
    (iso-tuple-snd-update isom o f) (g o iso-tuple-snd isom)
    (iso-tuple-cons isom a b) u (iso-tuple-cons isom a b') v
proof −
  from assms have f id = id
    by (auto simp add: iso-tuple-update-accessor-eq-assist-def
      intro: iso-tuple-update-accessor-cong-assist-id)
  with assms show ?thesis
    by (clarsimp simp: iso-tuple-update-accessor-eq-assist-def
      iso-tuple-snd-update-def iso-tuple-snd-def
      iso-tuple-update-accessor-cong-assist-def iso-tuple-cons-def simps)
```

qed

lemma *iso-tuple-cons-conj-eqI*:
  $a = c \wedge b = d \wedge P \longleftrightarrow Q \Longrightarrow$
    iso-tuple-cons isom a b = *iso-tuple-cons isom c d* $\wedge$ $P \longleftrightarrow Q$
  by (*clarsimp simp*: *iso-tuple-cons-def simps*)

lemmas *intros* =
  iso-tuple-access-update-fst-fst
  iso-tuple-access-update-snd-snd
  iso-tuple-access-update-fst-snd
  iso-tuple-access-update-snd-fst
  iso-tuple-update-swap-fst-fst
  iso-tuple-update-swap-snd-snd
  iso-tuple-update-swap-fst-snd
  iso-tuple-update-swap-snd-fst
  iso-tuple-update-compose-fst-fst
  iso-tuple-update-compose-snd-snd
  iso-tuple-surjective-proof-assist-step
  iso-tuple-fst-update-accessor-eq-assist
  iso-tuple-snd-update-accessor-eq-assist
  iso-tuple-fst-update-accessor-cong-assist
  iso-tuple-snd-update-accessor-cong-assist
  iso-tuple-cons-conj-eqI

end

lemma *isomorphic-tuple-intro*:
  fixes *repr abst*
  assumes *repr-inj*: $\bigwedge x\ y.$ *repr* x = *repr* $y \longleftrightarrow x = y$
    and *abst-inv*: $\bigwedge z.$ *repr* (*abst* z) $= z$
    and v: $v \equiv$ *Tuple-Isomorphism repr abst*
  shows *isomorphic-tuple v*
proof
  fix x **have** *repr* (*abst* (*repr* x)) = *repr* x
    by (*simp add*: *abst-inv*)
  then show *Record.abst v* (*Record.repr v* x) $= x$
    by (*simp add*: v *repr-inj*)
next
  fix y
  show *Record.repr v* (*Record.abst v* y) $= y$
    by (*simp add*: v) (*fact abst-inv*)
qed

definition
  tuple-iso-tuple $\equiv$ *Tuple-Isomorphism id id*

lemma *tuple-iso-tuple*:
  isomorphic-tuple tuple-iso-tuple

by (*simp add*: *isomorphic-tuple-intro* [*OF - - reflexive*] *tuple-iso-tuple-def*)

lemma *refl-conj-eq*: $Q = R \Longrightarrow P \wedge Q \longleftrightarrow P \wedge R$
by *simp*

lemma *iso-tuple-UNIV-I*: $x \in UNIV \equiv True$
by *simp*

lemma *iso-tuple-True-simp*: $(True \Longrightarrow PROP\ P) \equiv PROP\ P$
by *simp*

lemma *prop-subst*: $s = t \Longrightarrow PROP\ P\ t \Longrightarrow PROP\ P\ s$
by *simp*

lemma *K-record-comp*: $(\lambda x.\ c) \circ f = (\lambda x.\ c)$
by (*simp add*: *comp-def*)

lemma *o-eq-dest-lhs*: $a\ o\ b = c \Longrightarrow a\ (b\ v) = c\ v$
by *clarsimp*

lemma *o-eq-id-dest*: $a\ o\ b = id\ o\ c \Longrightarrow a\ (b\ v) = c\ v$
by *clarsimp*

17.4 Concrete record syntax

nonterminals
ident field-type field-types field fields field-update field-updates
syntax
-constify :: *id* => *ident* (-)
-constify :: *longid* => *ident* (-)

-field-type :: *ident* => *type* => *field-type* ((*2*- ::/ -))
:: *field-type* => *field-types* (-)
-field-types :: *field-type* => *field-types* => *field-types* (-,/ -)
-record-type :: *field-types* => *type* ((*3*'(| - |')))
-record-type-scheme :: *field-types* => *type* => *type* ((*3*'(| -,/ (*2*... ::/ -) |')))

-field :: *ident* => 'a => *field* ((*2*- =/ -))
:: *field* => *fields* (-)
-fields :: *field* => *fields* => *fields* (-,/ -)
-record :: *fields* => 'a ((*3*'(| - |')))
-record-scheme :: *fields* => 'a => 'a ((*3*'(| -,/ (*2*... =/ -) |')))

-field-update :: *ident* => 'a => *field-update* ((*2*- :=/ -))
:: *field-update* => *field-updates* (-)
-field-updates :: *field-update* => *field-updates* => *field-updates* (-,/ -)
-record-update :: 'a => *field-updates* => 'b (-/(*3*'(| - |')) [*900*, *0*] *900*)

syntax (*xsymbols*)

-record-type :: *field-types* => *type* ((*3*(|-|)))
-record-type-scheme :: *field-types* => *type* => *type* ((*3*(|-,/ (*2*... ::/ -)|)))
-record :: *fields* => *'a* ((*3*(|-|)))
-record-scheme :: *fields* => *'a* => *'a* ((*3*(|-,/ (*2*... =/ -)|)))
-record-update :: *'a* => *field-updates* => *'b* (-/(*3*(|-|)) [*900*, *0*] *900*)

17.5 Record package

use *Tools/record.ML*
setup *Record.setup*

hide-const (**open**) *Tuple-Isomorphism repr abst iso-tuple-fst iso-tuple-snd iso-tuple-fst-update iso-tuple-snd-update iso-tuple-cons iso-tuple-surjective-proof-assist iso-tuple-update-accessor-cong-assist iso-tuple-update-accessor-eq-assist tuple-iso-tuple*

end

18 Power: Exponentiation

theory *Power*
imports *Nat*
begin

18.1 Powers for Arbitrary Monoids

class *power* = *one* + *times*
begin

primrec *power* :: *'a* ⇒ *nat* ⇒ *'a* (**infixr** ^ *80*) **where**
  power-0: *a* ^ *0* = *1*
| *power-Suc*: *a* ^ *Suc n* = *a* ∗ *a* ^ *n*

notation (*latex* **output**)
  power ((-[^]) [*1000*] *1000*)

notation (*HTML* **output**)
  power ((-[^]) [*1000*] *1000*)

end

context *monoid-mult*
begin

subclass *power* **..**

lemma *power-one* [*simp*]:
  1 ^ *n* = *1*

by (*induct n*) *simp-all*

lemma *power-one-right* [*simp*]:
a ^ 1 = a
by *simp*

lemma *power-commutes*:
*a ^ n * a = a * a ^ n*
by (*induct n*) (*simp-all add*: *mult-assoc*)

lemma *power-Suc2*:
*a ^ Suc n = a ^ n * a*
by (*simp add*: *power-commutes*)

lemma *power-add*:
*a ^ (m + n) = a ^ m * a ^ n*
by (*induct m*) (*simp-all add*: *algebra-simps*)

lemma *power-mult*:
*a ^ (m * n) = (a ^ m) ^ n*
by (*induct n*) (*simp-all add*: *power-add*)

end

context *comm-monoid-mult*
begin

lemma *power-mult-distrib*:
*(a * b) ^ n = (a ^ n) * (b ^ n)*
by (*induct n*) (*simp-all add*: *mult-ac*)

end

context *semiring-1*
begin

lemma *of-nat-power*:
of-nat (m ^ n) = of-nat m ^ n
by (*induct n*) (*simp-all add*: *of-nat-mult*)

end

context *comm-semiring-1*
begin

The divides relation

lemma *le-imp-power-dvd*:
assumes $m \leq n$ **shows** *a ^ m dvd a ^ n*
proof

```
  have a ^ n = a ^ (m + (n - m))
    using ‹m ≤ n› by simp
  also have ... = a ^ m * a ^ (n - m)
    by (rule power-add)
  finally show a ^ n = a ^ m * a ^ (n - m) .
qed

lemma power-le-dvd:
  a ^ n dvd b ⟹ m ≤ n ⟹ a ^ m dvd b
  by (rule dvd-trans [OF le-imp-power-dvd])

lemma dvd-power-same:
  x dvd y ⟹ x ^ n dvd y ^ n
  by (induct n) (auto simp add: mult-dvd-mono)

lemma dvd-power-le:
  x dvd y ⟹ m ≥ n ⟹ x ^ n dvd y ^ m
  by (rule power-le-dvd [OF dvd-power-same])

lemma dvd-power [simp]:
  assumes n > (0::nat) ∨ x = 1
  shows x dvd (x ^ n)
using assms proof
  assume 0 < n
  then have x ^ n = x ^ Suc (n - 1) by simp
  then show x dvd (x ^ n) by simp
next
  assume x = 1
  then show x dvd (x ^ n) by simp
qed

end

context ring-1
begin

lemma power-minus:
  (- a) ^ n = (- 1) ^ n * a ^ n
proof (induct n)
  case 0 show ?case by simp
next
  case (Suc n) then show ?case
    by (simp del: power-Suc add: power-Suc2 mult-assoc)
qed

end

context linordered-semidom
begin
```

lemma *zero-less-power* [*simp*]:
0 < a ⟹ 0 < a ^ n
by (*induct n*) (*simp-all add*: *mult-pos-pos*)

lemma *zero-le-power* [*simp*]:
0 ≤ a ⟹ 0 ≤ a ^ n
by (*induct n*) (*simp-all add*: *mult-nonneg-nonneg*)

lemma *one-le-power*[*simp*]:
1 ≤ a ⟹ 1 ≤ a ^ n
apply (*induct n*)
apply *simp-all*
apply (*rule order-trans* [*OF - mult-mono* [*of 1 - 1*]])
apply (*simp-all add*: *order-trans* [*OF zero-le-one*])
done

lemma *power-gt1-lemma*:
assumes *gt1*: *1 < a*
shows *1 < a * a ^ n*
proof −
from *gt1* **have** *0 ≤ a*
by (*fact order-trans* [*OF zero-le-one less-imp-le*])
have *1 * 1 < a * 1* **using** *gt1* **by** *simp*
also have *... ≤ a * a ^ n* **using** *gt1*
by (*simp only*: *mult-mono ‹0 ≤ a› one-le-power order-less-imp-le*
zero-le-one order-refl)
finally show *?thesis* **by** *simp*
qed

lemma *power-gt1*:
1 < a ⟹ 1 < a ^ Suc n
by (*simp add*: *power-gt1-lemma*)

lemma *one-less-power* [*simp*]:
1 < a ⟹ 0 < n ⟹ 1 < a ^ n
by (*cases n*) (*simp-all add*: *power-gt1-lemma*)

lemma *power-le-imp-le-exp*:
assumes *gt1*: *1 < a*
shows *a ^ m ≤ a ^ n ⟹ m ≤ n*
proof (*induct m arbitrary*: *n*)
case *0*
show *?case* **by** *simp*
next
case (*Suc m*)
show *?case*
proof (*cases n*)
case *0*

```
    with Suc.prems Suc.hyps have a * a ^ m ≤ 1 by simp
    with gt1 show ?thesis
      by (force simp only: power-gt1-lemma
          not-less [symmetric])
  next
    case (Suc n)
    with Suc.prems Suc.hyps show ?thesis
      by (force dest: mult-left-le-imp-le
          simp add: less-trans [OF zero-less-one gt1])
  qed
qed
```

Surely we can strengthen this? It holds for $0<a<1$ too.

```
lemma power-inject-exp [simp]:
  1 < a ⟹ a ^ m = a ^ n ⟷ m = n
  by (force simp add: order-antisym power-le-imp-le-exp)
```

Can relax the first premise to $(0::'a) < a$ in the case of the natural numbers.

```
lemma power-less-imp-less-exp:
  1 < a ⟹ a ^ m < a ^ n ⟹ m < n
  by (simp add: order-less-le [of m n] less-le [of a^m a^n]
    power-le-imp-le-exp)

lemma power-mono:
  a ≤ b ⟹ 0 ≤ a ⟹ a ^ n ≤ b ^ n
  by (induct n)
    (auto intro: mult-mono order-trans [of 0 a b])

lemma power-strict-mono [rule-format]:
  a < b ⟹ 0 ≤ a ⟹ 0 < n ⟶ a ^ n < b ^ n
  by (induct n)
   (auto simp add: mult-strict-mono le-less-trans [of 0 a b])
```

Lemma for *power-strict-decreasing*

```
lemma power-Suc-less:
  0 < a ⟹ a < 1 ⟹ a * a ^ n < a ^ n
  by (induct n)
    (auto simp add: mult-strict-left-mono)

lemma power-strict-decreasing [rule-format]:
  n < N ⟹ 0 < a ⟹ a < 1 ⟶ a ^ N < a ^ n
proof (induct N)
  case 0 then show ?case by simp
next
  case (Suc N) then show ?case
  apply (auto simp add: power-Suc-less less-Suc-eq)
  apply (subgoal-tac a * a^N < 1 * a^n)
  apply simp
  apply (rule mult-strict-mono) apply auto
```

done
qed

Proof resembles that of *power-strict-decreasing*

lemma *power-decreasing* [*rule-format*]:
n ≤ *N* ⟹ *0* < *a* ⟹ *a* ≤ *1* ⟶ *a* ^ *N* ≤ *a* ^ *n*
proof (*induct N*)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc N*) **then show** *?case*
apply (*auto simp add*: *le-Suc-eq*)
apply (*subgoal-tac a* ∗ *a*^*N* ≤ *1* ∗ *a*^*n*, *simp*)
apply (*rule mult-mono*) **apply** *auto*
done
qed

lemma *power-Suc-less-one*:
0 < *a* ⟹ *a* < *1* ⟹ *a* ^ *Suc n* < *1*
using *power-strict-decreasing* [*of 0 Suc n a*] **by** *simp*

Proof again resembles that of *power-strict-decreasing*

lemma *power-increasing* [*rule-format*]:
n ≤ *N* ⟹ *1* ≤ *a* ⟹ *a* ^ *n* ≤ *a* ^ *N*
proof (*induct N*)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc N*) **then show** *?case*
apply (*auto simp add*: *le-Suc-eq*)
apply (*subgoal-tac 1* ∗ *a*^*n* ≤ *a* ∗ *a*^*N*, *simp*)
apply (*rule mult-mono*) **apply** (*auto simp add*: *order-trans* [*OF zero-le-one*])
done
qed

Lemma for *power-strict-increasing*

lemma *power-less-power-Suc*:
1 < *a* ⟹ *a* ^ *n* < *a* ∗ *a* ^ *n*
by (*induct n*) (*auto simp add*: *mult-strict-left-mono less-trans* [*OF zero-less-one*])

lemma *power-strict-increasing* [*rule-format*]:
n < *N* ⟹ *1* < *a* ⟶ *a* ^ *n* < *a* ^ *N*
proof (*induct N*)
case *0* **then show** *?case* **by** *simp*
next
case (*Suc N*) **then show** *?case*
apply (*auto simp add*: *power-less-power-Suc less-Suc-eq*)
apply (*subgoal-tac 1* ∗ *a*^*n* < *a* ∗ *a*^*N*, *simp*)
apply (*rule mult-strict-mono*) **apply** (*auto simp add*: *less-trans* [*OF zero-less-one*] *less-imp-le*)
done

qed

lemma *power-increasing-iff* [*simp*]:
1 < b ⟹ b ^ x ≤ b ^ y ⟷ x ≤ y
by (*blast intro*: *power-le-imp-le-exp power-increasing less-imp-le*)

lemma *power-strict-increasing-iff* [*simp*]:
1 < b ⟹ b ^ x < b ^ y ⟷ x < y
by (*blast intro*: *power-less-imp-less-exp power-strict-increasing*)

lemma *power-le-imp-le-base*:
assumes *le*: *a ^ Suc n ≤ b ^ Suc n*
and *ynonneg*: *0 ≤ b*
shows *a ≤ b*
proof (*rule ccontr*)
assume ~ *a ≤ b*
then have *b < a* **by** (*simp only*: *linorder-not-le*)
then have *b ^ Suc n < a ^ Suc n*
by (*simp only*: *prems power-strict-mono*)
from *le* **and** *this* **show** *False*
by (*simp add*: *linorder-not-less* [*symmetric*])
qed

lemma *power-less-imp-less-base*:
assumes *less*: *a ^ n < b ^ n*
assumes *nonneg*: *0 ≤ b*
shows *a < b*
proof (*rule contrapos-pp* [*OF less*])
assume ~ *a < b*
hence *b ≤ a* **by** (*simp only*: *linorder-not-less*)
hence *b ^ n ≤ a ^ n* **using** *nonneg* **by** (*rule power-mono*)
thus ¬ *a ^ n < b ^ n* **by** (*simp only*: *linorder-not-less*)
qed

lemma *power-inject-base*:
a ^ Suc n = b ^ Suc n ⟹ 0 ≤ a ⟹ 0 ≤ b ⟹ a = b
by (*blast intro*: *power-le-imp-le-base antisym eq-refl sym*)

lemma *power-eq-imp-eq-base*:
a ^ n = b ^ n ⟹ 0 ≤ a ⟹ 0 ≤ b ⟹ 0 < n ⟹ a = b
by (*cases n*) (*simp-all del*: *power-Suc*, *rule power-inject-base*)

end

context *linordered-idom*
begin

lemma *power-abs*:
abs (a ^ n) = abs a ^ n

by (*induct n*) (*auto simp add*: *abs-mult*)

lemma *abs-power-minus* [*simp*]:
abs ((−*a*) ^ *n*) = *abs* (*a* ^ *n*)
by (*simp add*: *power-abs*)

lemma *zero-less-power-abs-iff* [*simp*, *no-atp*]:
0 < *abs a* ^ *n* ⟷ *a* ≠ *0* ∨ *n* = *0*
proof (*induct n*)
case *0* **show** *?case* **by** *simp*
next
case (*Suc n*) **show** *?case* **by** (*auto simp add*: *Suc zero-less-mult-iff*)
qed

lemma *zero-le-power-abs* [*simp*]:
0 ≤ *abs a* ^ *n*
by (*rule zero-le-power* [*OF abs-ge-zero*])

end

context *ring-1-no-zero-divisors*
begin

lemma *field-power-not-zero*:
a ≠ *0* ⟹ *a* ^ *n* ≠ *0*
by (*induct n*) *auto*

end

context *division-ring*
begin

FIXME reorient or rename to *nonzero-inverse-power*

lemma *nonzero-power-inverse*:
a ≠ *0* ⟹ *inverse* (*a* ^ *n*) = (*inverse a*) ^ *n*
by (*induct n*)
(*simp-all add*: *nonzero-inverse-mult-distrib power-commutes field-power-not-zero*)

end

context *field*
begin

lemma *nonzero-power-divide*:
b ≠ *0* ⟹ (*a* / *b*) ^ *n* = *a* ^ *n* / *b* ^ *n*
by (*simp add*: *divide-inverse power-mult-distrib nonzero-power-inverse*)

end

lemma *power-0-Suc* [*simp*]:
(*0*::*'a*::{*power*, *semiring-0*}) ^ *Suc n* = *0*
by *simp*

It looks plausible as a simprule, but its effect can be strange.

lemma *power-0-left*:
0 ^ *n* = (*if n* = *0 then 1 else* (*0*::*'a*::{*power*, *semiring-0*}))
by (*induct n*) *simp-all*

lemma *power-eq-0-iff* [*simp*]:
a ^ *n* = *0* ⟷
a = (*0*::*'a*::{*mult-zero*,*zero-neq-one*,*no-zero-divisors*,*power*}) ∧ *n* ≠ *0*
by (*induct n*)
(*auto simp add*: *no-zero-divisors elim*: *contrapos-pp*)

lemma (**in** *field*) *power-diff*:
assumes *nz*: *a* ≠ *0*
shows *n* ≤ *m* ⟹ *a* ^ (*m* − *n*) = *a* ^ *m* / *a* ^ *n*
by (*induct m n rule*: *diff-induct*) (*simp-all add*: *nz field-power-not-zero*)

Perhaps these should be simprules.

lemma *power-inverse*:
fixes *a* :: *'a*::*division-ring-inverse-zero*
shows *inverse* (*a* ^ *n*) = *inverse a* ^ *n*
apply (*cases a* = *0*)
apply (*simp add*: *power-0-left*)
apply (*simp add*: *nonzero-power-inverse*)
done

lemma *power-one-over*:
1 / (*a*::*'a*::{*field-inverse-zero*, *power*}) ^ *n* = (*1* / *a*) ^ *n*
by (*simp add*: *divide-inverse*) (*rule power-inverse*)

lemma *power-divide*:
(*a* / *b*) ^ *n* = (*a*::*'a*::*field-inverse-zero*) ^ *n* / *b* ^ *n*
apply (*cases b* = *0*)
apply (*simp add*: *power-0-left*)
apply (*rule nonzero-power-divide*)
apply *assumption*
done

18.2 Exponentiation for the Natural Numbers

lemma *nat-one-le-power* [*simp*]:
Suc 0 ≤ *i* ⟹ *Suc 0* ≤ *i* ^ *n*
by (*rule one-le-power* [*of i n*, *unfolded One-nat-def*])

lemma *nat-zero-less-power-iff* [*simp*]:
x ^ *n* > *0* ⟷ *x* > (*0*::*nat*) ∨ *n* = *0*

by (*induct n*) *auto*

lemma *nat-power-eq-Suc-0-iff* [*simp*]:
x ^ m = Suc 0 ⟷ m = 0 ∨ x = Suc 0
by (*induct m*) *auto*

lemma *power-Suc-0* [*simp*]:
Suc 0 ^ n = Suc 0
by *simp*

Valid for the naturals, but what if *0<i<1*? Premises cannot be weakened: consider the case where $i = (0::'a)$, $m = (1::'a)$ and $n = (0::'a)$.

lemma *nat-power-less-imp-less*:
assumes *nonneg*: *0 < (i::nat)*
assumes *less*: *i ^ m < i ^ n*
shows *m < n*
proof (*cases i = 1*)
case *True* **with** *less power-one* [**where** *'a = nat*] **show** *?thesis* **by** *simp*
next
case *False* **with** *nonneg* **have** *1 < i* **by** *auto*
from *power-strict-increasing-iff* [*OF this*] *less* **show** *?thesis* **..**
qed

lemma *power-dvd-imp-le*:
i ^ m dvd i ^ n ⟹ (1::nat) < i ⟹ m ≤ n
apply (*rule power-le-imp-le-exp*, *assumption*)
apply (*erule dvd-imp-le*, *simp*)
done

18.3 Code generator tweak

lemma *power-power-power* [*code*, *code-unfold*, *code-inline del*]:
*power = power.power (1::'a::{power}) (op *)*
unfolding *power-def power.power-def* **..**

declare *power.power.simps* [*code*]

code-modulename *SML*
Power Arith

code-modulename *OCaml*
Power Arith

code-modulename *Haskell*
Power Arith

end

19 Option: Datatype option

theory *Option*
imports *Datatype*
begin

datatype *'a option = None | Some 'a*

lemma *not-None-eq* [*iff*]: *(x ~= None) = (EX y. x = Some y)*
by *(induct x) auto*

lemma *not-Some-eq* [*iff*]: *(ALL y. x ~= Some y) = (x = None)*
by *(induct x) auto*

Although it may appear that both of these equalities are helpful only when applied to assumptions, in practice it seems better to give them the uniform iff attribute.

lemma *inj-Some* [*simp*]: *inj-on Some A*
by *(rule inj-onI) simp*

lemma *option-caseE*:
assumes *c*: *(case x of None => P | Some y => Q y)*
obtains
(None) x = None **and** *P*
| *(Some) y* **where** *x = Some y* **and** *Q y*
using *c* **by** *(cases x) simp-all*

lemma *UNIV-option-conv*: *UNIV = insert None (range Some)*
by(*auto intro*: *classical*)

19.0.1 Operations

primrec *the* :: *'a option => 'a* **where**
the (Some x) = x

primrec *set* :: *'a option => 'a set* **where**
set None = {} |
set (Some x) = {x}

lemma *ospec* [*dest*]: *(ALL x:set A. P x) ==> A = Some x ==> P x*
by *simp*

declaration ⟪ *fn - =>*
Classical.map-cs (fn cs => cs addSD2 (ospec, thm ospec))
⟫

lemma *elem-set* [*iff*]: *(x : set xo) = (xo = Some x)*
by *(cases xo) auto*

lemma *set-empty-eq* [*simp*]: (*set xo* = {}) = (*xo* = *None*)
by (*cases xo*) *auto*

definition *map* :: ('*a* ⇒ '*b*) ⇒ '*a option* ⇒ '*b option* **where**
map = (%*f y*. *case y of None* => *None* | *Some x* => *Some* (*f x*))

lemma *option-map-None* [*simp*, *code*]: *map f None* = *None*
by (*simp add*: *map-def*)

lemma *option-map-Some* [*simp*, *code*]: *map f* (*Some x*) = *Some* (*f x*)
by (*simp add*: *map-def*)

lemma *option-map-is-None* [*iff*]:
(*map f opt* = *None*) = (*opt* = *None*)
by (*simp add*: *map-def split add*: *option.split*)

lemma *option-map-eq-Some* [*iff*]:
(*map f xo* = *Some y*) = (*EX z*. *xo* = *Some z* & *f z* = *y*)
by (*simp add*: *map-def split add*: *option.split*)

lemma *option-map-comp*:
map f (*map g opt*) = *map* (*f o g*) *opt*
by (*simp add*: *map-def split add*: *option.split*)

lemma *option-map-o-sum-case* [*simp*]:
map f o sum-case g h = *sum-case* (*map f o g*) (*map f o h*)
by (*rule ext*) (*simp split*: *sum.split*)

hide-const (**open**) *set map*

19.0.2 Code generator setup

definition *is-none* :: '*a option* ⇒ *bool* **where**
[*code-post*]: *is-none x* ⟷ *x* = *None*

lemma *is-none-code* [*code*]:
shows *is-none None* ⟷ *True*
and *is-none* (*Some x*) ⟷ *False*
unfolding *is-none-def* **by** *simp-all*

lemma *is-none-none*:
is-none x ⟷ *x* = *None*
by (*simp add*: *is-none-def*)

lemma [*code-unfold*]:
eq-class.eq x None ⟷ *is-none x*
by (*simp add*: *eq is-none-none*)

hide-const (**open**) *is-none*

code-type *option*
(*SML* - *option*)
(*OCaml* - *option*)
(*Haskell Maybe* -)
(*Scala* !*Option*[(-)])

code-const *None* **and** *Some*
(*SML NONE* **and** *SOME*)
(*OCaml None* **and** *Some* -)
(*Haskell Nothing* **and** *Just*)
(*Scala None* **and** !*Some*((-)))

code-instance *option* :: *eq*
(*Haskell* −)

code-const *eq-class.eq* :: 'a::*eq option* ⇒ 'a *option* ⇒ *bool*
(*Haskell* **infixl** 4 ==)

code-reserved *SML*
option NONE SOME

code-reserved *OCaml*
option None Some

code-reserved *Scala*
Option None Some

end

20 Finite-Set: Finite sets

theory *Finite-Set*
imports *Power Option*
begin

20.1 Predicate for finite sets

inductive *finite* :: 'a *set* => *bool*
where
emptyI [*simp*, *intro*!]: *finite* {}
| *insertI* [*simp*, *intro*!]: *finite A* ==> *finite* (*insert a A*)

lemma *ex-new-if-finite*: — does not depend on def of finite at all
assumes ¬ *finite* (*UNIV* :: 'a *set*) **and** *finite A*
shows ∃ a::'a. *a* ∉ *A*
proof −

from *assms* **have** $A \neq UNIV$ **by** *blast*
thus *?thesis* **by** *blast*
qed

lemma *finite-induct* [*case-names empty insert*, *induct set*: *finite*]:
finite F ==>
$P\ \{\}$ ==> (!!$x\ F$. *finite* F ==> $x \notin F$ ==> $P\ F$ ==> $P\ (insert\ x\ F)$) ==>
$P\ F$
— Discharging $x \notin F$ entails extra work.
proof −
assume $P\ \{\}$ **and**
insert: !!$x\ F$. *finite* F ==> $x \notin F$ ==> $P\ F$ ==> $P\ (insert\ x\ F)$
assume *finite* F
thus $P\ F$
proof *induct*
show $P\ \{\}$ **by** *fact*
fix $x\ F$ **assume** F: *finite* F **and** P: $P\ F$
show $P\ (insert\ x\ F)$
proof *cases*
assume $x \in F$
hence $insert\ x\ F = F$ **by** (*rule insert-absorb*)
with P **show** *?thesis* **by** (*simp only*:)
next
assume $x \notin F$
from F *this* P **show** *?thesis* **by** (*rule insert*)
qed
qed
qed

lemma *finite-ne-induct*[*case-names singleton insert*, *consumes 2*]:
assumes *fin*: *finite* F **shows** $F \neq \{\} \Longrightarrow$
$[\![\ \bigwedge x.\ P\{x\};$
$\bigwedge x\ F.\ [\![$ *finite* F; $F \neq \{\}$; $x \notin F$; $P\ F\ ]\!] \Longrightarrow P\ (insert\ x\ F)\ ]\!]$
$\Longrightarrow P\ F$
using *fin*
proof *induct*
case *empty* **thus** *?case* **by** *simp*
next
case (*insert* $x\ F$)
show *?case*
proof *cases*
assume $F = \{\}$
thus *?thesis* **using** ‹$P\ \{x\}$› **by** *simp*
next
assume $F \neq \{\}$
thus *?thesis* **using** *insert* **by** *blast*
qed
qed

lemma *finite-subset-induct* [*consumes 2*, *case-names empty insert*]:
assumes *finite F* **and** $F \subseteq A$
and *empty*: $P\ \{\}$
and *insert*: $!!a\ F.$ *finite* F ==> $a \in A$ ==> $a \notin F$ ==> $P\ F$ ==> P (*insert* $a\ F$)
shows $P\ F$
proof −
from ‹*finite F*› **and** ‹$F \subseteq A$›
show *?thesis*
proof *induct*
show $P\ \{\}$ **by** *fact*
next
fix $x\ F$
assume *finite F* **and** $x \notin F$ **and**
P: $F \subseteq A$ ==> $P\ F$ **and** i: *insert* $x\ F \subseteq A$
show P (*insert* $x\ F$)
proof (*rule insert*)
from i **show** $x \in A$ **by** *blast*
from i **have** $F \subseteq A$ **by** *blast*
with P **show** $P\ F$ **.**
show *finite F* **by** *fact*
show $x \notin F$ **by** *fact*
qed
qed
qed

A finite choice principle. Does not need the SOME choice operator.

lemma *finite-set-choice*:
finite $A \Longrightarrow$ *ALL* x:A. (*EX* y. $P\ x\ y$) $\Longrightarrow$ *EX* f. *ALL* x:A. $P\ x\ (f\ x)$
proof (*induct set*: *finite*)
case *empty* **thus** *?case* **by** *simp*
next
case (*insert a A*)
then obtain $f\ b$ **where** f: *ALL* x:A. $P\ x\ (f\ x)$ **and** ab: $P\ a\ b$ **by** *auto*
show *?case* (**is** *EX f*. *?P f*)
proof
show *?P*(%x. *if* $x = a$ *then* b *else* $f\ x$) **using** $f\ ab$ **by** *auto*
qed
qed

Finite sets are the images of initial segments of natural numbers:

lemma *finite-imp-nat-seg-image-inj-on*:
assumes *fin*: *finite A*
shows ∃ (n::*nat*) f. $A = f$ ‘ {i. i<n} & *inj-on f* {i. i<n}
using *fin*
proof *induct*
case *empty*
show *?case*
proof show ∃f. $\{\} = f$ ‘ {i::*nat*. $i < 0$} & *inj-on f* {i. i<0} **by** *simp*

```
  qed
next
  case (insert a A)
  have notinA: a ∉ A by fact
  from insert.hyps obtain n f
    where A = f ` {i::nat. i < n} inj-on f {i. i < n} by blast
  hence insert a A = f(n:=a) ` {i. i < Suc n}
       inj-on (f(n:=a)) {i. i < Suc n} using notinA
    by (auto simp add: image-def Ball-def inj-on-def less-Suc-eq)
  thus ?case by blast
qed

lemma nat-seg-image-imp-finite:
  !!f A. A = f ` {i::nat. i<n} ⟹ finite A
proof (induct n)
  case 0 thus ?case by simp
next
  case (Suc n)
  let ?B = f ` {i. i < n}
  have finB: finite ?B by(rule Suc.hyps[OF refl])
  show ?case
  proof cases
    assume ∃ k<n. f n = f k
    hence A = ?B using Suc.prems by(auto simp:less-Suc-eq)
    thus ?thesis using finB by simp
  next
    assume ¬(∃ k<n. f n = f k)
    hence A = insert (f n) ?B using Suc.prems by(auto simp:less-Suc-eq)
    thus ?thesis using finB by simp
  qed
qed

lemma finite-conv-nat-seg-image:
  finite A = (∃ (n::nat) f. A = f ` {i::nat. i<n})
by(blast intro: nat-seg-image-imp-finite dest: finite-imp-nat-seg-image-inj-on)

lemma finite-imp-inj-to-nat-seg:
assumes finite A
shows EX f n::nat. f`A = {i. i<n} & inj-on f A
proof -
  from finite-imp-nat-seg-image-inj-on[OF ‹finite A›]
  obtain f and n::nat where bij: bij-betw f {i. i<n} A
    by (auto simp:bij-betw-def)
  let ?f = the-inv-into {i. i<n} f
  have inj-on ?f A & ?f ` A = {i. i<n}
    by (fold bij-betw-def) (rule bij-betw-the-inv-into[OF bij])
  thus ?thesis by blast
qed
```

lemma *finite-Collect-less-nat*[*iff*]: *finite*{*n*::*nat*. *n*<*k*}
by(*fastsimp simp*: *finite-conv-nat-seg-image*)

Finiteness and set theoretic constructions

lemma *finite-UnI*: *finite F* ==> *finite G* ==> *finite* (*F Un G*)
by (*induct set*: *finite*) *simp-all*

lemma *finite-subset*: *A* ⊆ *B* ==> *finite B* ==> *finite A*
— Every subset of a finite set is finite.
proof −
assume *finite B*
thus !!*A*. *A* ⊆ *B* ==> *finite A*
proof *induct*
case *empty*
thus *?case* **by** *simp*
next
case (*insert x F A*)
have *A*: *A* ⊆ *insert x F* **and** *r*: *A* − {*x*} ⊆ *F* ==> *finite* (*A* − {*x*}) **by** *fact*+
show *finite A*
proof *cases*
assume *x*: *x* ∈ *A*
with *A* **have** *A* − {*x*} ⊆ *F* **by** (*simp add*: *subset-insert-iff*)
with *r* **have** *finite* (*A* − {*x*}) **.**
hence *finite* (*insert x* (*A* − {*x*})) **..**
also have *insert x* (*A* − {*x*}) = *A* **using** *x* **by** (*rule insert-Diff*)
finally show *?thesis* **.**
next
show *A* ⊆ *F* ==> *?thesis* **by** *fact*
assume *x* ∉ *A*
with *A* **show** *A* ⊆ *F* **by** (*simp add*: *subset-insert-iff*)
qed
qed
qed

lemma *rev-finite-subset*: *finite B* ==> *A* ⊆ *B* ==> *finite A*
by (*rule finite-subset*)

lemma *finite-Un* [*iff*]: *finite* (*F Un G*) = (*finite F* & *finite G*)
by (*blast intro*: *finite-subset* [*of* - *X Un Y*, *standard*] *finite-UnI*)

lemma *finite-Collect-disjI*[*simp*]:
finite{*x*. *P x* | *Q x*} = (*finite*{*x*. *P x*} & *finite*{*x*. *Q x*})
by(*simp add*:*Collect-disj-eq*)

lemma *finite-Int* [*simp*, *intro*]: *finite F* | *finite G* ==> *finite* (*F Int G*)
— The converse obviously fails.
by (*blast intro*: *finite-subset*)

lemma *finite-Collect-conjI* [*simp*, *intro*]:

finite{*x*. *P x*} | *finite*{*x*. *Q x*} ==> *finite*{*x*. *P x* & *Q x*}
— The converse obviously fails.
by(*simp add*:*Collect-conj-eq*)

lemma *finite-Collect-le-nat*[*iff*]: *finite*{*n*::*nat*. *n*<=*k*}
by(*simp add*: *le-eq-less-or-eq*)

lemma *finite-insert* [*simp*]: *finite* (*insert a A*) = *finite A*
apply (*subst insert-is-Un*)
apply (*simp only*: *finite-Un*, *blast*)
done

lemma *finite-Union*[*simp*, *intro*]:
⟦ *finite A*; !!*M*. *M* ∈ *A* ⟹ *finite M* ⟧ ⟹ *finite*(⋃ *A*)
by (*induct rule*:*finite-induct*) *simp-all*

lemma *finite-Inter*[*intro*]: *EX A*:*M*. *finite*(*A*) ⟹ *finite*(*Inter M*)
by (*blast intro*: *Inter-lower finite-subset*)

lemma *finite-INT*[*intro*]: *EX x*:*I*. *finite*(*A x*) ⟹ *finite*(*INT x*:*I*. *A x*)
by (*blast intro*: *INT-lower finite-subset*)

lemma *finite-empty-induct*:
assumes *finite A*
and *P A*
and !!*a A*. *finite A* ==> *a*:*A* ==> *P A* ==> *P* (*A* − {*a*})
shows *P* {}
proof −
have *P* (*A* − *A*)
proof −
{
fix *c b* :: *'a set*
assume *c*: *finite c* **and** *b*: *finite b*
and *P1*: *P b* **and** *P2*: !!*x y*. *finite y* ==> *x* ∈ *y* ==> *P y* ==> *P* (*y* − {*x*})
have *c* ⊆ *b* ==> *P* (*b* − *c*)
using *c*
proof *induct*
case *empty*
from *P1* **show** *?case* **by** *simp*
next
case (*insert x F*)
have *P* (*b* − *F* − {*x*})
proof (*rule P2*)
from - *b* **show** *finite* (*b* − *F*) **by** (*rule finite-subset*) *blast*
from *insert* **show** *x* ∈ *b* − *F* **by** *simp*
from *insert* **show** *P* (*b* − *F*) **by** *simp*
qed
also have *b* − *F* − {*x*} = *b* − *insert x F* **by** (*rule Diff-insert* [*symmetric*])

```
      finally show ?case .
    qed
  }
  then show ?thesis by this (simp-all add: assms)
 qed
 then show ?thesis by simp
qed

lemma finite-Diff [simp]: finite A ==> finite (A - B)
by (rule Diff-subset [THEN finite-subset])

lemma finite-Diff2 [simp]:
 assumes finite B shows finite (A - B) = finite A
proof -
 have finite A ⟷ finite((A-B) Un (A Int B)) by(simp add: Un-Diff-Int)
 also have ... ⟷ finite(A-B) using ‹finite B› by(simp)
 finally show ?thesis ..
qed

lemma finite-compl[simp]:
 finite(A::'a set) ⟹ finite(-A) = finite(UNIV::'a set)
by(simp add:Compl-eq-Diff-UNIV)

lemma finite-Collect-not[simp]:
 finite{x::'a. P x} ⟹ finite{x. ~P x} = finite(UNIV::'a set)
by(simp add:Collect-neg-eq)

lemma finite-Diff-insert [iff]: finite (A - insert a B) = finite (A - B)
 apply (subst Diff-insert)
 apply (case-tac a : A - B)
  apply (rule finite-insert [symmetric, THEN trans])
  apply (subst insert-Diff, simp-all)
 done
```

Image and Inverse Image over Finite Sets

```
lemma finite-imageI[simp]: finite F ==> finite (h ' F)
 — The image of a finite set is finite.
 by (induct set: finite) simp-all

lemma finite-image-set [simp]:
 finite {x. P x} ⟹ finite { f x | x. P x }
 by (simp add: image-Collect [symmetric])

lemma finite-surj: finite A ==> B <= f ' A ==> finite B
 apply (frule finite-imageI)
 apply (erule finite-subset, assumption)
 done

lemma finite-range-imageI:
```

```
    finite (range g) ==> finite (range (%x. f (g x)))
  apply (drule finite-imageI, simp add: range-composition)
  done

lemma finite-imageD: finite (f`A) ==> inj-on f A ==> finite A
proof −
  have aux: !!A. finite (A − {}) = finite A by simp
  fix B :: 'a set
  assume finite B
  thus !!A. f`A = B ==> inj-on f A ==> finite A
    apply induct
     apply simp
    apply (subgoal-tac EX y:A. f y = x & F = f ` (A − {y}))
     apply clarify
     apply (simp (no-asm-use) add: inj-on-def)
     apply (blast dest!: aux [THEN iffD1], atomize)
    apply (erule-tac V = ALL A. ?PP (A) in thin-rl)
    apply (frule subsetD [OF equalityD2 insertI1], clarify)
    apply (rule-tac x = xa in bexI)
     apply (simp-all add: inj-on-image-set-diff)
    done
qed (rule refl)


lemma inj-vimage-singleton: inj f ==> f−`{a} ⊆ {THE x. f x = a}
  — The inverse image of a singleton under an injective function is included in a
singleton.
  apply (auto simp add: inj-on-def)
  apply (blast intro: the-equality [symmetric])
  done

lemma finite-vimageI: [|finite F; inj h|] ==> finite (h −` F)
  — The inverse image of a finite set under an injective function is finite.
  apply (induct set: finite)
   apply simp-all
  apply (subst vimage-insert)
  apply (simp add: finite-subset [OF inj-vimage-singleton])
  done

lemma finite-vimageD:
  assumes fin: finite (h −` F) and surj: surj h
  shows finite F
proof −
  have finite (h ` (h −` F)) using fin by (rule finite-imageI)
  also have h ` (h −` F) = F using surj by (rule surj-image-vimage-eq)
  finally show finite F .
qed

lemma finite-vimage-iff: bij h ⟹ finite (h −` F) ⟷ finite F
```

unfolding *bij-def* **by** (*auto elim*: *finite-vimageD finite-vimageI*)

The finite UNION of finite sets

lemma *finite-UN-I*: *finite A* ==> (!!*a*. *a*:*A* ==> *finite* (*B a*)) ==> *finite* (*UN a*:*A*. *B a*)
by (*induct set*: *finite*) *simp-all*

Strengthen RHS to $(\forall x{\in}A.\ \mathit{finite}\ (B\ x)) \wedge \mathit{finite}\ \{x \in A.\ B\ x \neq \{\}\}$?
We'd need to prove $\mathit{finite}\ C \Longrightarrow \forall A\ B.\ \mathit{UNION}\ A\ B \subseteq C \longrightarrow \mathit{finite}\ \{x \in A.\ B\ x \neq \{\}\}$ by induction.

lemma *finite-UN* [*simp*]:
finite A ==> *finite* (*UNION A B*) = (*ALL x*:*A*. *finite* (*B x*))
by (*blast intro*: *finite-UN-I finite-subset*)

lemma *finite-Collect-bex*[*simp*]: *finite A* ⟹
finite{*x*. *EX y*:*A*. *Q x y*} = (*ALL y*:*A*. *finite*{*x*. *Q x y*})
apply(*subgoal-tac* {*x*. *EX y*:*A*. *Q x y*} = *UNION A* (%*y*. {*x*. *Q x y*}))
apply *auto*
done

lemma *finite-Collect-bounded-ex*[*simp*]: *finite*{*y*. *P y*} ⟹
finite{*x*. *EX y*. *P y* & *Q x y*} = (*ALL y*. *P y* ⟶ *finite*{*x*. *Q x y*})
apply(*subgoal-tac* {*x*. *EX y*. *P y* & *Q x y*} = *UNION* {*y*. *P y*} (%*y*. {*x*. *Q x y*}))
apply *auto*
done

lemma *finite-Plus*: [| *finite A*; *finite B* |] ==> *finite* (*A* <+> *B*)
by (*simp add*: *Plus-def*)

lemma *finite-PlusD*:
fixes *A* :: *'a set* **and** *B* :: *'b set*
assumes *fin*: *finite* (*A* <+> *B*)
shows *finite A finite B*
proof −
have *Inl* ` *A* ⊆ *A* <+> *B* **by** *auto*
hence *finite* (*Inl* ` *A* :: (*'a* + *'b*) *set*) **using** *fin* **by**(*rule finite-subset*)
thus *finite A* **by**(*rule finite-imageD*)(*auto intro*: *inj-onI*)
next
have *Inr* ` *B* ⊆ *A* <+> *B* **by** *auto*
hence *finite* (*Inr* ` *B* :: (*'a* + *'b*) *set*) **using** *fin* **by**(*rule finite-subset*)
thus *finite B* **by**(*rule finite-imageD*)(*auto intro*: *inj-onI*)
qed

lemma *finite-Plus-iff*[*simp*]: *finite* (*A* <+> *B*) ⟷ *finite A* ∧ *finite B*
by(*auto intro*: *finite-PlusD finite-Plus*)

lemma *finite-Plus-UNIV-iff*[*simp*]:
finite (*UNIV* :: (*'a* + *'b*) *set*) =

(*finite* (*UNIV* :: *'a set*) & *finite* (*UNIV* :: *'b set*))
by(*subst UNIV-Plus-UNIV*[*symmetric*])(*rule finite-Plus-iff*)

Sigma of finite sets

lemma *finite-SigmaI* [*simp*]:
finite A ==> (!!*a*. *a*:*A* ==> *finite* (*B a*)) ==> *finite* (*SIGMA a*:*A*. *B a*)
by (*unfold Sigma-def*) (*blast intro*!: *finite-UN-I*)

lemma *finite-cartesian-product*: [| *finite A*; *finite B* |] ==>
finite (*A* <*>* *B*)
by (*rule finite-SigmaI*)

lemma *finite-Prod-UNIV*:
finite (*UNIV*::*'a set*) ==> *finite* (*UNIV*::*'b set*) ==> *finite* (*UNIV*::(*'a* * *'b*) *set*)
apply (*subgoal-tac* (*UNIV*:: (*'a* * *'b*) *set*) = *Sigma UNIV* (%*x*. *UNIV*))
apply (*erule ssubst*)
apply (*erule finite-SigmaI*, *auto*)
done

lemma *finite-cartesian-productD1*:
[| *finite* (*A* <*>* *B*); *B* ≠ {} |] ==> *finite A*
apply (*auto simp add*: *finite-conv-nat-seg-image*)
apply (*drule-tac x*=*n* **in** *spec*)
apply (*drule-tac x*=*fst o f* **in** *spec*)
apply (*auto simp add*: *o-def*)
prefer *2* **apply** (*force dest*!: *equalityD2*)
apply (*drule equalityD1*)
apply (*rename-tac y x*)
apply (*subgoal-tac* ∃*k*. *k*<*n* & *f k* = (*x*,*y*))
prefer *2* **apply** *force*
apply *clarify*
apply (*rule-tac x*=*k* **in** *image-eqI*, *auto*)
done

lemma *finite-cartesian-productD2*:
[| *finite* (*A* <*>* *B*); *A* ≠ {} |] ==> *finite B*
apply (*auto simp add*: *finite-conv-nat-seg-image*)
apply (*drule-tac x*=*n* **in** *spec*)
apply (*drule-tac x*=*snd o f* **in** *spec*)
apply (*auto simp add*: *o-def*)
prefer *2* **apply** (*force dest*!: *equalityD2*)
apply (*drule equalityD1*)
apply (*rename-tac x y*)
apply (*subgoal-tac* ∃*k*. *k*<*n* & *f k* = (*x*,*y*))
prefer *2* **apply** *force*
apply *clarify*
apply (*rule-tac x*=*k* **in** *image-eqI*, *auto*)
done

The powerset of a finite set

lemma *finite-Pow-iff* [*iff*]: *finite* (*Pow A*) = *finite A*
proof
  assume *finite* (*Pow A*)
  with - **have** *finite* ((%*x*. {*x*}) ‘ *A*) **by** (*rule finite-subset*) *blast*
  thus *finite A* **by** (*rule finite-imageD* [*unfolded inj-on-def*]) *simp*
next
  assume *finite A*
  thus *finite* (*Pow A*)
    by *induct* (*simp-all add*: *Pow-insert*)
qed

lemma *finite-Collect-subsets*[*simp*,*intro*]: *finite A* ⟹ *finite*{*B*. *B* ⊆ *A*}
by(*simp add*: *Pow-def*[*symmetric*])

lemma *finite-UnionD*: *finite*(⋃ *A*) ⟹ *finite A*
by(*blast intro*: *finite-subset*[*OF subset-Pow-Union*])

lemma *finite-subset-image*:
  assumes *finite B*
  shows *B* ⊆ *f* ‘ *A* ⟹ ∃ *C*⊆*A*. *finite C* ∧ *B* = *f* ‘ *C*
using *assms* **proof**(*induct*)
  case *empty* **thus** *?case* **by** *simp*
next
  case *insert* **thus** *?case*
    by (*clarsimp simp del*: *image-insert simp add*: *image-insert*[*symmetric*])
      blast
qed

20.2 Class *finite*

class *finite* =
  assumes *finite-UNIV*: *finite* (*UNIV* :: *'a set*)
begin

lemma *finite* [*simp*]: *finite* (*A* :: *'a set*)
  by (*rule subset-UNIV finite-UNIV finite-subset*)+

end

lemma *UNIV-unit* [*no-atp*]:
  UNIV = {()} **by** *auto*

instance *unit* :: *finite* **proof**
qed (*simp add*: *UNIV-unit*)

lemma *UNIV-bool* [*no-atp*]:

UNIV = {*False*, *True*} **by** *auto*

instance *bool* :: *finite* **proof**
qed (*simp add*: *UNIV-bool*)

instance ∗ :: (*finite*, *finite*) *finite* **proof**
qed (*simp only*: *UNIV-Times-UNIV* [*symmetric*] *finite-cartesian-product finite*)

lemma *finite-option-UNIV* [*simp*]:
finite (*UNIV* :: *'a option set*) = *finite* (*UNIV* :: *'a set*)
by (*auto simp add*: *UNIV-option-conv elim*: *finite-imageD intro*: *inj-Some*)

instance *option* :: (*finite*) *finite* **proof**
qed (*simp add*: *UNIV-option-conv*)

lemma *inj-graph*: *inj* (%*f*. {(*x*, *y*). *y* = *f x*})
by (*rule inj-onI*, *auto simp add*: *expand-set-eq expand-fun-eq*)

instance *fun* :: (*finite*, *finite*) *finite*
proof
show *finite* (*UNIV* :: (*'a* => *'b*) *set*)
proof (*rule finite-imageD*)
let *?graph* = %*f*::*'a* => *'b*. {(*x*, *y*). *y* = *f x*}
have *range ?graph* ⊆ *Pow UNIV* **by** *simp*
moreover have *finite* (*Pow* (*UNIV* :: (*'a* ∗ *'b*) *set*))
by (*simp only*: *finite-Pow-iff finite*)
ultimately show *finite* (*range ?graph*)
by (*rule finite-subset*)
show *inj ?graph* **by** (*rule inj-graph*)
qed
qed

instance + :: (*finite*, *finite*) *finite* **proof**
qed (*simp only*: *UNIV-Plus-UNIV* [*symmetric*] *finite-Plus finite*)

20.3 A basic fold functional for finite sets

The intended behaviour is *fold f z* {x_1, ..., x_n} = $f\ x_1$ (... ($f\ x_n\ z$)...) if f is "left-commutative":

locale *fun-left-comm* =
fixes *f* :: *'a* ⇒ *'b* ⇒ *'b*
assumes *fun-left-comm*: *f x* (*f y z*) = *f y* (*f x z*)
begin

On a functional level it looks much nicer:

lemma *fun-comp-comm*: *f x* ∘ *f y* = *f y* ∘ *f x*
by (*simp add*: *fun-left-comm expand-fun-eq*)

end

inductive *fold-graph* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b \Rightarrow bool$
for $f :: 'a \Rightarrow 'b \Rightarrow 'b$ **and** $z :: 'b$ **where**
emptyI [*intro*]: *fold-graph f z* {} *z* |
insertI [*intro*]: $x \notin A \Longrightarrow$ *fold-graph f z A y*
$\Longrightarrow$ *fold-graph f z* (*insert x A*) (*f x y*)

inductive-cases *empty-fold-graphE* [*elim*!]: *fold-graph f z* {} *x*

definition *fold* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b$ **where**
[*code del*]: *fold f z A* = (*THE y*. *fold-graph f z A y*)

A tempting alternative for the definiens is *if finite A then THE y*. *fold-graph f z A y else e*. It allows the removal of finiteness assumptions from the theorems *fold-comm*, *fold-reindex* and *fold-distrib*. The proofs become ugly. It is not worth the effort. (???)

lemma *Diff1-fold-graph*:
fold-graph f z (*A* − {*x*}) *y* $\Longrightarrow$ *x* ∈ *A* $\Longrightarrow$ *fold-graph f z A* (*f x y*)
by (*erule insert-Diff* [*THEN subst*], *rule fold-graph.intros*, *auto*)

lemma *fold-graph-imp-finite*: *fold-graph f z A x* $\Longrightarrow$ *finite A*
by (*induct set*: *fold-graph*) *auto*

lemma *finite-imp-fold-graph*: *finite A* $\Longrightarrow \exists x$. *fold-graph f z A x*
by (*induct set*: *finite*) *auto*

20.3.1 From *fold-graph* to *fold*

context *fun-left-comm*
begin

lemma *fold-graph-insertE-aux*:
fold-graph f z A y $\Longrightarrow$ *a* ∈ *A* $\Longrightarrow \exists y'$. *y* = *f a y'* ∧ *fold-graph f z* (*A* − {*a*}) *y'*
proof (*induct set*: *fold-graph*)
case (*insertI x A y*) **show** *?case*
proof (*cases x* = *a*)
assume *x* = *a* **with** *insertI* **show** *?case* **by** *auto*
next
assume *x* ≠ *a*
then obtain *y'* **where** *y*: *y* = *f a y'* **and** *y'*: *fold-graph f z* (*A* − {*a*}) *y'*
using *insertI* **by** *auto*
have *1*: *f x y* = *f a* (*f x y'*)
unfolding *y* **by** (*rule fun-left-comm*)
have *2*: *fold-graph f z* (*insert x A* − {*a*}) (*f x y'*)
using *y'* **and** ‹*x* ≠ *a*› **and** ‹*x* ∉ *A*›
by (*simp add*: *insert-Diff-if fold-graph.insertI*)
from *1 2* **show** *?case* **by** *fast*
qed

qed *simp*

lemma *fold-graph-insertE*:
  assumes *fold-graph f z* (*insert x A*) *v* **and** $x \notin A$
  obtains *y* **where** $v = f\ x\ y$ **and** *fold-graph f z A y*
using *assms* **by** (*auto dest*: *fold-graph-insertE-aux* [*OF - insertI1*])

lemma *fold-graph-determ*:
  fold-graph f z A x $\Longrightarrow$ *fold-graph f z A y* $\Longrightarrow$ $y = x$
proof (*induct arbitrary*: *y set*: *fold-graph*)
  case (*insertI x A y v*)
  from ‹*fold-graph f z* (*insert x A*) *v*› **and** ‹$x \notin A$›
  obtain y' **where** $v = f\ x\ y'$ **and** *fold-graph f z A* y'
    by (*rule fold-graph-insertE*)
  from ‹*fold-graph f z A* y'› **have** $y' = y$ **by** (*rule insertI*)
  with ‹$v = f\ x\ y'$› **show** $v = f\ x\ y$ **by** *simp*
qed *fast*

lemma *fold-equality*:
  fold-graph f z A y $\Longrightarrow$ *fold f z A* $= y$
by (*unfold fold-def*) (*blast intro*: *fold-graph-determ*)

lemma *fold-graph-fold*: *finite A* $\Longrightarrow$ *fold-graph f z A* (*fold f z A*)
unfolding *fold-def*
apply (*rule theI'*)
apply (*rule ex-ex1I*)
apply (*erule finite-imp-fold-graph*)
apply (*erule* (*1*) *fold-graph-determ*)
done

The base case for *fold*:

lemma (**in** −) *fold-empty* [*simp*]: *fold f z* $\{\} = z$
by (*unfold fold-def*) *blast*

The various recursion equations for *fold*:

lemma *fold-insert* [*simp*]:
  finite A ==> $x \notin A$ ==> *fold f z* (*insert x A*) $= f\ x$ (*fold f z A*)
apply (*rule fold-equality*)
apply (*erule fold-graph.insertI*)
apply (*erule fold-graph-fold*)
done

lemma *fold-fun-comm*:
  finite A $\Longrightarrow$ $f\ x$ (*fold f z A*) = *fold f* (*f x z*) *A*
proof (*induct rule*: *finite-induct*)
  case *empty* **then show** *?case* **by** *simp*
next
  case (*insert y A*) **then show** *?case*
    by (*simp add*: *fun-left-comm*[*of x*])

qed

lemma *fold-insert2*:
finite A $\Longrightarrow$ $x \notin A$ $\Longrightarrow$ *fold f z* (*insert x A*) = *fold f* (*f x z*) *A*
by (*simp add*: *fold-fun-comm*)

lemma *fold-rec*:
assumes *finite A* **and** $x \in A$
shows *fold f z A* = *f x* (*fold f z* ($A - \{x\}$))
proof −
have *A*: *A* = *insert x* ($A - \{x\}$) **using** ‹$x \in A$› **by** *blast*
then have *fold f z A* = *fold f z* (*insert x* ($A - \{x\}$)) **by** *simp*
also have ... = *f x* (*fold f z* ($A - \{x\}$))
by (*rule fold-insert*) (*simp add*: ‹*finite A*›)+
finally show *?thesis* **.**
qed

lemma *fold-insert-remove*:
assumes *finite A*
shows *fold f z* (*insert x A*) = *f x* (*fold f z* ($A - \{x\}$))
proof −
from ‹*finite A*› **have** *finite* (*insert x A*) **by** *auto*
moreover have $x \in$ *insert x A* **by** *auto*
ultimately have *fold f z* (*insert x A*) = *f x* (*fold f z* (*insert x A* − $\{x\}$))
by (*rule fold-rec*)
then show *?thesis* **by** *simp*
qed

end

A simplified version for idempotent functions:

locale *fun-left-comm-idem* = *fun-left-comm* +
assumes *fun-left-idem*: *f x* (*f x z*) = *f x z*
begin

The nice version:

lemma *fun-comp-idem* : *f x o f x* = *f x*
by (*simp add*: *fun-left-idem expand-fun-eq*)

lemma *fold-insert-idem*:
assumes *fin*: *finite A*
shows *fold f z* (*insert x A*) = *f x* (*fold f z A*)
proof *cases*
assume $x \in A$
then obtain *B* **where** *A* = *insert x B* **and** $x \notin B$ **by** (*rule set-insert*)
then show *?thesis* **using** *assms* **by** (*simp add*:*fun-left-idem*)
next
assume $x \notin A$ **then show** *?thesis* **using** *assms* **by** *simp*
qed

declare *fold-insert*[*simp del*] *fold-insert-idem*[*simp*]

lemma *fold-insert-idem2*:
  finite A $\Longrightarrow$ *fold f z* (*insert x A*) = *fold f* (*f x z*) *A*
by(*simp add*:*fold-fun-comm*)

end

20.3.2 Expressing set operations via *fold*

lemma (**in** *fun-left-comm*) *fun-left-comm-apply*:
  fun-left-comm (λx. *f* (*g x*))
proof
qed (*simp-all add*: *fun-left-comm*)

lemma (**in** *fun-left-comm-idem*) *fun-left-comm-idem-apply*:
  fun-left-comm-idem (λx. *f* (*g x*))
  by (*rule fun-left-comm-idem.intro*, *rule fun-left-comm-apply*, *unfold-locales*)
    (*simp-all add*: *fun-left-idem*)

lemma *fun-left-comm-idem-insert*:
  fun-left-comm-idem insert
proof
qed *auto*

lemma *fun-left-comm-idem-remove*:
  fun-left-comm-idem (λx *A*. *A* − {*x*})
proof
qed *auto*

lemma (**in** *semilattice-inf*) *fun-left-comm-idem-inf*:
  fun-left-comm-idem inf
proof
qed (*auto simp add*: *inf-left-commute*)

lemma (**in** *semilattice-sup*) *fun-left-comm-idem-sup*:
  fun-left-comm-idem sup
proof
qed (*auto simp add*: *sup-left-commute*)

lemma *union-fold-insert*:
  assumes *finite A*
  shows *A* ∪ *B* = *fold insert B A*
proof −
  interpret *fun-left-comm-idem insert* **by** (*fact fun-left-comm-idem-insert*)
  from ‹*finite A*› **show** *?thesis* **by** (*induct A arbitrary*: *B*) *simp-all*
qed

lemma *minus-fold-remove*:
assumes *finite A*
shows $B - A = fold\ (\lambda x\ A.\ A - \{x\})\ B\ A$
proof −
interpret *fun-left-comm-idem* $\lambda x\ A.\ A - \{x\}$ **by** (*fact fun-left-comm-idem-remove*)
from ‹*finite A*› **show** *?thesis* **by** (*induct A arbitrary*: *B*) *auto*
qed

context *complete-lattice*
begin

lemma *inf-Inf-fold-inf*:
assumes *finite A*
shows *inf B* (*Inf A*) = *fold inf B A*
proof −
interpret *fun-left-comm-idem inf* **by** (*fact fun-left-comm-idem-inf*)
from ‹*finite A*› **show** *?thesis* **by** (*induct A arbitrary*: *B*)
(*simp-all add*: *Inf-empty Inf-insert inf-commute fold-fun-comm*)
qed

lemma *sup-Sup-fold-sup*:
assumes *finite A*
shows *sup B* (*Sup A*) = *fold sup B A*
proof −
interpret *fun-left-comm-idem sup* **by** (*fact fun-left-comm-idem-sup*)
from ‹*finite A*› **show** *?thesis* **by** (*induct A arbitrary*: *B*)
(*simp-all add*: *Sup-empty Sup-insert sup-commute fold-fun-comm*)
qed

lemma *Inf-fold-inf*:
assumes *finite A*
shows *Inf A* = *fold inf top A*
using *assms inf-Inf-fold-inf* [*of A top*] **by** (*simp add*: *inf-absorb2*)

lemma *Sup-fold-sup*:
assumes *finite A*
shows *Sup A* = *fold sup bot A*
using *assms sup-Sup-fold-sup* [*of A bot*] **by** (*simp add*: *sup-absorb2*)

lemma *inf-INFI-fold-inf*:
assumes *finite A*
shows *inf B* (*INFI A f*) = *fold* ($\lambda A.$ *inf* (*f A*)) *B A* (**is** *?inf* = *?fold*)
proof (*rule sym*)
interpret *fun-left-comm-idem inf* **by** (*fact fun-left-comm-idem-inf*)
interpret *fun-left-comm-idem* $\lambda A.$ *inf* (*f A*) **by** (*fact fun-left-comm-idem-apply*)
from ‹*finite A*› **show** *?fold* = *?inf*
by (*induct A arbitrary*: *B*)
(*simp-all add*: *INFI-def Inf-empty Inf-insert inf-left-commute*)
qed

lemma *sup-SUPR-fold-sup*:
assumes *finite A*
shows *sup B* (*SUPR A f*) = *fold* (λ*A*. *sup* (*f A*)) *B A* (**is** *?sup* = *?fold*)
proof (*rule sym*)
interpret *fun-left-comm-idem sup* **by** (*fact fun-left-comm-idem-sup*)
interpret *fun-left-comm-idem* λ*A*. *sup* (*f A*) **by** (*fact fun-left-comm-idem-apply*)
from ‹*finite A*› **show** *?fold* = *?sup*
by (*induct A arbitrary*: *B*)
(*simp-all add*: *SUPR-def Sup-empty Sup-insert sup-left-commute*)
qed

lemma *INFI-fold-inf*:
assumes *finite A*
shows *INFI A f* = *fold* (λ*A*. *inf* (*f A*)) *top A*
using *assms inf-INFI-fold-inf* [*of A top*] **by** *simp*

lemma *SUPR-fold-sup*:
assumes *finite A*
shows *SUPR A f* = *fold* (λ*A*. *sup* (*f A*)) *bot A*
using *assms sup-SUPR-fold-sup* [*of A bot*] **by** *simp*

end

20.4 The derived combinator *fold-image*

definition *fold-image* :: ('*b* ⇒ '*b* ⇒ '*b*) ⇒ ('*a* ⇒ '*b*) ⇒ '*b* ⇒ '*a set* ⇒ '*b*
where *fold-image f g* = *fold* (%*x y*. *f* (*g x*) *y*)

lemma *fold-image-empty*[*simp*]: *fold-image f g z* {} = *z*
by(*simp add*:*fold-image-def*)

context *ab-semigroup-mult*
begin

lemma *fold-image-insert*[*simp*]:
assumes *finite A* **and** *a* ∉ *A*
shows *fold-image times g z* (*insert a A*) = *g a* ∗ (*fold-image times g z A*)
proof −
interpret *I*: *fun-left-comm* %*x y*. (*g x*) ∗ *y*
by *unfold-locales* (*simp add*: *mult-ac*)
show *?thesis* **using** *assms* **by**(*simp add*:*fold-image-def*)
qed

lemma *fold-image-reindex*:
assumes *fin*: *finite A*
shows *inj-on h A* ⟹ *fold-image times g z* (*h*‘*A*) = *fold-image times* (*g*∘*h*) *z A*

using *fin* **by** *induct auto*

lemma *fold-image-cong*:
finite A $\Longrightarrow$
(!!*x*. *x*:*A* ==> *g x* = *h x*) ==> *fold-image times g z A* = *fold-image times h z A*
apply (*subgoal-tac ALL C*. *C* <= *A* --> (*ALL x*:*C*. *g x* = *h x*) --> *fold-image times g z C* = *fold-image times h z C*)
apply *simp*
apply (*erule finite-induct*, *simp*)
apply (*simp add*: *subset-insert-iff*, *clarify*)
apply (*subgoal-tac finite C*)
prefer *2* **apply** (*blast dest*: *finite-subset* [*COMP swap-prems-rl*])
apply (*subgoal-tac C* = *insert x* (*C* − {*x*}))
prefer *2* **apply** *blast*
apply (*erule ssubst*)
apply (*drule spec*)
apply (*erule* (*1*) *notE impE*)
apply (*simp add*: *Ball-def del*: *insert-Diff-single*)
done

end

context *comm-monoid-mult*
begin

lemma *fold-image-1*:
finite S $\Longrightarrow$ ($\forall$ *x*$\in$*S*. *f x* = *1*) $\Longrightarrow$ *fold-image op* ∗ *f 1 S* = *1*
apply (*induct set*: *finite*)
apply *simp* **by** *auto*

lemma *fold-image-Un-Int*:
finite A ==> *finite B* ==>
fold-image times g 1 A ∗ *fold-image times g 1 B* =
fold-image times g 1 (*A Un B*) ∗ *fold-image times g 1* (*A Int B*)
by (*induct set*: *finite*)
(*auto simp add*: *mult-ac insert-absorb Int-insert-left*)

lemma *fold-image-Un-one*:
assumes *fS*: *finite S* **and** *fT*: *finite T*
and *I0*: $\forall$ *x* ∈ *S*∩*T*. *f x* = *1*
shows *fold-image* (*op* ∗) *f 1* (*S* ∪ *T*) = *fold-image* (*op* ∗) *f 1 S* ∗ *fold-image* (*op* ∗) *f 1 T*
proof−
have *fold-image op* ∗ *f 1* (*S* ∩ *T*) = *1*
apply (*rule fold-image-1*)
using *fS fT I0* **by** *auto*
with *fold-image-Un-Int*[*OF fS fT*] **show** *?thesis* **by** *simp*

qed

corollary *fold-Un-disjoint*:
finite A ==> finite B ==> A Int B = {} ==>
fold-image times g 1 (A Un B) =
*fold-image times g 1 A * fold-image times g 1 B*
by (*simp add*: *fold-image-Un-Int*)

lemma *fold-image-UN-disjoint*:
⟦ *finite I*; *ALL i:I. finite (A i)*;
ALL i:I. ALL j:I. i ≠ j --> A i Int A j = {} ⟧
⟹ *fold-image times g 1 (UNION I A) =*
fold-image times (%i. fold-image times g 1 (A i)) 1 I
apply (*induct set*: *finite*, *simp*, *atomize*)
apply (*subgoal-tac ALL i:F. x ≠ i*)
prefer *2* **apply** *blast*
apply (*subgoal-tac A x Int UNION F A = {}*)
prefer *2* **apply** *blast*
apply (*simp add*: *fold-Un-disjoint*)
done

lemma *fold-image-Sigma*: *finite A ==> ALL x:A. finite (B x) ==>*
fold-image times (%x. fold-image times (g x) 1 (B x)) 1 A =
fold-image times (split g) 1 (SIGMA x:A. B x)
apply (*subst Sigma-def*)
apply (*subst fold-image-UN-disjoint*, *assumption*, *simp*)
apply *blast*
apply (*erule fold-image-cong*)
apply (*subst fold-image-UN-disjoint*, *simp*, *simp*)
apply *blast*
apply *simp*
done

lemma *fold-image-distrib*: *finite A ⟹*
*fold-image times (%x. g x * h x) 1 A =*
*fold-image times g 1 A * fold-image times h 1 A*
by (*erule finite-induct*) (*simp-all add*: *mult-ac*)

lemma *fold-image-related*:
assumes *Re*: *R e e*
and *Rop*: ∀ *x1 y1 x2 y2*. *R x1 x2* ∧ *R y1 y2* ⟶ *R (x1 * y1) (x2 * y2)*
and *fS*: *finite S* **and** *Rfg*: ∀ *x*∈*S*. *R (h x) (g x)*
shows *R (fold-image (op *) h e S) (fold-image (op *) g e S)*
using *fS* **by** (*rule finite-subset-induct*) (*insert assms*, *auto*)

lemma *fold-image-eq-general*:
assumes *fS*: *finite S*
and *h*: ∀ *y*∈*S'*. ∃!*x*. *x*∈ *S* ∧ *h*(*x*) = *y*
and *f12*: ∀ *x*∈*S*. *h x* ∈ *S'* ∧ *f2*(*h x*) = *f1 x*

shows *fold-image* (*op* ∗) *f1 e S* = *fold-image* (*op* ∗) *f2 e S′*
proof−
from *h f12* **have** *hS*: *h* ‘ *S* = *S′* **by** *auto*
{fix *x y* **assume** *H*: *x* ∈ *S y* ∈ *S h x* = *h y*
from *f12 h H* **have** *x* = *y* **by** *auto* **}**
hence *hinj*: *inj-on h S* **unfolding** *inj-on-def Ex1-def* **by** *blast*
from *f12* **have** *th*: ⋀*x*. *x* ∈ *S* ⟹ (*f2* ∘ *h*) *x* = *f1 x* **by** *auto*
from *hS* **have** *fold-image* (*op* ∗) *f2 e S′* = *fold-image* (*op* ∗) *f2 e* (*h* ‘ *S*) **by** *simp*
also have . . . = *fold-image* (*op* ∗) (*f2 o h*) *e S*
using *fold-image-reindex*[*OF fS hinj*, *of f2 e*] **.**
also have . . . = *fold-image* (*op* ∗) *f1 e S* **using** *th fold-image-cong*[*OF fS*, *of f2 o h f1 e*]
by *blast*
finally show *?thesis* **..**
qed

lemma *fold-image-eq-general-inverses*:
assumes *fS*: *finite S*
and *kh*: ⋀*y*. *y* ∈ *T* ⟹ *k y* ∈ *S* ∧ *h* (*k y*) = *y*
and *hk*: ⋀*x*. *x* ∈ *S* ⟹ *h x* ∈ *T* ∧ *k* (*h x*) = *x* ∧ *g* (*h x*) = *f x*
shows *fold-image* (*op* ∗) *f e S* = *fold-image* (*op* ∗) *g e T*

apply (*rule fold-image-eq-general*[*OF fS*, *of T h g f e*])
apply (*rule ballI*)
apply (*frule kh*)
apply (*rule ex1I*[])
apply *blast*
apply *clarsimp*
apply (*drule hk*) **apply** *simp*
apply (*rule sym*)
apply (*erule conjunct1*[*OF conjunct2*[*OF hk*]])
apply (*rule ballI*)
apply (*drule hk*)
apply *blast*
done

end

20.5 A fold functional for non-empty sets

Does not require start value.

inductive
fold1Set :: ('*a* => '*a* => '*a*) => '*a set* => '*a* => *bool*
for *f* :: '*a* => '*a* => '*a*
where
fold1Set-insertI [*intro*]:
⟦ *fold-graph f a A x*; *a* ∉ *A* ⟧ ⟹ *fold1Set f* (*insert a A*) *x*

definition *fold1* :: ('*a* => '*a* => '*a*) => '*a set* => '*a* **where**
fold1 f A == *THE x*. *fold1Set f A x*

lemma *fold1Set-nonempty*:
fold1Set f A x ⟹ *A* ≠ {}
by(*erule fold1Set.cases*, *simp-all*)

inductive-cases *empty-fold1SetE* [*elim*!]: *fold1Set f* {} *x*

inductive-cases *insert-fold1SetE* [*elim*!]: *fold1Set f* (*insert a X*) *x*

lemma *fold1Set-sing* [*iff*]: (*fold1Set f* {*a*} *b*) = (*a* = *b*)
by (*blast elim*: *fold-graph.cases*)

lemma *fold1-singleton* [*simp*]: *fold1 f* {*a*} = *a*
by (*unfold fold1-def*) *blast*

lemma *finite-nonempty-imp-fold1Set*:
⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *EX x*. *fold1Set f A x*
apply (*induct A rule*: *finite-induct*)
apply (*auto dest*: *finite-imp-fold-graph* [*of* - *f*])
done

First, some lemmas about *fold-graph*.

context *ab-semigroup-mult*
begin

lemma *fun-left-comm*: *fun-left-comm*(*op* ∗)
by *unfold-locales* (*simp add*: *mult-ac*)

lemma *fold-graph-insert-swap*:
assumes *fold*: *fold-graph times* (*b*::'*a*) *A y* **and** *b* ∉ *A*
shows *fold-graph times z* (*insert b A*) (*z* ∗ *y*)
proof −
interpret *fun-left-comm op* ∗::'*a* ⇒ '*a* ⇒ '*a* **by** (*rule fun-left-comm*)
from *assms* **show** *?thesis*
proof (*induct rule*: *fold-graph.induct*)
case *emptyI* **show** *?case* **by** (*subst mult-commute* [*of z b*], *fast*)
next
case (*insertI x A y*)
have *fold-graph times z* (*insert x* (*insert b A*)) (*x* ∗ (*z* ∗ *y*))
using *insertI* **by** *force* — how does *id* get unfolded?
thus *?case* **by** (*simp add*: *insert-commute mult-ac*)
qed
qed

lemma *fold-graph-permute-diff*:
assumes *fold*: *fold-graph times b A x*

shows !!a. $[\![a \in A;\ b \notin A]\!] \Longrightarrow$ *fold-graph times a* (*insert b* (A−{a})) x
using *fold*
proof (*induct rule*: *fold-graph.induct*)
case *emptyI* **thus** *?case* **by** *simp*
next
case (*insertI x A y*)
have $a = x \vee a \in A$ **using** *insertI* **by** *simp*
thus *?case*
proof
assume $a = x$
with *insertI* **show** *?thesis*
by (*simp add*: *id-def* [*symmetric*], *blast intro*: *fold-graph-insert-swap*)
next
assume *ainA*: $a \in A$
hence *fold-graph times a* (*insert x* (*insert b* (A − {a}))) (x ∗ y)
using *insertI* **by** *force*
moreover
have *insert x* (*insert b* (A − {a})) = *insert b* (*insert x A* − {a})
using *ainA insertI* **by** *blast*
ultimately show *?thesis* **by** *simp*
qed
qed

lemma *fold1-eq-fold*:
assumes *finite A a* ∉ *A* **shows** *fold1 times* (*insert a A*) = *fold times a A*
proof −
interpret *fun-left-comm op* ∗::'a ⇒ 'a ⇒ 'a **by** (*rule fun-left-comm*)
from *assms* **show** *?thesis*
apply (*simp add*: *fold1-def fold-def*)
apply (*rule the-equality*)
apply (*best intro*: *fold-graph-determ theI dest*: *finite-imp-fold-graph* [*of* - *times*])
apply (*rule sym*, *clarify*)
apply (*case-tac Aa*=A)
apply (*best intro*: *fold-graph-determ*)
apply (*subgoal-tac fold-graph times a A x*)
apply (*best intro*: *fold-graph-determ*)
apply (*subgoal-tac insert aa* (Aa − {a}) = A)
prefer *2* **apply** (*blast elim*: *equalityE*)
apply (*auto dest*: *fold-graph-permute-diff* [**where** a=a])
done
qed

lemma *nonempty-iff*: ($A \neq \{\}$) = (∃ x B. A = *insert x B* & $x \notin B$)
apply *safe*
apply *simp*
apply (*drule-tac x*=x **in** *spec*)
apply (*drule-tac x*=A−{x} **in** *spec*, *auto*)
done

lemma *fold1-insert*:
  assumes *nonempty*: $A \neq \{\}$ **and** A: *finite* A $x \notin A$
  shows *fold1 times* (*insert x A*) = x ∗ *fold1 times* A
proof −
  interpret *fun-left-comm op* ∗::'a ⇒ 'a ⇒ 'a **by** (*rule fun-left-comm*)
  from *nonempty* **obtain** a A' **where** A = *insert a A'* & a ~: A'
    by (*auto simp add*: *nonempty-iff*)
  with A **show** *?thesis*
    by (*simp add*: *insert-commute* [*of x*] *fold1-eq-fold eq-commute*)
qed

end

context *ab-semigroup-idem-mult*
begin

lemma *fun-left-comm-idem*: *fun-left-comm-idem*(*op* ∗)
apply *unfold-locales*
 apply (*rule mult-left-commute*)
apply (*rule mult-left-idem*)
done

lemma *fold1-insert-idem* [*simp*]:
  assumes *nonempty*: $A \neq \{\}$ **and** A: *finite* A
  shows *fold1 times* (*insert x A*) = x ∗ *fold1 times* A
proof −
  interpret *fun-left-comm-idem op* ∗::'a ⇒ 'a ⇒ 'a
    by (*rule fun-left-comm-idem*)
  from *nonempty* **obtain** a A' **where** A': A = *insert a A'* & a ~: A'
    by (*auto simp add*: *nonempty-iff*)
  show *?thesis*
  proof *cases*
    assume $a = x$
    thus *?thesis*
    proof *cases*
      assume $A' = \{\}$
      with *prems* **show** *?thesis* **by** *simp*
    next
      assume $A' \neq \{\}$
      with *prems* **show** *?thesis*
        by (*simp add*: *fold1-insert mult-assoc* [*symmetric*])
    qed
  next
    assume $a \neq x$
    with *prems* **show** *?thesis*
      by (*simp add*: *insert-commute fold1-eq-fold*)
  qed
qed

lemma *hom-fold1-commute*:
assumes *hom*: !!$x\ y.\ h\ (x * y) = h\ x * h\ y$
and N: *finite* N $N \neq \{\}$ **shows** h (*fold1 times* N) = *fold1 times* (h ‘ N)
using N **proof** (*induct rule*: *finite-ne-induct*)
case *singleton* **thus** *?case* **by** *simp*
next
case (*insert n N*)
then have h (*fold1 times* (*insert n N*)) = h (n * *fold1 times* N) **by** *simp*
also have ... = $h\ n * h$ (*fold1 times* N) **by**(*rule hom*)
also have h (*fold1 times* N) = *fold1 times* (h ‘ N) **by**(*rule insert*)
also have *times* ($h\ n$) ... = *fold1 times* (*insert* ($h\ n$) (h ‘ N))
using *insert* **by**(*simp*)
also have *insert* ($h\ n$) (h ‘ N) = h ‘ *insert n N* **by** *simp*
finally show *?case* **.**
qed

lemma *fold1-eq-fold-idem*:
assumes *finite A*
shows *fold1 times* (*insert a A*) = *fold times a A*
proof (*cases* $a \in A$)
case *False*
with *assms* **show** *?thesis* **by** (*simp add*: *fold1-eq-fold*)
next
interpret *fun-left-comm-idem times* **by** (*fact fun-left-comm-idem*)
case *True* **then obtain** *b B*
where A: A = *insert a B* **and** $a \notin B$ **by** (*rule set-insert*)
with *assms* **have** *finite B* **by** *auto*
then have *fold times a* (*insert a B*) = *fold times* ($a * a$) B
using ‹$a \notin B$› **by** (*rule fold-insert2*)
then show *?thesis*
using ‹$a \notin B$› ‹*finite B*› **by** (*simp add*: *fold1-eq-fold A*)
qed

end

Now the recursion rules for definitions:

lemma *fold1-singleton-def*: g = *fold1* $f \Longrightarrow g\ \{a\} = a$
by *simp*

lemma (**in** *ab-semigroup-mult*) *fold1-insert-def*:
⟦ g = *fold1 times*; *finite A*; $x \notin A$; $A \neq \{\}$ ⟧ $\Longrightarrow g$ (*insert x A*) = $x * g\ A$
by (*simp add*:*fold1-insert*)

lemma (**in** *ab-semigroup-idem-mult*) *fold1-insert-idem-def*:
⟦ g = *fold1 times*; *finite A*; $A \neq \{\}$ ⟧ $\Longrightarrow g$ (*insert x A*) = $x * g\ A$
by *simp*

20.5.1 Determinacy for *fold1Set*

declare
empty-fold-graphE [*rule del*] *fold-graph.intros* [*rule del*]
empty-fold1SetE [*rule del*] *insert-fold1SetE* [*rule del*]
— No more proofs involve these relations.

20.5.2 Lemmas about *fold1*

context *ab-semigroup-mult*
begin

lemma *fold1-Un*:
assumes *A*: *finite A A* $\neq \{\}$
shows *finite B* $\Longrightarrow$ *B* $\neq \{\} \Longrightarrow$ *A Int B* $= \{\} \Longrightarrow$
fold1 times (*A Un B*) = *fold1 times A* $*$ *fold1 times B*
using *A* **by** (*induct rule*: *finite-ne-induct*)
(*simp-all add*: *fold1-insert mult-assoc*)

lemma *fold1-in*:
assumes *A*: *finite* (*A*) *A* $\neq \{\}$ **and** *elem*: $\bigwedge x\ y.\ x * y \in \{x,y\}$
shows *fold1 times A* $\in$ *A*
using *A*
proof (*induct rule*:*finite-ne-induct*)
case *singleton* **thus** *?case* **by** *simp*
next
case *insert* **thus** *?case* **using** *elem* **by** (*force simp add*:*fold1-insert*)
qed

end

lemma (**in** *ab-semigroup-idem-mult*) *fold1-Un2*:
assumes *A*: *finite A A* $\neq \{\}$
shows *finite B* $\Longrightarrow$ *B* $\neq \{\} \Longrightarrow$
fold1 times (*A Un B*) = *fold1 times A* $*$ *fold1 times B*
using *A*
proof(*induct rule*:*finite-ne-induct*)
case *singleton* **thus** *?case* **by** *simp*
next
case *insert* **thus** *?case* **by** (*simp add*: *mult-assoc*)
qed

20.6 Locales as mini-packages for fold operations

20.6.1 The natural case

locale *folding* =
fixes *f* :: $'a \Rightarrow 'b \Rightarrow 'b$
fixes *F* :: $'a$ *set* $\Rightarrow 'b \Rightarrow 'b$
assumes *commute-comp*: *f y* $\circ$ *f x* = *f x* $\circ$ *f y*

assumes *eq-fold*: *finite A* $\Longrightarrow$ *F A s* = *fold f s A*
begin

lemma *empty* [*simp*]:
F {} = *id*
by (*simp add*: *eq-fold expand-fun-eq*)

lemma *insert* [*simp*]:
assumes *finite A* **and** $x \notin A$
shows *F* (*insert x A*) = *F A* ∘ *f x*
proof −
interpret *fun-left-comm f* **proof**
qed (*insert commute-comp*, *simp add*: *expand-fun-eq*)
from *fold-insert2 assms*
have $\bigwedge s$. *fold f s* (*insert x A*) = *fold f* (*f x s*) *A* **.**
with ‹*finite A*› **show** *?thesis* **by** (*simp add*: *eq-fold expand-fun-eq*)
qed

lemma *remove*:
assumes *finite A* **and** $x \in A$
shows *F A* = *F* (*A* − {*x*}) ∘ *f x*
proof −
from ‹$x \in A$› **obtain** *B* **where** *A*: *A* = *insert x B* **and** $x \notin B$
by (*auto dest*: *mk-disjoint-insert*)
moreover from ‹*finite A*› *this* **have** *finite B* **by** *simp*
ultimately show *?thesis* **by** *simp*
qed

lemma *insert-remove*:
assumes *finite A*
shows *F* (*insert x A*) = *F* (*A* − {*x*}) ∘ *f x*
using *assms* **by** (*cases* $x \in A$) (*simp-all add*: *remove insert-absorb*)

lemma *commute-left-comp*:
f y ∘ (*f x* ∘ *g*) = *f x* ∘ (*f y* ∘ *g*)
by (*simp add*: *o-assoc commute-comp*)

lemma *commute-comp′*:
assumes *finite A*
shows *f x* ∘ *F A* = *F A* ∘ *f x*
using *assms* **by** (*induct A*)
(*simp*, *simp del*: *o-apply add*: *o-assoc*, *simp del*: *o-apply add*: *o-assoc* [*symmetric*] *commute-comp*)

lemma *commute-left-comp′*:
assumes *finite A*
shows *f x* ∘ (*F A* ∘ *g*) = *F A* ∘ (*f x* ∘ *g*)
using *assms* **by** (*simp add*: *o-assoc commute-comp′*)

lemma *commute-comp''*:
assumes *finite A* **and** *finite B*
shows $F\ B \circ F\ A = F\ A \circ F\ B$
using *assms* **by** (*induct A*)
(*simp-all add*: *o-assoc*, *simp add*: *o-assoc* [*symmetric*] *commute-comp'*)

lemma *commute-left-comp''*:
assumes *finite A* **and** *finite B*
shows $F\ B \circ (F\ A \circ g) = F\ A \circ (F\ B \circ g)$
using *assms* **by** (*simp add*: *o-assoc commute-comp''*)

lemmas *commute-comps* = *o-assoc* [*symmetric*] *commute-comp commute-left-comp commute-comp' commute-left-comp' commute-comp'' commute-left-comp''*

lemma *union-inter*:
assumes *finite A* **and** *finite B*
shows $F\ (A \cup B) \circ F\ (A \cap B) = F\ A \circ F\ B$
using *assms* **by** (*induct A*)
(*simp-all del*: *o-apply add*: *insert-absorb Int-insert-left commute-comps*,
simp add: *o-assoc*)

lemma *union*:
assumes *finite A* **and** *finite B*
and $A \cap B = \{\}$
shows $F\ (A \cup B) = F\ A \circ F\ B$
proof −
from *union-inter* ‹*finite A*› ‹*finite B*› **have** $F\ (A \cup B) \circ F\ (A \cap B) = F\ A \circ F\ B$ **.**
with ‹$A \cap B = \{\}$› **show** *?thesis* **by** *simp*
qed

end

20.6.2 The natural case with idempotency

locale *folding-idem* = *folding* +
assumes *idem-comp*: $f\ x \circ f\ x = f\ x$
begin

lemma *idem-left-comp*:
$f\ x \circ (f\ x \circ g) = f\ x \circ g$
by (*simp add*: *o-assoc idem-comp*)

lemma *in-comp-idem*:
assumes *finite A* **and** $x \in A$
shows $F\ A \circ f\ x = F\ A$
using *assms* **by** (*induct A*)
(*auto simp add*: *commute-comps idem-comp*, *simp add*: *commute-left-comp'* [*symmetric*] *commute-comp'*)

lemma *subset-comp-idem*:
assumes *finite A* **and** $B \subseteq A$
shows $F\ A \circ F\ B = F\ A$
proof −
from *assms* **have** *finite B* **by** (*blast dest*: *finite-subset*)
then show *?thesis* **using** ‹$B \subseteq A$› **by** (*induct B*)
(*simp-all add*: *o-assoc in-comp-idem* ‹*finite A*›)
qed

declare *insert* [*simp del*]

lemma *insert-idem* [*simp*]:
assumes *finite A*
shows $F\ (\mathit{insert}\ x\ A) = F\ A \circ f\ x$
using *assms* **by** (*cases* $x \in A$) (*simp-all add*: *insert in-comp-idem insert-absorb*)

lemma *union-idem*:
assumes *finite A* **and** *finite B*
shows $F\ (A \cup B) = F\ A \circ F\ B$
proof −
from *assms* **have** *finite* $(A \cup B)$ **and** $A \cap B \subseteq A \cup B$ **by** *auto*
then have $F\ (A \cup B) \circ F\ (A \cap B) = F\ (A \cup B)$ **by** (*rule subset-comp-idem*)
with *assms* **show** *?thesis* **by** (*simp add*: *union-inter*)
qed

end

20.6.3 The image case with fixed function

no-notation *times* (**infixl** $*$ *70*)
no-notation *Groups.one* (*1*)

locale *folding-image-simple* = *comm-monoid* +
fixes $g :: ('b \Rightarrow 'a)$
fixes $F :: 'b\ set \Rightarrow 'a$
assumes *eq-fold-g*: *finite* $A \Longrightarrow F\ A = \mathit{fold\text{-}image}\ f\ g\ 1\ A$
begin

lemma *empty* [*simp*]:
$F\ \{\} = 1$
by (*simp add*: *eq-fold-g*)

lemma *insert* [*simp*]:
assumes *finite A* **and** $x \notin A$
shows $F\ (\mathit{insert}\ x\ A) = g\ x * F\ A$
proof −
interpret *fun-left-comm* $\%x\ y.\ (g\ x) * y$ **proof**
qed (*simp add*: *ac-simps*)

with *assms* **have** *fold-image* (*op* ∗) *g 1* (*insert x A*) = *g x* ∗ *fold-image* (*op* ∗) *g 1 A*
by (*simp add*: *fold-image-def*)
with ‹*finite A*› **show** *?thesis* **by** (*simp add*: *eq-fold-g*)
qed

lemma *remove*:
assumes *finite A* **and** $x \in A$
shows *F A* = *g x* ∗ *F* (*A* − {*x*})
proof −
from ‹$x \in A$› **obtain** *B* **where** *A*: *A* = *insert x B* **and** $x \notin B$
by (*auto dest*: *mk-disjoint-insert*)
moreover from ‹*finite A*› *this* **have** *finite B* **by** *simp*
ultimately show *?thesis* **by** *simp*
qed

lemma *insert-remove*:
assumes *finite A*
shows *F* (*insert x A*) = *g x* ∗ *F* (*A* − {*x*})
using *assms* **by** (*cases* $x \in A$) (*simp-all add*: *remove insert-absorb*)

lemma *neutral*:
assumes *finite A* **and** ∀ *x*∈*A*. *g x* = *1*
shows *F A* = *1*
using *assms* **by** (*induct A*) *simp-all*

lemma *union-inter*:
assumes *finite A* **and** *finite B*
shows *F* (*A* ∪ *B*) ∗ *F* (*A* ∩ *B*) = *F A* ∗ *F B*
using *assms* **proof** (*induct A*)
case *empty* **then show** *?case* **by** *simp*
next
case (*insert x A*) **then show** *?case*
by (*auto simp add*: *insert-absorb Int-insert-left commute* [*of* - *g x*] *assoc left-commute*)
qed

corollary *union-inter-neutral*:
assumes *finite A* **and** *finite B*
and *I0*: ∀ *x* ∈ *A*∩*B*. *g x* = *1*
shows *F* (*A* ∪ *B*) = *F A* ∗ *F B*
using *assms* **by** (*simp add*: *union-inter* [*symmetric*] *neutral*)

corollary *union-disjoint*:
assumes *finite A* **and** *finite B*
assumes *A* ∩ *B* = {}
shows *F* (*A* ∪ *B*) = *F A* ∗ *F B*
using *assms* **by** (*simp add*: *union-inter-neutral*)

end

20.6.4 The image case with flexible function

locale *folding-image* = *comm-monoid* +
  fixes $F :: ('b \Rightarrow 'a) \Rightarrow 'b\ set \Rightarrow 'a$
  assumes *eq-fold*: $\bigwedge g.\ finite\ A \Longrightarrow F\ g\ A = fold\text{-}image\ f\ g\ 1\ A$

sublocale *folding-image* < *folding-image-simple op* $*$ *1 g F g* **proof**
qed (*fact eq-fold*)

context *folding-image*
begin

lemma *reindex*:
  assumes *finite A* **and** *inj-on h A*
  shows $F\ g\ (h\ `\ A) = F\ (g \circ h)\ A$
  using *assms* **by** (*induct A*) *auto*

lemma *cong*:
  assumes *finite A* **and** $\bigwedge x.\ x \in A \Longrightarrow g\ x = h\ x$
  shows $F\ g\ A = F\ h\ A$
proof −
  from *assms* **have** *ALL C. C <= A −−> (ALL x:C. g x = h x) −−> F g C = F h C*
  apply − **apply** (*erule finite-induct*) **apply** *simp*
  apply (*simp add*: *subset-insert-iff*, *clarify*)
  apply (*subgoal-tac finite C*)
  prefer *2* **apply** (*blast dest*: *finite-subset* [*COMP swap-prems-rl*])
  apply (*subgoal-tac C = insert x* $(C - \{x\})$)
  prefer *2* **apply** *blast*
  apply (*erule ssubst*)
  apply (*drule spec*)
  apply (*erule* (*1*) *notE impE*)
  apply (*simp add*: *Ball-def del*: *insert-Diff-single*)
  done
  with *assms* **show** *?thesis* **by** *simp*
qed

lemma *UNION-disjoint*:
  assumes *finite I* **and** $\forall i{\in}I.\ finite\ (A\ i)$
  and $\forall i{\in}I.\ \forall j{\in}I.\ i \neq j \longrightarrow A\ i \cap A\ j = \{\}$
  shows $F\ g\ (UNION\ I\ A) = F\ (F\ g \circ A)\ I$
apply (*insert assms*)
apply (*induct set*: *finite*, *simp*, *atomize*)
apply (*subgoal-tac* $\forall i{\in}Fa.\ x \neq i$)
 prefer *2* **apply** *blast*
apply (*subgoal-tac A x Int UNION Fa A* $= \{\}$)
 prefer *2* **apply** *blast*

apply (*simp add*: *union-disjoint*)
done

lemma *distrib*:
assumes *finite A*
shows $F\ (\lambda x.\ g\ x * h\ x)\ A = F\ g\ A * F\ h\ A$
using *assms* **by** (*rule finite-induct*) (*simp-all add*: *assoc commute left-commute*)

lemma *related*:
assumes *Re*: $R\ 1\ 1$
and *Rop*: $\forall x1\ y1\ x2\ y2.\ R\ x1\ x2 \wedge R\ y1\ y2 \longrightarrow R\ (x1 * y1)\ (x2 * y2)$
and *fS*: *finite S* **and** *Rfg*: $\forall x{\in}S.\ R\ (h\ x)\ (g\ x)$
shows $R\ (F\ h\ S)\ (F\ g\ S)$
using *fS* **by** (*rule finite-subset-induct*) (*insert assms*, *auto*)

lemma *eq-general*:
assumes *fS*: *finite S*
and *h*: $\forall y{\in}S'.\ \exists!x.\ x \in S \wedge h\ x = y$
and *f12*: $\forall x{\in}S.\ h\ x \in S' \wedge f2\ (h\ x) = f1\ x$
shows $F\ f1\ S = F\ f2\ S'$
proof−
from *h f12* **have** *hS*: $h\ `\ S = S'$ **by** *blast*
{fix *x y* **assume** *H*: $x \in S\ y \in S\ h\ x = h\ y$
from *f12 h H* **have** $x = y$ **by** *auto* **}**
hence *hinj*: *inj-on h S* **unfolding** *inj-on-def Ex1-def* **by** *blast*
from *f12* **have** *th*: $\bigwedge x.\ x \in S \Longrightarrow (f2 \circ h)\ x = f1\ x$ **by** *auto*
from *hS* **have** $F\ f2\ S' = F\ f2\ (h\ `\ S)$ **by** *simp*
also have $\ldots = F\ (f2\ o\ h)\ S$ **using** *reindex* [*OF fS hinj*, *of f2*] **.**
also have $\ldots = F\ f1\ S$ **using** *th cong* [*OF fS*, *of f2 o h f1*]
by *blast*
finally show *?thesis* **..**
qed

lemma *eq-general-inverses*:
assumes *fS*: *finite S*
and *kh*: $\bigwedge y.\ y \in T \Longrightarrow k\ y \in S \wedge h\ (k\ y) = y$
and *hk*: $\bigwedge x.\ x \in S \Longrightarrow h\ x \in T \wedge k\ (h\ x) = x \wedge g\ (h\ x) = j\ x$
shows $F\ j\ S = F\ g\ T$

apply (*rule eq-general* [*OF fS*, *of T h g j*])
apply (*rule ballI*)
apply (*frule kh*)
apply (*rule ex1I*[])
apply *blast*
apply *clarsimp*
apply (*drule hk*) **apply** *simp*
apply (*rule sym*)
apply (*erule conjunct1*[*OF conjunct2*[*OF hk*]])
apply (*rule ballI*)

apply (*drule hk*)
apply *blast*
done

end

20.6.5 The image case with fixed function and idempotency

locale *folding-image-simple-idem* = *folding-image-simple* +
assumes *idem*: $x * x = x$

sublocale *folding-image-simple-idem* < *semilattice* **proof**
qed (*fact idem*)

context *folding-image-simple-idem*
begin

lemma *in-idem*:
assumes *finite A* **and** $x \in A$
shows $g\ x * F\ A = F\ A$
using *assms* **by** (*induct A*) (*auto simp add*: *left-commute*)

lemma *subset-idem*:
assumes *finite A* **and** $B \subseteq A$
shows $F\ B * F\ A = F\ A$
proof −
from *assms* **have** *finite B* **by** (*blast dest*: *finite-subset*)
then show *?thesis* **using** ‹$B \subseteq A$› **by** (*induct B*)
(*auto simp add*: *assoc in-idem* ‹*finite A*›)
qed

declare *insert* [*simp del*]

lemma *insert-idem* [*simp*]:
assumes *finite A*
shows $F\ (\mathit{insert}\ x\ A) = g\ x * F\ A$
using *assms* **by** (*cases* $x \in A$) (*simp-all add*: *insert in-idem insert-absorb*)

lemma *union-idem*:
assumes *finite A* **and** *finite B*
shows $F\ (A \cup B) = F\ A * F\ B$
proof −
from *assms* **have** *finite* $(A \cup B)$ **and** $A \cap B \subseteq A \cup B$ **by** *auto*
then have $F\ (A \cap B) * F\ (A \cup B) = F\ (A \cup B)$ **by** (*rule subset-idem*)
with *assms* **show** *?thesis* **by** (*simp add*: *union-inter* [*of A B*, *symmetric*] *commute*)
qed

end

20.6.6 The image case with flexible function and idempotency

locale *folding-image-idem* = *folding-image* +
  assumes *idem*: $x * x = x$

sublocale *folding-image-idem* < *folding-image-simple-idem op* $*$ *1 g F g* **proof qed** (*fact idem*)

20.6.7 The neutral-less case

locale *folding-one* = *abel-semigroup* +
  fixes $F :: 'a\ set \Rightarrow 'a$
  assumes *eq-fold*: *finite* $A \Longrightarrow F\ A = fold1\ f\ A$
begin

lemma *singleton* [*simp*]:
  $F\ \{x\} = x$
  by (*simp add*: *eq-fold*)

lemma *eq-fold'*:
  assumes *finite* A **and** $x \notin A$
  shows $F\ (insert\ x\ A) = fold\ (op\ *)\ x\ A$
proof −
  interpret *ab-semigroup-mult op* $*$ **proof qed** (*simp-all add*: *ac-simps*)
  with *assms* **show** *?thesis* **by** (*simp add*: *eq-fold fold1-eq-fold*)
qed

lemma *insert* [*simp*]:
  assumes *finite* A **and** $x \notin A$ **and** $A \neq \{\}$
  shows $F\ (insert\ x\ A) = x * F\ A$
proof −
  from ‹$A \neq \{\}$› **obtain** b **where** $b \in A$ **by** *blast*
 then obtain B **where** $*$: $A = insert\ b\ B\ b \notin B$ **by** (*blast dest*: *mk-disjoint-insert*)
  with ‹*finite A*› **have** *finite B* **by** *simp*
  interpret *fold*: *folding op* $*$ $\lambda a\ b.\ fold\ (op\ *)\ b\ a$ **proof**
  qed (*simp-all add*: *expand-fun-eq ac-simps*)
  thm *fold.commute-comp'* [*of B b*, *simplified expand-fun-eq*, *simplified*]
  from ‹*finite B*› *fold.commute-comp'* [*of B x*]
    have *op* $*$ $x \circ (\lambda b.\ fold\ op\ *\ b\ B) = (\lambda b.\ fold\ op\ *\ b\ B) \circ op\ *\ x$ **by** *simp*
 then have *A*: $x * fold\ op\ *\ b\ B = fold\ op\ *\ (b * x)\ B$ **by** (*simp add*: *expand-fun-eq commute*)
  from ‹*finite B*› $*$ *fold.insert* [*of B b*]
    have $(\lambda x.\ fold\ op\ *\ x\ (insert\ b\ B)) = (\lambda x.\ fold\ op\ *\ x\ B) \circ op\ *\ b$ **by** *simp*
  then have *B*: *fold op* $*$ x $(insert\ b\ B) = fold\ op\ *\ (b * x)\ B$ **by** (*simp add*: *expand-fun-eq*)
  from *A B assms* $*$ **show** *?thesis* **by** (*simp add*: *eq-fold' del*: *fold.insert*)
qed

lemma *remove*:
  assumes *finite* A **and** $x \in A$

shows $F\ A = (\mathit{if}\ A - \{x\} = \{\}\ \mathit{then}\ x\ \mathit{else}\ x * F\ (A - \{x\}))$
proof −
from *assms* **obtain** B **where** $A = \mathit{insert}\ x\ B$ **and** $x \notin B$ **by** (*blast dest*: *mk-disjoint-insert*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma *insert-remove*:
assumes *finite A*
shows $F\ (\mathit{insert}\ x\ A) = (\mathit{if}\ A - \{x\} = \{\}\ \mathit{then}\ x\ \mathit{else}\ x * F\ (A - \{x\}))$
using *assms* **by** (*cases* $x \in A$) (*simp-all add*: *insert-absorb remove*)

lemma *union-disjoint*:
assumes *finite A* $A \neq \{\}$ **and** *finite B* $B \neq \{\}$ **and** $A \cap B = \{\}$
shows $F\ (A \cup B) = F\ A * F\ B$
using *assms* **by** (*induct A rule*: *finite-ne-induct*) (*simp-all add*: *ac-simps*)

lemma *union-inter*:
assumes *finite A* **and** *finite B* **and** $A \cap B \neq \{\}$
shows $F\ (A \cup B) * F\ (A \cap B) = F\ A * F\ B$
proof −
from *assms* **have** $A \neq \{\}$ **and** $B \neq \{\}$ **by** *auto*
from ‹*finite A*› ‹$A \neq \{\}$› ‹$A \cap B \neq \{\}$› **show** *?thesis* **proof** (*induct A rule*: *finite-ne-induct*)
case (*singleton x*) **then show** *?case* **by** (*simp add*: *insert-absorb ac-simps*)
next
case (*insert x A*) **show** *?case* **proof** (*cases* $x \in B$)
case *True* **then have** $B \neq \{\}$ **by** *auto*
with *insert True* ‹*finite B*› **show** *?thesis* **by** (*cases* $A \cap B = \{\}$)
(*simp-all add*: *insert-absorb ac-simps union-disjoint*)
next
case *False* **with** *insert* **have** $F\ (A \cup B) * F\ (A \cap B) = F\ A * F\ B$ **by** *simp*
moreover from *False* ‹*finite B*› *insert* **have** *finite* $(A \cup B)$ $x \notin A \cup B$ $A \cup B \neq \{\}$
by *auto*
ultimately show *?thesis* **using** *False* ‹*finite A*› ‹$x \notin A$› ‹$A \neq \{\}$› **by** (*simp add*: *assoc*)
qed
qed
qed

lemma *closed*:
assumes *finite A* $A \neq \{\}$ **and** *elem*: $\bigwedge x\ y.\ x * y \in \{x, y\}$
shows $F\ A \in A$
using ‹*finite A*› ‹$A \neq \{\}$› **proof** (*induct rule*: *finite-ne-induct*)
case *singleton* **then show** *?case* **by** *simp*
next
case *insert* **with** *elem* **show** *?case* **by** *force*
qed

end

20.6.8 The neutral-less case with idempotency

locale *folding-one-idem* = *folding-one* +
  assumes *idem*: $x * x = x$

sublocale *folding-one-idem* < *semilattice* **proof**
qed (*fact idem*)

context *folding-one-idem*
begin

lemma *in-idem*:
  assumes *finite* A **and** $x \in A$
  shows $x * F\ A = F\ A$
proof −
  from *assms* **have** $A \neq \{\}$ **by** *auto*
  with ‹*finite* A› **show** *?thesis* **using** ‹$x \in A$› **by** (*induct* A *rule*: *finite-ne-induct*)
(*auto simp add*: *ac-simps*)
qed

lemma *subset-idem*:
  assumes *finite* A $B \neq \{\}$ **and** $B \subseteq A$
  shows $F\ B * F\ A = F\ A$
proof −
  from *assms* **have** *finite* B **by** (*blast dest*: *finite-subset*)
  then show *?thesis* **using** ‹$B \neq \{\}$› ‹$B \subseteq A$› **by** (*induct* B *rule*: *finite-ne-induct*)
    (*simp-all add*: *assoc in-idem* ‹*finite* A›)
qed

lemma *eq-fold-idem'*:
  assumes *finite* A
  shows $F\ (\mathit{insert}\ a\ A) = \mathit{fold}\ (\mathit{op}\ *)\ a\ A$
proof −
  interpret *ab-semigroup-idem-mult op* $*$ **proof qed** (*simp-all add*: *ac-simps*)
  with *assms* **show** *?thesis* **by** (*simp add*: *eq-fold fold1-eq-fold-idem*)
qed

lemma *insert-idem* [*simp*]:
  assumes *finite* A **and** $A \neq \{\}$
  shows $F\ (\mathit{insert}\ x\ A) = x * F\ A$
proof (*cases* $x \in A$)
  case *False* **from** ‹*finite* A› ‹$x \notin A$› ‹$A \neq \{\}$› **show** *?thesis* **by** (*rule insert*)
next
  case *True*
  from ‹*finite* A› ‹$A \neq \{\}$› **show** *?thesis* **by** (*simp add*: *in-idem insert-absorb True*)
qed

lemma *union-idem*:
assumes *finite A* $A \neq \{\}$ **and** *finite B* $B \neq \{\}$
shows $F\ (A \cup B) = F\ A * F\ B$
proof (*cases* $A \cap B = \{\}$)
case *True* **with** *assms* **show** *?thesis* **by** (*simp add*: *union-disjoint*)
next
case *False*
from *assms* **have** *finite* $(A \cup B)$ **and** $A \cap B \subseteq A \cup B$ **by** *auto*
with *False* **have** $F\ (A \cap B) * F\ (A \cup B) = F\ (A \cup B)$ **by** (*auto intro*: *subset-idem*)
with *assms False* **show** *?thesis* **by** (*simp add*: *union-inter* [*of A B*, *symmetric*] *commute*)
qed

lemma *hom-commute*:
assumes *hom*: $\bigwedge x\ y.\ h\ (x * y) = h\ x * h\ y$
and *N*: *finite N* $N \neq \{\}$ **shows** $h\ (F\ N) = F\ (h\ `\ N)$
using *N* **proof** (*induct rule*: *finite-ne-induct*)
case *singleton* **thus** *?case* **by** *simp*
next
case (*insert n N*)
then have $h\ (F\ (\mathit{insert}\ n\ N)) = h\ (n * F\ N)$ **by** *simp*
also have $\ldots = h\ n * h\ (F\ N)$ **by** (*rule hom*)
also have $h\ (F\ N) = F\ (h\ `\ N)$ **by**(*rule insert*)
also have $h\ n * \ldots = F\ (\mathit{insert}\ (h\ n)\ (h\ `\ N))$
using *insert* **by**(*simp*)
also have *insert* $(h\ n)\ (h\ `\ N) = h\ `\ \mathit{insert}\ n\ N$ **by** *simp*
finally show *?case* **.**
qed

end

notation *times* (**infixl** $*$ *70*)
notation *Groups.one* (*1*)

20.7 Finite cardinality

This definition, although traditional, is ugly to work with: *card A* == *LEAST n. EX f. A* = {*f i* | *i. i* < *n*}. But now that we have *fold-image* things are easy:

definition *card* :: *'a set* ⇒ *nat* **where**
card A = (*if finite A then fold-image* (*op* +) (λ*x. 1*) *0 A else 0*)

interpretation *card*!: *folding-image-simple op + 0* λ*x. 1 card* **proof**
qed (*simp add*: *card-def*)

lemma *card-infinite* [*simp*]:
¬ *finite A* ⟹ *card A* = *0*

by (*simp add*: *card-def*)

lemma *card-empty*:
card {} = *0*
by (*fact card.empty*)

lemma *card-insert-disjoint*:
finite A ==> *x* ∉ *A* ==> *card* (*insert x A*) = *Suc* (*card A*)
by *simp*

lemma *card-insert-if*:
finite A ==> *card* (*insert x A*) = (*if x* ∈ *A then card A else Suc* (*card A*))
by *auto* (*simp add*: *card.insert-remove card.remove*)

lemma *card-ge-0-finite*:
card A > *0* ⟹ *finite A*
by (*rule ccontr*) *simp*

lemma *card-0-eq* [*simp*, *no-atp*]:
finite A ⟹ *card A* = *0* ⟷ *A* = {}
by (*auto dest*: *mk-disjoint-insert*)

lemma *finite-UNIV-card-ge-0*:
finite (*UNIV* :: *'a set*) ⟹ *card* (*UNIV* :: *'a set*) > *0*
by (*rule ccontr*) *simp*

lemma *card-eq-0-iff*:
card A = *0* ⟷ *A* = {} ∨ ¬ *finite A*
by *auto*

lemma *card-gt-0-iff*:
0 < *card A* ⟷ *A* ≠ {} ∧ *finite A*
by (*simp add*: *neq0-conv* [*symmetric*] *card-eq-0-iff*)

lemma *card-Suc-Diff1*: *finite A* ==> *x*: *A* ==> *Suc* (*card* (*A* − {*x*})) = *card A*
apply(*rule-tac t* = *A* **in** *insert-Diff* [*THEN subst*], *assumption*)
apply(*simp del*:*insert-Diff-single*)
done

lemma *card-Diff-singleton*:
finite A ==> *x*: *A* ==> *card* (*A* − {*x*}) = *card A* − *1*
by (*simp add*: *card-Suc-Diff1* [*symmetric*])

lemma *card-Diff-singleton-if*:
finite A ==> *card* (*A*−{*x*}) = (*if x* : *A then card A* − *1 else card A*)
by (*simp add*: *card-Diff-singleton*)

lemma *card-Diff-insert*[*simp*]:
assumes *finite A* **and** *a*:*A* **and** *a* ~: *B*

shows *card*(*A* − *insert a B*) = *card*(*A* − *B*) − *1*
proof −
have *A* − *insert a B* = (*A* − *B*) − {*a*} **using** *assms* **by** *blast*
then show *?thesis* **using** *assms* **by**(*simp add*:*card-Diff-singleton*)
qed

lemma *card-insert*: *finite A* ==> *card* (*insert x A*) = *Suc* (*card* (*A* − {*x*}))
by (*simp add*: *card-insert-if card-Suc-Diff1 del*:*card-Diff-insert*)

lemma *card-insert-le*: *finite A* ==> *card A* <= *card* (*insert x A*)
by (*simp add*: *card-insert-if*)

lemma *card-mono*:
assumes *finite B* **and** *A* ⊆ *B*
shows *card A* ≤ *card B*
proof −
from *assms* **have** *finite A* **by** (*auto intro*: *finite-subset*)
then show *?thesis* **using** *assms* **proof** (*induct A arbitrary*: *B*)
case *empty* **then show** *?case* **by** *simp*
next
case (*insert x A*)
then have *x* ∈ *B* **by** *simp*
from *insert* **have** *A* ⊆ *B* − {*x*} **and** *finite* (*B* − {*x*}) **by** *auto*
with *insert.hyps* **have** *card A* ≤ *card* (*B* − {*x*}) **by** *auto*
with ‹*finite A*› ‹*x* ∉ *A*› ‹*finite B*› ‹*x* ∈ *B*› **show** *?case* **by** *simp* (*simp only*: *card.remove*)
qed
qed

lemma *card-seteq*: *finite B* ==> (!!*A*. *A* <= *B* ==> *card B* <= *card A* ==> *A* = *B*)
apply (*induct set*: *finite*, *simp*, *clarify*)
apply (*subgoal-tac finite A* & *A* − {*x*} <= *F*)
prefer *2* **apply** (*blast intro*: *finite-subset*, *atomize*)
apply (*drule-tac x* = *A* − {*x*} **in** *spec*)
apply (*simp add*: *card-Diff-singleton-if split add*: *split-if-asm*)
apply (*case-tac card A*, *auto*)
done

lemma *psubset-card-mono*: *finite B* ==> *A* < *B* ==> *card A* < *card B*
apply (*simp add*: *psubset-eq linorder-not-le* [*symmetric*])
apply (*blast dest*: *card-seteq*)
done

lemma *card-Un-Int*: *finite A* ==> *finite B*
==> *card A* + *card B* = *card* (*A Un B*) + *card* (*A Int B*)
by (*fact card.union-inter* [*symmetric*])

lemma *card-Un-disjoint*: *finite A* ==> *finite B*

```
  ==> A Int B = {} ==> card (A Un B) = card A + card B
 by (fact card.union-disjoint)

lemma card-Diff-subset:
 assumes finite B and B ⊆ A
 shows card (A − B) = card A − card B
proof (cases finite A)
 case False with assms show ?thesis by simp
next
 case True with assms show ?thesis by (induct B arbitrary: A) simp-all
qed

lemma card-Diff-subset-Int:
 assumes AB: finite (A ∩ B) shows card (A − B) = card A − card (A ∩ B)
proof −
 have A − B = A − A ∩ B by auto
 thus ?thesis
  by (simp add: card-Diff-subset AB)
qed

lemma card-Diff1-less: finite A ==> x: A ==> card (A − {x}) < card A
apply (rule Suc-less-SucD)
apply (simp add: card-Suc-Diff1 del:card-Diff-insert)
done

lemma card-Diff2-less:
 finite A ==> x: A ==> y: A ==> card (A − {x} − {y}) < card A
apply (case-tac x = y)
 apply (simp add: card-Diff1-less del:card-Diff-insert)
apply (rule less-trans)
 prefer 2 apply (auto intro!: card-Diff1-less simp del:card-Diff-insert)
done

lemma card-Diff1-le: finite A ==> card (A − {x}) <= card A
apply (case-tac x : A)
 apply (simp-all add: card-Diff1-less less-imp-le)
done

lemma card-psubset: finite B ==> A ⊆ B ==> card A < card B ==> A < B
by (erule psubsetI, blast)

lemma insert-partition:
 ⟦ x ∉ F; ∀ c1 ∈ insert x F. ∀ c2 ∈ insert x F. c1 ≠ c2 ⟶ c1 ∩ c2 = {} ⟧
 ⟹ x ∩ ⋃ F = {}
by auto

lemma finite-psubset-induct[consumes 1, case-names psubset]:
 assumes fin: finite A
 and     major: ⋀A. finite A ⟹ (⋀B. B ⊂ A ⟹ P B) ⟹ P A
```

```
  shows P A
using fin
proof (induct A taking: card rule: measure-induct-rule)
  case (less A)
  have fin: finite A by fact
  have ih: ⋀B. ⟦card B < card A; finite B⟧ ⟹ P B by fact
  { fix B
    assume asm: B ⊂ A
    from asm have card B < card A using psubset-card-mono fin by blast
    moreover
    from asm have B ⊆ A by auto
    then have finite B using fin finite-subset by blast
    ultimately
    have P B using ih by simp
  }
  with fin show P A using major by blast
qed
```

main cardinality theorem

```
lemma card-partition [rule-format]:
  finite C ==>
     finite (⋃ C) -->
     (∀ c∈C. card c = k) -->
     (∀ c1 ∈ C. ∀ c2 ∈ C. c1 ≠ c2 --> c1 ∩ c2 = {}) -->
     k * card(C) = card (⋃ C)
apply (erule finite-induct, simp)
apply (simp add: card-Un-disjoint insert-partition
       finite-subset [of - ⋃ (insert x F)])
done

lemma card-eq-UNIV-imp-eq-UNIV:
  assumes fin: finite (UNIV :: 'a set)
  and card: card A = card (UNIV :: 'a set)
  shows A = (UNIV :: 'a set)
proof
  show A ⊆ UNIV by simp
  show UNIV ⊆ A
  proof
    fix x
    show x ∈ A
    proof (rule ccontr)
      assume x ∉ A
      then have A ⊂ UNIV by auto
      with fin have card A < card (UNIV :: 'a set) by (fact psubset-card-mono)
      with card show False by simp
    qed
  qed
qed
```

The form of a finite set of given cardinality

lemma *card-eq-SucD*:
assumes *card A = Suc k*
shows $\exists b\ B.\ A = insert\ b\ B\ \&\ b \notin B\ \&\ card\ B = k\ \&\ (k{=}0 \longrightarrow B{=}\{\})$
proof −
  have *fin*: *finite A* **using** *assms* **by** (*auto intro*: *ccontr*)
  moreover have *card A* ≠ *0* **using** *assms* **by** *auto*
  ultimately obtain *b* **where** *b*: *b* ∈ *A* **by** *auto*
  show *?thesis*
  proof (*intro exI conjI*)
    show *A* = *insert b* (*A*−{*b*}) **using** *b* **by** *blast*
    show *b* ∉ *A* − {*b*} **by** *blast*
    show *card* (*A* − {*b*}) = *k* **and** *k* = *0* ⟶ *A* − {*b*} = {}
      using *assms b fin* **by**(*fastsimp dest*:*mk-disjoint-insert*)+
  qed
qed

lemma *card-Suc-eq*:
  (*card A* = *Suc k*) =
   (∃ *b B*. *A* = *insert b B* & *b* ∉ *B* & *card B* = *k* & (*k*=*0* ⟶ *B*={}))
apply(*rule iffI*)
 apply(*erule card-eq-SucD*)
apply(*auto*)
apply(*subst card-insert*)
 apply(*auto intro*:*ccontr*)
done

lemma *finite-fun-UNIVD2*:
  assumes *fin*: *finite* (*UNIV* :: (*'a* ⇒ *'b*) *set*)
  shows *finite* (*UNIV* :: *'b set*)
proof −
  from *fin* **have** *finite* (*range* (λ*f* :: *'a* ⇒ *'b*. *f arbitrary*))
    by(*rule finite-imageI*)
  moreover have *UNIV* = *range* (λ*f* :: *'a* ⇒ *'b*. *f arbitrary*)
    by(*rule UNIV-eq-I*) *auto*
  ultimately show *finite* (*UNIV* :: *'b set*) **by** *simp*
qed

lemma *card-UNIV-unit*: *card* (*UNIV* :: *unit set*) = *1*
  unfolding *UNIV-unit* **by** *simp*

20.7.1 Cardinality of image

lemma *card-image-le*: *finite A* ==> *card* (*f* ‘ *A*) <= *card A*
apply (*induct set*: *finite*)
 apply *simp*
apply (*simp add*: *le-SucI card-insert-if*)
done

lemma *card-image*:

assumes *inj-on f A*
shows *card* (*f* ‘ *A*) = *card A*
proof (*cases finite A*)
case *True* **then show** *?thesis* **using** *assms* **by** (*induct A*) *simp-all*
next
case *False* **then have** ¬ *finite* (*f* ‘ *A*) **using** *assms* **by** (*auto dest*: *finite-imageD*)
with *False* **show** *?thesis* **by** *simp*
qed

lemma *bij-betw-same-card*: *bij-betw f A B* ⟹ *card A* = *card B*
by(*auto simp*: *card-image bij-betw-def*)

lemma *endo-inj-surj*: *finite A* ==> *f* ‘ *A* ⊆ *A* ==> *inj-on f A* ==> *f* ‘ *A* = *A*
by (*simp add*: *card-seteq card-image*)

lemma *eq-card-imp-inj-on*:
[| *finite A*; *card*(*f* ‘ *A*) = *card A* |] ==> *inj-on f A*
apply (*induct rule*:*finite-induct*)
apply *simp*
apply(*frule card-image-le*[**where** *f* = *f*])
apply(*simp add*:*card-insert-if split*:*if-splits*)
done

lemma *inj-on-iff-eq-card*:
finite A ==> *inj-on f A* = (*card*(*f* ‘ *A*) = *card A*)
by(*blast intro*: *card-image eq-card-imp-inj-on*)

lemma *card-inj-on-le*:
[|*inj-on f A*; *f* ‘ *A* ⊆ *B*; *finite B* |] ==> *card A* ≤ *card B*
apply (*subgoal-tac finite A*)
apply (*force intro*: *card-mono simp add*: *card-image* [*symmetric*])
apply (*blast intro*: *finite-imageD dest*: *finite-subset*)
done

lemma *card-bij-eq*:
[|*inj-on f A*; *f* ‘ *A* ⊆ *B*; *inj-on g B*; *g* ‘ *B* ⊆ *A*;
finite A; *finite B* |] ==> *card A* = *card B*
by (*auto intro*: *le-antisym card-inj-on-le*)

20.7.2 Cardinality of sums

lemma *card-Plus*:
assumes *finite A* **and** *finite B*
shows *card* (*A* <+> *B*) = *card A* + *card B*
proof −
have *Inl*‘*A* ∩ *Inr*‘*B* = {} **by** *fast*
with *assms* **show** *?thesis*
unfolding *Plus-def*

by (*simp add*: *card-Un-disjoint card-image*)
qed

lemma *card-Plus-conv-if*:
card (*A* <+> *B*) = (*if finite A* ∧ *finite B then card A* + *card B else* 0)
by (*auto simp add*: *card-Plus*)

20.7.3 Cardinality of the Powerset

lemma *card-Pow*: *finite A* ==> *card* (*Pow A*) = *Suc* (*Suc* 0) ^ *card A*
apply (*induct set*: *finite*)
apply (*simp-all add*: *Pow-insert*)
apply (*subst card-Un-disjoint*, *blast*)
apply (*blast intro*: *finite-imageI*, *blast*)
apply (*subgoal-tac inj-on* (*insert x*) (*Pow F*))
apply (*simp add*: *card-image Pow-insert*)
apply (*unfold inj-on-def*)
apply (*blast elim*!: *equalityE*)
done

Relates to equivalence classes. Based on a theorem of F. KammÃ¼ller.

lemma *dvd-partition*:
finite (*Union C*) ==>
ALL c : *C*. *k dvd card c* ==>
(*ALL c1*: *C*. *ALL c2*: *C*. *c1* ≠ *c2* −−> *c1 Int c2* = {}) ==>
k dvd card (*Union C*)
apply(*frule finite-UnionD*)
apply(*rotate-tac* −1)
apply (*induct set*: *finite*, *simp-all*, *clarify*)
apply (*subst card-Un-disjoint*)
apply (*auto simp add*: *disjoint-eq-subset-Compl*)
done

20.7.4 Relating injectivity and surjectivity

lemma *finite-surj-inj*: *finite*(*A*) ⟹ *A* <= *f*‘*A* ⟹ *inj-on f A*
apply(*rule eq-card-imp-inj-on*, *assumption*)
apply(*frule finite-imageI*)
apply(*drule* (1) *card-seteq*)
apply(*erule card-image-le*)
apply *simp*
done

lemma *finite-UNIV-surj-inj*: **fixes** *f* :: *'a* ⇒ *'a*
shows *finite*(*UNIV*:: *'a set*) ⟹ *surj f* ⟹ *inj f*
by (*blast intro*: *finite-surj-inj subset-UNIV dest*:*surj-range*)

lemma *finite-UNIV-inj-surj*: **fixes** *f* :: *'a* ⇒ *'a*
shows *finite*(*UNIV*:: *'a set*) ⟹ *inj f* ⟹ *surj f*
by(*fastsimp simp*:*surj-def dest*!: *endo-inj-surj*)

corollary *infinite-UNIV-nat*[*iff*]: ~*finite*(*UNIV*::*nat set*)
proof
assume *finite*(*UNIV*::*nat set*)
with *finite-UNIV-inj-surj*[*of Suc*]
show *False* **by** *simp* (*blast dest*: *Suc-neq-Zero surjD*)
qed

lemma *infinite-UNIV-char-0*[*no-atp*]:
¬ *finite* (*UNIV*::'*a*::*semiring-char-0 set*)
proof
assume *finite* (*UNIV*::'*a set*)
with *subset-UNIV* **have** *finite* (*range of-nat*::'*a set*)
by (*rule finite-subset*)
moreover have *inj* (*of-nat*::*nat* ⇒ '*a*)
by (*simp add*: *inj-on-def*)
ultimately have *finite* (*UNIV*::*nat set*)
by (*rule finite-imageD*)
then show *False*
by *simp*
qed

end

21 Relation: Relations

theory *Relation*
imports *Datatype Finite-Set*
begin

21.1 Definitions

definition
converse :: ('*a* * '*b*) *set* => ('*b* * '*a*) *set*
((-^−1) [*1000*] *999*) **where**
r^−*1* == {(*y*, *x*). (*x*, *y*) : *r*}

notation (*xsymbols*)
converse ((-$^{-1}$) [*1000*] *999*)

definition
rel-comp :: [('*a* * '*b*) *set*, ('*b* * '*c*) *set*] => ('*a* * '*c*) *set*
(**infixr** *O* *75*) **where**
r O s == {(*x*,*z*). *EX y*. (*x*, *y*) : *r* & (*y*, *z*) : *s*}

definition
Image :: [('*a* * '*b*) *set*, '*a set*] => '*b set*

(**infixl** *``* *90*) **where**
r `` s == {*y*. *EX x*:*s*. (*x*,*y*):*r*}

definition
Id :: (*'a* ∗ *'a*) *set* **where** — the identity relation
Id == {*p*. *EX x*. *p* = (*x*,*x*)}

definition
Id-on :: *'a set* => (*'a* ∗ *'a*) *set* **where** — diagonal: identity over a set
Id-on A == ⋃*x*∈*A*. {(*x*,*x*)}

definition
Domain :: (*'a* ∗ *'b*) *set* => *'a set* **where**
Domain r == {*x*. *EX y*. (*x*,*y*):*r*}

definition
Range :: (*'a* ∗ *'b*) *set* => *'b set* **where**
Range r == *Domain*(*r*^−*1*)

definition
Field :: (*'a* ∗ *'a*) *set* => *'a set* **where**
Field r == *Domain r* ∪ *Range r*

definition
refl-on :: [*'a set*, (*'a* ∗ *'a*) *set*] => *bool* **where** — reflexivity over a set
refl-on A r == *r* ⊆ *A* × *A* & (*ALL x*: *A*. (*x*,*x*) : *r*)

abbreviation
refl :: (*'a* ∗ *'a*) *set* => *bool* **where** — reflexivity over a type
refl == *refl-on UNIV*

definition
sym :: (*'a* ∗ *'a*) *set* => *bool* **where** — symmetry predicate
sym r == *ALL x y*. (*x*,*y*): *r* −−> (*y*,*x*): *r*

definition
antisym :: (*'a* ∗ *'a*) *set* => *bool* **where** — antisymmetry predicate
antisym r == *ALL x y*. (*x*,*y*):*r* −−> (*y*,*x*):*r* −−> *x*=*y*

definition
trans :: (*'a* ∗ *'a*) *set* => *bool* **where** — transitivity predicate
trans r == (*ALL x y z*. (*x*,*y*):*r* −−> (*y*,*z*):*r* −−> (*x*,*z*):*r*)

definition
irrefl :: (*'a* ∗ *'a*) *set* => *bool* **where**
irrefl r ≡ ∀*x*. (*x*,*x*) ∉ *r*

definition
total-on :: *'a set* => (*'a* ∗ *'a*) *set* => *bool* **where**

total-on A r ≡ ∀ *x*∈*A*.∀ *y*∈*A*. *x*≠*y* ⟶ (*x*,*y*)∈*r* ∨ (*y*,*x*)∈*r*

abbreviation *total* ≡ *total-on UNIV*

definition
single-valued :: ('*a* ∗ '*b*) *set* => *bool* **where**
single-valued r == *ALL x y*. (*x*,*y*):*r* −−> (*ALL z*. (*x*,*z*):*r* −−> *y*=*z*)

definition
inv-image :: ('*b* ∗ '*b*) *set* => ('*a* => '*b*) => ('*a* ∗ '*a*) *set* **where**
inv-image r f == {(*x*, *y*). (*f x*, *f y*) : *r*}

21.2 The identity relation

lemma *IdI* [*intro*]: (*a*, *a*) : *Id*
by (*simp add*: *Id-def*)

lemma *IdE* [*elim*!]: *p* : *Id* ==> (!!*x*. *p* = (*x*, *x*) ==> *P*) ==> *P*
by (*unfold Id-def*) (*iprover elim*: *CollectE*)

lemma *pair-in-Id-conv* [*iff*]: ((*a*, *b*) : *Id*) = (*a* = *b*)
by (*unfold Id-def*) *blast*

lemma *refl-Id*: *refl Id*
by (*simp add*: *refl-on-def*)

lemma *antisym-Id*: *antisym Id*
— A strange result, since *Id* is also symmetric.
by (*simp add*: *antisym-def*)

lemma *sym-Id*: *sym Id*
by (*simp add*: *sym-def*)

lemma *trans-Id*: *trans Id*
by (*simp add*: *trans-def*)

21.3 Diagonal: identity over a set

lemma *Id-on-empty* [*simp*]: *Id-on* {} = {}
by (*simp add*: *Id-on-def*)

lemma *Id-on-eqI*: *a* = *b* ==> *a* : *A* ==> (*a*, *b*) : *Id-on A*
by (*simp add*: *Id-on-def*)

lemma *Id-onI* [*intro*!,*no-atp*]: *a* : *A* ==> (*a*, *a*) : *Id-on A*
by (*rule Id-on-eqI*) (*rule refl*)

lemma *Id-onE* [*elim*!]:
c : *Id-on A* ==> (!!*x*. *x* : *A* ==> *c* = (*x*, *x*) ==> *P*) ==> *P*
— The general elimination rule.

by (*unfold Id-on-def*) (*iprover elim*!: *UN-E singletonE*)

lemma *Id-on-iff*: $((x, y) : Id\text{-}on\ A) = (x = y\ \&\ x : A)$
by *blast*

lemma *Id-on-subset-Times*: $Id\text{-}on\ A \subseteq A \times A$
by *blast*

21.4 Composition of two relations

lemma *rel-compI* [*intro*]:
$(a, b) : r \Longrightarrow (b, c) : s \Longrightarrow (a, c) : r\ O\ s$
by (*unfold rel-comp-def*) *blast*

lemma *rel-compE* [*elim*!]: $xz : r\ O\ s \Longrightarrow$
$(!!x\ y\ z.\ xz = (x, z) \Longrightarrow (x, y) : r \Longrightarrow (y, z) : s \Longrightarrow P) \Longrightarrow P$
by (*unfold rel-comp-def*) (*iprover elim*!: *CollectE splitE exE conjE*)

lemma *rel-compEpair*:
$(a, c) : r\ O\ s \Longrightarrow (!!y.\ (a, y) : r \Longrightarrow (y, c) : s \Longrightarrow P) \Longrightarrow P$
by (*iprover elim*: *rel-compE Pair-inject ssubst*)

lemma *R-O-Id* [*simp*]: $R\ O\ Id = R$
by *fast*

lemma *Id-O-R* [*simp*]: $Id\ O\ R = R$
by *fast*

lemma *rel-comp-empty1*[*simp*]: $\{\}\ O\ R = \{\}$
by *blast*

lemma *rel-comp-empty2*[*simp*]: $R\ O\ \{\} = \{\}$
by *blast*

lemma *O-assoc*: $(R\ O\ S)\ O\ T = R\ O\ (S\ O\ T)$
by *blast*

lemma *trans-O-subset*: $trans\ r \Longrightarrow r\ O\ r \subseteq r$
by (*unfold trans-def*) *blast*

lemma *rel-comp-mono*: $r' \subseteq r \Longrightarrow s' \subseteq s \Longrightarrow (r'\ O\ s') \subseteq (r\ O\ s)$
by *blast*

lemma *rel-comp-subset-Sigma*:
$r \subseteq A \times B \Longrightarrow s \subseteq B \times C \Longrightarrow (r\ O\ s) \subseteq A \times C$
by *blast*

lemma *rel-comp-distrib*[*simp*]: $R\ O\ (S \cup T) = (R\ O\ S) \cup (R\ O\ T)$
by *auto*

lemma *rel-comp-distrib2*[*simp*]: $(S \cup T)\ O\ R = (S\ O\ R) \cup (T\ O\ R)$
by *auto*

lemma *rel-comp-UNION-distrib*: $s\ O\ UNION\ I\ r = UNION\ I\ (\%i.\ s\ O\ r\ i)$
by *auto*

lemma *rel-comp-UNION-distrib2*: $UNION\ I\ r\ O\ s = UNION\ I\ (\%i.\ r\ i\ O\ s)$
by *auto*

21.5 Reflexivity

lemma *refl-onI*: $r \subseteq A \times A \Longrightarrow (!!x.\ x : A \Longrightarrow (x, x) : r) \Longrightarrow refl\text{-}on\ A\ r$
by (*unfold refl-on-def*) (*iprover intro*!: *ballI*)

lemma *refl-onD*: $refl\text{-}on\ A\ r \Longrightarrow a : A \Longrightarrow (a, a) : r$
by (*unfold refl-on-def*) *blast*

lemma *refl-onD1*: $refl\text{-}on\ A\ r \Longrightarrow (x, y) : r \Longrightarrow x : A$
by (*unfold refl-on-def*) *blast*

lemma *refl-onD2*: $refl\text{-}on\ A\ r \Longrightarrow (x, y) : r \Longrightarrow y : A$
by (*unfold refl-on-def*) *blast*

lemma *refl-on-Int*: $refl\text{-}on\ A\ r \Longrightarrow refl\text{-}on\ B\ s \Longrightarrow refl\text{-}on\ (A \cap B)\ (r \cap s)$
by (*unfold refl-on-def*) *blast*

lemma *refl-on-Un*: $refl\text{-}on\ A\ r \Longrightarrow refl\text{-}on\ B\ s \Longrightarrow refl\text{-}on\ (A \cup B)\ (r \cup s)$
by (*unfold refl-on-def*) *blast*

lemma *refl-on-INTER*:
$ALL\ x{:}S.\ refl\text{-}on\ (A\ x)\ (r\ x) \Longrightarrow refl\text{-}on\ (INTER\ S\ A)\ (INTER\ S\ r)$
by (*unfold refl-on-def*) *fast*

lemma *refl-on-UNION*:
$ALL\ x{:}S.\ refl\text{-}on\ (A\ x)\ (r\ x) \Longrightarrow refl\text{-}on\ (UNION\ S\ A)\ (UNION\ S\ r)$
by (*unfold refl-on-def*) *blast*

lemma *refl-on-empty*[*simp*]: $refl\text{-}on\ \{\}\ \{\}$
by(*simp add*:*refl-on-def*)

lemma *refl-on-Id-on*: $refl\text{-}on\ A\ (Id\text{-}on\ A)$
by (*rule refl-onI* [*OF Id-on-subset-Times Id-onI*])

21.6 Antisymmetry

lemma *antisymI*:
$(!!x\ y.\ (x, y) : r \Longrightarrow (y, x) : r \Longrightarrow x{=}y) \Longrightarrow antisym\ r$
by (*unfold antisym-def*) *iprover*

lemma *antisymD*: *antisym r ==> (a, b) : r ==> (b, a) : r ==> a = b*
by (*unfold antisym-def*) *iprover*

lemma *antisym-subset*: *r ⊆ s ==> antisym s ==> antisym r*
by (*unfold antisym-def*) *blast*

lemma *antisym-empty* [*simp*]: *antisym {}*
by (*unfold antisym-def*) *blast*

lemma *antisym-Id-on* [*simp*]: *antisym (Id-on A)*
by (*unfold antisym-def*) *blast*

21.7 Symmetry

lemma *symI*: *(!!a b. (a, b) : r ==> (b, a) : r) ==> sym r*
by (*unfold sym-def*) *iprover*

lemma *symD*: *sym r ==> (a, b) : r ==> (b, a) : r*
by (*unfold sym-def*, *blast*)

lemma *sym-Int*: *sym r ==> sym s ==> sym (r ∩ s)*
by (*fast intro*: *symI dest*: *symD*)

lemma *sym-Un*: *sym r ==> sym s ==> sym (r ∪ s)*
by (*fast intro*: *symI dest*: *symD*)

lemma *sym-INTER*: *ALL x:S. sym (r x) ==> sym (INTER S r)*
by (*fast intro*: *symI dest*: *symD*)

lemma *sym-UNION*: *ALL x:S. sym (r x) ==> sym (UNION S r)*
by (*fast intro*: *symI dest*: *symD*)

lemma *sym-Id-on* [*simp*]: *sym (Id-on A)*
by (*rule symI*) *clarify*

21.8 Transitivity

lemma *transI*:
(!!x y z. (x, y) : r ==> (y, z) : r ==> (x, z) : r) ==> trans r
by (*unfold trans-def*) *iprover*

lemma *transD*: *trans r ==> (a, b) : r ==> (b, c) : r ==> (a, c) : r*
by (*unfold trans-def*) *iprover*

lemma *trans-Int*: *trans r ==> trans s ==> trans (r ∩ s)*
by (*fast intro*: *transI elim*: *transD*)

lemma *trans-INTER*: *ALL x:S. trans (r x) ==> trans (INTER S r)*
by (*fast intro*: *transI elim*: *transD*)

lemma *trans-Id-on* [*simp*]: *trans* (*Id-on A*)
by (*fast intro*: *transI elim*: *transD*)

lemma *trans-diff-Id*: *trans r* ⟹ *antisym r* ⟹ *trans* (*r*−*Id*)
unfolding *antisym-def trans-def* **by** *blast*

21.9 Irreflexivity

lemma *irrefl-diff-Id*[*simp*]: *irrefl*(*r*−*Id*)
by(*simp add*:*irrefl-def*)

21.10 Totality

lemma *total-on-empty*[*simp*]: *total-on* {} *r*
by(*simp add*:*total-on-def*)

lemma *total-on-diff-Id*[*simp*]: *total-on A* (*r*−*Id*) = *total-on A r*
by(*simp add*: *total-on-def*)

21.11 Converse

lemma *converse-iff* [*iff*]: ((*a*,*b*): *r*^−*1*) = ((*b*,*a*) : *r*)
by (*simp add*: *converse-def*)

lemma *converseI*[*sym*]: (*a*, *b*) : *r* ==> (*b*, *a*) : *r*^−*1*
by (*simp add*: *converse-def*)

lemma *converseD*[*sym*]: (*a*,*b*) : *r*^−*1* ==> (*b*, *a*) : *r*
by (*simp add*: *converse-def*)

lemma *converseE* [*elim*!]:
yx : *r*^−*1* ==> (!!*x y*. *yx* = (*y*, *x*) ==> (*x*, *y*) : *r* ==> *P*) ==> *P*
— More general than *converseD*, as it "splits" the member of the relation.
by (*unfold converse-def*) (*iprover elim*!: *CollectE splitE bexE*)

lemma *converse-converse* [*simp*]: (*r*^−*1*)^−*1* = *r*
by (*unfold converse-def*) *blast*

lemma *converse-rel-comp*: (*r O s*)^−*1* = *s*^−*1 O r*^−*1*
by *blast*

lemma *converse-Int*: (*r* ∩ *s*)^−*1* = *r*^−*1* ∩ *s*^−*1*
by *blast*

lemma *converse-Un*: (*r* ∪ *s*)^−*1* = *r*^−*1* ∪ *s*^−*1*
by *blast*

lemma *converse-INTER*: (*INTER S r*)^−*1* = (*INT x*:*S*. (*r x*)^−*1*)
by *fast*

lemma *converse-UNION*: $(UNION\ S\ r)\hat{}-1 = (UN\ x{:}S.\ (r\ x)\hat{}-1)$
by *blast*

lemma *converse-Id* [*simp*]: $Id\hat{}-1 = Id$
by *blast*

lemma *converse-Id-on* [*simp*]: $(Id\text{-}on\ A)\hat{}-1 = Id\text{-}on\ A$
by *blast*

lemma *refl-on-converse* [*simp*]: *refl-on A* (*converse r*) = *refl-on A r*
by (*unfold refl-on-def*) *auto*

lemma *sym-converse* [*simp*]: *sym* (*converse r*) = *sym r*
by (*unfold sym-def*) *blast*

lemma *antisym-converse* [*simp*]: *antisym* (*converse r*) = *antisym r*
by (*unfold antisym-def*) *blast*

lemma *trans-converse* [*simp*]: *trans* (*converse r*) = *trans r*
by (*unfold trans-def*) *blast*

lemma *sym-conv-converse-eq*: *sym r* = $(r\hat{}-1 = r)$
by (*unfold sym-def*) *fast*

lemma *sym-Un-converse*: *sym* $(r \cup r\hat{}-1)$
by (*unfold sym-def*) *blast*

lemma *sym-Int-converse*: *sym* $(r \cap r\hat{}-1)$
by (*unfold sym-def*) *blast*

lemma *total-on-converse*[*simp*]: *total-on A* $(r\hat{}-1)$ = *total-on A r*
by (*auto simp*: *total-on-def*)

21.12 Domain

declare *Domain-def* [*no-atp*]

lemma *Domain-iff*: $(a : Domain\ r) = (EX\ y.\ (a,\ y) : r)$
by (*unfold Domain-def*) *blast*

lemma *DomainI* [*intro*]: $(a,\ b) : r \Longrightarrow a : Domain\ r$
by (*iprover intro*!: *iffD2* [*OF Domain-iff*])

lemma *DomainE* [*elim*!]:
$a : Domain\ r \Longrightarrow (!!y.\ (a,\ y) : r \Longrightarrow P) \Longrightarrow P$
by (*iprover dest*!: *iffD1* [*OF Domain-iff*])

lemma *Domain-empty* [*simp*]: $Domain\ \{\} = \{\}$
by *blast*

lemma *Domain-empty-iff*: $Domain\ r = \{\} \longleftrightarrow r = \{\}$
by *auto*

lemma *Domain-insert*: $Domain\ (insert\ (a,\ b)\ r) = insert\ a\ (Domain\ r)$
by *blast*

lemma *Domain-Id* [*simp*]: $Domain\ Id = UNIV$
by *blast*

lemma *Domain-Id-on* [*simp*]: $Domain\ (Id\text{-}on\ A) = A$
by *blast*

lemma *Domain-Un-eq*: $Domain(A \cup B) = Domain(A) \cup Domain(B)$
by *blast*

lemma *Domain-Int-subset*: $Domain(A \cap B) \subseteq Domain(A) \cap Domain(B)$
by *blast*

lemma *Domain-Diff-subset*: $Domain(A) - Domain(B) \subseteq Domain(A - B)$
by *blast*

lemma *Domain-Union*: $Domain\ (Union\ S) = (\bigcup A{\in}S.\ Domain\ A)$
by *blast*

lemma *Domain-converse*[*simp*]: $Domain(r\hat{}-1) = Range\ r$
by(*auto simp*:*Range-def*)

lemma *Domain-mono*: $r \subseteq s \Longrightarrow Domain\ r \subseteq Domain\ s$
by *blast*

lemma *fst-eq-Domain*: $fst\ `\ R = Domain\ R$
by (*auto intro*!:*image-eqI*)

lemma *Domain-dprod* [*simp*]: $Domain\ (dprod\ r\ s) = uprod\ (Domain\ r)\ (Domain\ s)$
by *auto*

lemma *Domain-dsum* [*simp*]: $Domain\ (dsum\ r\ s) = usum\ (Domain\ r)\ (Domain\ s)$
by *auto*

21.13 Range

lemma *Range-iff*: $(a : Range\ r) = (EX\ y.\ (y,\ a) : r)$
by (*simp add*: *Domain-def Range-def*)

lemma *RangeI* [*intro*]: $(a,\ b) : r \Longrightarrow b : Range\ r$
by (*unfold Range-def*) (*iprover intro*!: *converseI DomainI*)

lemma *RangeE* [*elim*!]: *b* : *Range r* ==> (!!*x*. (*x*, *b*) : *r* ==> *P*) ==> *P*
by (*unfold Range-def*) (*iprover elim*!: *DomainE dest*!: *converseD*)

lemma *Range-empty* [*simp*]: *Range* {} = {}
by *blast*

lemma *Range-empty-iff*: *Range r* = {} ⟷ *r* = {}
by *auto*

lemma *Range-insert*: *Range* (*insert* (*a*, *b*) *r*) = *insert b* (*Range r*)
by *blast*

lemma *Range-Id* [*simp*]: *Range Id* = *UNIV*
by *blast*

lemma *Range-Id-on* [*simp*]: *Range* (*Id-on A*) = *A*
by *auto*

lemma *Range-Un-eq*: *Range*(*A* ∪ *B*) = *Range*(*A*) ∪ *Range*(*B*)
by *blast*

lemma *Range-Int-subset*: *Range*(*A* ∩ *B*) ⊆ *Range*(*A*) ∩ *Range*(*B*)
by *blast*

lemma *Range-Diff-subset*: *Range*(*A*) − *Range*(*B*) ⊆ *Range*(*A* − *B*)
by *blast*

lemma *Range-Union*: *Range* (*Union S*) = (⋃ *A*∈*S*. *Range A*)
by *blast*

lemma *Range-converse*[*simp*]: *Range*(*r*^−*1*) = *Domain r*
by *blast*

lemma *snd-eq-Range*: *snd* ' *R* = *Range R*
by (*auto intro*!:*image-eqI*)

21.14 Field

lemma *mono-Field*: *r* ⊆ *s* ⟹ *Field r* ⊆ *Field s*
by(*auto simp*:*Field-def Domain-def Range-def*)

lemma *Field-empty*[*simp*]: *Field* {} = {}
by(*auto simp*:*Field-def*)

lemma *Field-insert*[*simp*]: *Field* (*insert* (*a*,*b*) *r*) = {*a*,*b*} ∪ *Field r*
by(*auto simp*:*Field-def*)

lemma *Field-Un*[*simp*]: *Field* (*r* ∪ *s*) = *Field r* ∪ *Field s*

by(*auto simp*:*Field-def*)

lemma *Field-Union*[*simp*]: *Field* (⋃ *R*) = ⋃(*Field* ‘ *R*)
by(*auto simp*:*Field-def*)

lemma *Field-converse*[*simp*]: *Field*(*r*^−1) = *Field r*
by(*auto simp*:*Field-def*)

21.15 Image of a set under a relation

declare *Image-def* [*no-atp*]

lemma *Image-iff*: (*b* : *r*‘‘*A*) = (*EX x*:*A*. (*x*, *b*) : *r*)
by (*simp add*: *Image-def*)

lemma *Image-singleton*: *r*‘‘{*a*} = {*b*. (*a*, *b*) : *r*}
by (*simp add*: *Image-def*)

lemma *Image-singleton-iff* [*iff*]: (*b* : *r*‘‘{*a*}) = ((*a*, *b*) : *r*)
by (*rule Image-iff* [*THEN trans*]) *simp*

lemma *ImageI* [*intro*,*no-atp*]: (*a*, *b*) : *r* ==> *a* : *A* ==> *b* : *r*‘‘*A*
by (*unfold Image-def*) *blast*

lemma *ImageE* [*elim*!]:
b : *r* ‘‘ *A* ==> (!!*x*. (*x*, *b*) : *r* ==> *x* : *A* ==> *P*) ==> *P*
by (*unfold Image-def*) (*iprover elim*!: *CollectE bexE*)

lemma *rev-ImageI*: *a* : *A* ==> (*a*, *b*) : *r* ==> *b* : *r* ‘‘ *A*
— This version's more effective when we already have the required *a*
by *blast*

lemma *Image-empty* [*simp*]: *R*‘‘{} = {}
by *blast*

lemma *Image-Id* [*simp*]: *Id* ‘‘ *A* = *A*
by *blast*

lemma *Image-Id-on* [*simp*]: *Id-on A* ‘‘ *B* = *A* ∩ *B*
by *blast*

lemma *Image-Int-subset*: *R* ‘‘ (*A* ∩ *B*) ⊆ *R* ‘‘ *A* ∩ *R* ‘‘ *B*
by *blast*

lemma *Image-Int-eq*:
single-valued (*converse R*) ==> *R* ‘‘ (*A* ∩ *B*) = *R* ‘‘ *A* ∩ *R* ‘‘ *B*
by (*simp add*: *single-valued-def*, *blast*)

lemma *Image-Un*: *R* ‘‘ (*A* ∪ *B*) = *R* ‘‘ *A* ∪ *R* ‘‘ *B*

by *blast*

lemma *Un-Image*: $(R \cup S)$ `` $A = R$ `` $A \cup S$ `` A
by *blast*

lemma *Image-subset*: $r \subseteq A \times B$ ==> $r``C \subseteq B$
by (*iprover intro*!: *subsetI elim*!: *ImageE dest*!: *subsetD SigmaD2*)

lemma *Image-eq-UN*: $r``B = (\bigcup y{\in}\ B.\ r``\{y\})$
— NOT suitable for rewriting
by *blast*

lemma *Image-mono*: $r' \subseteq r$ ==> $A' \subseteq A$ ==> $(r' `` A') \subseteq (r `` A)$
by *blast*

lemma *Image-UN*: $(r `` (UNION\ A\ B)) = (\bigcup x{\in}A.\ r `` (B\ x))$
by *blast*

lemma *Image-INT-subset*: $(r `` INTER\ A\ B) \subseteq (\bigcap x{\in}A.\ r `` (B\ x))$
by *blast*

Converse inclusion requires some assumptions

lemma *Image-INT-eq*:
$[|single\text{-}valued\ (r^{-1}); A{\neq}\{\}|]$ ==> $r `` INTER\ A\ B = (\bigcap x{\in}A.\ r `` B\ x)$
apply (*rule equalityI*)
apply (*rule Image-INT-subset*)
apply (*simp add*: *single-valued-def*, *blast*)
done

lemma *Image-subset-eq*: $(r``A \subseteq B) = (A \subseteq -((r\hat{}{-}1) `` (-B)))$
by *blast*

21.16 Single valued relations

lemma *single-valuedI*:
$ALL\ x\ y.\ (x,y){:}r \longrightarrow (ALL\ z.\ (x,z){:}r \longrightarrow y{=}z)$ ==> *single-valued* r
by (*unfold single-valued-def*)

lemma *single-valuedD*:
single-valued r ==> $(x, y) : r$ ==> $(x, z) : r$ ==> $y = z$
by (*simp add*: *single-valued-def*)

lemma *single-valued-rel-comp*:
single-valued r ==> *single-valued* s ==> *single-valued* $(r\ O\ s)$
by (*unfold single-valued-def*) *blast*

lemma *single-valued-subset*:
$r \subseteq s$ ==> *single-valued* s ==> *single-valued* r
by (*unfold single-valued-def*) *blast*

lemma *single-valued-Id* [*simp*]: *single-valued Id*
by (*unfold single-valued-def*) *blast*

lemma *single-valued-Id-on* [*simp*]: *single-valued* (*Id-on A*)
by (*unfold single-valued-def*) *blast*

21.17 Graphs given by *Collect*

lemma *Domain-Collect-split* [*simp*]: *Domain*{(*x*,*y*). *P x y*} = {*x*. *EX y*. *P x y*}
by *auto*

lemma *Range-Collect-split* [*simp*]: *Range*{(*x*,*y*). *P x y*} = {*y*. *EX x*. *P x y*}
by *auto*

lemma *Image-Collect-split* [*simp*]: {(*x*,*y*). *P x y*} `` *A* = {*y*. *EX x*:*A*. *P x y*}
by *auto*

21.18 Inverse image

lemma *sym-inv-image*: *sym r* ==> *sym* (*inv-image r f*)
by (*unfold sym-def inv-image-def*) *blast*

lemma *trans-inv-image*: *trans r* ==> *trans* (*inv-image r f*)
apply (*unfold trans-def inv-image-def*)
apply (*simp* (*no-asm*))
apply *blast*
done

lemma *in-inv-image*[*simp*]: ((*x*,*y*) : *inv-image r f*) = ((*f x*, *f y*) : *r*)
by (*auto simp*:*inv-image-def*)

lemma *converse-inv-image*[*simp*]: (*inv-image R f*)^-1 = *inv-image* (*R*^-1) *f*
unfolding *inv-image-def converse-def* **by** *auto*

21.19 Finiteness

lemma *finite-converse* [*iff*]: *finite* (*r*^-1) = *finite r*
apply (*subgoal-tac r*^-1 = (%(*x*,*y*). (*y*,*x*))`*r*)
apply *simp*
apply (*rule iffI*)
apply (*erule finite-imageD* [*unfolded inj-on-def*])
apply (*simp split add*: *split-split*)
apply (*erule finite-imageI*)
apply (*simp add*: *converse-def image-def*, *auto*)
apply (*rule bexI*)
prefer *2* **apply** *assumption*
apply *simp*
done

lemma *finite-Domain*: *finite r ==> finite* (*Domain r*)
by (*induct set*: *finite*) (*auto simp add*: *Domain-insert*)

lemma *finite-Range*: *finite r ==> finite* (*Range r*)
by (*induct set*: *finite*) (*auto simp add*: *Range-insert*)

lemma *finite-Field*: *finite r ==> finite* (*Field r*)
— A finite relation has a finite field (= *domain* ∪ *range*.
apply (*induct set*: *finite*)
apply (*auto simp add*: *Field-def Domain-insert Range-insert*)
done

21.20 Miscellaneous

Version of *lfp-induct* for binary relations

lemmas *lfp-induct2* =
lfp-induct-set [*of* (*a*, *b*), *split-format* (*complete*)]

Version of *subsetI* for binary relations

lemma *subrelI*: $(\bigwedge x\ y.\ (x,\ y) \in r \Longrightarrow (x,\ y) \in s) \Longrightarrow r \subseteq s$
by *auto*

end

22 Predicate: Predicates as relations and enumerations

theory *Predicate*
imports *Inductive Relation*
begin

notation
inf (**infixl** ⊓ *70*) **and**
sup (**infixl** ⊔ *65*) **and**
Inf (⨅- [*900*] *900*) **and**
Sup (⨆- [*900*] *900*) **and**
top (⊤) **and**
bot (⊥)

22.1 Predicates as (complete) lattices

Handy introduction and elimination rules for ≤ on unary and binary predicates

lemma *predicate1I*:
assumes *PQ*: $\bigwedge x.\ P\ x \Longrightarrow Q\ x$
shows $P \leq Q$

apply (*rule le-funI*)
apply (*rule le-boolI*)
apply (*rule PQ*)
apply *assumption*
done

lemma *predicate1D* [*Pure.dest?*, *dest?*]:
$P \leq Q \Longrightarrow P\ x \Longrightarrow Q\ x$
apply (*erule le-funE*)
apply (*erule le-boolE*)
apply *assumption+*
done

lemma *rev-predicate1D*:
$P\ x$ ==> P <= Q ==> $Q\ x$
by (*rule predicate1D*)

lemma *predicate2I* [*Pure.intro!*, *intro!*]:
assumes *PQ*: $\bigwedge x\ y.\ P\ x\ y \Longrightarrow Q\ x\ y$
shows $P \leq Q$
apply (*rule le-funI*)+
apply (*rule le-boolI*)
apply (*rule PQ*)
apply *assumption*
done

lemma *predicate2D* [*Pure.dest*, *dest*]:
$P \leq Q \Longrightarrow P\ x\ y \Longrightarrow Q\ x\ y$
apply (*erule le-funE*)+
apply (*erule le-boolE*)
apply *assumption+*
done

lemma *rev-predicate2D*:
$P\ x\ y$ ==> P <= Q ==> $Q\ x\ y$
by (*rule predicate2D*)

22.1.1 Equality

lemma *pred-equals-eq*: $((\lambda x.\ x \in R) = (\lambda x.\ x \in S)) = (R = S)$
by (*simp add*: *mem-def*)

lemma *pred-equals-eq2* [*pred-set-conv*]: $((\lambda x\ y.\ (x,\ y) \in R) = (\lambda x\ y.\ (x,\ y) \in S))$
$= (R = S)$
by (*simp add*: *expand-fun-eq mem-def*)

22.1.2 Order relation

lemma *pred-subset-eq*: $((\lambda x.\ x \in R) <= (\lambda x.\ x \in S)) = (R <= S)$
by (*simp add*: *mem-def*)

lemma *pred-subset-eq2* [*pred-set-conv*]: $((\lambda x\ y.\ (x,\ y) \in R) <= (\lambda x\ y.\ (x,\ y) \in S)) = (R <= S)$
by *fast*

22.1.3 Top and bottom elements

lemma *top1I* [*intro*!]: *top x*
by (*simp add*: *top-fun-eq top-bool-eq*)

lemma *top2I* [*intro*!]: *top x y*
by (*simp add*: *top-fun-eq top-bool-eq*)

lemma *bot1E* [*elim*!]: *bot x* $\Longrightarrow$ *P*
by (*simp add*: *bot-fun-eq bot-bool-eq*)

lemma *bot2E* [*elim*!]: *bot x y* $\Longrightarrow$ *P*
by (*simp add*: *bot-fun-eq bot-bool-eq*)

lemma *bot-empty-eq*: *bot* $= (\lambda x.\ x \in \{\})$
by (*auto simp add*: *expand-fun-eq*)

lemma *bot-empty-eq2*: *bot* $= (\lambda x\ y.\ (x,\ y) \in \{\})$
by (*auto simp add*: *expand-fun-eq*)

22.1.4 Binary union

lemma *sup1I1* [*elim?*]: *A x* $\Longrightarrow$ *sup A B x*
by (*simp add*: *sup-fun-eq sup-bool-eq*)

lemma *sup2I1* [*elim?*]: *A x y* $\Longrightarrow$ *sup A B x y*
by (*simp add*: *sup-fun-eq sup-bool-eq*)

lemma *sup1I2* [*elim?*]: *B x* $\Longrightarrow$ *sup A B x*
by (*simp add*: *sup-fun-eq sup-bool-eq*)

lemma *sup2I2* [*elim?*]: *B x y* $\Longrightarrow$ *sup A B x y*
by (*simp add*: *sup-fun-eq sup-bool-eq*)

lemma *sup1E* [*elim*!]: *sup A B x* ==> (*A x* ==> *P*) ==> (*B x* ==> *P*) ==> *P*
by (*simp add*: *sup-fun-eq sup-bool-eq*) *iprover*

lemma *sup2E* [*elim*!]: *sup A B x y* ==> (*A x y* ==> *P*) ==> (*B x y* ==> *P*) ==> *P*
by (*simp add*: *sup-fun-eq sup-bool-eq*) *iprover*

Classical introduction rule: no commitment to *A* vs *B*.

lemma *sup1CI* [*intro*!]: ($\sim$ *B x* ==> *A x*) ==> *sup A B x*

by (*auto simp add*: *sup-fun-eq sup-bool-eq*)

lemma *sup2CI* [*intro*!]: (~ *B x y* ==> *A x y*) ==> *sup A B x y*
by (*auto simp add*: *sup-fun-eq sup-bool-eq*)

lemma *sup-Un-eq*: *sup* (λ*x*. *x* ∈ *R*) (λ*x*. *x* ∈ *S*) = (λ*x*. *x* ∈ *R* ∪ *S*)
by (*simp add*: *sup-fun-eq sup-bool-eq mem-def*)

lemma *sup-Un-eq2* [*pred-set-conv*]: *sup* (λ*x y*. (*x*, *y*) ∈ *R*) (λ*x y*. (*x*, *y*) ∈ *S*) = (λ*x y*. (*x*, *y*) ∈ *R* ∪ *S*)
by (*simp add*: *sup-fun-eq sup-bool-eq mem-def*)

22.1.5 Binary intersection

lemma *inf1I* [*intro*!]: *A x* ==> *B x* ==> *inf A B x*
by (*simp add*: *inf-fun-eq inf-bool-eq*)

lemma *inf2I* [*intro*!]: *A x y* ==> *B x y* ==> *inf A B x y*
by (*simp add*: *inf-fun-eq inf-bool-eq*)

lemma *inf1E* [*elim*!]: *inf A B x* ==> (*A x* ==> *B x* ==> *P*) ==> *P*
by (*simp add*: *inf-fun-eq inf-bool-eq*)

lemma *inf2E* [*elim*!]: *inf A B x y* ==> (*A x y* ==> *B x y* ==> *P*) ==> *P*
by (*simp add*: *inf-fun-eq inf-bool-eq*)

lemma *inf1D1*: *inf A B x* ==> *A x*
by (*simp add*: *inf-fun-eq inf-bool-eq*)

lemma *inf2D1*: *inf A B x y* ==> *A x y*
by (*simp add*: *inf-fun-eq inf-bool-eq*)

lemma *inf1D2*: *inf A B x* ==> *B x*
by (*simp add*: *inf-fun-eq inf-bool-eq*)

lemma *inf2D2*: *inf A B x y* ==> *B x y*
by (*simp add*: *inf-fun-eq inf-bool-eq*)

lemma *inf-Int-eq*: *inf* (λ*x*. *x* ∈ *R*) (λ*x*. *x* ∈ *S*) = (λ*x*. *x* ∈ *R* ∩ *S*)
by (*simp add*: *inf-fun-eq inf-bool-eq mem-def*)

lemma *inf-Int-eq2* [*pred-set-conv*]: *inf* (λ*x y*. (*x*, *y*) ∈ *R*) (λ*x y*. (*x*, *y*) ∈ *S*) = (λ*x y*. (*x*, *y*) ∈ *R* ∩ *S*)
by (*simp add*: *inf-fun-eq inf-bool-eq mem-def*)

22.1.6 Unions of families

lemma *SUP1-iff*: (*SUP x*:*A*. *B x*) *b* = (*EX x*:*A*. *B x b*)
by (*simp add*: *SUPR-def Sup-fun-def Sup-bool-def*) *blast*

lemma *SUP2-iff*: $(SUP\ x{:}A.\ B\ x)\ b\ c = (EX\ x{:}A.\ B\ x\ b\ c)$
by (*simp add*: *SUPR-def Sup-fun-def Sup-bool-def*) *blast*

lemma *SUP1-I* [*intro*]: $a : A \Longrightarrow B\ a\ b \Longrightarrow (SUP\ x{:}A.\ B\ x)\ b$
by (*auto simp add*: *SUP1-iff*)

lemma *SUP2-I* [*intro*]: $a : A \Longrightarrow B\ a\ b\ c \Longrightarrow (SUP\ x{:}A.\ B\ x)\ b\ c$
by (*auto simp add*: *SUP2-iff*)

lemma *SUP1-E* [*elim*!]: $(SUP\ x{:}A.\ B\ x)\ b \Longrightarrow (!!x.\ x : A \Longrightarrow B\ x\ b \Longrightarrow R) \Longrightarrow R$
by (*auto simp add*: *SUP1-iff*)

lemma *SUP2-E* [*elim*!]: $(SUP\ x{:}A.\ B\ x)\ b\ c \Longrightarrow (!!x.\ x : A \Longrightarrow B\ x\ b\ c \Longrightarrow R) \Longrightarrow R$
by (*auto simp add*: *SUP2-iff*)

lemma *SUP-UN-eq*: $(SUP\ i.\ (\lambda x.\ x \in r\ i)) = (\lambda x.\ x \in (UN\ i.\ r\ i))$
by (*simp add*: *SUP1-iff expand-fun-eq*)

lemma *SUP-UN-eq2*: $(SUP\ i.\ (\lambda x\ y.\ (x, y) \in r\ i)) = (\lambda x\ y.\ (x, y) \in (UN\ i.\ r\ i))$
by (*simp add*: *SUP2-iff expand-fun-eq*)

22.1.7 Intersections of families

lemma *INF1-iff*: $(INF\ x{:}A.\ B\ x)\ b = (ALL\ x{:}A.\ B\ x\ b)$
by (*simp add*: *INFI-def Inf-fun-def Inf-bool-def*) *blast*

lemma *INF2-iff*: $(INF\ x{:}A.\ B\ x)\ b\ c = (ALL\ x{:}A.\ B\ x\ b\ c)$
by (*simp add*: *INFI-def Inf-fun-def Inf-bool-def*) *blast*

lemma *INF1-I* [*intro*!]: $(!!x.\ x : A \Longrightarrow B\ x\ b) \Longrightarrow (INF\ x{:}A.\ B\ x)\ b$
by (*auto simp add*: *INF1-iff*)

lemma *INF2-I* [*intro*!]: $(!!x.\ x : A \Longrightarrow B\ x\ b\ c) \Longrightarrow (INF\ x{:}A.\ B\ x)\ b\ c$
by (*auto simp add*: *INF2-iff*)

lemma *INF1-D* [*elim*]: $(INF\ x{:}A.\ B\ x)\ b \Longrightarrow a : A \Longrightarrow B\ a\ b$
by (*auto simp add*: *INF1-iff*)

lemma *INF2-D* [*elim*]: $(INF\ x{:}A.\ B\ x)\ b\ c \Longrightarrow a : A \Longrightarrow B\ a\ b\ c$
by (*auto simp add*: *INF2-iff*)

lemma *INF1-E* [*elim*]: $(INF\ x{:}A.\ B\ x)\ b \Longrightarrow (B\ a\ b \Longrightarrow R) \Longrightarrow (a \sim: A \Longrightarrow R) \Longrightarrow R$
by (*auto simp add*: *INF1-iff*)

lemma *INF2-E* [*elim*]: $(INF\ x{:}A.\ B\ x)\ b\ c \Longrightarrow (B\ a\ b\ c \Longrightarrow R) \Longrightarrow (a \sim: A \Longrightarrow R) \Longrightarrow R$

by (*auto simp add*: *INF2-iff*)

lemma *INF-INT-eq*: (*INF i*. (λ*x*. *x* ∈ *r i*)) = (λ*x*. *x* ∈ (*INT i*. *r i*))
by (*simp add*: *INF1-iff expand-fun-eq*)

lemma *INF-INT-eq2*: (*INF i*. (λ*x y*. (*x*, *y*) ∈ *r i*)) = (λ*x y*. (*x*, *y*) ∈ (*INT i*. *r i*))
by (*simp add*: *INF2-iff expand-fun-eq*)

22.2 Predicates as relations

22.2.1 Composition

inductive
pred-comp :: ['*a* => '*b* => *bool*, '*b* => '*c* => *bool*] => '*a* => '*c* => *bool*
(**infixr** *OO* 75)
for *r* :: '*a* => '*b* => *bool* **and** *s* :: '*b* => '*c* => *bool*
where
pred-compI [*intro*]: *r a b* ==> *s b c* ==> (*r OO s*) *a c*

inductive-cases *pred-compE* [*elim*!]: (*r OO s*) *a c*

lemma *pred-comp-rel-comp-eq* [*pred-set-conv*]:
((λ*x y*. (*x*, *y*) ∈ *r*) *OO* (λ*x y*. (*x*, *y*) ∈ *s*)) = (λ*x y*. (*x*, *y*) ∈ *r O s*)
by (*auto simp add*: *expand-fun-eq elim*: *pred-compE*)

22.2.2 Converse

inductive
conversep :: ('*a* => '*b* => *bool*) => '*b* => '*a* => *bool*
((-^--1) [*1000*] *1000*)
for *r* :: '*a* => '*b* => *bool*
where
conversepI: *r a b* ==> *r*^--1 *b a*

notation (*xsymbols*)
conversep ((-^-1-1) [*1000*] *1000*)

lemma *conversepD*:
assumes *ab*: *r*^--1 *a b*
shows *r b a* **using** *ab*
by *cases simp*

lemma *conversep-iff* [*iff*]: *r*^--1 *a b* = *r b a*
by (*iprover intro*: *conversepI dest*: *conversepD*)

lemma *conversep-converse-eq* [*pred-set-conv*]:
(λ*x y*. (*x*, *y*) ∈ *r*)^--1 = (λ*x y*. (*x*, *y*) ∈ *r*^-1)
by (*auto simp add*: *expand-fun-eq*)

lemma *conversep-conversep* [*simp*]: (*r*^--1)^--1 = *r*

by (*iprover intro*: *order-antisym conversepI dest*: *conversepD*)

lemma *converse-pred-comp*: (*r OO s*) ^−−1 = *s* ^−−1 *OO r* ^−−1
by (*iprover intro*: *order-antisym conversepI pred-compI*
elim: *pred-compE dest*: *conversepD*)

lemma *converse-meet*: (*inf r s*) ^−−1 = *inf r* ^−−1 *s* ^−−1
by (*simp add*: *inf-fun-eq inf-bool-eq*)
(*iprover intro*: *conversepI ext dest*: *conversepD*)

lemma *converse-join*: (*sup r s*) ^−−1 = *sup r* ^−−1 *s* ^−−1
by (*simp add*: *sup-fun-eq sup-bool-eq*)
(*iprover intro*: *conversepI ext dest*: *conversepD*)

lemma *conversep-noteq* [*simp*]: (*op* ~=) ^−−1 = *op* ~=
by (*auto simp add*: *expand-fun-eq*)

lemma *conversep-eq* [*simp*]: (*op* =) ^−−1 = *op* =
by (*auto simp add*: *expand-fun-eq*)

22.2.3 Domain

inductive
DomainP :: ('*a* => '*b* => *bool*) => '*a* => *bool*
for *r* :: '*a* => '*b* => *bool*
where
DomainPI [*intro*]: *r a b* ==> *DomainP r a*

inductive-cases *DomainPE* [*elim*!]: *DomainP r a*

lemma *DomainP-Domain-eq* [*pred-set-conv*]: *DomainP* (λ*x y*. (*x*, *y*) ∈ *r*) = (λ*x*. *x* ∈ *Domain r*)
by (*blast intro*!: *Orderings.order-antisym predicate1I*)

22.2.4 Range

inductive
RangeP :: ('*a* => '*b* => *bool*) => '*b* => *bool*
for *r* :: '*a* => '*b* => *bool*
where
RangePI [*intro*]: *r a b* ==> *RangeP r b*

inductive-cases *RangePE* [*elim*!]: *RangeP r b*

lemma *RangeP-Range-eq* [*pred-set-conv*]: *RangeP* (λ*x y*. (*x*, *y*) ∈ *r*) = (λ*x*. *x* ∈ *Range r*)
by (*blast intro*!: *Orderings.order-antisym predicate1I*)

22.2.5 Inverse image

definition
inv-imagep :: ('*b* => '*b* => *bool*) => ('*a* => '*b*) => '*a* => '*a* => *bool* **where**
inv-imagep r f == %*x y*. *r* (*f x*) (*f y*)

lemma [*pred-set-conv*]: *inv-imagep* (λ*x y*. (*x*, *y*) ∈ *r*) *f* = (λ*x y*. (*x*, *y*) ∈ *inv-image r f*)
by (*simp add*: *inv-image-def inv-imagep-def*)

lemma *in-inv-imagep* [*simp*]: *inv-imagep r f x y* = *r* (*f x*) (*f y*)
by (*simp add*: *inv-imagep-def*)

22.2.6 Powerset

definition *Powp* :: ('*a* ⇒ *bool*) ⇒ '*a set* ⇒ *bool* **where**
Powp A == λ*B*. ∀ *x* ∈ *B*. *A x*

lemma *Powp-Pow-eq* [*pred-set-conv*]: *Powp* (λ*x*. *x* ∈ *A*) = (λ*x*. *x* ∈ *Pow A*)
by (*auto simp add*: *Powp-def expand-fun-eq*)

lemmas *Powp-mono* [*mono*] = *Pow-mono* [*to-pred pred-subset-eq*]

22.2.7 Properties of relations

abbreviation *antisymP* :: ('*a* => '*a* => *bool*) => *bool* **where**
antisymP r == *antisym* {(*x*, *y*). *r x y*}

abbreviation *transP* :: ('*a* => '*a* => *bool*) => *bool* **where**
transP r == *trans* {(*x*, *y*). *r x y*}

abbreviation *single-valuedP* :: ('*a* => '*b* => *bool*) => *bool* **where**
single-valuedP r == *single-valued* {(*x*, *y*). *r x y*}

22.3 Predicates as enumerations

22.3.1 The type of predicate enumerations (a monad)

datatype '*a pred* = *Pred* '*a* ⇒ *bool*

primrec *eval* :: '*a pred* ⇒ '*a* ⇒ *bool* **where**
eval-pred: *eval* (*Pred f*) = *f*

lemma *Pred-eval* [*simp*]:
Pred (*eval x*) = *x*
by (*cases x*) *simp*

lemma *eval-inject*: *eval x* = *eval y* ⟷ *x* = *y*
by (*cases x*) *auto*

definition *single* :: '*a* ⇒ '*a pred* **where**

single x = Pred ((op =) x)

definition *bind* :: *'a pred ⇒ ('a ⇒ 'b pred) ⇒ 'b pred* (**infixl** *⤜* *70*) **where**
P ⤜ f = Pred (λx. (∃ y. eval P y ∧ eval (f y) x))

instantiation *pred* :: (*type*) {*complete-lattice*, *boolean-algebra*}
begin

definition
P ≤ Q ⟷ eval P ≤ eval Q

definition
P < Q ⟷ eval P < eval Q

definition
⊥ = Pred ⊥

definition
⊤ = Pred ⊤

definition
P ⊓ Q = Pred (eval P ⊓ eval Q)

definition
P ⊔ Q = Pred (eval P ⊔ eval Q)

definition
[*code del*]: *⨅ A = Pred (INFI A eval)*

definition
[*code del*]: *⨆ A = Pred (SUPR A eval)*

definition
- P = Pred (- eval P)

definition
P - Q = Pred (eval P - eval Q)

instance proof
qed (*auto simp add*: *less-eq-pred-def less-pred-def*
inf-pred-def sup-pred-def bot-pred-def top-pred-def
Inf-pred-def Sup-pred-def uminus-pred-def minus-pred-def fun-Compl-def bool-Compl-def,
auto simp add: *le-fun-def less-fun-def le-bool-def less-bool-def*
eval-inject mem-def)

end

lemma *bind-bind*:
(P ⤜ Q) ⤜ R = P ⤜ (λx. Q x ⤜ R)

by (*auto simp add*: *bind-def expand-fun-eq*)

lemma *bind-single*:
$P \gg= single = P$
by (*simp add*: *bind-def single-def*)

lemma *single-bind*:
$single\ x \gg= P = P\ x$
by (*simp add*: *bind-def single-def*)

lemma *bottom-bind*:
$\bot \gg= P = \bot$
by (*auto simp add*: *bot-pred-def bind-def expand-fun-eq*)

lemma *sup-bind*:
$(P \sqcup Q) \gg= R = P \gg= R \sqcup Q \gg= R$
by (*auto simp add*: *bind-def sup-pred-def expand-fun-eq*)

lemma *Sup-bind*: $(\bigsqcup A \gg= f) = \bigsqcup ((\lambda x.\ x \gg= f)\ `\ A)$
by (*auto simp add*: *bind-def Sup-pred-def SUP1-iff expand-fun-eq*)

lemma *pred-iffI*:
assumes $\bigwedge x.\ eval\ A\ x \Longrightarrow eval\ B\ x$
and $\bigwedge x.\ eval\ B\ x \Longrightarrow eval\ A\ x$
shows $A = B$
proof −
from *assms* **have** $\bigwedge x.\ eval\ A\ x \longleftrightarrow eval\ B\ x$ **by** *blast*
then show *?thesis* **by** (*cases A*, *cases B*) (*simp add*: *expand-fun-eq*)
qed

lemma *singleI*: $eval\ (single\ x)\ x$
unfolding *single-def* **by** *simp*

lemma *singleI-unit*: $eval\ (single\ ())\ x$
by *simp* (*rule singleI*)

lemma *singleE*: $eval\ (single\ x)\ y \Longrightarrow (y = x \Longrightarrow P) \Longrightarrow P$
unfolding *single-def* **by** *simp*

lemma *singleE'*: $eval\ (single\ x)\ y \Longrightarrow (x = y \Longrightarrow P) \Longrightarrow P$
by (*erule singleE*) *simp*

lemma *bindI*: $eval\ P\ x \Longrightarrow eval\ (Q\ x)\ y \Longrightarrow eval\ (P \gg= Q)\ y$
unfolding *bind-def* **by** *auto*

lemma *bindE*: $eval\ (R \gg= Q)\ y \Longrightarrow (\bigwedge x.\ eval\ R\ x \Longrightarrow eval\ (Q\ x)\ y \Longrightarrow P) \Longrightarrow P$
unfolding *bind-def* **by** *auto*

lemma *botE*: *eval* $\bot$ *x* $\Longrightarrow$ *P*
unfolding *bot-pred-def* **by** *auto*

lemma *supI1*: *eval A x* $\Longrightarrow$ *eval* (*A* $\sqcup$ *B*) *x*
unfolding *sup-pred-def* **by** (*simp add*: *sup-fun-eq sup-bool-eq*)

lemma *supI2*: *eval B x* $\Longrightarrow$ *eval* (*A* $\sqcup$ *B*) *x*
unfolding *sup-pred-def* **by** (*simp add*: *sup-fun-eq sup-bool-eq*)

lemma *supE*: *eval* (*A* $\sqcup$ *B*) *x* $\Longrightarrow$ (*eval A x* $\Longrightarrow$ *P*) $\Longrightarrow$ (*eval B x* $\Longrightarrow$ *P*) $\Longrightarrow$ *P*
unfolding *sup-pred-def* **by** *auto*

lemma *single-not-bot* [*simp*]:
single x $\neq$ $\bot$
by (*auto simp add*: *single-def bot-pred-def expand-fun-eq*)

lemma *not-bot*:
assumes *A* $\neq$ $\bot$
obtains *x* **where** *eval A x*
using *assms* **by** (*cases A*)
(*auto simp add*: *bot-pred-def*, *auto simp add*: *mem-def*)

22.3.2 Emptiness check and definite choice

definition *is-empty* :: *'a pred* $\Rightarrow$ *bool* **where**
is-empty A $\longleftrightarrow$ *A* = $\bot$

lemma *is-empty-bot*:
is-empty $\bot$
by (*simp add*: *is-empty-def*)

lemma *not-is-empty-single*:
$\neg$ *is-empty* (*single x*)
by (*auto simp add*: *is-empty-def single-def bot-pred-def expand-fun-eq*)

lemma *is-empty-sup*:
is-empty (*A* $\sqcup$ *B*) $\longleftrightarrow$ *is-empty A* $\wedge$ *is-empty B*
by (*auto simp add*: *is-empty-def*)

definition *singleton* :: (*unit* => *'a*) $\Rightarrow$ *'a pred* $\Rightarrow$ *'a* **where**
singleton dfault A = (*if* $\exists!x$. *eval A x then THE x*. *eval A x else dfault* ())

lemma *singleton-eqI*:
$\exists!x$. *eval A x* $\Longrightarrow$ *eval A x* $\Longrightarrow$ *singleton dfault A* = *x*
by (*auto simp add*: *singleton-def*)

lemma *eval-singletonI*:
$\exists!x$. *eval A x* $\Longrightarrow$ *eval A* (*singleton dfault A*)
proof −

assume *assm*: $\exists!x.\ eval\ A\ x$
then obtain x **where** *eval A x* **..**
moreover with *assm* **have** *singleton dfault A* $= x$ **by** (*rule singleton-eqI*)
ultimately show *?thesis* **by** *simp*
qed

lemma *single-singleton*:
$\exists!x.\ eval\ A\ x \Longrightarrow single\ (singleton\ dfault\ A) = A$
proof −
assume *assm*: $\exists!x.\ eval\ A\ x$
then have *eval A* (*singleton dfault A*)
by (*rule eval-singletonI*)
moreover from *assm* **have** $\bigwedge x.\ eval\ A\ x \Longrightarrow singleton\ dfault\ A = x$
by (*rule singleton-eqI*)
ultimately have *eval* (*single* (*singleton dfault A*)) = *eval A*
by (*simp* (*no-asm-use*) *add*: *single-def expand-fun-eq*) *blast*
then show *?thesis* **by** (*simp add*: *eval-inject*)
qed

lemma *singleton-undefinedI*:
$\neg\ (\exists!x.\ eval\ A\ x) \Longrightarrow singleton\ dfault\ A = dfault\ ()$
by (*simp add*: *singleton-def*)

lemma *singleton-bot*:
singleton dfault $\bot$ = *dfault* ()
by (*auto simp add*: *bot-pred-def intro*: *singleton-undefinedI*)

lemma *singleton-single*:
singleton dfault (*single x*) = *x*
by (*auto simp add*: *intro*: *singleton-eqI singleI elim*: *singleE*)

lemma *singleton-sup-single-single*:
singleton dfault (*single x* $\sqcup$ *single y*) = (*if x* = *y then x else dfault* ())
proof (*cases x* = *y*)
case *True* **then show** *?thesis* **by** (*simp add*: *singleton-single*)
next
case *False*
have *eval* (*single x* $\sqcup$ *single y*) *x*
and *eval* (*single x* $\sqcup$ *single y*) *y*
by (*auto intro*: *supI1 supI2 singleI*)
with *False* **have** $\neg\ (\exists!z.\ eval\ (single\ x \sqcup single\ y)\ z)$
by *blast*
then have *singleton dfault* (*single x* $\sqcup$ *single y*) = *dfault* ()
by (*rule singleton-undefinedI*)
with *False* **show** *?thesis* **by** *simp*
qed

lemma *singleton-sup-aux*:
singleton dfault (*A* $\sqcup$ *B*) = (*if A* = $\bot$ *then singleton dfault B*

```
    else if B = ⊥ then singleton dfault A
    else singleton dfault
      (single (singleton dfault A) ⊔ single (singleton dfault B)))
proof (cases (∃!x. eval A x) ∧ (∃!y. eval B y))
  case True then show ?thesis by (simp add: single-singleton)
next
  case False
  from False have A-or-B:
    singleton dfault A = dfault () ∨ singleton dfault B = dfault ()
    by (auto intro!: singleton-undefinedI)
  then have rhs: singleton dfault
    (single (singleton dfault A) ⊔ single (singleton dfault B)) = dfault ()
    by (auto simp add: singleton-sup-single-single singleton-single)
  from False have not-unique:
    ¬ (∃!x. eval A x) ∨ ¬ (∃!y. eval B y) by simp
  show ?thesis proof (cases A ≠ ⊥ ∧ B ≠ ⊥)
    case True
    then obtain a b where a: eval A a and b: eval B b
      by (blast elim: not-bot)
    with True not-unique have ¬ (∃!x. eval (A ⊔ B) x)
      by (auto simp add: sup-pred-def bot-pred-def)
    then have singleton dfault (A ⊔ B) = dfault () by (rule singleton-undefinedI)
    with True rhs show ?thesis by simp
  next
    case False then show ?thesis by auto
  qed
qed

lemma singleton-sup:
  singleton dfault (A ⊔ B) = (if A = ⊥ then singleton dfault B
    else if B = ⊥ then singleton dfault A
    else if singleton dfault A = singleton dfault B then singleton dfault A else dfault
())
using singleton-sup-aux [of dfault A B] by (simp only: singleton-sup-single-single)
```

22.3.3 Derived operations

```
definition if-pred :: bool ⇒ unit pred where
  if-pred-eq: if-pred b = (if b then single () else ⊥)

definition holds :: unit pred ⇒ bool where
  holds-eq: holds P = eval P ()

definition not-pred :: unit pred ⇒ unit pred where
  not-pred-eq: not-pred P = (if eval P () then ⊥ else single ())

lemma if-predI: P ⟹ eval (if-pred P) ()
  unfolding if-pred-eq by (auto intro: singleI)
```

lemma *if-predE*: *eval* (*if-pred b*) *x* ⟹ (*b* ⟹ *x* = () ⟹ *P*) ⟹ *P*
unfolding *if-pred-eq* **by** (*cases b*) (*auto elim*: *botE*)

lemma *not-predI*: ¬ *P* ⟹ *eval* (*not-pred* (*Pred* (λ*u*. *P*))) ()
unfolding *not-pred-eq eval-pred* **by** (*auto intro*: *singleI*)

lemma *not-predI'*: ¬ *eval P* () ⟹ *eval* (*not-pred P*) ()
unfolding *not-pred-eq* **by** (*auto intro*: *singleI*)

lemma *not-predE*: *eval* (*not-pred* (*Pred* (λ*u*. *P*))) *x* ⟹ (¬ *P* ⟹ *thesis*) ⟹ *thesis*
unfolding *not-pred-eq*
by (*auto split*: *split-if-asm elim*: *botE*)

lemma *not-predE'*: *eval* (*not-pred P*) *x* ⟹ (¬ *eval P x* ⟹ *thesis*) ⟹ *thesis*
unfolding *not-pred-eq*
by (*auto split*: *split-if-asm elim*: *botE*)
lemma *f* () = *False* ∨ *f* () = *True*
by *simp*

lemma *closure-of-bool-cases*:
assumes (*f* :: *unit* ⇒ *bool*) = (%*u*. *False*) ⟹ *P f*
assumes *f* = (%*u*. *True*) ⟹ *P f*
shows *P f*
proof −
have *f* = (%*u*. *False*) ∨ *f* = (%*u*. *True*)
apply (*cases f* ())
apply (*rule disjI2*)
apply (*rule ext*)
apply (*simp add*: *unit-eq*)
apply (*rule disjI1*)
apply (*rule ext*)
apply (*simp add*: *unit-eq*)
done
from *this prems* **show** *?thesis* **by** *blast*
qed

lemma *unit-pred-cases*:
assumes *P* ⊥
assumes *P* (*single* ())
shows *P Q*
using *assms*
unfolding *bot-pred-def Collect-def empty-def single-def*
apply (*cases Q*)
apply *simp*
apply (*rule-tac f*=*fun* **in** *closure-of-bool-cases*)
apply *auto*
apply (*subgoal-tac* (%*x*. () = *x*) = (%*x*. *True*))
apply *auto*

done

lemma *holds-if-pred*:
holds (*if-pred b*) = *b*
unfolding *if-pred-eq holds-eq*
by (*cases b*) (*auto intro*: *singleI elim*: *botE*)

lemma *if-pred-holds*:
if-pred (*holds P*) = *P*
unfolding *if-pred-eq holds-eq*
by (*rule unit-pred-cases*) (*auto intro*: *singleI elim*: *botE*)

lemma *is-empty-holds*:
is-empty P ⟷ ¬ *holds P*
unfolding *is-empty-def holds-eq*
by (*rule unit-pred-cases*) (*auto elim*: *botE intro*: *singleI*)

22.3.4 Implementation

datatype *'a seq* = *Empty* | *Insert 'a 'a pred* | *Join 'a pred 'a seq*

primrec *pred-of-seq* :: *'a seq* ⇒ *'a pred* **where**
pred-of-seq Empty = ⊥
| *pred-of-seq* (*Insert x P*) = *single x* ⊔ *P*
| *pred-of-seq* (*Join P xq*) = *P* ⊔ *pred-of-seq xq*

definition *Seq* :: (*unit* ⇒ *'a seq*) ⇒ *'a pred* **where**
Seq f = *pred-of-seq* (*f* ())

code-datatype *Seq*

primrec *member* :: *'a seq* ⇒ *'a* ⇒ *bool* **where**
member Empty x ⟷ *False*
| *member* (*Insert y P*) *x* ⟷ *x* = *y* ∨ *eval P x*
| *member* (*Join P xq*) *x* ⟷ *eval P x* ∨ *member xq x*

lemma *eval-member*:
member xq = *eval* (*pred-of-seq xq*)
proof (*induct xq*)
case *Empty* **show** *?case*
by (*auto simp add*: *expand-fun-eq elim*: *botE*)
next
case *Insert* **show** *?case*
by (*auto simp add*: *expand-fun-eq elim*: *supE singleE intro*: *supI1 supI2 singleI*)
next
case *Join* **then show** *?case*
by (*auto simp add*: *expand-fun-eq elim*: *supE intro*: *supI1 supI2*)
qed

lemma *eval-code* [*code*]: *eval* (*Seq f*) = *member* (*f* ())
unfolding *Seq-def* **by** (*rule sym*, *rule eval-member*)

lemma *single-code* [*code*]:
single x = *Seq* (λ*u*. *Insert x* ⊥)
unfolding *Seq-def* **by** *simp*

primrec *apply* :: ('*a* ⇒ '*b Predicate.pred*) ⇒ '*a seq* ⇒ '*b seq* **where**
apply f Empty = *Empty*
| *apply f* (*Insert x P*) = *Join* (*f x*) (*Join* (*P* ⤜ *f*) *Empty*)
| *apply f* (*Join P xq*) = *Join* (*P* ⤜ *f*) (*apply f xq*)

lemma *apply-bind*:
pred-of-seq (*apply f xq*) = *pred-of-seq xq* ⤜ *f*
proof (*induct xq*)
case *Empty* **show** *?case*
by (*simp add*: *bottom-bind*)
next
case *Insert* **show** *?case*
by (*simp add*: *single-bind sup-bind*)
next
case *Join* **then show** *?case*
by (*simp add*: *sup-bind*)
qed

lemma *bind-code* [*code*]:
Seq g ⤜ *f* = *Seq* (λ*u*. *apply f* (*g* ()))
unfolding *Seq-def* **by** (*rule sym*, *rule apply-bind*)

lemma *bot-set-code* [*code*]:
⊥ = *Seq* (λ*u*. *Empty*)
unfolding *Seq-def* **by** *simp*

primrec *adjunct* :: '*a pred* ⇒ '*a seq* ⇒ '*a seq* **where**
adjunct P Empty = *Join P Empty*
| *adjunct P* (*Insert x Q*) = *Insert x* (*Q* ⊔ *P*)
| *adjunct P* (*Join Q xq*) = *Join Q* (*adjunct P xq*)

lemma *adjunct-sup*:
pred-of-seq (*adjunct P xq*) = *P* ⊔ *pred-of-seq xq*
by (*induct xq*) (*simp-all add*: *sup-assoc sup-commute sup-left-commute*)

lemma *sup-code* [*code*]:
Seq f ⊔ *Seq g* = *Seq* (λ*u*. *case f* ()
of Empty ⇒ *g* ()
| *Insert x P* ⇒ *Insert x* (*P* ⊔ *Seq g*)
| *Join P xq* ⇒ *adjunct* (*Seq g*) (*Join P xq*))
proof (*cases f* ())
case *Empty*

```
  thus ?thesis
    unfolding Seq-def by (simp add: sup-commute [of ⊥])
next
  case Insert
  thus ?thesis
    unfolding Seq-def by (simp add: sup-assoc)
next
  case Join
  thus ?thesis
    unfolding Seq-def
    by (simp add: adjunct-sup sup-assoc sup-commute sup-left-commute)
qed

primrec contained :: 'a seq ⇒ 'a pred ⇒ bool where
    contained Empty Q ⟷ True
  | contained (Insert x P) Q ⟷ eval Q x ∧ P ≤ Q
  | contained (Join P xq) Q ⟷ P ≤ Q ∧ contained xq Q

lemma single-less-eq-eval:
  single x ≤ P ⟷ eval P x
  by (auto simp add: single-def less-eq-pred-def mem-def)

lemma contained-less-eq:
  contained xq Q ⟷ pred-of-seq xq ≤ Q
  by (induct xq) (simp-all add: single-less-eq-eval)

lemma less-eq-pred-code [code]:
  Seq f ≤ Q = (case f ()
   of Empty ⇒ True
    | Insert x P ⇒ eval Q x ∧ P ≤ Q
    | Join P xq ⇒ P ≤ Q ∧ contained xq Q)
  by (cases f ())
    (simp-all add: Seq-def single-less-eq-eval contained-less-eq)

lemma eq-pred-code [code]:
  fixes P Q :: 'a pred
  shows eq-class.eq P Q ⟷ P ≤ Q ∧ Q ≤ P
  unfolding eq by auto

lemma [code]:
  pred-case f P = f (eval P)
  by (cases P) simp

lemma [code]:
  pred-rec f P = f (eval P)
  by (cases P) simp

inductive eq :: 'a ⇒ 'a ⇒ bool where eq x x
```

lemma *eq-is-eq*: *eq x y* ≡ (*x* = *y*)
by (*rule eq-reflection*) (*auto intro*: *eq.intros elim*: *eq.cases*)

definition *map* :: (*'a* ⇒ *'b*) ⇒ *'a pred* ⇒ *'b pred* **where**
map f P = *P* ⤜ (*single* ∘ *f*)

primrec *null* :: *'a seq* ⇒ *bool* **where**
null Empty ⟷ *True*
| *null* (*Insert x P*) ⟷ *False*
| *null* (*Join P xq*) ⟷ *is-empty P* ∧ *null xq*

lemma *null-is-empty*:
null xq ⟷ *is-empty* (*pred-of-seq xq*)
by (*induct xq*) (*simp-all add*: *is-empty-bot not-is-empty-single is-empty-sup*)

lemma *is-empty-code* [*code*]:
is-empty (*Seq f*) ⟷ *null* (*f* ())
by (*simp add*: *null-is-empty Seq-def*)

primrec *the-only* :: (*unit* ⇒ *'a*) ⇒ *'a seq* ⇒ *'a* **where**
[*code del*]: *the-only dfault Empty* = *dfault* ()
| *the-only dfault* (*Insert x P*) = (*if is-empty P then x else let y* = *singleton dfault P in if x* = *y then x else dfault* ())
| *the-only dfault* (*Join P xq*) = (*if is-empty P then the-only dfault xq else if null xq then singleton dfault P*
else let x = *singleton dfault P*; *y* = *the-only dfault xq in*
if x = *y then x else dfault* ())

lemma *the-only-singleton*:
the-only dfault xq = *singleton dfault* (*pred-of-seq xq*)
by (*induct xq*)
(*auto simp add*: *singleton-bot singleton-single is-empty-def*
null-is-empty Let-def singleton-sup)

lemma *singleton-code* [*code*]:
singleton dfault (*Seq f*) = (*case f* ()
of Empty ⇒ *dfault* ()
| *Insert x P* ⇒ *if is-empty P then x*
else let y = *singleton dfault P in*
if x = *y then x else dfault* ()
| *Join P xq* ⇒ *if is-empty P then the-only dfault xq*
else if null xq then singleton dfault P
else let x = *singleton dfault P*; *y* = *the-only dfault xq in*
if x = *y then x else dfault* ())
by (*cases f* ())
(*auto simp add*: *Seq-def the-only-singleton is-empty-def*
null-is-empty singleton-bot singleton-single singleton-sup Let-def)

definition *not-unique* :: *'a pred* => *'a*

where
  [*code del*]: *not-unique A* = (*THE x*. *eval A x*)

definition *the* :: *'a pred* => *'a*
where
  [*code del*]: *the A* = (*THE x*. *eval A x*)

lemma *the-eq*[*code*]: *the A* = *singleton* (λ*x*. *not-unique A*) *A*
by (*auto simp add*: *the-def singleton-def not-unique-def*)

code-abort *not-unique*

code-reflect *Predicate*
  datatypes *pred* = *Seq* **and** *seq* = *Empty* | *Insert* | *Join*
  functions *map*

ML ⟪
```
signature PREDICATE =
sig
  datatype 'a pred = Seq of (unit -> 'a seq)
  and 'a seq = Empty | Insert of 'a * 'a pred | Join of 'a pred * 'a seq
  val yield: 'a pred -> ('a * 'a pred) option
  val yieldn: int -> 'a pred -> 'a list * 'a pred
  val map: ('a -> 'b) -> 'a pred -> 'b pred
end;

structure Predicate : PREDICATE =
struct

datatype pred = datatype Predicate.pred
datatype seq = datatype Predicate.seq

fun map f = Predicate.map f;

fun yield (Seq f) = next (f ())
and next Empty = NONE
  | next (Insert (x, P)) = SOME (x, P)
  | next (Join (P, xq)) = (case yield P
     of NONE => next xq
      | SOME (x, Q) => SOME (x, Seq (fn - => Join (Q, xq))));

fun anamorph f k x = (if k = 0 then ([], x)
  else case f x
   of NONE => ([], x)
    | SOME (v, y) => let
        val (vs, z) = anamorph f (k - 1) y
      in (v :: vs, z) end);

fun yieldn P = anamorph yield P;
```

end;
⟩⟩

no-notation
inf (**infixl** ⊓ *70*) **and**
sup (**infixl** ⊔ *65*) **and**
Inf (⨅- [*900*] *900*) **and**
Sup (⨆- [*900*] *900*) **and**
top (⊤) **and**
bot (⊥) **and**
bind (**infixl** ⤜ *70*)

hide-type (**open**) *pred seq*
hide-const (**open**) *Pred eval single bind is-empty singleton if-pred not-pred holds Empty Insert Join Seq member pred-of-seq apply adjunct null the-only eq map not-unique the*

end

23 Transitive-Closure: Reflexive and Transitive closure of a relation

theory *Transitive-Closure*
imports *Predicate*
uses ~~/*src/Provers/trancl.ML*
begin

rtrancl is reflexive/transitive closure, *trancl* is transitive closure, *reflcl* is reflexive closure.

These postfix operators have *maximum priority*, forcing their operands to be atomic.

inductive-set
rtrancl :: ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set* ((-^*) [*1000*] *999*)
for *r* :: ('*a* × '*a*) *set*
where
rtrancl-refl [*intro*!, *Pure.intro*!, *simp*]: (*a*, *a*) : *r*^*
| *rtrancl-into-rtrancl* [*Pure.intro*]: (*a*, *b*) : *r*^* ==> (*b*, *c*) : *r* ==> (*a*, *c*) : *r*^*

inductive-set
trancl :: ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set* ((-^+) [*1000*] *999*)
for *r* :: ('*a* × '*a*) *set*
where
r-into-trancl [*intro*, *Pure.intro*]: (*a*, *b*) : *r* ==> (*a*, *b*) : *r*^+
| *trancl-into-trancl* [*Pure.intro*]: (*a*, *b*) : *r*^+ ==> (*b*, *c*) : *r* ==> (*a*, *c*) : *r*^+

declare *rtrancl-def* [*nitpick-def del*]

rtranclp-def [*nitpick-def del*]
trancl-def [*nitpick-def del*]
tranclp-def [*nitpick-def del*]

notation
rtranclp ((-^**) [*1000*] *1000*) **and**
tranclp ((-^++) [*1000*] *1000*)

abbreviation
reflclp :: ('*a* => '*a* => *bool*) => '*a* => '*a* => *bool* ((-^==) [*1000*] *1000*)
where
r^== == *sup r op* =

abbreviation
reflcl :: ('*a* × '*a*) *set* => ('*a* × '*a*) *set* ((-^=) [*1000*] *999*) **where**
r^= == *r* ∪ *Id*

notation (*xsymbols*)
rtranclp ((-**) [*1000*] *1000*) **and**
tranclp ((-++) [*1000*] *1000*) **and**
reflclp ((-==) [*1000*] *1000*) **and**
rtrancl ((-*) [*1000*] *999*) **and**
trancl ((-+) [*1000*] *999*) **and**
reflcl ((-=) [*1000*] *999*)

notation (*HTML* **output**)
rtranclp ((-**) [*1000*] *1000*) **and**
tranclp ((-++) [*1000*] *1000*) **and**
reflclp ((-==) [*1000*] *1000*) **and**
rtrancl ((-*) [*1000*] *999*) **and**
trancl ((-+) [*1000*] *999*) **and**
reflcl ((-=) [*1000*] *999*)

23.1 Reflexive closure

lemma *refl-reflcl*[*simp*]: *refl*(*r*^=)
by(*simp add*:*refl-on-def*)

lemma *antisym-reflcl*[*simp*]: *antisym*(*r*^=) = *antisym r*
by(*simp add*:*antisym-def*)

lemma *trans-reflclI*[*simp*]: *trans r* ⟹ *trans*(*r*^=)
unfolding *trans-def* **by** *blast*

23.2 Reflexive-transitive closure

lemma *reflcl-set-eq* [*pred-set-conv*]: (*sup* (λ*x y*. (*x*, *y*) ∈ *r*) *op* =) = (λ*x y*. (*x*, *y*) ∈ *r* ∪ *Id*)
by (*auto simp add*: *expand-fun-eq*)

lemma *r-into-rtrancl* [*intro*]: !!*p*. *p* ∈ *r* ==> *p* ∈ *r*^*
— *rtrancl* of *r* contains *r*
apply (*simp only*: *split-tupled-all*)
apply (*erule rtrancl-refl* [*THEN rtrancl-into-rtrancl*])
done

lemma *r-into-rtranclp* [*intro*]: *r x y* ==> *r*^** *x y*
— *rtrancl* of *r* contains *r*
by (*erule rtranclp.rtrancl-refl* [*THEN rtranclp.rtrancl-into-rtrancl*])

lemma *rtranclp-mono*: *r* ≤ *s* ==> *r*^** ≤ *s*^**
— monotonicity of *rtrancl*
apply (*rule predicate2I*)
apply (*erule rtranclp.induct*)
apply (*rule-tac* [*2*] *rtranclp.rtrancl-into-rtrancl*, *blast*+)
done

lemmas *rtrancl-mono* = *rtranclp-mono* [*to-set*]

theorem *rtranclp-induct* [*consumes 1*, *case-names base step*, *induct set*: *rtranclp*]:
assumes *a*: *r*^** *a b*
and *cases*: *P a* !!*y z*. [| *r*^** *a y*; *r y z*; *P y* |] ==> *P z*
shows *P b* **using** *a*
by (*induct x*≡*a b*) (*rule cases*)+

lemmas *rtrancl-induct* [*induct set*: *rtrancl*] = *rtranclp-induct* [*to-set*]

lemmas *rtranclp-induct2* =
rtranclp-induct[*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*,
consumes 1, *case-names refl step*]

lemmas *rtrancl-induct2* =
rtrancl-induct[*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*),
consumes 1, *case-names refl step*]

lemma *refl-rtrancl*: *refl* (*r*^*)
by (*unfold refl-on-def*) *fast*

Transitivity of transitive closure.

lemma *trans-rtrancl*: *trans* (*r*^*)
proof (*rule transI*)
fix *x y z*
assume (*x*, *y*) ∈ *r**
assume (*y*, *z*) ∈ *r**
then show (*x*, *z*) ∈ *r**
proof *induct*
case *base*
show (*x*, *y*) ∈ *r** **by** *fact*
next

```
    case (step u v)
    from ‹(x, u) ∈ r*› and ‹(u, v) ∈ r›
    show (x, v) ∈ r* ..
  qed
qed

lemmas rtrancl-trans = trans-rtrancl [THEN transD, standard]

lemma rtranclp-trans:
  assumes xy: r^** x y
  and yz: r^** y z
  shows r^** x z using yz xy
  by induct iprover+

lemma rtranclE [cases set: rtrancl]:
  assumes major: (a::'a, b) : r^*
  obtains
    (base) a = b
  | (step) y where (a, y) : r^* and (y, b) : r
  — elimination of rtrancl – by induction on a special formula
  apply (subgoal-tac (a::'a) = b | (EX y. (a,y) : r^* & (y,b) : r))
   apply (rule-tac [2] major [THEN rtrancl-induct])
    prefer 2 apply blast
   prefer 2 apply blast
  apply (erule asm-rl exE disjE conjE base step)+
  done

lemma rtrancl-Int-subset: [| Id ⊆ s; (r^* ∩ s) O r ⊆ s|] ==> r^* ⊆ s
  apply (rule subsetI)
  apply (rule-tac p=x in PairE, clarify)
  apply (erule rtrancl-induct, auto)
  done

lemma converse-rtranclp-into-rtranclp:
  r a b ⟹ r** b c ⟹ r** a c
  by (rule rtranclp-trans) iprover+

lemmas converse-rtrancl-into-rtrancl = converse-rtranclp-into-rtranclp [to-set]
```

More r^* equations and inclusions.

```
lemma rtranclp-idemp [simp]: (r^**) ^** = r^**
  apply (auto intro!: order-antisym)
  apply (erule rtranclp-induct)
   apply (rule rtranclp.rtrancl-refl)
  apply (blast intro: rtranclp-trans)
  done

lemmas rtrancl-idemp [simp] = rtranclp-idemp [to-set]
```

lemma *rtrancl-idemp-self-comp* [*simp*]: $R\hat{}* \; O \; R\hat{}* = R\hat{}*$
apply (*rule set-ext*)
apply (*simp only*: *split-tupled-all*)
apply (*blast intro*: *rtrancl-trans*)
done

lemma *rtrancl-subset-rtrancl*: $r \subseteq s\hat{}* \Longrightarrow r\hat{}* \subseteq s\hat{}*$
apply (*drule rtrancl-mono*)
apply *simp*
done

lemma *rtranclp-subset*: $R \leq S \Longrightarrow S \leq R\hat{}** \Longrightarrow S\hat{}** = R\hat{}**$
apply (*drule rtranclp-mono*)
apply (*drule rtranclp-mono*)
apply *simp*
done

lemmas *rtrancl-subset* = *rtranclp-subset* [*to-set*]

lemma *rtranclp-sup-rtranclp*: $(sup \; (R\hat{}**) \; (S\hat{}**))\hat{}** = (sup \; R \; S)\hat{}**$
by (*blast intro*!: *rtranclp-subset intro*: *rtranclp-mono* [*THEN predicate2D*])

lemmas *rtrancl-Un-rtrancl* = *rtranclp-sup-rtranclp* [*to-set*]

lemma *rtranclp-reflcl* [*simp*]: $(R\hat{}==)\hat{}** = R\hat{}**$
by (*blast intro*!: *rtranclp-subset*)

lemmas *rtrancl-reflcl* [*simp*] = *rtranclp-reflcl* [*to-set*]

lemma *rtrancl-r-diff-Id*: $(r - Id)\hat{}* = r\hat{}*$
apply (*rule sym*)
apply (*rule rtrancl-subset*, *blast*, *clarify*)
apply (*rename-tac a b*)
apply (*case-tac a = b*)
apply *blast*
apply (*blast intro*!: *r-into-rtrancl*)
done

lemma *rtranclp-r-diff-Id*: $(inf \; r \; op \; \sim=)\hat{}** = r\hat{}**$
apply (*rule sym*)
apply (*rule rtranclp-subset*)
apply *blast+*
done

theorem *rtranclp-converseD*:
assumes *r*: $(r\hat{}--1)\hat{}** \; x \; y$
shows $r\hat{}** \; y \; x$
proof −
from *r* **show** *?thesis*

```
  by induct (iprover intro: rtranclp-trans dest!: conversepD)+
qed

lemmas rtrancl-converseD = rtranclp-converseD [to-set]

theorem rtranclp-converseI:
  assumes r^** y x
  shows (r^--1) ^** x y
  using assms
  by induct (iprover intro: rtranclp-trans conversepI)+

lemmas rtrancl-converseI = rtranclp-converseI [to-set]

lemma rtrancl-converse: (r^-1) ^* = (r^*) ^-1
  by (fast dest!: rtrancl-converseD intro!: rtrancl-converseI)

lemma sym-rtrancl: sym r ==> sym (r^*)
  by (simp only: sym-conv-converse-eq rtrancl-converse [symmetric])

theorem converse-rtranclp-induct [consumes 1, case-names base step]:
  assumes major: r^** a b
    and cases: P b !!y z. [| r y z; r^** z b; P z |] ==> P y
  shows P a
  using rtranclp-converseI [OF major]
  by induct (iprover intro: cases dest!: conversepD rtranclp-converseD)+

lemmas converse-rtrancl-induct = converse-rtranclp-induct [to-set]

lemmas converse-rtranclp-induct2 =
  converse-rtranclp-induct [of - (ax,ay) (bx,by), split-rule,
                consumes 1, case-names refl step]

lemmas converse-rtrancl-induct2 =
  converse-rtrancl-induct [of (ax,ay) (bx,by), split-format (complete),
                consumes 1, case-names refl step]

lemma converse-rtranclpE [consumes 1, case-names base step]:
  assumes major: r^** x z
    and cases: x=z ==> P
      !!y. [| r x y; r^** y z |] ==> P
  shows P
  apply (subgoal-tac x = z | (EX y. r x y & r^** y z))
   apply (rule-tac [2] major [THEN converse-rtranclp-induct])
    prefer 2 apply iprover
   prefer 2 apply iprover
  apply (erule asm-rl exE disjE conjE cases)+
  done

lemmas converse-rtranclE = converse-rtranclpE [to-set]
```

lemmas *converse-rtranclpE2 = converse-rtranclpE* [*of* - (*xa*,*xb*) (*za*,*zb*), *split-rule*]

lemmas *converse-rtranclE2 = converse-rtranclE* [*of* (*xa*,*xb*) (*za*,*zb*), *split-rule*]

lemma *r-comp-rtrancl-eq*: *r O r^* = r^* O r*
by (*blast elim*: *rtranclE converse-rtranclE*
intro: *rtrancl-into-rtrancl converse-rtrancl-into-rtrancl*)

lemma *rtrancl-unfold*: *r^* = Id Un r^* O r*
by (*auto intro*: *rtrancl-into-rtrancl elim*: *rtranclE*)

lemma *rtrancl-Un-separatorE*:
$(a,b) : (P \cup Q)^* \Longrightarrow \forall x\ y.\ (a,x) : P^* \longrightarrow (x,y) : Q \longrightarrow x{=}y \Longrightarrow (a,b) : P^*$
apply (*induct rule*:*rtrancl.induct*)
apply *blast*
apply (*blast intro*:*rtrancl-trans*)
done

lemma *rtrancl-Un-separator-converseE*:
$(a,b) : (P \cup Q)^* \Longrightarrow \forall x\ y.\ (x,b) : P^* \longrightarrow (y,x) : Q \longrightarrow y{=}x \Longrightarrow (a,b) : P^*$
apply (*induct rule*:*converse-rtrancl-induct*)
apply *blast*
apply (*blast intro*:*rtrancl-trans*)
done

lemma *Image-closed-trancl*:
assumes $r\ ``\ X \subseteq X$ **shows** $r^*\ ``\ X = X$
proof −
from *assms* **have** $**$: $\{y.\ \exists x{\in}X.\ (x,\ y) \in r\} \subseteq X$ **by** *auto*
have $\bigwedge x\ y.\ (y,\ x) \in r^* \Longrightarrow y \in X \Longrightarrow x \in X$
proof −
fix *x y*
assume $*$: $y \in X$
assume $(y,\ x) \in r^*$
then show $x \in X$
proof *induct*
case *base* **show** *?case* **by** (*fact* $*$)
next
case *step* **with** $**$ **show** *?case* **by** *auto*
qed
qed
then show *?thesis* **by** *auto*
qed

23.3 Transitive closure

lemma *trancl-mono*: !!*p*. *p* ∈ *r^*+* ==> *r* ⊆ *s* ==> *p* ∈ *s^*+*
apply (*simp add*: *split-tupled-all*)

apply (*erule trancl.induct*)
apply (*iprover dest*: *subsetD*)+
done

lemma *r-into-trancl'*: !!*p*. *p* : *r* ==> *p* : *r* ^+
by (*simp only*: *split-tupled-all*) (*erule r-into-trancl*)

Conversions between *trancl* and *rtrancl*.

lemma *tranclp-into-rtranclp*: *r* ^++ *a b* ==> *r* ^** *a b*
by (*erule tranclp.induct*) *iprover*+

lemmas *trancl-into-rtrancl* = *tranclp-into-rtranclp* [*to-set*]

lemma *rtranclp-into-tranclp1*: **assumes** *r*: *r* ^** *a b*
shows !!*c*. *r b c* ==> *r* ^++ *a c* **using** *r*
by *induct iprover*+

lemmas *rtrancl-into-trancl1* = *rtranclp-into-tranclp1* [*to-set*]

lemma *rtranclp-into-tranclp2*: [| *r a b*; *r* ^** *b c* |] ==> *r* ^++ *a c*
— intro rule from *r* and *rtrancl*
apply (*erule rtranclp.cases*)
apply *iprover*
apply (*rule rtranclp-trans* [*THEN rtranclp-into-tranclp1*])
apply (*simp* | *rule r-into-rtranclp*)+
done

lemmas *rtrancl-into-trancl2* = *rtranclp-into-tranclp2* [*to-set*]

Nice induction rule for *trancl*

lemma *tranclp-induct* [*consumes 1*, *case-names base step*, *induct pred*: *tranclp*]:
assumes *a*: *r* ^++ *a b*
and *cases*: !!*y*. *r a y* ==> *P y*
!!*y z*. *r* ^++ *a y* ==> *r y z* ==> *P y* ==> *P z*
shows *P b* **using** *a*
by (*induct x*≡*a b*) (*iprover intro*: *cases*)+

lemmas *trancl-induct* [*induct set*: *trancl*] = *tranclp-induct* [*to-set*]

lemmas *tranclp-induct2* =
tranclp-induct [*of* - (*ax*,*ay*) (*bx*,*by*), *split-rule*,
consumes 1, *case-names base step*]

lemmas *trancl-induct2* =
trancl-induct [*of* (*ax*,*ay*) (*bx*,*by*), *split-format* (*complete*),
consumes 1, *case-names base step*]

lemma *tranclp-trans-induct*:
assumes *major*: *r* ^++ *x y*

and *cases*: !!$x\ y$. $r\ x\ y$ ==> $P\ x\ y$
!!$x\ y\ z$. [| $r\hat{}{+}{+}\ x\ y$; $P\ x\ y$; $r\hat{}{+}{+}\ y\ z$; $P\ y\ z$ |] ==> $P\ x\ z$
shows $P\ x\ y$
— Another induction rule for trancl, incorporating transitivity
by (*iprover intro*: *major* [*THEN tranclp-induct*] *cases*)

lemmas *trancl-trans-induct* = *tranclp-trans-induct* [*to-set*]

lemma *tranclE* [*cases set*: *trancl*]:
assumes $(a,\ b) : r\hat{}{+}$
obtains
(*base*) $(a,\ b) : r$
| (*step*) c **where** $(a,\ c) : r\hat{}{+}$ **and** $(c,\ b) : r$
using *assms* **by** *cases simp-all*

lemma *trancl-Int-subset*: [| $r \subseteq s$; $(r\hat{}{+} \cap s)\ O\ r \subseteq s$ |] ==> $r\hat{}{+} \subseteq s$
apply (*rule subsetI*)
apply (*rule-tac* $p = x$ **in** *PairE*)
apply *clarify*
apply (*erule trancl-induct*)
apply *auto*
done

lemma *trancl-unfold*: $r\hat{}{+} = r\ Un\ r\hat{}{+}\ O\ r$
by (*auto intro*: *trancl-into-trancl elim*: *tranclE*)

Transitivity of r^+

lemma *trans-trancl* [*simp*]: *trans* $(r\hat{}{+})$
proof (*rule transI*)
fix $x\ y\ z$
assume $(x,\ y) \in r\hat{}{+}$
assume $(y,\ z) \in r\hat{}{+}$
then show $(x,\ z) \in r\hat{}{+}$
proof *induct*
case (*base u*)
from ‹$(x,\ y) \in r\hat{}{+}$› **and** ‹$(y,\ u) \in r$›
show $(x,\ u) \in r\hat{}{+}$ **..**
next
case (*step u v*)
from ‹$(x,\ u) \in r\hat{}{+}$› **and** ‹$(u,\ v) \in r$›
show $(x,\ v) \in r\hat{}{+}$ **..**
qed
qed

lemmas *trancl-trans* = *trans-trancl* [*THEN transD*, *standard*]

lemma *tranclp-trans*:
assumes *xy*: $r\hat{}{+}{+}\ x\ y$
and *yz*: $r\hat{}{+}{+}\ y\ z$

shows *r^++ x z* **using** *yz xy*
by *induct iprover+*

lemma *trancl-id* [*simp*]: *trans r ⟹ r^+ = r*
apply *auto*
apply (*erule trancl-induct*)
apply *assumption*
apply (*unfold trans-def*)
apply *blast*
done

lemma *rtranclp-tranclp-tranclp*:
assumes *r^** x y*
shows *!!z. r^++ y z ==> r^++ x z* **using** *assms*
by *induct* (*iprover intro*: *tranclp-trans*)+

lemmas *rtrancl-trancl-trancl = rtranclp-tranclp-tranclp* [*to-set*]

lemma *tranclp-into-tranclp2*: *r a b ==> r^++ b c ==> r^++ a c*
by (*erule tranclp-trans* [*OF tranclp.r-into-trancl*])

lemmas *trancl-into-trancl2 = tranclp-into-tranclp2* [*to-set*]

lemma *trancl-insert*:
(*insert* (*y*, *x*) *r*) *^+ = r^+ ∪ {(a, b). (a, y) ∈ r^* ∧ (x, b) ∈ r^**}
— primitive recursion for *trancl* over finite relations
apply (*rule equalityI*)
apply (*rule subsetI*)
apply (*simp only*: *split-tupled-all*)
apply (*erule trancl-induct*, *blast*)
apply (*blast intro*: *rtrancl-into-trancl1 trancl-into-rtrancl trancl-trans*)
apply (*rule subsetI*)
apply (*blast intro*: *trancl-mono rtrancl-mono*
[*THEN* [*2*] *rev-subsetD*] *rtrancl-trancl-trancl rtrancl-into-trancl2*)
done

lemma *tranclp-converseI*: *(r^++) ^--1 x y ==> (r^--1) ^++ x y*
apply (*drule conversepD*)
apply (*erule tranclp-induct*)
apply (*iprover intro*: *conversepI tranclp-trans*)+
done

lemmas *trancl-converseI = tranclp-converseI* [*to-set*]

lemma *tranclp-converseD*: *(r^--1) ^++ x y ==> (r^++) ^--1 x y*
apply (*rule conversepI*)
apply (*erule tranclp-induct*)
apply (*iprover dest*: *conversepD intro*: *tranclp-trans*)+
done

lemmas *trancl-converseD* = *tranclp-converseD* [*to-set*]

lemma *tranclp-converse*: $(r\hat{}--1)\hat{}++ = (r\hat{}++)\hat{}--1$
by (*fastsimp simp add*: *expand-fun-eq*
intro!: *tranclp-converseI dest*!: *tranclp-converseD*)

lemmas *trancl-converse* = *tranclp-converse* [*to-set*]

lemma *sym-trancl*: *sym r* ==> *sym* $(r\hat{}+)$
by (*simp only*: *sym-conv-converse-eq trancl-converse* [*symmetric*])

lemma *converse-tranclp-induct* [*consumes 1*, *case-names base step*]:
assumes *major*: $r\hat{}++$ *a b*
and *cases*: !!*y*. *r y b* ==> *P*(*y*)
!!*y z*.[| *r y z*; $r\hat{}++$ *z b*; *P*(*z*) |] ==> *P*(*y*)
shows *P a*
apply (*rule tranclp-induct* [*OF tranclp-converseI*, *OF conversepI*, *OF major*])
apply (*rule cases*)
apply (*erule conversepD*)
apply (*blast intro*: *assms dest*!: *tranclp-converseD*)
done

lemmas *converse-trancl-induct* = *converse-tranclp-induct* [*to-set*]

lemma *tranclpD*: $R\hat{}++$ *x y* ==> *EX z*. *R x z* ∧ $R\hat{}**$ *z y*
apply (*erule converse-tranclp-induct*)
apply *auto*
apply (*blast intro*: *rtranclp-trans*)
done

lemmas *tranclD* = *tranclpD* [*to-set*]

lemma *converse-tranclpE*:
assumes *major*: *tranclp r x z*
assumes *base*: *r x z* ==> *P*
assumes *step*: ⋀ *y*. [| *r x y*; *tranclp r y z* |] ==> *P*
shows *P*
proof −
from *tranclpD*[*OF major*]
obtain *y* **where** *r x y* **and** *rtranclp r y z* **by** *iprover*
from *this*(*2*) **show** *P*
proof (*cases rule*: *rtranclp.cases*)
case *rtrancl-refl*
with ‹*r x y*› *base* **show** *P* **by** *iprover*
next
case *rtrancl-into-rtrancl*
from *this* **have** *tranclp r y z*
by (*iprover intro*: *rtranclp-into-tranclp1*)

```
    with ‹r x y› step show P by iprover
  qed
qed

lemmas converse-tranclE = converse-tranclpE [to-set]

lemma tranclD2:
  (x, y) ∈ R^+ ⟹ ∃ z. (x, z) ∈ R^* ∧ (z, y) ∈ R
  by (blast elim: tranclE intro: trancl-into-rtrancl)

lemma irrefl-tranclI: r^-1 ∩ r^* = {} ==> (x, x) ∉ r^+
  by (blast elim: tranclE dest: trancl-into-rtrancl)

lemma irrefl-trancl-rD: !!X. ALL x. (x, x) ∉ r^+ ==> (x, y) ∈ r ==> x ≠ y
  by (blast dest: r-into-trancl)

lemma trancl-subset-Sigma-aux:
    (a, b) ∈ r^* ==> r ⊆ A × A ==> a = b ∨ a ∈ A
  by (induct rule: rtrancl-induct) auto

lemma trancl-subset-Sigma: r ⊆ A × A ==> r^+ ⊆ A × A
  apply (rule subsetI)
  apply (simp only: split-tupled-all)
  apply (erule tranclE)
   apply (blast dest!: trancl-into-rtrancl trancl-subset-Sigma-aux)+
  done

lemma reflcl-tranclp [simp]: (r^++)^== = r^**
  apply (safe intro!: order-antisym)
   apply (erule tranclp-into-rtranclp)
  apply (blast elim: rtranclp.cases dest: rtranclp-into-tranclp1)
  done

lemmas reflcl-trancl [simp] = reflcl-tranclp [to-set]

lemma trancl-reflcl [simp]: (r^=)^+ = r^*
  apply safe
   apply (drule trancl-into-rtrancl, simp)
  apply (erule rtranclE, safe)
   apply (rule r-into-trancl, simp)
  apply (rule rtrancl-into-trancl1)
   apply (erule rtrancl-reflcl [THEN equalityD2, THEN subsetD], fast)
  done

lemma trancl-empty [simp]: {}^+ = {}
  by (auto elim: trancl-induct)

lemma rtrancl-empty [simp]: {}^* = Id
  by (rule subst [OF reflcl-trancl]) simp
```

lemma *rtranclpD*: $R\hat{}** a\ b ==> a = b \vee a \neq b \wedge R\hat{}++ a\ b$
by (*force simp add*: *reflcl-tranclp* [*symmetric*] *simp del*: *reflcl-tranclp*)

lemmas *rtranclD* = *rtranclpD* [*to-set*]

lemma *rtrancl-eq-or-trancl*:
$(x,y) \in R^* = (x=y \vee x\neq y \wedge (x,y) \in R^+)$
by (*fast elim*: *trancl-into-rtrancl dest*: *rtranclD*)

lemma *trancl-unfold-right*: $r\hat{}+ = r\hat{}* \ O\ r$
by (*auto dest*: *tranclD2 intro*: *rtrancl-into-trancl1*)

lemma *trancl-unfold-left*: $r\hat{}+ = r \ O\ r\hat{}*$
by (*auto dest*: *tranclD intro*: *rtrancl-into-trancl2*)

Simplifying nested closures

lemma *rtrancl-trancl-absorb*[*simp*]: $(R\hat{}*)\hat{}+ = R\hat{}*$
by (*simp add*: *trans-rtrancl*)

lemma *trancl-rtrancl-absorb*[*simp*]: $(R\hat{}+)\hat{}* = R\hat{}*$
by (*subst reflcl-trancl*[*symmetric*]) *simp*

lemma *rtrancl-reflcl-absorb*[*simp*]: $(R\hat{}*)\hat{}= = R\hat{}*$
by *auto*

Domain and *Range*

lemma *Domain-rtrancl* [*simp*]: *Domain* $(R\hat{}*) = UNIV$
by *blast*

lemma *Range-rtrancl* [*simp*]: *Range* $(R\hat{}*) = UNIV$
by *blast*

lemma *rtrancl-Un-subset*: $(R\hat{}* \cup S\hat{}*) \subseteq (R\ Un\ S)\hat{}*$
by (*rule rtrancl-Un-rtrancl* [*THEN subst*]) *fast*

lemma *in-rtrancl-UnI*: $x \in R\hat{}* \vee x \in S\hat{}* ==> x \in (R \cup S)\hat{}*$
by (*blast intro*: *subsetD* [*OF rtrancl-Un-subset*])

lemma *trancl-domain* [*simp*]: *Domain* $(r\hat{}+) =$ *Domain* r
by (*unfold Domain-def*) (*blast dest*: *tranclD*)

lemma *trancl-range* [*simp*]: *Range* $(r\hat{}+) =$ *Range* r
unfolding *Range-def* **by**(*simp add*: *trancl-converse* [*symmetric*])

lemma *Not-Domain-rtrancl*:
$x \sim$: *Domain* $R ==> ((x,\ y) : R\hat{}*) = (x = y)$
apply *auto*
apply (*erule rev-mp*)

apply (*erule rtrancl-induct*)
apply *auto*
done

lemma *trancl-subset-Field2*: $r\hat{}+ <= Field\ r \times Field\ r$
apply *clarify*
apply (*erule trancl-induct*)
apply (*auto simp add*: *Field-def*)
done

lemma *finite-trancl*: *finite* $(r\hat{}+) = finite\ r$
apply *auto*
prefer *2*
apply (*rule trancl-subset-Field2* [*THEN finite-subset*])
apply (*rule finite-SigmaI*)
prefer *3*
apply (*blast intro*: *r-into-trancl'* *finite-subset*)
apply (*auto simp add*: *finite-Field*)
done

More about converse *rtrancl* and *trancl*, should be merged with main body.

lemma *single-valued-confluent*:
$[\![\ single\text{-}valued\ r;\ (x,y) \in r\hat{}*;\ (x,z) \in r\hat{}*\ ]\!]$
$\Longrightarrow (y,z) \in r\hat{}* \vee (z,y) \in r\hat{}*$
apply (*erule rtrancl-induct*)
apply *simp*
apply (*erule disjE*)
apply (*blast elim*:*converse-rtranclE dest*:*single-valuedD*)
apply(*blast intro*:*rtrancl-trans*)
done

lemma *r-r-into-trancl*: $(a,\ b) \in R ==> (b,\ c) \in R ==> (a,\ c) \in R\hat{}+$
by (*fast intro*: *trancl-trans*)

lemma *trancl-into-trancl* [*rule-format*]:
$(a,\ b) \in r^{+} ==> (b,\ c) \in r --> (a,c) \in r^{+}$
apply (*erule trancl-induct*)
apply (*fast intro*: *r-r-into-trancl*)
apply (*fast intro*: *r-r-into-trancl trancl-trans*)
done

lemma *tranclp-rtranclp-tranclp*:
$r^{++}\ a\ b ==> r^{**}\ b\ c ==> r^{++}\ a\ c$
apply (*drule tranclpD*)
apply (*elim exE conjE*)
apply (*drule rtranclp-trans*, *assumption*)
apply (*drule rtranclp-into-tranclp2*, *assumption*, *assumption*)
done

lemmas *trancl-rtrancl-trancl = tranclp-rtranclp-tranclp* [*to-set*]

lemmas *transitive-closure-trans* [*trans*] =
r-r-into-trancl trancl-trans rtrancl-trans
trancl.trancl-into-trancl trancl-into-trancl2
rtrancl.rtrancl-into-rtrancl converse-rtrancl-into-rtrancl
rtrancl-trancl-trancl trancl-rtrancl-trancl

lemmas *transitive-closurep-trans'* [*trans*] =
tranclp-trans rtranclp-trans
tranclp.trancl-into-trancl tranclp-into-tranclp2
rtranclp.rtrancl-into-rtrancl converse-rtranclp-into-rtranclp
rtranclp-tranclp-tranclp tranclp-rtranclp-tranclp

declare *trancl-into-rtrancl* [*elim*]

23.4 The power operation on relations

$R \hat{\ }\hat{\ } n = R\ O\ ...\ O\ R$, the n-fold composition of R

overloading
relpow == *compow* :: *nat* ⇒ ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set*
begin

primrec *relpow* :: *nat* ⇒ ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set* **where**
relpow 0 R = Id
| *relpow* (*Suc n*) *R* = (*R* ^^ *n*) *O R*

end

lemma *rel-pow-1* [*simp*]:
fixes *R* :: ('*a* × '*a*) *set*
shows *R* ^^ *1* = *R*
by *simp*

lemma *rel-pow-0-I*:
(*x*, *x*) ∈ *R* ^^ *0*
by *simp*

lemma *rel-pow-Suc-I*:
(*x*, *y*) ∈ *R* ^^ *n* ⟹ (*y*, *z*) ∈ *R* ⟹ (*x*, *z*) ∈ *R* ^^ *Suc n*
by *auto*

lemma *rel-pow-Suc-I2*:
(*x*, *y*) ∈ *R* ⟹ (*y*, *z*) ∈ *R* ^^ *n* ⟹ (*x*, *z*) ∈ *R* ^^ *Suc n*
by (*induct n arbitrary*: *z*) (*simp*, *fastsimp*)

lemma *rel-pow-0-E*:
(*x*, *y*) ∈ *R* ^^ *0* ⟹ (*x* = *y* ⟹ *P*) ⟹ *P*
by *simp*

lemma *rel-pow-Suc-E*:
$(x, z) \in R \mathbin{\hat{}\hat{}} Suc\ n \Longrightarrow (\bigwedge y.\ (x, y) \in R \mathbin{\hat{}\hat{}} n \Longrightarrow (y, z) \in R \Longrightarrow P) \Longrightarrow P$
by *auto*

lemma *rel-pow-E*:
$(x, z) \in R \mathbin{\hat{}\hat{}} n \Longrightarrow (n = 0 \Longrightarrow x = z \Longrightarrow P)$
$\Longrightarrow (\bigwedge y\ m.\ n = Suc\ m \Longrightarrow (x, y) \in R \mathbin{\hat{}\hat{}} m \Longrightarrow (y, z) \in R \Longrightarrow P)$
$\Longrightarrow P$
by (*cases n*) *auto*

lemma *rel-pow-Suc-D2*:
$(x, z) \in R \mathbin{\hat{}\hat{}} Suc\ n \Longrightarrow (\exists y.\ (x, y) \in R \wedge (y, z) \in R \mathbin{\hat{}\hat{}} n)$
apply (*induct n arbitrary*: *x z*)
apply (*blast intro*: *rel-pow-0-I elim*: *rel-pow-0-E rel-pow-Suc-E*)
apply (*blast intro*: *rel-pow-Suc-I elim*: *rel-pow-0-E rel-pow-Suc-E*)
done

lemma *rel-pow-Suc-E2*:
$(x, z) \in R \mathbin{\hat{}\hat{}} Suc\ n \Longrightarrow (\bigwedge y.\ (x, y) \in R \Longrightarrow (y, z) \in R \mathbin{\hat{}\hat{}} n \Longrightarrow P) \Longrightarrow P$
by (*blast dest*: *rel-pow-Suc-D2*)

lemma *rel-pow-Suc-D2'*:
$\forall x\ y\ z.\ (x, y) \in R \mathbin{\hat{}\hat{}} n \wedge (y, z) \in R \longrightarrow (\exists w.\ (x, w) \in R \wedge (w, z) \in R \mathbin{\hat{}\hat{}} n)$
by (*induct n*) (*simp-all*, *blast*)

lemma *rel-pow-E2*:
$(x, z) \in R \mathbin{\hat{}\hat{}} n \Longrightarrow (n = 0 \Longrightarrow x = z \Longrightarrow P)$
$\Longrightarrow (\bigwedge y\ m.\ n = Suc\ m \Longrightarrow (x, y) \in R \Longrightarrow (y, z) \in R \mathbin{\hat{}\hat{}} m \Longrightarrow P)$
$\Longrightarrow P$
apply (*cases n*, *simp*)
apply (*cut-tac n*=*nat* **and** *R*=*R* **in** *rel-pow-Suc-D2'*, *simp*, *blast*)
done

lemma *rel-pow-add*: $R \mathbin{\hat{}\hat{}} (m+n) = R \hat{}\hat{} m\ O\ R \hat{}\hat{} n$
by(*induct n*) *auto*

lemma *rel-pow-commute*: $R\ O\ R \mathbin{\hat{}\hat{}} n = R \mathbin{\hat{}\hat{}} n\ O\ R$
by (*induct n*) (*simp*, *simp add*: *O-assoc* [*symmetric*])

lemma *rtrancl-imp-UN-rel-pow*:
assumes $p \in R \hat{}*$
shows $p \in (\bigcup n.\ R \mathbin{\hat{}\hat{}} n)$
proof (*cases p*)
case (*Pair x y*)
with *assms* **have** $(x, y) \in R \hat{}*$ **by** *simp*
then have $(x, y) \in (\bigcup n.\ R \mathbin{\hat{}\hat{}} n)$ **proof** *induct*
case *base* **show** *?case* **by** (*blast intro*: *rel-pow-0-I*)
next

```
    case step then show ?case by (blast intro: rel-pow-Suc-I)
  qed
  with Pair show ?thesis by simp
qed

lemma rel-pow-imp-rtrancl:
  assumes p ∈ R ^^ n
  shows p ∈ R^*
proof (cases p)
  case (Pair x y)
  with assms have (x, y) ∈ R ^^ n by simp
  then have (x, y) ∈ R^* proof (induct n arbitrary: x y)
    case 0 then show ?case by simp
  next
    case Suc then show ?case
      by (blast elim: rel-pow-Suc-E intro: rtrancl-into-rtrancl)
  qed
  with Pair show ?thesis by simp
qed

lemma rtrancl-is-UN-rel-pow:
  R^* = (⋃n. R ^^ n)
  by (blast intro: rtrancl-imp-UN-rel-pow rel-pow-imp-rtrancl)

lemma rtrancl-power:
  p ∈ R^* ⟷ (∃n. p ∈ R ^^ n)
  by (simp add: rtrancl-is-UN-rel-pow)

lemma trancl-power:
  p ∈ R^+ ⟷ (∃n > 0. p ∈ R ^^ n)
  apply (cases p)
  apply simp
  apply (rule iffI)
   apply (drule tranclD2)
   apply (clarsimp simp: rtrancl-is-UN-rel-pow)
   apply (rule-tac x=Suc n in exI)
   apply (clarsimp simp: rel-comp-def)
   apply fastsimp
  apply clarsimp
  apply (case-tac n, simp)
  apply clarsimp
  apply (drule rel-pow-imp-rtrancl)
  apply (drule rtrancl-into-trancl1) apply auto
  done

lemma rtrancl-imp-rel-pow:
  p ∈ R^* ⟹ ∃n. p ∈ R ^^ n
  by (auto dest: rtrancl-imp-UN-rel-pow)
```

lemma *single-valued-rel-pow*:
fixes $R :: ('a * 'a)$ *set*
shows *single-valued* $R \Longrightarrow$ *single-valued* $(R \hat{} n)$
apply (*induct n arbitrary*: *R*)
apply *simp-all*
apply (*rule single-valuedI*)
apply (*fast dest*: *single-valuedD elim*: *rel-pow-Suc-E*)
done

23.5 Setup of transitivity reasoner

ML ⟪

```
structure Trancl-Tac = Trancl-Tac
(
  val r-into-trancl = @{thm trancl.r-into-trancl};
  val trancl-trans  = @{thm trancl-trans};
  val rtrancl-refl = @{thm rtrancl.rtrancl-refl};
  val r-into-rtrancl = @{thm r-into-rtrancl};
  val trancl-into-rtrancl = @{thm trancl-into-rtrancl};
  val rtrancl-trancl-trancl = @{thm rtrancl-trancl-trancl};
  val trancl-rtrancl-trancl = @{thm trancl-rtrancl-trancl};
  val rtrancl-trans = @{thm rtrancl-trans};

  fun decomp (@{const Trueprop} $ t) =
    let fun dec (Const (op :, -) $ (Const (Pair, -) $ a $ b) $ rel ) =
        let fun decr (Const (Transitive-Closure.rtrancl, - ) $ r) = (r,r*)
              | decr (Const (Transitive-Closure.trancl, - ) $ r)  = (r,r+)
              | decr r = (r,r);
            val (rel,r) = decr (Envir.beta-eta-contract rel);
        in SOME (a,b,rel,r) end
      | dec - =  NONE
    in dec t end
    | decomp - = NONE;
);

structure Tranclp-Tac = Trancl-Tac
(
  val r-into-trancl = @{thm tranclp.r-into-trancl};
  val trancl-trans  = @{thm tranclp-trans};
  val rtrancl-refl = @{thm rtranclp.rtrancl-refl};
  val r-into-rtrancl = @{thm r-into-rtranclp};
  val trancl-into-rtrancl = @{thm tranclp-into-rtranclp};
  val rtrancl-trancl-trancl = @{thm rtranclp-tranclp-tranclp};
  val trancl-rtrancl-trancl = @{thm tranclp-rtranclp-tranclp};
  val rtrancl-trans = @{thm rtranclp-trans};

  fun decomp (@{const Trueprop} $ t) =
    let fun dec (rel $ a $ b) =
```

```
      let fun decr (Const (Transitive-Closure.rtranclp, - ) $ r) = (r,r*)
            | decr (Const (Transitive-Closure.tranclp, - ) $ r)  = (r,r+)
            | decr r = (r,r);
          val (rel,r) = decr rel;
      in SOME (a, b, rel, r) end
    | dec - =  NONE
  in dec t end
  | decomp - = NONE;
);
⟫
```

declaration ⟪ *fn - =>*
Simplifier.map-ss (fn ss => ss
addSolver (mk-solver' Trancl (Trancl-Tac.trancl-tac o Simplifier.the-context))
addSolver (mk-solver' Rtrancl (Trancl-Tac.rtrancl-tac o Simplifier.the-context))
addSolver (mk-solver' Tranclp (Tranclp-Tac.trancl-tac o Simplifier.the-context))
addSolver (mk-solver' Rtranclp (Tranclp-Tac.rtrancl-tac o Simplifier.the-context)))
⟫

Optional methods.

method-setup *trancl* =
⟪ *Scan.succeed (SIMPLE-METHOD' o Trancl-Tac.trancl-tac)* ⟫
⟪ *simple transitivity reasoner* ⟫
method-setup *rtrancl* =
⟪ *Scan.succeed (SIMPLE-METHOD' o Trancl-Tac.rtrancl-tac)* ⟫
⟪ *simple transitivity reasoner* ⟫
method-setup *tranclp* =
⟪ *Scan.succeed (SIMPLE-METHOD' o Tranclp-Tac.trancl-tac)* ⟫
⟪ *simple transitivity reasoner (predicate version)* ⟫
method-setup *rtranclp* =
⟪ *Scan.succeed (SIMPLE-METHOD' o Tranclp-Tac.rtrancl-tac)* ⟫
⟪ *simple transitivity reasoner (predicate version)* ⟫

end

24 Wellfounded: Well-founded Recursion

theory *Wellfounded*
imports *Transitive-Closure*
uses (*Tools/Function/size.ML*)
begin

24.1 Basic Definitions

definition *wf* :: ('*a* * '*a*) *set* => *bool* **where**
wf(r) == (!*P*. (!*x*. (!*y*. (*y*,*x*):*r* --> *P(y)*) --> *P(x)*) --> (!*x*. *P(x)*)))

definition *wfP* :: ('*a* => '*a* => *bool*) => *bool* **where**

wfP r == wf {(x, y). r x y}

lemma *wfP-wf-eq* [*pred-set-conv*]: *wfP* (λ*x y*. (*x*, *y*) ∈ *r*) = *wf r*
by (*simp add*: *wfP-def*)

lemma *wfUNIVI*:
(!!*P x*. (*ALL x*. (*ALL y*. (*y*,*x*) : *r* −−> *P*(*y*)) −−> *P*(*x*)) ==> *P*(*x*)) ==> *wf*(*r*)
unfolding *wf-def* **by** *blast*

lemmas *wfPUNIVI* = *wfUNIVI* [*to-pred*]

Restriction to domain *A* and range *B*. If *r* is well-founded over their intersection, then *wf r*

lemma *wfI*:
[| *r* ⊆ *A* <*> *B*;
!!*x P*. [|∀ *x*. (∀ *y*. (*y*,*x*) : *r* −−> *P y*) −−> *P x*; *x* : *A*; *x* : *B* |] ==> *P x* |]
==> *wf r*
unfolding *wf-def* **by** *blast*

lemma *wf-induct*:
[| *wf*(*r*);
!!*x*.[| *ALL y*. (*y*,*x*): *r* −−> *P*(*y*) |] ==> *P*(*x*)
|] ==> *P*(*a*)
unfolding *wf-def* **by** *blast*

lemmas *wfP-induct* = *wf-induct* [*to-pred*]

lemmas *wf-induct-rule* = *wf-induct* [*rule-format*, *consumes 1*, *case-names less*, *induct set*: *wf*]

lemmas *wfP-induct-rule* = *wf-induct-rule* [*to-pred*, *induct set*: *wfP*]

lemma *wf-not-sym*: *wf r* ==> (*a*, *x*) : *r* ==> (*x*, *a*) ~: *r*
by (*induct a arbitrary*: *x set*: *wf*) *blast*

lemma *wf-asym*:
assumes *wf r* (*a*, *x*) ∈ *r*
obtains (*x*, *a*) ∉ *r*
by (*drule wf-not-sym*[*OF assms*])

lemma *wf-not-refl* [*simp*]: *wf r* ==> (*a*, *a*) ~: *r*
by (*blast elim*: *wf-asym*)

lemma *wf-irrefl*: **assumes** *wf r* **obtains** (*a*, *a*) ∉ *r*
by (*drule wf-not-refl*[*OF assms*])

lemma *wf-wellorderI*:
assumes *wf*: *wf* {(*x*::'*a*::*ord*, *y*). *x* < *y*}

assumes *lin*: *OFCLASS*('*a*::*ord*, *linorder-class*)
shows *OFCLASS*('*a*::*ord*, *wellorder-class*)
using *lin* **by** (*rule wellorder-class.intro*)
(*blast intro*: *class.wellorder-axioms.intro wf-induct-rule* [*OF wf*])

lemma (**in** *wellorder*) *wf*:
wf {(*x*, *y*). *x* < *y*}
unfolding *wf-def* **by** (*blast intro*: *less-induct*)

24.2 Basic Results

Point-free characterization of well-foundedness

lemma *wfE-pf*:
assumes *wf*: *wf R*
assumes *a*: *A* ⊆ *R* `` *A*
shows *A* = {}
proof −
{ fix *x*
from *wf* **have** *x* ∉ *A*
proof *induct*
fix *x* **assume** ⋀*y*. (*y*, *x*) ∈ *R* ⟹ *y* ∉ *A*
then have *x* ∉ *R* `` *A* **by** *blast*
with *a* **show** *x* ∉ *A* **by** *blast*
qed
} thus *?thesis* **by** *auto*
qed

lemma *wfI-pf*:
assumes *a*: ⋀*A*. *A* ⊆ *R* `` *A* ⟹ *A* = {}
shows *wf R*
proof (*rule wfUNIVI*)
fix *P* :: '*a* ⇒ *bool* **and** *x*
let *?A* = {*x*. ¬ *P x*}
assume ∀*x*. (∀*y*. (*y*, *x*) ∈ *R* ⟶ *P y*) ⟶ *P x*
then have *?A* ⊆ *R* `` *?A* **by** *blast*
with *a* **show** *P x* **by** *blast*
qed

Minimal-element characterization of well-foundedness

lemma *wfE-min*:
assumes *wf*: *wf R* **and** *Q*: *x* ∈ *Q*
obtains *z* **where** *z* ∈ *Q* ⋀*y*. (*y*, *z*) ∈ *R* ⟹ *y* ∉ *Q*
using *Q wfE-pf*[*OF wf*, *of Q*] **by** *blast*

lemma *wfI-min*:
assumes *a*: ⋀*x Q*. *x* ∈ *Q* ⟹ ∃ *z*∈*Q*. ∀*y*. (*y*, *z*) ∈ *R* ⟶ *y* ∉ *Q*
shows *wf R*
proof (*rule wfI-pf*)
fix *A* **assume** *b*: *A* ⊆ *R* `` *A*

```
  { fix x assume x ∈ A
    from a[OF this] b have False by blast
  }
  thus A = {} by blast
qed

lemma wf-eq-minimal: wf r = (∀ Q x. x∈Q --> (∃ z∈Q. ∀ y. (y,z)∈r -->
y∉Q))
apply auto
apply (erule wfE-min, assumption, blast)
apply (rule wfI-min, auto)
done

lemmas wfP-eq-minimal = wf-eq-minimal [to-pred]
```

Well-foundedness of transitive closure

```
lemma wf-trancl:
  assumes wf r
  shows wf (r^+)
proof -
  {
    fix P and x
    assume induct-step: !!x. (!!y. (y, x) : r^+ ==> P y) ==> P x
    have P x
    proof (rule induct-step)
      fix y assume (y, x) : r^+
      with ‹wf r› show P y
      proof (induct x arbitrary: y)
        case (less x)
        note hyp = ‹⋀x' y'. (x', x) : r ==> (y', x') : r^+ ==> P y'›
        from ‹(y, x) : r^+› show P y
        proof cases
          case base
          show P y
          proof (rule induct-step)
            fix y' assume (y', y) : r^+
            with ‹(y, x) : r› show P y' by (rule hyp [of y y'])
          qed
        next
          case step
          then obtain x' where (x', x) : r and (y, x') : r^+ by simp
          then show P y by (rule hyp [of x' y])
        qed
      qed
    qed
  } then show ?thesis unfolding wf-def by blast
qed

lemmas wfP-trancl = wf-trancl [to-pred]
```

lemma *wf-converse-trancl*: *wf* (*r* ^ −1) ==> *wf* ((*r* ^ +) ^ −1)
apply (*subst trancl-converse* [*symmetric*])
apply (*erule wf-trancl*)
done

Well-foundedness of subsets

lemma *wf-subset*: [| *wf*(*r*); *p*<=*r* |] ==> *wf*(*p*)
apply (*simp* (*no-asm-use*) *add*: *wf-eq-minimal*)
apply *fast*
done

lemmas *wfP-subset* = *wf-subset* [*to-pred*]

Well-foundedness of the empty relation

lemma *wf-empty* [*iff*]: *wf* {}
by (*simp add*: *wf-def*)

lemma *wfP-empty* [*iff*]:
wfP (λ*x y*. *False*)
proof −
have *wfP bot* **by** (*fact wf-empty* [*to-pred bot-empty-eq2*])
then show *?thesis* **by** (*simp add*: *bot-fun-eq bot-bool-eq*)
qed

lemma *wf-Int1*: *wf r* ==> *wf* (*r Int r'*)
apply (*erule wf-subset*)
apply (*rule Int-lower1*)
done

lemma *wf-Int2*: *wf r* ==> *wf* (*r' Int r*)
apply (*erule wf-subset*)
apply (*rule Int-lower2*)
done

Exponentiation

lemma *wf-exp*:
assumes *wf* (*R* ^^ *n*)
shows *wf R*
proof (*rule wfI-pf*)
fix *A* **assume** *A* ⊆ *R* `` *A*
then have *A* ⊆ (*R* ^^ *n*) `` *A* **by** (*induct n*) *force*+
with ‹*wf* (*R* ^^ *n*)›
show *A* = {} **by** (*rule wfE-pf*)
qed

Well-foundedness of insert

lemma *wf-insert* [*iff*]: *wf*(*insert* (*y*,*x*) *r*) = (*wf*(*r*) & (*x*,*y*) ~: *r* ^ *)

apply (*rule iffI*)
apply (*blast elim*: *wf-trancl* [*THEN wf-irrefl*]
intro: *rtrancl-into-trancl1 wf-subset*
rtrancl-mono [*THEN* [*2*] *rev-subsetD*])
apply (*simp add*: *wf-eq-minimal*, *safe*)
apply (*rule allE*, *assumption*, *erule impE*, *blast*)
apply (*erule bexE*)
apply (*rename-tac a*, *case-tac a* = *x*)
prefer *2*
apply *blast*
apply (*case-tac y*:*Q*)
prefer *2* **apply** *blast*
apply (*rule-tac x* = {*z*. *z*:*Q* & (*z*,*y*) : *r*^*}} **in** *allE*)
apply *assumption*
apply (*erule-tac V* = *ALL Q*. (*EX x*. *x* : *Q*) −−> *?P Q* **in** *thin-rl*)
— essential for speed

Blast with new substOccur fails

apply (*fast intro*: *converse-rtrancl-into-rtrancl*)
done

Well-foundedness of image

lemma *wf-prod-fun-image*: [| *wf r*; *inj f* |] ==> *wf*(*prod-fun f f* ‘ *r*)
apply (*simp only*: *wf-eq-minimal*, *clarify*)
apply (*case-tac EX p*. *f p* : *Q*)
apply (*erule-tac x* = {*p*. *f p* : *Q*} **in** *allE*)
apply (*fast dest*: *inj-onD*, *blast*)
done

24.3 Well-Foundedness Results for Unions

lemma *wf-union-compatible*:
assumes *wf R wf S*
assumes *R O S* ⊆ *R*
shows *wf* (*R* ∪ *S*)
proof (*rule wfI-min*)
fix *x* :: *'a* **and** *Q*
let *?Q'* = {*x* ∈ *Q*. ∀ *y*. (*y*, *x*) ∈ *R* ⟶ *y* ∉ *Q*}
assume *x* ∈ *Q*
obtain *a* **where** *a* ∈ *?Q'*
by (*rule wfE-min* [*OF* ‹*wf R*› ‹*x* ∈ *Q*›]) *blast*
with ‹*wf S*›
obtain *z* **where** *z* ∈ *?Q'* **and** *zmin*: ⋀*y*. (*y*, *z*) ∈ *S* ⟹ *y* ∉ *?Q'* **by** (*erule wfE-min*)
{
fix *y* **assume** (*y*, *z*) ∈ *S*
then have *y* ∉ *?Q'* **by** (*rule zmin*)

have *y* ∉ *Q*

```
    proof
      assume y ∈ Q
      with ‹y ∉ ?Q'›
      obtain w where (w, y) ∈ R and w ∈ Q by auto
      from ‹(w, y) ∈ R› ‹(y, z) ∈ S› have (w, z) ∈ R O S by (rule rel-compI)
      with ‹R O S ⊆ R› have (w, z) ∈ R ..
      with ‹z ∈ ?Q'› have w ∉ Q by blast
      with ‹w ∈ Q› show False by contradiction
    qed
  }
  with ‹z ∈ ?Q'› show ∃z∈Q. ∀y. (y, z) ∈ R ∪ S ⟶ y ∉ Q by blast
qed
```

Well-foundedness of indexed union with disjoint domains and ranges

```
lemma wf-UN: [| ALL i:I. wf(r i);
        ALL i:I. ALL j:I. r i ~= r j --> Domain(r i) Int Range(r j) = {}
     |] ==> wf(UN i:I. r i)
apply (simp only: wf-eq-minimal, clarify)
apply (rename-tac A a, case-tac EX i:I. EX a:A. EX b:A. (b,a) : r i)
 prefer 2
 apply force
apply clarify
apply (drule bspec, assumption)
apply (erule-tac x={a. a:A & (EX b:A. (b,a) : r i) } in allE)
apply (blast elim!: allE)
done

lemma wfP-SUP:
  ∀i. wfP (r i) ⟹ ∀i j. r i ≠ r j ⟶ inf (DomainP (r i)) (RangeP (r j)) = bot
⟹ wfP (SUPR UNIV r)
  by (rule wf-UN [where I=UNIV and r=λi. {(x, y). r i x y}, to-pred SUP-UN-eq2])
    (simp-all add: Collect-def)

lemma wf-Union:
 [| ALL r:R. wf r;
     ALL r:R. ALL s:R. r ~= s --> Domain r Int Range s = {}
  |] ==> wf(Union R)
apply (simp add: Union-def)
apply (blast intro: wf-UN)
done


lemma wf-Un:
     [| wf r; wf s; Domain r Int Range s = {} |] ==> wf(r Un s)
  using wf-union-compatible[of s r]
  by (auto simp: Un-ac)

lemma wf-union-merge:
  wf (R ∪ S) = wf (R O R ∪ S O R ∪ S) (is wf ?A = wf ?B)
```

proof
 assume *wf ?A*
 with *wf-trancl* **have** *wfT*: *wf* (*?A* ^+) **.**
 moreover have *?B* ⊆ *?A* ^+
  by (*subst trancl-unfold*, *subst trancl-unfold*) *blast*
 ultimately show *wf ?B* **by** (*rule wf-subset*)
next
 assume *wf ?B*

 show *wf ?A*
 proof (*rule wfI-min*)
  fix *Q* :: *'a set* **and** *x*
  assume *x* ∈ *Q*

  with ‹*wf ?B*›
  obtain *z* **where** *z* ∈ *Q* **and** ⋀*y*. (*y*, *z*) ∈ *?B* ⟹ *y* ∉ *Q*
   by (*erule wfE-min*)
  then have *A1*: ⋀*y*. (*y*, *z*) ∈ *R O R* ⟹ *y* ∉ *Q*
   and *A2*: ⋀*y*. (*y*, *z*) ∈ *S O R* ⟹ *y* ∉ *Q*
   and *A3*: ⋀*y*. (*y*, *z*) ∈ *S* ⟹ *y* ∉ *Q*
   by *auto*

  show ∃ *z*∈*Q*. ∀ *y*. (*y*, *z*) ∈ *?A* ⟶ *y* ∉ *Q*
  proof (*cases* ∀ *y*. (*y*, *z*) ∈ *R* ⟶ *y* ∉ *Q*)
   case *True*
   with ‹*z* ∈ *Q*› *A3* **show** *?thesis* **by** *blast*
  next
   case *False*
   then obtain *z'* **where** *z'*∈*Q* (*z'*, *z*) ∈ *R* **by** *blast*

   have ∀ *y*. (*y*, *z'*) ∈ *?A* ⟶ *y* ∉ *Q*
   proof (*intro allI impI*)
    fix *y* **assume** (*y*, *z'*) ∈ *?A*
    then show *y* ∉ *Q*
    proof
     assume (*y*, *z'*) ∈ *R*
     then have (*y*, *z*) ∈ *R O R* **using** ‹(*z'*, *z*) ∈ *R*› **..**
     with *A1* **show** *y* ∉ *Q* **.**
    next
     assume (*y*, *z'*) ∈ *S*
     then have (*y*, *z*) ∈ *S O R* **using** ‹(*z'*, *z*) ∈ *R*› **..**
     with *A2* **show** *y* ∉ *Q* **.**
    qed
   qed
   with ‹*z'* ∈ *Q*› **show** *?thesis* **..**
  qed
 qed
qed

lemma *wf-comp-self*: *wf R = wf (R O R)* — special case
by (*rule wf-union-merge* [**where** *S* = {}, *simplified*])

24.4 Acyclic relations

definition *acyclic* :: ('*a* * '*a*) *set* => *bool* **where**
acyclic r == !*x*. (*x*,*x*) ~: *r*^+

abbreviation *acyclicP* :: ('*a* => '*a* => *bool*) => *bool* **where**
acyclicP r == *acyclic* {(*x*, *y*). *r x y*}

lemma *acyclicI*: *ALL x*. (*x*, *x*) ~: *r*^+ ==> *acyclic r*
by (*simp add*: *acyclic-def*)

lemma *wf-acyclic*: *wf r* ==> *acyclic r*
apply (*simp add*: *acyclic-def*)
apply (*blast elim*: *wf-trancl* [*THEN wf-irrefl*])
done

lemmas *wfP-acyclicP* = *wf-acyclic* [*to-pred*]

lemma *acyclic-insert* [*iff*]:
acyclic(*insert* (*y*,*x*) *r*) = (*acyclic r* & (*x*,*y*) ~: *r*^*)
apply (*simp add*: *acyclic-def trancl-insert*)
apply (*blast intro*: *rtrancl-trans*)
done

lemma *acyclic-converse* [*iff*]: *acyclic*(*r*^−1) = *acyclic r*
by (*simp add*: *acyclic-def trancl-converse*)

lemmas *acyclicP-converse* [*iff*] = *acyclic-converse* [*to-pred*]

lemma *acyclic-impl-antisym-rtrancl*: *acyclic r* ==> *antisym*(*r*^*)
apply (*simp add*: *acyclic-def antisym-def*)
apply (*blast elim*: *rtranclE intro*: *rtrancl-into-trancl1 rtrancl-trancl-trancl*)
done

lemma *acyclic-subset*: [| *acyclic s*; *r* <= *s* |] ==> *acyclic r*
apply (*simp add*: *acyclic-def*)
apply (*blast intro*: *trancl-mono*)
done

Wellfoundedness of finite acyclic relations

lemma *finite-acyclic-wf* [*rule-format*]: *finite r* ==> *acyclic r* −−> *wf r*
apply (*erule finite-induct*, *blast*)
apply (*simp* (*no-asm-simp*) *only*: *split-tupled-all*)
apply *simp*

done

lemma *finite-acyclic-wf-converse*: [|*finite r*; *acyclic r*|] ==> *wf* (*r*^−*1*)
apply (*erule finite-converse* [*THEN iffD2*, *THEN finite-acyclic-wf*])
apply (*erule acyclic-converse* [*THEN iffD2*])
done

lemma *wf-iff-acyclic-if-finite*: *finite r* ==> *wf r* = *acyclic r*
by (*blast intro*: *finite-acyclic-wf wf-acyclic*)

24.5 *nat* is well-founded

lemma *less-nat-rel*: *op* < = (λ*m n*. *n* = *Suc m*) ^++
proof (*rule ext*, *rule ext*, *rule iffI*)
  fix *n m* :: *nat*
  assume *m* < *n*
  then show (λ*m n*. *n* = *Suc m*) ^++ *m n*
  proof (*induct n*)
    case *0* **then show** *?case* **by** *auto*
  next
    case (*Suc n*) **then show** *?case*
      by (*auto simp add*: *less-Suc-eq-le le-less intro*: *tranclp.trancl-into-trancl*)
  qed
next
  fix *n m* :: *nat*
  assume (λ*m n*. *n* = *Suc m*) ^++ *m n*
  then show *m* < *n*
    by (*induct n*)
      (*simp-all add*: *less-Suc-eq-le reflexive le-less*)
qed

definition
  pred-nat :: (*nat* ∗ *nat*) *set* **where**
  pred-nat = {(*m*, *n*). *n* = *Suc m*}

definition
  less-than :: (*nat* ∗ *nat*) *set* **where**
  less-than = *pred-nat*^+

lemma *less-eq*: (*m*, *n*) ∈ *pred-nat*^+ ⟷ *m* < *n*
  unfolding *less-nat-rel pred-nat-def trancl-def* **by** *simp*

lemma *pred-nat-trancl-eq-le*:
  (*m*, *n*) ∈ *pred-nat*^∗ ⟷ *m* ≤ *n*
  unfolding *less-eq rtrancl-eq-or-trancl* **by** *auto*

lemma *wf-pred-nat*: *wf pred-nat*
  apply (*unfold wf-def pred-nat-def*, *clarify*)
  apply (*induct-tac x*, *blast*+)

done

lemma *wf-less-than* [*iff*]: *wf less-than*
by (*simp add*: *less-than-def wf-pred-nat* [*THEN wf-trancl*])

lemma *trans-less-than* [*iff*]: *trans less-than*
by (*simp add*: *less-than-def*)

lemma *less-than-iff* [*iff*]: ((*x*,*y*): *less-than*) = (*x*<*y*)
by (*simp add*: *less-than-def less-eq*)

lemma *wf-less*: *wf* {(*x*, *y*::*nat*). *x* < *y*}
using *wf-less-than* **by** (*simp add*: *less-than-def less-eq* [*symmetric*])

24.6 Accessible Part

Inductive definition of the accessible part *acc r* of a relation; see also [**?**].

inductive-set
acc :: ('*a* ∗ '*a*) *set* => '*a set*
for *r* :: ('*a* ∗ '*a*) *set*
where
accI: (!!*y*. (*y*, *x*) : *r* ==> *y* : *acc r*) ==> *x* : *acc r*

abbreviation
termip :: ('*a* => '*a* => *bool*) => '*a* => *bool* **where**
termip r == *accp* (r^{-1-1})

abbreviation
termi :: ('*a* ∗ '*a*) *set* => '*a set* **where**
termi r == *acc* (r^{-1})

lemmas *accpI* = *accp.accI*

Induction rules

theorem *accp-induct*:
assumes *major*: *accp r a*
assumes *hyp*: !!*x*. *accp r x* ==> ∀ *y*. *r y x* −−> *P y* ==> *P x*
shows *P a*
apply (*rule major* [*THEN accp.induct*])
apply (*rule hyp*)
apply (*rule accp.accI*)
apply *fast*
apply *fast*
done

theorems *accp-induct-rule* = *accp-induct* [*rule-format*, *induct set*: *accp*]

theorem *accp-downward*: *accp r b* ==> *r a b* ==> *accp r a*
apply (*erule accp.cases*)

apply *fast*
done

lemma *not-accp-down*:
assumes *na*: ¬ *accp R x*
obtains *z* **where** *R z x* **and** ¬ *accp R z*
proof −
assume *a*: ⋀*z*. ⟦*R z x*; ¬ *accp R z*⟧ ⟹ *thesis*

show *thesis*
proof (*cases* ∀ *z*. *R z x* ⟶ *accp R z*)
case *True*
hence ⋀*z*. *R z x* ⟹ *accp R z* **by** *auto*
hence *accp R x*
by (*rule accp.accI*)
with *na* **show** *thesis* **..**
next
case *False* **then obtain** *z* **where** *R z x* **and** ¬ *accp R z*
by *auto*
with *a* **show** *thesis* **.**
qed
qed

lemma *accp-downwards-aux*: *r*** *b a* ==> *accp r a* −−> *accp r b*
apply (*erule rtranclp-induct*)
apply *blast*
apply (*blast dest*: *accp-downward*)
done

theorem *accp-downwards*: *accp r a* ==> *r*** *b a* ==> *accp r b*
apply (*blast dest*: *accp-downwards-aux*)
done

theorem *accp-wfPI*: ∀ *x*. *accp r x* ==> *wfP r*
apply (*rule wfPUNIVI*)
apply (*induct-tac P x rule*: *accp-induct*)
apply *blast*
apply *blast*
done

theorem *accp-wfPD*: *wfP r* ==> *accp r x*
apply (*erule wfP-induct-rule*)
apply (*rule accp.accI*)
apply *blast*
done

theorem *wfP-accp-iff*: *wfP r* = (∀ *x*. *accp r x*)
apply (*blast intro*: *accp-wfPI dest*: *accp-wfPD*)
done

Smaller relations have bigger accessible parts:

lemma *accp-subset*:
  assumes *sub*: $R1 \leq R2$
  shows $accp\ R2 \leq accp\ R1$
proof (*rule predicate1I*)
  fix x **assume** $accp\ R2\ x$
  then show $accp\ R1\ x$
  proof (*induct x*)
    fix x
    assume *ih*: $\bigwedge y.\ R2\ y\ x \Longrightarrow accp\ R1\ y$
    with *sub* **show** $accp\ R1\ x$
      by (*blast intro*: *accp.accI*)
  qed
qed

This is a generalized induction theorem that works on subsets of the accessible part.

lemma *accp-subset-induct*:
  assumes *subset*: $D \leq accp\ R$
    and *dcl*: $\bigwedge x\ z.\ [\![D\ x;\ R\ z\ x]\!] \Longrightarrow D\ z$
    and $D\ x$
    and *istep*: $\bigwedge x.\ [\![D\ x;\ (\bigwedge z.\ R\ z\ x \Longrightarrow P\ z)]\!] \Longrightarrow P\ x$
  shows $P\ x$
proof −
  from *subset* **and** ‹$D\ x$›
  have $accp\ R\ x$ **..**
  then show $P\ x$ **using** ‹$D\ x$›
  proof (*induct x*)
    fix x
    assume $D\ x$
      and $\bigwedge y.\ R\ y\ x \Longrightarrow D\ y \Longrightarrow P\ y$
    with *dcl* **and** *istep* **show** $P\ x$ **by** *blast*
  qed
qed

Set versions of the above theorems

lemmas *acc-induct* = *accp-induct* [*to-set*]

lemmas *acc-induct-rule* = *acc-induct* [*rule-format*, *induct set*: *acc*]

lemmas *acc-downward* = *accp-downward* [*to-set*]

lemmas *not-acc-down* = *not-accp-down* [*to-set*]

lemmas *acc-downwards-aux* = *accp-downwards-aux* [*to-set*]

lemmas *acc-downwards* = *accp-downwards* [*to-set*]

lemmas *acc-wfI* = *accp-wfPI* [*to-set*]

lemmas *acc-wfD* = *accp-wfPD* [*to-set*]

lemmas *wf-acc-iff* = *wfP-accp-iff* [*to-set*]

lemmas *acc-subset* = *accp-subset* [*to-set pred-subset-eq*]

lemmas *acc-subset-induct* = *accp-subset-induct* [*to-set pred-subset-eq*]

24.7 Tools for building wellfounded relations

Inverse Image

lemma *wf-inv-image* [*simp*,*intro*!]: *wf*(*r*) ==> *wf*(*inv-image r* (*f*::'*a*=>'*b*))
apply (*simp* (*no-asm-use*) *add*: *inv-image-def wf-eq-minimal*)
apply *clarify*
apply (*subgoal-tac EX* (*w*::'*b*) . *w* : {*w*. *EX* (*x*::'*a*) . *x*: *Q* & (*f x* = *w*) })
prefer *2* **apply** (*blast del*: *allE*)
apply (*erule allE*)
apply (*erule* (*1*) *notE impE*)
apply *blast*
done

Measure functions into *nat*

definition *measure* :: ('*a* => *nat*) => ('*a* * '*a*)*set*
where *measure* == *inv-image less-than*

lemma *in-measure*[*simp*]: ((*x*,*y*) : *measure f*) = (*f x* < *f y*)
by (*simp add*:*measure-def*)

lemma *wf-measure* [*iff*]: *wf* (*measure f*)
apply (*unfold measure-def*)
apply (*rule wf-less-than* [*THEN wf-inv-image*])
done

Lexicographic combinations

definition
lex-prod :: [('*a**'*a*)*set*, ('*b**'*b*)*set*] => (('*a**'*b*)*('*a**'*b*))*set*
(**infixr** <**lex**> *80*)
where
ra <**lex**> *rb* == {((*a*,*b*),(*a*',*b*')). (*a*,*a*') : *ra* | *a*=*a*' & (*b*,*b*') : *rb*}

lemma *wf-lex-prod* [*intro*!]: [| *wf*(*ra*); *wf*(*rb*) |] ==> *wf*(*ra* <**lex**> *rb*)
apply (*unfold wf-def lex-prod-def*)
apply (*rule allI*, *rule impI*)
apply (*simp* (*no-asm-use*) *only*: *split-paired-All*)
apply (*drule spec*, *erule mp*)
apply (*rule allI*, *rule impI*)

apply (*drule spec*, *erule mp*, *blast*)
done

lemma *in-lex-prod*[*simp*]:
(((*a*,*b*),(*a'*,*b'*)): *r* <∗*lex*∗> *s*) = ((*a*,*a'*): *r* ∨ (*a* = *a'* ∧ (*b*, *b'*) : *s*))
by (*auto simp*:*lex-prod-def*)

op <∗*lex*∗> preserves transitivity

lemma *trans-lex-prod* [*intro*!]:
[| *trans R1*; *trans R2* |] ==> *trans* (*R1* <∗*lex*∗> *R2*)
by (*unfold trans-def lex-prod-def*, *blast*)

lexicographic combinations with measure functions

definition
mlex-prod :: ('*a* ⇒ *nat*) ⇒ ('*a* × '*a*) *set* ⇒ ('*a* × '*a*) *set* (**infixr** <∗*mlex*∗> *80*)
where
f <∗*mlex*∗> *R* = *inv-image* (*less-than* <∗*lex*∗> *R*) (%*x*. (*f x*, *x*))

lemma *wf-mlex*: *wf R* ⟹ *wf* (*f* <∗*mlex*∗> *R*)
unfolding *mlex-prod-def*
by *auto*

lemma *mlex-less*: *f x* < *f y* ⟹ (*x*, *y*) ∈ *f* <∗*mlex*∗> *R*
unfolding *mlex-prod-def* **by** *simp*

lemma *mlex-leq*: *f x* ≤ *f y* ⟹ (*x*, *y*) ∈ *R* ⟹ (*x*, *y*) ∈ *f* <∗*mlex*∗> *R*
unfolding *mlex-prod-def* **by** *auto*

proper subset relation on finite sets

definition *finite-psubset* :: ('*a set* ∗ '*a set*) *set*
where *finite-psubset* == {(*A*,*B*). *A* < *B* & *finite B*}

lemma *wf-finite-psubset*[*simp*]: *wf*(*finite-psubset*)
apply (*unfold finite-psubset-def*)
apply (*rule wf-measure* [*THEN wf-subset*])
apply (*simp add*: *measure-def inv-image-def less-than-def less-eq*)
apply (*fast elim*!: *psubset-card-mono*)
done

lemma *trans-finite-psubset*: *trans finite-psubset*
by (*simp add*: *finite-psubset-def less-le trans-def*, *blast*)

lemma *in-finite-psubset*[*simp*]: (*A*, *B*) ∈ *finite-psubset* = (*A* < *B* & *finite B*)
unfolding *finite-psubset-def* **by** *auto*

max- and min-extension of order to finite sets

inductive-set *max-ext* :: ('*a* × '*a*) *set* ⇒ ('*a set* × '*a set*) *set*
for *R* :: ('*a* × '*a*) *set*

where
max-extI[*intro*]: *finite* $X \Longrightarrow$ *finite* $Y \Longrightarrow Y \neq \{\} \Longrightarrow (\bigwedge x.\ x \in X \Longrightarrow \exists\, y{\in}Y.\ (x,\ y) \in R) \Longrightarrow (X,\ Y) \in$ *max-ext* R

lemma *max-ext-wf*:
assumes *wf*: *wf r*
shows *wf* (*max-ext r*)
proof (*rule acc-wfI*, *intro allI*)
fix M **show** $M \in$ *acc* (*max-ext r*) (**is** - $\in$ *?W*)
proof *cases*
assume *finite M*
thus *?thesis*
proof (*induct M*)
show $\{\} \in$ *?W*
by (*rule accI*) (*auto elim*: *max-ext.cases*)
next
fix $M\ a$ **assume** $M \in$ *?W finite M*
with *wf* **show** *insert a M* $\in$ *?W*
proof (*induct arbitrary*: M)
fix $M\ a$
assume $M \in$ *?W* **and** [*intro*]: *finite M*
assume *hyp*: $\bigwedge b\ M.\ (b,\ a) \in r \Longrightarrow M \in$ *?W* $\Longrightarrow$ *finite M* $\Longrightarrow$ *insert b M* $\in$ *?W*
{
fix $N\ M$ :: *'a set*
assume *finite N finite M*
then
have $[\![M \in$ *?W* ; $(\bigwedge y.\ y \in N \Longrightarrow (y,\ a) \in r)]\!] \Longrightarrow\ N \cup M \in$ *?W*
by (*induct N arbitrary*: M) (*auto simp*: *hyp*)
}
note *add-less* = *this*

show *insert a M* $\in$ *?W*
proof (*rule accI*)
fix N **assume** *Nless*: $(N,$ *insert a M*$) \in$ *max-ext r*
hence *asm1*: $\bigwedge x.\ x \in N \Longrightarrow (x,\ a) \in r \vee (\exists\, y \in M.\ (x,\ y) \in r)$
by (*auto elim*!: *max-ext.cases*)

let *?N1* = $\{\ n \in N.\ (n,\ a) \in r\ \}$
let *?N2* = $\{\ n \in N.\ (n,\ a) \notin r\ \}$
have *N*: *?N1* $\cup$ *?N2* = N **by** (*rule set-ext*) *auto*
from *Nless* **have** *finite N* **by** (*auto elim*: *max-ext.cases*)
then have *finites*: *finite ?N1 finite ?N2* **by** *auto*

have *?N2* $\in$ *?W*
proof *cases*
assume [*simp*]: $M = \{\}$
have *Mw*: $\{\} \in$ *?W* **by** (*rule accI*) (*auto elim*: *max-ext.cases*)

```
        from asm1 have ?N2 = {} by auto
        with Mw show ?N2 ∈ ?W by (simp only:)
       next
        assume M ≠ {}
        have N2: (?N2, M) ∈ max-ext r
         by (rule max-extI[OF - - ‹M ≠ {}›]) (insert asm1, auto intro: finites)

        with ‹M ∈ ?W› show ?N2 ∈ ?W by (rule acc-downward)
       qed
       with finites have ?N1 ∪ ?N2 ∈ ?W
        by (rule add-less) simp
       then show N ∈ ?W by (simp only: N)
      qed
     qed
    qed
  next
    assume [simp]: ¬ finite M
    show ?thesis
     by (rule accI) (auto elim: max-ext.cases)
  qed
qed

lemma max-ext-additive:
 (A, B) ∈ max-ext R ⟹ (C, D) ∈ max-ext R ⟹
  (A ∪ C, B ∪ D) ∈ max-ext R
by (force elim!: max-ext.cases)


definition
  min-ext :: ('a × 'a) set ⇒ ('a set × 'a set) set
where
  [code del]: min-ext r = {(X, Y) | X Y. X ≠ {} ∧ (∀ y ∈ Y. (∃ x ∈ X. (x, y) ∈
r))}

lemma min-ext-wf:
  assumes wf r
  shows wf (min-ext r)
proof (rule wfI-min)
  fix Q :: 'a set set
  fix x
  assume nonempty: x ∈ Q
  show ∃ m ∈ Q. (∀ n. (n, m) ∈ min-ext r ⟶ n ∉ Q)
  proof cases
    assume Q = {{}} thus ?thesis by (simp add: min-ext-def)
  next
    assume Q ≠ {{}}
    with nonempty
    obtain e x where x ∈ Q e ∈ x by force
    then have eU: e ∈ ⋃ Q by auto
```

```
    with ‹wf r›
    obtain z where z: z ∈ ⋃ Q ⋀y. (y, z) ∈ r ⟹ y ∉ ⋃ Q
      by (erule wfE-min)
    from z obtain m where m ∈ Q z ∈ m by auto
    from ‹m ∈ Q›
    show ?thesis
    proof (rule, intro bexI allI impI)
      fix n
      assume smaller: (n, m) ∈ min-ext r
     with ‹z ∈ m› obtain y where y: y ∈ n (y, z) ∈ r by (auto simp: min-ext-def)
      then show n ∉ Q using z(2) by auto
    qed
  qed
qed
```

24.8 Weakly decreasing sequences (w.r.t. some well-founded order) stabilize.

This material does not appear to be used any longer.

```
lemma sequence-trans: [| ALL i. (f (Suc i), f i) : r^* |] ==> (f (i+k), f i) : r^*
by (induct k) (auto intro: rtrancl-trans)

lemma wf-weak-decr-stable:
  assumes as: ALL i. (f (Suc i), f i) : r^* wf (r^+)
  shows EX i. ALL k. f (i+k) = f i
proof −
  have lem: !!x. [| ALL i. (f (Suc i), f i) : r^*; wf (r^+) |]
     ==> ALL m. f m = x −−> (EX i. ALL k. f (m+i+k) = f (m+i))
  apply (erule wf-induct, clarify)
  apply (case-tac EX j. (f (m+j), f m) : r^+)
   apply clarify
   apply (subgoal-tac EX i. ALL k. f ((m+j) +i+k) = f ( (m+j) +i) )
    apply clarify
    apply (rule-tac x = j+i in exI)
    apply (simp add: add-ac, blast)
  apply (rule-tac x = 0 in exI, clarsimp)
  apply (drule-tac i = m and k = k in sequence-trans)
  apply (blast elim: rtranclE dest: rtrancl-into-trancl1)
  done

  from lem[OF as, THEN spec, of 0, simplified]
  show ?thesis by auto
qed


lemma weak-decr-stable:
     ALL i. f (Suc i) <= ((f i)::nat) ==> EX i. ALL k. f (i+k) = f i
apply (rule-tac r = pred-nat in wf-weak-decr-stable)
apply (simp add: pred-nat-trancl-eq-le)
```

apply (*intro wf-trancl wf-pred-nat*)
done

24.9 size of a datatype value

use *Tools/Function/size.ML*

setup *Size.setup*

lemma *size-bool* [*code*]:
size (*b*::*bool*) = *0* **by** (*cases b*) *auto*

lemma *nat-size* [*simp*, *code*]: *size* (*n*::*nat*) = *n*
by (*induct n*) *simp-all*

declare *prod.size* [*no-atp*]

lemma [*code*]:
size (*P* :: *'a Predicate.pred*) = *0* **by** (*cases P*) *simp*

lemma [*code*]:
pred-size f P = *0* **by** (*cases P*) *simp*

end

25 FunDef: Function Definitions and Termination Proofs

theory *FunDef*
imports *Wellfounded*
uses
Tools/prop-logic.ML
Tools/sat-solver.ML
(*Tools/Function/function-lib.ML*)
(*Tools/Function/function-common.ML*)
(*Tools/Function/context-tree.ML*)
(*Tools/Function/function-core.ML*)
(*Tools/Function/sum-tree.ML*)
(*Tools/Function/mutual.ML*)
(*Tools/Function/pattern-split.ML*)
(*Tools/Function/function.ML*)
(*Tools/Function/relation.ML*)
(*Tools/Function/measure-functions.ML*)
(*Tools/Function/lexicographic-order.ML*)
(*Tools/Function/pat-completeness.ML*)
(*Tools/Function/fun.ML*)
(*Tools/Function/induction-schema.ML*)

(*Tools/Function/termination.ML*)
(*Tools/Function/scnp-solve.ML*)
(*Tools/Function/scnp-reconstruct.ML*)
begin

25.1 Definitions with default value.

definition
THE-default :: $'a \Rightarrow ('a \Rightarrow bool) \Rightarrow 'a$ **where**
THE-default d P = (*if* $(\exists!x.\ P\ x)$ *then* (*THE x. P x*) *else d*)

lemma *THE-defaultI'*: $\exists!x.\ P\ x \Longrightarrow P$ (*THE-default d P*)
by (*simp add*: *theI' THE-default-def*)

lemma *THE-default1-equality*:
$[\![\exists!x.\ P\ x;\ P\ a]\!] \Longrightarrow$ *THE-default d P* = *a*
by (*simp add*: *the1-equality THE-default-def*)

lemma *THE-default-none*:
$\neg(\exists!x.\ P\ x) \Longrightarrow$ *THE-default d P* = *d*
by (*simp add*:*THE-default-def*)

lemma *fundef-ex1-existence*:
assumes *f-def*: *f* == $(\lambda x::'a.$ *THE-default* (*d x*) $(\lambda y.$ *G x y*))
assumes *ex1*: $\exists!y.$ *G x y*
shows *G x* (*f x*)
apply (*simp only*: *f-def*)
apply (*rule THE-defaultI'*)
apply (*rule ex1*)
done

lemma *fundef-ex1-uniqueness*:
assumes *f-def*: *f* == $(\lambda x::'a.$ *THE-default* (*d x*) $(\lambda y.$ *G x y*))
assumes *ex1*: $\exists!y.$ *G x y*
assumes *elm*: *G x* (*h x*)
shows *h x* = *f x*
apply (*simp only*: *f-def*)
apply (*rule THE-default1-equality* [*symmetric*])
apply (*rule ex1*)
apply (*rule elm*)
done

lemma *fundef-ex1-iff*:
assumes *f-def*: *f* == $(\lambda x::'a.$ *THE-default* (*d x*) $(\lambda y.$ *G x y*))
assumes *ex1*: $\exists!y.$ *G x y*
shows (*G x y*) = (*f x* = *y*)
apply (*auto simp*:*ex1 f-def THE-default1-equality*)
apply (*rule THE-defaultI'*)

```
  apply (rule ex1)
  done

lemma fundef-default-value:
  assumes f-def: f == (λx::'a. THE-default (d x) (λy. G x y))
  assumes graph: ⋀x y. G x y ⟹ D x
  assumes ¬ D x
  shows f x = d x
proof −
  have ¬(∃ y. G x y)
  proof
    assume ∃ y. G x y
    hence D x using graph ..
    with ‹¬ D x› show False ..
  qed
  hence ¬(∃!y. G x y) by blast

  thus ?thesis
    unfolding f-def
    by (rule THE-default-none)
qed

definition in-rel-def[simp]:
  in-rel R x y == (x, y) ∈ R

lemma wf-in-rel:
  wf R ⟹ wfP (in-rel R)
  by (simp add: wfP-def)

use Tools/Function/function-lib.ML
use Tools/Function/function-common.ML
use Tools/Function/context-tree.ML
use Tools/Function/function-core.ML
use Tools/Function/sum-tree.ML
use Tools/Function/mutual.ML
use Tools/Function/pattern-split.ML
use Tools/Function/relation.ML
use Tools/Function/function.ML
use Tools/Function/pat-completeness.ML
use Tools/Function/fun.ML
use Tools/Function/induction-schema.ML

setup ⟪
  Function.setup
  #> Pat-Completeness.setup
  #> Function-Fun.setup
  #> Induction-Schema.setup
⟫
```

25.2 Measure Functions

inductive *is-measure* :: $('a \Rightarrow nat) \Rightarrow bool$
where *is-measure-trivial*: *is-measure f*

use *Tools/Function/measure-functions.ML*
setup *MeasureFunctions.setup*

lemma *measure-size*[*measure-function*]: *is-measure size*
by (*rule is-measure-trivial*)

lemma *measure-fst*[*measure-function*]: *is-measure f* $\Longrightarrow$ *is-measure* $(\lambda p.\ f\ (fst\ p))$
by (*rule is-measure-trivial*)
lemma *measure-snd*[*measure-function*]: *is-measure f* $\Longrightarrow$ *is-measure* $(\lambda p.\ f\ (snd\ p))$
by (*rule is-measure-trivial*)

use *Tools/Function/lexicographic-order.ML*
setup *Lexicographic-Order.setup*

25.3 Congruence Rules

lemma *let-cong* [*fundef-cong*]:
$M = N \Longrightarrow (\bigwedge x.\ x = N \Longrightarrow f\ x = g\ x) \Longrightarrow Let\ M\ f = Let\ N\ g$
unfolding *Let-def* **by** *blast*

lemmas [*fundef-cong*] =
if-cong image-cong INT-cong UN-cong
bex-cong ball-cong imp-cong

lemma *split-cong* [*fundef-cong*]:
$(\bigwedge x\ y.\ (x, y) = q \Longrightarrow f\ x\ y = g\ x\ y) \Longrightarrow p = q$
$\Longrightarrow split\ f\ p = split\ g\ q$
by (*auto simp*: *split-def*)

lemma *comp-cong* [*fundef-cong*]:
$f\ (g\ x) = f'\ (g'\ x') \Longrightarrow (f\ o\ g)\ x = (f'\ o\ g')\ x'$
unfolding *o-apply* **.**

25.4 Simp rules for termination proofs

lemma *termination-basic-simps*[*termination-simp*]:
$x < (y::nat) \Longrightarrow x < y + z$
$x < z \Longrightarrow x < y + z$
$x \leq y \Longrightarrow x \leq y + (z::nat)$
$x \leq z \Longrightarrow x \leq y + (z::nat)$
$x < y \Longrightarrow x \leq (y::nat)$
by *arith+*

declare *le-imp-less-Suc*[*termination-simp*]

lemma *prod-size-simp*[*termination-simp*]:
prod-size f g p = *f* (*fst p*) + *g* (*snd p*) + *Suc 0*
by (*induct p*) *auto*

25.5 Decomposition

lemma *less-by-empty*:
$A = \{\} \Longrightarrow A \subseteq B$
and *union-comp-emptyL*:
$[\![A\ O\ C = \{\};\ B\ O\ C = \{\}]\!] \Longrightarrow (A \cup B)\ O\ C = \{\}$
and *union-comp-emptyR*:
$[\![A\ O\ B = \{\};\ A\ O\ C = \{\}]\!] \Longrightarrow A\ O\ (B \cup C) = \{\}$
and *wf-no-loop*:
$R\ O\ R = \{\} \Longrightarrow wf\ R$
by (*auto simp add*: *wf-comp-self*[*of R*])

25.6 Reduction Pairs

definition
reduction-pair P = (*wf* (*fst P*) ∧ *fst P O snd P* ⊆ *fst P*)

lemma *reduction-pairI*[*intro*]: *wf R* ⟹ *R O S* ⊆ *R* ⟹ *reduction-pair* (*R*, *S*)
unfolding *reduction-pair-def* **by** *auto*

lemma *reduction-pair-lemma*:
assumes *rp*: *reduction-pair P*
assumes *R* ⊆ *fst P*
assumes *S* ⊆ *snd P*
assumes *wf S*
shows *wf* (*R* ∪ *S*)
proof −
from *rp* ‹*S* ⊆ *snd P*› **have** *wf* (*fst P*) *fst P O S* ⊆ *fst P*
unfolding *reduction-pair-def* **by** *auto*
with ‹*wf S*› **have** *wf* (*fst P* ∪ *S*)
by (*auto intro*: *wf-union-compatible*)
moreover from ‹*R* ⊆ *fst P*› **have** *R* ∪ *S* ⊆ *fst P* ∪ *S* **by** *auto*
ultimately show *?thesis* **by** (*rule wf-subset*)
qed

definition
rp-inv-image = (λ(*R*,*S*) *f*. (*inv-image R f*, *inv-image S f*))

lemma *rp-inv-image-rp*:
reduction-pair P ⟹ *reduction-pair* (*rp-inv-image P f*)
unfolding *reduction-pair-def rp-inv-image-def split-def*
by *force*

25.7 Concrete orders for SCNP termination proofs

definition *pair-less* = *less-than* <∗*lex*∗> *less-than*
definition [*code del*]: *pair-leq* = *pair-less*^=
definition *max-strict* = *max-ext pair-less*
definition [*code del*]: *max-weak* = *max-ext pair-leq* ∪ {({}, {})}
definition [*code del*]: *min-strict* = *min-ext pair-less*
definition [*code del*]: *min-weak* = *min-ext pair-leq* ∪ {({}, {})}

lemma *wf-pair-less*[*simp*]: *wf pair-less*
by (*auto simp*: *pair-less-def*)

Introduction rules for *pair-less*/*pair-leq*

lemma *pair-leqI1*: $a < b \Longrightarrow ((a, s), (b, t)) \in$ *pair-leq*
and *pair-leqI2*: $a \leq b \Longrightarrow s \leq t \Longrightarrow ((a, s), (b, t)) \in$ *pair-leq*
and *pair-lessI1*: $a < b \Longrightarrow ((a, s), (b, t)) \in$ *pair-less*
and *pair-lessI2*: $a \leq b \Longrightarrow s < t \Longrightarrow ((a, s), (b, t)) \in$ *pair-less*
unfolding *pair-leq-def pair-less-def* **by** *auto*

Introduction rules for max

lemma *smax-emptyI*:
finite $Y \Longrightarrow Y \neq \{\} \Longrightarrow (\{\}, Y) \in$ *max-strict*
and *smax-insertI*:
$[\![y \in Y; (x, y) \in$ *pair-less*; $(X, Y) \in$ *max-strict*$]\!] \Longrightarrow (insert\ x\ X, Y) \in$ *max-strict*
and *wmax-emptyI*:
finite $X \Longrightarrow (\{\}, X) \in$ *max-weak*
and *wmax-insertI*:
$[\![y \in YS; (x, y) \in$ *pair-leq*; $(XS, YS) \in$ *max-weak*$]\!] \Longrightarrow (insert\ x\ XS, YS) \in$ *max-weak*
unfolding *max-strict-def max-weak-def* **by** (*auto elim*!: *max-ext.cases*)

Introduction rules for min

lemma *smin-emptyI*:
$X \neq \{\} \Longrightarrow (X, \{\}) \in$ *min-strict*
and *smin-insertI*:
$[\![x \in XS; (x, y) \in$ *pair-less*; $(XS, YS) \in$ *min-strict*$]\!] \Longrightarrow (XS, insert\ y\ YS) \in$ *min-strict*
and *wmin-emptyI*:
$(X, \{\}) \in$ *min-weak*
and *wmin-insertI*:
$[\![x \in XS; (x, y) \in$ *pair-leq*; $(XS, YS) \in$ *min-weak*$]\!] \Longrightarrow (XS, insert\ y\ YS) \in$ *min-weak*
by (*auto simp*: *min-strict-def min-weak-def min-ext-def*)

Reduction Pairs

lemma *max-ext-compat*:
assumes *R O S* ⊆ *R*
shows *max-ext R O* (*max-ext S* ∪ {({},{})}) ⊆ *max-ext R*
using *assms*

apply *auto*
apply (*elim max-ext.cases*)
apply *rule*
apply *auto*[*3*]
apply (*drule-tac x=xa* **in** *meta-spec*)
apply *simp*
apply (*erule bexE*)
apply (*drule-tac x=xb* **in** *meta-spec*)
by *auto*

lemma *max-rpair-set*: *reduction-pair* (*max-strict*, *max-weak*)
unfolding *max-strict-def max-weak-def*
apply (*intro reduction-pairI max-ext-wf*)
apply *simp*
apply (*rule max-ext-compat*)
by (*auto simp*: *pair-less-def pair-leq-def*)

lemma *min-ext-compat*:
assumes *R O S* ⊆ *R*
shows *min-ext R O* (*min-ext S* ∪ {({},{})}) ⊆ *min-ext R*
using *assms*
apply (*auto simp*: *min-ext-def*)
apply (*drule-tac x=ya* **in** *bspec*, *assumption*)
apply (*erule bexE*)
apply (*drule-tac x=xc* **in** *bspec*)
apply *assumption*
by *auto*

lemma *min-rpair-set*: *reduction-pair* (*min-strict*, *min-weak*)
unfolding *min-strict-def min-weak-def*
apply (*intro reduction-pairI min-ext-wf*)
apply *simp*
apply (*rule min-ext-compat*)
by (*auto simp*: *pair-less-def pair-leq-def*)

25.8 Tool setup

use *Tools/Function/termination.ML*
use *Tools/Function/scnp-solve.ML*
use *Tools/Function/scnp-reconstruct.ML*

setup ⟪ *ScnpReconstruct.setup* ⟫

ML-val — setup inactive
⟪
Context.theory-map (*Function-Common.set-termination-prover*
(*ScnpReconstruct.decomp-scnp-tac* [*ScnpSolve.MAX*, *ScnpSolve.MIN*, *ScnpSolve.MS*]))
⟫

end

26 Extraction: Program extraction for HOL

theory *Extraction*
imports *Option*
uses *Tools/rewrite-hol-proof.ML*
begin

26.1 Setup

setup ⟪
  Extraction.add-types
    [(*bool*, ([], *NONE*))] #>
  Extraction.set-preprocessor (*fn thy* =>
    Proofterm.rewrite-proof-notypes
      ([], *RewriteHOLProof.elim-cong* :: *ProofRewriteRules.rprocs true*) *o*
    Proofterm.rewrite-proof thy
      (*RewriteHOLProof.rews*,
       ProofRewriteRules.rprocs true @ [*ProofRewriteRules.expand-of-class thy*]) *o*
    ProofRewriteRules.elim-vars (*curry Const* @{*const-name default*}))
⟫

lemmas [*extraction-expand*] =
  meta-spec atomize-eq atomize-all atomize-imp atomize-conj
  allE rev-mp conjE Eq-TrueI Eq-FalseI eqTrueI eqTrueE eq-cong2
  notE′ impE′ impE iffE imp-cong simp-thms eq-True eq-False
  induct-forall-eq induct-implies-eq induct-equal-eq induct-conj-eq
  induct-atomize induct-atomize′ induct-rulify induct-rulify′
  induct-rulify-fallback induct-trueI
  True-implies-equals TrueE

lemmas [*extraction-expand-def*] =
  induct-forall-def induct-implies-def induct-equal-def induct-conj-def
  induct-true-def induct-false-def

datatype *sumbool* = *Left* | *Right*

26.2 Type of extracted program

extract-type
  typeof (*Trueprop P*) ≡ *typeof P*

  typeof P ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof Q* ≡ *Type* (*TYPE*('*Q*)) ⟹
    typeof (*P* ⟶ *Q*) ≡ *Type* (*TYPE*('*Q*))

  typeof Q ≡ *Type* (*TYPE*(*Null*)) ⟹ *typeof* (*P* ⟶ *Q*) ≡ *Type* (*TYPE*(*Null*))

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$typeof\ (P \longrightarrow Q) \equiv Type\ (TYPE('P \Rightarrow 'Q))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$typeof\ (\forall x.\ P\ x) \equiv Type\ (TYPE(Null))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE('P))) \Longrightarrow$
$typeof\ (\forall x{::}'a.\ P\ x) \equiv Type\ (TYPE('a \Rightarrow 'P))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$typeof\ (\exists x{::}'a.\ P\ x) \equiv Type\ (TYPE('a))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE('P))) \Longrightarrow$
$typeof\ (\exists x{::}'a.\ P\ x) \equiv Type\ (TYPE('a \times 'P))$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ (P \vee Q) \equiv Type\ (TYPE(sumbool))$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$typeof\ (P \vee Q) \equiv Type\ (TYPE('Q\ option))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ (P \vee Q) \equiv Type\ (TYPE('P\ option))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$typeof\ (P \vee Q) \equiv Type\ (TYPE('P + 'Q))$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$typeof\ (P \wedge Q) \equiv Type\ (TYPE('Q))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ (P \wedge Q) \equiv Type\ (TYPE('P))$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow typeof\ Q \equiv Type\ (TYPE('Q)) \Longrightarrow$
$typeof\ (P \wedge Q) \equiv Type\ (TYPE('P \times 'Q))$

$typeof\ (P = Q) \equiv typeof\ ((P \longrightarrow Q) \wedge (Q \longrightarrow P))$

$typeof\ (x \in P) \equiv typeof\ P$

26.3 Realizability

realizability
$(realizes\ t\ (Trueprop\ P)) \equiv (Trueprop\ (realizes\ t\ P))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \longrightarrow Q)) \equiv (realizes\ Null\ P \longrightarrow realizes\ t\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE('P))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x::'P.\ realizes\ x\ P \longrightarrow realizes\ Null\ Q)$

$(realizes\ t\ (P \longrightarrow Q)) \equiv (\forall x.\ realizes\ x\ P \longrightarrow realizes\ (t\ x)\ Q)$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ Null\ (P\ x))$

$(realizes\ t\ (\forall x.\ P\ x)) \equiv (\forall x.\ realizes\ (t\ x)\ (P\ x))$

$(\lambda x.\ typeof\ (P\ x)) \equiv (\lambda x.\ Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ Null\ (P\ t))$

$(realizes\ t\ (\exists x.\ P\ x)) \equiv (realizes\ (snd\ t)\ (P\ (fst\ t)))$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ Left \Rightarrow realizes\ Null\ P \mid Right \Rightarrow realizes\ Null\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ None \Rightarrow realizes\ Null\ P \mid Some\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ None \Rightarrow realizes\ Null\ Q \mid Some\ p \Rightarrow realizes\ p\ P)$

$(realizes\ t\ (P \vee Q)) \equiv$
$(case\ t\ of\ Inl\ p \Rightarrow realizes\ p\ P \mid Inr\ q \Rightarrow realizes\ q\ Q)$

$(typeof\ P) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ Null\ P \wedge realizes\ t\ Q)$

$(typeof\ Q) \equiv (Type\ (TYPE(Null))) \Longrightarrow$
$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ t\ P \wedge realizes\ Null\ Q)$

$(realizes\ t\ (P \wedge Q)) \equiv (realizes\ (fst\ t)\ P \wedge realizes\ (snd\ t)\ Q)$

$typeof\ P \equiv Type\ (TYPE(Null)) \Longrightarrow$
$realizes\ t\ (\neg P) \equiv \neg\ realizes\ Null\ P$

$typeof\ P \equiv Type\ (TYPE('P)) \Longrightarrow$
$realizes\ t\ (\neg P) \equiv (\forall x::'P.\ \neg\ realizes\ x\ P)$

$typeof\ (P::bool) \equiv Type\ (TYPE(Null)) \Longrightarrow$
$typeof\ Q \equiv Type\ (TYPE(Null)) \Longrightarrow$
$realizes\ t\ (P = Q) \equiv realizes\ Null\ P = realizes\ Null\ Q$

(*realizes t* (*P* = *Q*)) ≡ (*realizes t* ((*P* ⟶ *Q*) ∧ (*Q* ⟶ *P*)))

26.4 Computational content of basic inference rules

theorem *disjE-realizer*:
assumes *r*: *case x of Inl p ⇒ P p | Inr q ⇒ Q q*
and *r1*: ⋀*p*. *P p* ⟹ *R* (*f p*) **and** *r2*: ⋀*q*. *Q q* ⟹ *R* (*g q*)
shows *R* (*case x of Inl p ⇒ f p | Inr q ⇒ g q*)
proof (*cases x*)
case *Inl*
with *r* **show** *?thesis* **by** *simp* (*rule r1*)
next
case *Inr*
with *r* **show** *?thesis* **by** *simp* (*rule r2*)
qed

theorem *disjE-realizer2*:
assumes *r*: *case x of None ⇒ P | Some q ⇒ Q q*
and *r1*: *P* ⟹ *R f* **and** *r2*: ⋀*q*. *Q q* ⟹ *R* (*g q*)
shows *R* (*case x of None ⇒ f | Some q ⇒ g q*)
proof (*cases x*)
case *None*
with *r* **show** *?thesis* **by** *simp* (*rule r1*)
next
case *Some*
with *r* **show** *?thesis* **by** *simp* (*rule r2*)
qed

theorem *disjE-realizer3*:
assumes *r*: *case x of Left ⇒ P | Right ⇒ Q*
and *r1*: *P* ⟹ *R f* **and** *r2*: *Q* ⟹ *R g*
shows *R* (*case x of Left ⇒ f | Right ⇒ g*)
proof (*cases x*)
case *Left*
with *r* **show** *?thesis* **by** *simp* (*rule r1*)
next
case *Right*
with *r* **show** *?thesis* **by** *simp* (*rule r2*)
qed

theorem *conjI-realizer*:
P p ⟹ *Q q* ⟹ *P* (*fst* (*p*, *q*)) ∧ *Q* (*snd* (*p*, *q*))
by *simp*

theorem *exI-realizer*:
P y x ⟹ *P* (*snd* (*x*, *y*)) (*fst* (*x*, *y*)) **by** *simp*

theorem *exE-realizer*: *P* (*snd p*) (*fst p*) ⟹

$(\bigwedge x\ y.\ P\ y\ x \Longrightarrow Q\ (f\ x\ y)) \Longrightarrow Q\ (let\ (x,\ y) = p\ in\ f\ x\ y)$
by (*cases p*) (*simp add*: *Let-def*)

theorem *exE-realizer'*: $P\ (snd\ p)\ (fst\ p) \Longrightarrow$
$(\bigwedge x\ y.\ P\ y\ x \Longrightarrow Q) \Longrightarrow Q$ **by** (*cases p*) *simp*

realizers
impI (*P*, *Q*): $\lambda pq.\ pq$
Λ (*c*: -) (*d*: -) *P Q pq* (*h*: -). *allI* · - · *c* · (Λ *x*. *impI* · - · - · - · (*h* · *x*))

impI (*P*): *Null*
Λ (*c*: -) *P Q* (*h*: -). *allI* · - · *c* · (Λ *x*. *impI* · - · - · - · (*h* · *x*))

impI (*Q*): $\lambda q.\ q\ \Lambda$ (*c*: -) *P Q q*. *impI* · - · -

impI: *Null impI*

mp (*P*, *Q*): $\lambda pq.\ pq$
Λ (*c*: -) (*d*: -) *P Q pq* (*h*: -) *p*. *mp* · - · - · (*spec* · - · *p* · *c* · *h*)

mp (*P*): *Null*
Λ (*c*: -) *P Q* (*h*: -) *p*. *mp* · - · - · (*spec* · - · *p* · *c* · *h*)

mp (*Q*): $\lambda q.\ q\ \Lambda$ (*c*: -) *P Q q*. *mp* · - · -

mp: *Null mp*

allI (*P*): $\lambda p.\ p\ \Lambda$ (*c*: -) *P* (*d*: -) *p*. *allI* · - · *d*

allI: *Null allI*

spec (*P*): $\lambda x\ p.\ p\ x\ \Lambda$ (*c*: -) *P x* (*d*: -) *p*. *spec* · - · *x* · *d*

spec: *Null spec*

exI (*P*): $\lambda x\ p.\ (x,\ p)\ \Lambda$ (*c*: -) *P x* (*d*: -) *p*. *exI-realizer* · *P* · *p* · *x* · *c* · *d*

exI: $\lambda x.\ x\ \Lambda\ P\ x$ (*c*: -) (*h*: -). *h*

exE (*P*, *Q*): $\lambda p\ pq.\ let\ (x,\ y) = p\ in\ pq\ x\ y$
Λ (*c*: -) (*d*: -) *P Q* (*e*: -) *p* (*h*: -) *pq*. *exE-realizer* · *P* · *p* · *Q* · *pq* · *c* · *e* · *d* · *h*

exE (*P*): *Null*
Λ (*c*: -) *P Q* (*d*: -) *p*. *exE-realizer'* · - · - · - · - · *c* · *d*

exE (*Q*): $\lambda x\ pq.\ pq\ x$
Λ (*c*: -) *P Q* (*d*: -) *x* (*h1*: -) *pq* (*h2*: -). *h2* · *x* · *h1*

exE: *Null*

Λ *P Q* (*c*: -) *x* (*h1*: -) (*h2*: -). *h2* · *x* • *h1*

conjI (*P*, *Q*): *Pair*
Λ (*c*: -) (*d*: -) *P Q p* (*h*: -) *q*. *conjI-realizer* · *P* · *p* · *Q* · *q* • *c* • *d* • *h*

conjI (*P*): λ*p*. *p*
Λ (*c*: -) *P Q p*. *conjI* · - · -

conjI (*Q*): λ*q*. *q*
Λ (*c*: -) *P Q* (*h*: -) *q*. *conjI* · - · - • *h*

conjI: *Null conjI*

conjunct1 (*P*, *Q*): *fst*
Λ (*c*: -) (*d*: -) *P Q pq*. *conjunct1* · - · -

conjunct1 (*P*): λ*p*. *p*
Λ (*c*: -) *P Q p*. *conjunct1* · - · -

conjunct1 (*Q*): *Null*
Λ (*c*: -) *P Q q*. *conjunct1* · - · -

conjunct1: *Null conjunct1*

conjunct2 (*P*, *Q*): *snd*
Λ (*c*: -) (*d*: -) *P Q pq*. *conjunct2* · - · -

conjunct2 (*P*): *Null*
Λ (*c*: -) *P Q p*. *conjunct2* · - · -

conjunct2 (*Q*): λ*p*. *p*
Λ (*c*: -) *P Q p*. *conjunct2* · - · -

conjunct2: *Null conjunct2*

disjI1 (*P*, *Q*): *Inl*
Λ (*c*: -) (*d*: -) *P Q p*. *iffD2* · - · - • (*sum.cases-1* · *P* · - · *p* • *arity-type-bool* • *c* • *d*)

disjI1 (*P*): *Some*
Λ (*c*: -) *P Q p*. *iffD2* · - · - • (*option.cases-2* · - · *P* · *p* • *arity-type-bool* • *c*)

disjI1 (*Q*): *None*
Λ (*c*: -) *P Q*. *iffD2* · - · - • (*option.cases-1* · - · - • *arity-type-bool* • *c*)

disjI1: *Left*
Λ *P Q*. *iffD2* · - · - • (*sumbool.cases-1* · - · - • *arity-type-bool*)

disjI2 (*P*, *Q*): *Inr*

Λ (*d*: -) (*c*: -) *Q P q*. *iffD2* · - · - · (*sum.cases-2* · - · *Q* · *q* • *arity-type-bool* • *c* • *d*)

disjI2 (*P*): *None*
Λ (*c*: -) *Q P*. *iffD2* · - · - · (*option.cases-1* · - · - · *arity-type-bool* • *c*)

disjI2 (*Q*): *Some*
Λ (*c*: -) *Q P q*. *iffD2* · - · - · (*option.cases-2* · - · *Q* · *q* • *arity-type-bool* • *c*)

disjI2: *Right*
Λ *Q P*. *iffD2* · - · - · (*sumbool.cases-2* · - · - · *arity-type-bool*)

disjE (*P*, *Q*, *R*): λ*pq pr qr*.
(*case pq of Inl p* ⇒ *pr p* | *Inr q* ⇒ *qr q*)
Λ (*c*: -) (*d*: -) (*e*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.
disjE-realizer · - · - · *pq* · *R* · *pr* · *qr* • *c* • *d* • *e* • *h1* • *h2*

disjE (*Q*, *R*): λ*pq pr qr*.
(*case pq of None* ⇒ *pr* | *Some q* ⇒ *qr q*)
Λ (*c*: -) (*d*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.
disjE-realizer2 · - · - · *pq* · *R* · *pr* · *qr* • *c* • *d* • *h1* • *h2*

disjE (*P*, *R*): λ*pq pr qr*.
(*case pq of None* ⇒ *qr* | *Some p* ⇒ *pr p*)
Λ (*c*: -) (*d*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr* (*h3*: -).
disjE-realizer2 · - · - · *pq* · *R* · *qr* · *pr* • *c* • *d* • *h1* • *h3* • *h2*

disjE (*R*): λ*pq pr qr*.
(*case pq of Left* ⇒ *pr* | *Right* ⇒ *qr*)
Λ (*c*: -) *P Q R pq* (*h1*: -) *pr* (*h2*: -) *qr*.
disjE-realizer3 · - · - · *pq* · *R* · *pr* · *qr* • *c* • *h1* • *h2*

disjE (*P*, *Q*): *Null*
Λ (*c*: -) (*d*: -) *P Q R pq*. *disjE-realizer* · - · - · *pq* · (λ*x*. *R*) · - · - · *c* • *d* • *arity-type-bool*

disjE (*Q*): *Null*
Λ (*c*: -) *P Q R pq*. *disjE-realizer2* · - · - · *pq* · (λ*x*. *R*) · - · - · *c* • *arity-type-bool*

disjE (*P*): *Null*
Λ (*c*: -) *P Q R pq* (*h1*: -) (*h2*: -) (*h3*: -).
disjE-realizer2 · - · - · *pq* · (λ*x*. *R*) · - · - · *c* • *arity-type-bool* • *h1* • *h3* • *h2*

disjE: *Null*
Λ *P Q R pq*. *disjE-realizer3* · - · - · *pq* · (λ*x*. *R*) · - · - · *arity-type-bool*

FalseE (*P*): *default*
Λ (*c*: -) *P*. *FalseE* · -

FalseE: *Null FalseE*

notI (*P*): *Null*
  Λ (*c*: -) *P* (*h*: -). *allI* · - · *c* · (Λ *x*. *notI* · - · (*h* · *x*))

notI: *Null notI*

notE (*P*, *R*): λ*p*. *default*
  Λ (*c*: -) (*d*: -) *P R* (*h*: -) *p*. *notE* · - · - · (*spec* · - · *p* · *c* · *h*)

notE (*P*): *Null*
  Λ (*c*: -) *P R* (*h*: -) *p*. *notE* · - · - · (*spec* · - · *p* · *c* · *h*)

notE (*R*): *default*
  Λ (*c*: -) *P R*. *notE* · - · -

notE: *Null notE*

subst (*P*): λ*s t ps*. *ps*
  Λ (*c*: -) *s t P* (*d*: -) (*h*: -) *ps*. *subst* · *s* · *t* · *P ps* · *d* · *h*

subst: *Null subst*

iffD1 (*P*, *Q*): *fst*
  Λ (*d*: -) (*c*: -) *Q P pq* (*h*: -) *p*.
    mp · - · - · (*spec* · - · *p* · *d* · (*conjunct1* · - · - · *h*))

iffD1 (*P*): λ*p*. *p*
  Λ (*c*: -) *Q P p* (*h*: -). *mp* · - · - · (*conjunct1* · - · - · *h*)

iffD1 (*Q*): *Null*
  Λ (*c*: -) *Q P q1* (*h*: -) *q2*.
    mp · - · - · (*spec* · - · *q2* · *c* · (*conjunct1* · - · - · *h*))

iffD1: *Null iffD1*

iffD2 (*P*, *Q*): *snd*
  Λ (*c*: -) (*d*: -) *P Q pq* (*h*: -) *q*.
    mp · - · - · (*spec* · - · *q* · *d* · (*conjunct2* · - · - · *h*))

iffD2 (*P*): λ*p*. *p*
  Λ (*c*: -) *P Q p* (*h*: -). *mp* · - · - · (*conjunct2* · - · - · *h*)

iffD2 (*Q*): *Null*
  Λ (*c*: -) *P Q q1* (*h*: -) *q2*.
    mp · - · - · (*spec* · - · *q2* · *c* · (*conjunct2* · - · - · *h*))

iffD2: *Null iffD2*

iffI (*P*, *Q*): *Pair*
  Λ (*c*: -) (*d*: -) *P Q pq* (*h1* : -) *qp* (*h2* : -). *conjI-realizer* ·
    (λ*pq*. ∀ *x*. *P x* ⟶ *Q* (*pq x*)) · *pq* ·
    (λ*qp*. ∀ *x*. *Q x* ⟶ *P* (*qp x*)) · *qp* ·
    (*arity-type-fun* · *c* · *d*) ·
    (*arity-type-fun* · *d* · *c*) ·
    (*allI* · - · *c* · (Λ *x*. *impI* · - · - · (*h1* · *x*))) ·
    (*allI* · - · *d* · (Λ *x*. *impI* · - · - · (*h2* · *x*)))

iffI (*P*): λ*p*. *p*
  Λ (*c*: -) *P Q* (*h1* : -) *p* (*h2* : -). *conjI* · - · - ·
    (*allI* · - · *c* · (Λ *x*. *impI* · - · - · (*h1* · *x*))) ·
    (*impI* · - · - · *h2*)

iffI (*Q*): λ*q*. *q*
  Λ (*c*: -) *P Q q* (*h1* : -) (*h2* : -). *conjI* · - · - ·
    (*impI* · - · - · *h1*) ·
    (*allI* · - · *c* · (Λ *x*. *impI* · - · - · (*h2* · *x*)))

iffI: *Null iffI*

end

27 Plain: Plain HOL

theory *Plain*
imports *Datatype Record FunDef Extraction*
begin

Plain bootstrap of fundamental HOL tools and packages; does not include *Hilbert-Choice*.

ML ⟪ *path-add* ~~/*src*/*HOL*/*Library* ⟫

end

28 Big-Operators: Big operators and finite (non-empty) sets

theory *Big-Operators*
imports *Plain*
begin

28.1 Generic monoid operation over a set

no-notation *times* (**infixl** ∗ *70*)

no-notation *Groups.one* (*1*)

locale *comm-monoid-big* = *comm-monoid* +
  fixes *F* :: ('*b* ⇒ '*a*) ⇒ '*b set* ⇒ '*a*
  assumes *F-eq*: *F g A* = (*if finite A then fold-image* (*op* ∗) *g 1 A else 1*)

sublocale *comm-monoid-big* < *folding-image* **proof**
qed (*simp add*: *F-eq*)

context *comm-monoid-big*
begin

lemma *infinite* [*simp*]:
  ¬ *finite A* ⟹ *F g A* = *1*
  by (*simp add*: *F-eq*)

end

for ad-hoc proofs for *fold-image*

lemma (**in** *comm-monoid-add*) *comm-monoid-mult*:
  class.comm-monoid-mult (*op* +) *0*
proof qed (*auto intro*: *add-assoc add-commute*)

notation *times* (**infixl** ∗ *70*)
notation *Groups.one* (*1*)

28.2 Generalized summation over a set

definition (**in** *comm-monoid-add*) *setsum* :: ('*b* ⇒ '*a*) => '*b set* => '*a* **where**
  setsum f A = (*if finite A then fold-image* (*op* +) *f 0 A else 0*)

sublocale *comm-monoid-add* < *setsum*!: *comm-monoid-big op* + *0 setsum* **proof**
qed (*fact setsum-def*)

abbreviation
  Setsum (∑ - [*1000*] *999*) **where**
  ∑ *A* == *setsum* (%*x*. *x*) *A*

Now: lot's of fancy syntax. First, *setsum* (λ*x*. *e*) *A* is written ∑ *x*∈*A*. *e*.

syntax
  -setsum :: *pttrn* => '*a set* => '*b* => '*b*::*comm-monoid-add* ((*3SUM* -:-. -) [*0*, *51*, *10*] *10*)
syntax (*xsymbols*)
  -setsum :: *pttrn* => '*a set* => '*b* => '*b*::*comm-monoid-add* ((*3*∑ -∈-. -) [*0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
  -setsum :: *pttrn* => '*a set* => '*b* => '*b*::*comm-monoid-add* ((*3*∑ -∈-. -) [*0*, *51*, *10*] *10*)

translations — Beware of argument permutation!
SUM i:A. b == CONST setsum (%i. b) A
∑ i∈A. b == CONST setsum (%i. b) A

Instead of *∑ x∈{x. P}. e* we introduce the shorter *∑ x|P. e*.

syntax
-qsetsum :: pttrn ⇒ bool ⇒ 'a ⇒ 'a ((3SUM - |/ -./ -) [0,0,10] 10)
syntax (*xsymbols*)
-qsetsum :: pttrn ⇒ bool ⇒ 'a ⇒ 'a ((3∑ - | (-)./ -) [0,0,10] 10)
syntax (*HTML* **output**)
-qsetsum :: pttrn ⇒ bool ⇒ 'a ⇒ 'a ((3∑ - | (-)./ -) [0,0,10] 10)

translations
SUM x|P. t => CONST setsum (%x. t) {x. P}
∑ x|P. t => CONST setsum (%x. t) {x. P}

print-translation ⟪
let
fun setsum-tr' [Abs (x, Tx, t), Const (@{const-syntax Collect}, -) $ Abs (y, Ty, P)] =
if x <> y then raise Match
else
let
val x' = Syntax.mark-bound x;
val t' = subst-bound (x', t);
val P' = subst-bound (x', P);
in Syntax.const @{syntax-const -qsetsum} $ Syntax.mark-bound x $ P' $ t' end
| setsum-tr' - = raise Match;
in [(@{const-syntax setsum}, setsum-tr')] end
⟫

lemma *setsum-empty*:
setsum f {} = 0
by (*fact setsum.empty*)

lemma *setsum-insert*:
finite F ==> a ∉ F ==> setsum f (insert a F) = f a + setsum f F
by (*fact setsum.insert*)

lemma *setsum-infinite*:
~ finite A ==> setsum f A = 0
by (*fact setsum.infinite*)

lemma (**in** *comm-monoid-add*) *setsum-reindex*:
assumes *inj-on f B* **shows** *setsum h (f ' B) = setsum (h ∘ f) B*
proof −
interpret *comm-monoid-mult op + 0* **by** (*fact comm-monoid-mult*)
from *assms* **show** *?thesis* **by** (*auto simp add*: *setsum-def fold-image-reindex*

dest!:*finite-imageD*)
qed

lemma (**in** *comm-monoid-add*) *setsum-reindex-id*:
inj-on f B ==> *setsum f B* = *setsum id* (*f* ‘ *B*)
by (*simp add*: *setsum-reindex*)

lemma (**in** *comm-monoid-add*) *setsum-reindex-nonzero*:
assumes *fS*: *finite S*
and *nz*: ⋀ *x y*. *x* ∈ *S* ⟹ *y* ∈ *S* ⟹ *x* ≠ *y* ⟹ *f x* = *f y* ⟹ *h* (*f x*) = *0*
shows *setsum h* (*f* ‘ *S*) = *setsum* (*h o f*) *S*
using *nz*
proof(*induct rule*: *finite-induct*[*OF fS*])
case *1* **thus** *?case* **by** *simp*
next
case (*2 x F*)
{assume *fxF*: *f x* ∈ *f* ‘ *F* **hence** ∃ *y* ∈ *F* . *f y* = *f x* **by** *auto*
then obtain *y* **where** *y*: *y* ∈ *F f x* = *f y* **by** *auto*
from *2.hyps y* **have** *xy*: *x* ≠ *y* **by** *auto*

from *2.prems*[*of x y*] *2.hyps xy y* **have** *h0*: *h* (*f x*) = *0* **by** *simp*
have *setsum h* (*f* ‘ *insert x F*) = *setsum h* (*f* ‘ *F*) **using** *fxF* **by** *auto*
also have ... = *setsum* (*h o f*) (*insert x F*)
unfolding *setsum.insert*[*OF* ‹*finite F*› ‹*x*∉*F*›]
using *h0*
apply *simp*
apply (*rule 2.hyps*(*3*))
apply (*rule-tac y*=*y* **in** *2.prems*)
apply *simp-all*
done
finally have *?case* **.}**
moreover
{assume *fxF*: *f x* ∉ *f* ‘ *F*
have *setsum h* (*f* ‘ *insert x F*) = *h* (*f x*) + *setsum h* (*f* ‘ *F*)
using *fxF 2.hyps* **by** *simp*
also have ... = *setsum* (*h o f*) (*insert x F*)
unfolding *setsum.insert*[*OF* ‹*finite F*› ‹*x*∉*F*›]
apply *simp*
apply (*rule cong* [*OF refl* [*of op* + (*h* (*f x*))]])
apply (*rule 2.hyps*(*3*))
apply (*rule-tac y*=*y* **in** *2.prems*)
apply *simp-all*
done
finally have *?case* **.}**
ultimately show *?case* **by** *blast*
qed

lemma (**in** *comm-monoid-add*) *setsum-cong*:
A = *B* ==> (!!*x*. *x*:*B* ==> *f x* = *g x*) ==> *setsum f A* = *setsum g B*

by (*cases finite A*) (*auto intro*: *setsum.cong*)

lemma (**in** *comm-monoid-add*) *strong-setsum-cong* [*cong*]:
A = B ==> (!!x. x:B =simp=> f x = g x)
==> setsum (%x. f x) A = setsum (%x. g x) B
by (*rule setsum-cong*) (*simp-all add*: *simp-implies-def*)

lemma (**in** *comm-monoid-add*) *setsum-cong2*: *⟦⋀x. x ∈ A ⟹ f x = g x⟧ ⟹ setsum f A = setsum g A*
by (*auto intro*: *setsum-cong*)

lemma (**in** *comm-monoid-add*) *setsum-reindex-cong*:
[|inj-on f A; B = f ' A; !!a. a:A ⟹ g a = h (f a)|]
==> setsum h B = setsum g A
by (*simp add*: *setsum-reindex cong*: *setsum-cong*)

lemma (**in** *comm-monoid-add*) *setsum-0*[*simp*]: *setsum (%i. 0) A = 0*
by (*cases finite A*) (*erule finite-induct*, *auto*)

lemma (**in** *comm-monoid-add*) *setsum-0'*: *ALL a:A. f a = 0 ==> setsum f A = 0*
by (*simp add*:*setsum-cong*)

lemma (**in** *comm-monoid-add*) *setsum-Un-Int*: *finite A ==> finite B ==>*
setsum g (A Un B) + setsum g (A Int B) = setsum g A + setsum g B
— The reversed orientation looks more natural, but LOOPS as a simprule!
by (*fact setsum.union-inter*)

lemma (**in** *comm-monoid-add*) *setsum-Un-disjoint*: *finite A ==> finite B*
==> A Int B = {} ==> setsum g (A Un B) = setsum g A + setsum g B
by (*fact setsum.union-disjoint*)

lemma *setsum-mono-zero-left*:
assumes *fT*: *finite T* **and** *ST*: *S ⊆ T*
and *z*: *∀ i ∈ T − S. f i = 0*
shows *setsum f S = setsum f T*
proof−
have *eq*: *T = S ∪ (T − S)* **using** *ST* **by** *blast*
have *d*: *S ∩ (T − S) = {}* **using** *ST* **by** *blast*
from *fT ST* **have** *f*: *finite S finite (T − S)* **by** (*auto intro*: *finite-subset*)
show *?thesis*
by (*simp add*: *setsum-Un-disjoint*[*OF f d*, *unfolded eq*[*symmetric*]] *setsum-0'*[*OF z*])
qed

lemma *setsum-mono-zero-right*:
finite T ⟹ S ⊆ T ⟹ ∀ i ∈ T − S. f i = 0 ⟹ setsum f T = setsum f S
by(*blast intro*!: *setsum-mono-zero-left*[*symmetric*])

lemma *setsum-mono-zero-cong-left*:
assumes *fT*: *finite T* **and** *ST*: $S \subseteq T$
and *z*: $\forall i \in T - S.\ g\ i = 0$
and *fg*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ x$
shows *setsum f S* = *setsum g T*
proof−
have *eq*: $T = S \cup (T - S)$ **using** *ST* **by** *blast*
have *d*: $S \cap (T - S) = \{\}$ **using** *ST* **by** *blast*
from *fT ST* **have** *f*: *finite S finite* $(T - S)$ **by** (*auto intro*: *finite-subset*)
show *?thesis*
using *fg* **by** (*simp add*: *setsum-Un-disjoint*[*OF f d*, *unfolded eq*[*symmetric*]]
setsum-0'[*OF z*])
qed

lemma *setsum-mono-zero-cong-right*:
assumes *fT*: *finite T* **and** *ST*: $S \subseteq T$
and *z*: $\forall i \in T - S.\ f\ i = 0$
and *fg*: $\bigwedge x.\ x \in S \Longrightarrow f\ x = g\ x$
shows *setsum f T* = *setsum g S*
using *setsum-mono-zero-cong-left*[*OF fT ST z*] *fg*[*symmetric*] **by** *auto*

lemma *setsum-delta*:
assumes *fS*: *finite S*
shows *setsum* $(\lambda k.$ *if k*=*a then b k else 0*$)$ *S* = (*if a* $\in$ *S then b a else 0*)
proof−
let *?f* = $(\lambda k.$ *if k*=*a then b k else 0*$)$
{assume *a*: $a \notin S$
hence $\forall\ k \in S.$ *?f k* = *0* **by** *simp*
hence *?thesis* **using** *a* **by** *simp***}**
moreover
{assume *a*: $a \in S$
let *?A* = $S - \{a\}$
let *?B* = $\{a\}$
have *eq*: *S* = *?A* $\cup$ *?B* **using** *a* **by** *blast*
have *dj*: *?A* $\cap$ *?B* = $\{\}$ **by** *simp*
from *fS* **have** *fAB*: *finite ?A finite ?B* **by** *auto*
have *setsum ?f S* = *setsum ?f ?A* + *setsum ?f ?B*
using *setsum-Un-disjoint*[*OF fAB dj*, *of ?f*, *unfolded eq*[*symmetric*]]
by *simp*
then have *?thesis* **using** *a* **by** *simp***}**
ultimately show *?thesis* **by** *blast*
qed
lemma *setsum-delta'*:
assumes *fS*: *finite S* **shows**
setsum $(\lambda k.$ *if a* = *k then b k else 0*$)$ *S* =
(*if a*$\in$ *S then b a else 0*)
using *setsum-delta*[*OF fS*, *of a b*, *symmetric*]
by (*auto intro*: *setsum-cong*)

lemma *setsum-restrict-set*:
assumes *fA*: *finite A*
shows *setsum f* $(A \cap B)$ = *setsum* $(\lambda x.$ *if* $x \in B$ *then f x else 0*$)$ *A*
proof−
from *fA* **have** *fab*: *finite* $(A \cap B)$ **by** *auto*
have *aba*: $A \cap B \subseteq A$ **by** *blast*
let *?g* = $\lambda x.$ *if* $x \in A \cap B$ *then f x else 0*
from *setsum-mono-zero-left*[*OF fA aba*, *of ?g*]
show *?thesis* **by** *simp*
qed

lemma *setsum-cases*:
assumes *fA*: *finite A*
shows *setsum* $(\lambda x.$ *if P x then f x else g x*$)$ *A* =
setsum f $(A \cap \{x.\ P\ x\})$ + *setsum g* $(A \cap -\{x.\ P\ x\})$
proof−
have *a*: $A = A \cap \{x.\ P\ x\} \cup A \cap -\{x.\ P\ x\}$
$(A \cap \{x.\ P\ x\}) \cap (A \cap -\{x.\ P\ x\}) = \{\}$
by *blast*+
from *fA*
have *f*: *finite* $(A \cap \{x.\ P\ x\})$ *finite* $(A \cap -\{x.\ P\ x\})$ **by** *auto*
let *?g* = $\lambda x.$ *if P x then f x else g x*
from *setsum-Un-disjoint*[*OF f a*(*2*), *of ?g*] *a*(*1*)
show *?thesis* **by** *simp*
qed

lemma (**in** *comm-monoid-add*) *setsum-UN-disjoint*:
assumes *finite I* **and** *ALL i*:*I*. *finite* (*A i*)
and *ALL i*:*I*. *ALL j*:*I*. $i \neq j$ −−> *A i Int A j* = {}
shows *setsum f* (*UNION I A*) = $(\sum i \in I.$ *setsum f* (*A i*))
proof −
interpret *comm-monoid-mult op* + *0* **by** (*fact comm-monoid-mult*)
from *assms* **show** *?thesis* **by** (*simp add*: *setsum-def fold-image-UN-disjoint cong*: *setsum-cong*)
qed

No need to assume that C is finite. If infinite, the rhs is directly 0, and $\bigcup C$ is also infinite, hence the lhs is also 0.

lemma *setsum-Union-disjoint*:
[| (*ALL A*:*C*. *finite A*);
(*ALL A*:*C*. *ALL B*:*C*. $A \neq B$ −−> *A Int B* = {}) |]
==> *setsum f* (*Union C*) = *setsum* (*setsum f*) *C*
apply (*cases finite C*)
prefer *2* **apply** (*force dest*: *finite-UnionD simp add*: *setsum-def*)
apply (*frule setsum-UN-disjoint* [*of C id f*])
apply (*unfold Union-def id-def*, *assumption*+)
done

lemma (**in** *comm-monoid-add*) *setsum-Sigma*:
assumes *finite A* **and** *ALL x:A. finite* (*B x*)
shows $(\sum x{\in}A.\ (\sum y{\in}B\ x.\ f\ x\ y)) = (\sum (x,y){\in}(SIGMA\ x{:}A.\ B\ x).\ f\ x\ y)$
proof −
interpret *comm-monoid-mult op + 0* **by** (*fact comm-monoid-mult*)
from *assms* **show** *?thesis* **by** (*simp add*: *setsum-def fold-image-Sigma split-def cong*: *setsum-cong*)
qed

Here we can eliminate the finiteness assumptions, by cases.

lemma *setsum-cartesian-product*:
$(\sum x{\in}A.\ (\sum y{\in}B.\ f\ x\ y)) = (\sum (x,y) \in A <*> B.\ f\ x\ y)$
apply (*cases finite A*)
apply (*cases finite B*)
apply (*simp add*: *setsum-Sigma*)
apply (*cases A={}*, *simp*)
apply (*simp*)
apply (*auto simp add*: *setsum-def*
dest: *finite-cartesian-productD1 finite-cartesian-productD2*)
done

lemma (**in** *comm-monoid-add*) *setsum-addf*: *setsum* (%*x*. *f x* + *g x*) *A* = (*setsum f A* + *setsum g A*)
by (*cases finite A*) (*simp-all add*: *setsum.distrib*)

28.2.1 Properties in more restricted classes of structures

lemma *setsum-SucD*: *setsum f A = Suc n ==> EX a:A. 0 < f a*
apply (*case-tac finite A*)
prefer *2* **apply** (*simp add*: *setsum-def*)
apply (*erule rev-mp*)
apply (*erule finite-induct*, *auto*)
done

lemma *setsum-eq-0-iff* [*simp*]:
finite F ==> (setsum f F = 0) = (ALL a:F. f a = (0::nat))
by (*induct set*: *finite*) *auto*

lemma *setsum-eq-Suc0-iff*: *finite A* ⟹
(*setsum f A = Suc 0*) = (*EX a:A. f a = Suc 0* & (*ALL b:A. a≠b* ⟶ *f b = 0*))
apply(*erule finite-induct*)
apply (*auto simp add*:*add-is-1*)
done

lemmas *setsum-eq-1-iff = setsum-eq-Suc0-iff*[*simplified One-nat-def*[*symmetric*]]

lemma *setsum-Un-nat*: *finite A ==> finite B ==>*

(*setsum f* (*A Un B*) :: *nat*) = *setsum f A* + *setsum f B* − *setsum f* (*A Int B*)
— For the natural numbers, we have subtraction.
by (*subst setsum-Un-Int* [*symmetric*], *auto simp add*: *algebra-simps*)

lemma *setsum-Un*: *finite A* ==> *finite B* ==>
(*setsum f* (*A Un B*) :: *'a* :: *ab-group-add*) =
setsum f A + *setsum f B* − *setsum f* (*A Int B*)
by (*subst setsum-Un-Int* [*symmetric*], *auto simp add*: *algebra-simps*)

lemma (**in** *comm-monoid-add*) *setsum-eq-general-reverses*:
assumes *fS*: *finite S* **and** *fT*: *finite T*
and *kh*: $\bigwedge y.\ y \in T \Longrightarrow k\ y \in S \wedge h\ (k\ y) = y$
and *hk*: $\bigwedge x.\ x \in S \Longrightarrow h\ x \in T \wedge k\ (h\ x) = x \wedge g\ (h\ x) = f\ x$
shows *setsum f S* = *setsum g T*
proof −
interpret *comm-monoid-mult op* + *0* **by** (*fact comm-monoid-mult*)
show *?thesis*
apply (*simp add*: *setsum-def fS fT*)
apply (*rule fold-image-eq-general-inverses*)
apply (*rule fS*)
apply (*erule kh*)
apply (*erule hk*)
done
qed

lemma (**in** *comm-monoid-add*) *setsum-Un-zero*:
assumes *fS*: *finite S* **and** *fT*: *finite T*
and *I0*: $\forall x \in S \cap T.\ f\ x = 0$
shows *setsum f* (*S* ∪ *T*) = *setsum f S* + *setsum f T*
proof −
interpret *comm-monoid-mult op* + *0* **by** (*fact comm-monoid-mult*)
show *?thesis*
using *fS fT*
apply (*simp add*: *setsum-def*)
apply (*rule fold-image-Un-one*)
using *I0* **by** *auto*
qed

lemma *setsum-UNION-zero*:
assumes *fS*: *finite S* **and** *fSS*: ∀ *T* ∈ *S*. *finite T*
and *f0*: $\bigwedge T1\ T2\ x.\ T1 \in S \Longrightarrow T2 \in S \Longrightarrow T1 \neq T2 \Longrightarrow x \in T1 \Longrightarrow x \in T2 \Longrightarrow f\ x = 0$
shows *setsum f* (⋃ *S*) = *setsum* (λ*T*. *setsum f T*) *S*
using *fSS f0*
proof(*induct rule*: *finite-induct*[*OF fS*])
case *1* **thus** *?case* **by** *simp*
next
case (*2 T F*)
then have *fTF*: *finite T* ∀ *T*∈*F*. *finite T finite F* **and** *TF*: *T* ∉ *F*

```
    and H: setsum f (⋃ F) = setsum (setsum f) F by auto
  from fTF have fUF: finite (⋃F) by auto
  from 2.prems TF fTF
  show ?case
    by (auto simp add: H[symmetric] intro: setsum-Un-zero[OF fTF(1) fUF, of
f])
qed

lemma setsum-diff1-nat: (setsum f (A - {a}) :: nat) =
  (if a:A then setsum f A - f a else setsum f A)
apply (case-tac finite A)
 prefer 2 apply (simp add: setsum-def)
apply (erule finite-induct)
 apply (auto simp add: insert-Diff-if)
apply (drule-tac a = a in mk-disjoint-insert, auto)
done

lemma setsum-diff1: finite A ⟹
  (setsum f (A - {a}) :: ('a::ab-group-add)) =
  (if a:A then setsum f A - f a else setsum f A)
by (erule finite-induct) (auto simp add: insert-Diff-if)

lemma setsum-diff1'[rule-format]:
  finite A ⟹ a ∈ A ⟶ (∑ x ∈ A. f x) = f a + (∑ x ∈ (A - {a}). f x)
apply (erule finite-induct[where F=A and P=% A. (a ∈ A ⟶ (∑ x ∈ A. f x)
= f a + (∑ x ∈ (A - {a}). f x))])
apply (auto simp add: insert-Diff-if add-ac)
done

lemma setsum-diff1-ring: assumes finite A a ∈ A
  shows setsum f (A - {a}) = setsum f A - (f a::'a::ring)
unfolding setsum-diff1'[OF assms] by auto



lemma setsum-diff-nat:
assumes finite B and B ⊆ A
shows (setsum f (A - B) :: nat) = (setsum f A) - (setsum f B)
using assms
proof induct
  show setsum f (A - {}) = (setsum f A) - (setsum f {}) by simp
next
  fix F x assume finF: finite F and xnotinF: x ∉ F
    and xFinA: insert x F ⊆ A
    and IH: F ⊆ A ⟹ setsum f (A - F) = setsum f A - setsum f F
  from xnotinF xFinA have xinAF: x ∈ (A - F) by simp
  from xinAF have A: setsum f ((A - F) - {x}) = setsum f (A - F) - f x
    by (simp add: setsum-diff1-nat)
  from xFinA have F ⊆ A by simp
```

```
  with IH have setsum f (A − F) = setsum f A − setsum f F by simp
  with A have B: setsum f ((A − F) − {x}) = setsum f A − setsum f F − f x
    by simp
  from xnotinF have A − insert x F = (A − F) − {x} by auto
  with B have C: setsum f (A − insert x F) = setsum f A − setsum f F − f x
    by simp
  from finF xnotinF have setsum f (insert x F) = setsum f F + f x by simp
  with C have setsum f (A − insert x F) = setsum f A − setsum f (insert x F)
    by simp
  thus setsum f (A − insert x F) = setsum f A − setsum f (insert x F) by simp
qed

lemma setsum-diff:
  assumes le: finite A B ⊆ A
  shows setsum f (A − B) = setsum f A − ((setsum f B)::('a::ab-group-add))
proof −
  from le have finiteB: finite B using finite-subset by auto
  show ?thesis using finiteB le
  proof induct
    case empty
    thus ?case by auto
  next
    case (insert x F)
    thus ?case using le finiteB
      by (simp add: Diff-insert[where a=x and B=F] setsum-diff1 insert-absorb)
  qed
qed

lemma setsum-mono:
 assumes le: ⋀i. i∈K ⟹ f (i::'a) ≤ ((g i)::('b::{comm-monoid-add, ordered-ab-semigroup-add}))
  shows (∑ i∈K. f i) ≤ (∑ i∈K. g i)
proof (cases finite K)
  case True
  thus ?thesis using le
  proof induct
    case empty
    thus ?case by simp
  next
    case insert
    thus ?case using add-mono by fastsimp
  qed
next
  case False
  thus ?thesis
    by (simp add: setsum-def)
qed

lemma setsum-strict-mono:
  fixes f :: 'a ⇒ 'b::{ordered-cancel-ab-semigroup-add,comm-monoid-add}
```

assumes *finite A* $A \neq \{\}$
and $!!x.\ x{:}A \Longrightarrow f\ x < g\ x$
shows *setsum f A* < *setsum g A*
using *prems*
proof (*induct rule*: *finite-ne-induct*)
case *singleton* **thus** *?case* **by** *simp*
next
case *insert* **thus** *?case* **by** (*auto simp*: *add-strict-mono*)
qed

lemma *setsum-negf*:
setsum (%*x*. − (*f x*)::'*a*::*ab-group-add*) *A* = − *setsum f A*
proof (*cases finite A*)
case *True* **thus** *?thesis* **by** (*induct set*: *finite*) *auto*
next
case *False* **thus** *?thesis* **by** (*simp add*: *setsum-def*)
qed

lemma *setsum-subtractf*:
setsum (%*x*. ((*f x*)::'*a*::*ab-group-add*) − *g x*) *A* =
setsum f A − *setsum g A*
proof (*cases finite A*)
case *True* **thus** *?thesis* **by** (*simp add*: *diff-minus setsum-addf setsum-negf*)
next
case *False* **thus** *?thesis* **by** (*simp add*: *setsum-def*)
qed

lemma *setsum-nonneg*:
assumes *nn*: ∀ *x*∈*A*. (*0*::'*a*::{*ordered-ab-semigroup-add*,*comm-monoid-add*}) ≤ *f x*
shows *0* ≤ *setsum f A*
proof (*cases finite A*)
case *True* **thus** *?thesis* **using** *nn*
proof *induct*
case *empty* **then show** *?case* **by** *simp*
next
case (*insert x F*)
then have *0* + *0* ≤ *f x* + *setsum f F* **by** (*blast intro*: *add-mono*)
with *insert* **show** *?case* **by** *simp*
qed
next
case *False* **thus** *?thesis* **by** (*simp add*: *setsum-def*)
qed

lemma *setsum-nonpos*:
assumes *np*: ∀ *x*∈*A*. *f x* ≤ (*0*::'*a*::{*ordered-ab-semigroup-add*,*comm-monoid-add*})
shows *setsum f A* ≤ *0*
proof (*cases finite A*)
case *True* **thus** *?thesis* **using** *np*

```
  proof induct
    case empty then show ?case by simp
  next
    case (insert x F)
    then have f x + setsum f F ≤ 0 + 0 by (blast intro: add-mono)
    with insert show ?case by simp
  qed
next
  case False thus ?thesis by (simp add: setsum-def)
qed

lemma setsum-nonneg-leq-bound:
  fixes f :: 'a ⇒ 'b::{ordered-ab-group-add}
  assumes finite s ⋀i. i ∈ s ⟹ f i ≥ 0 (∑ i ∈ s. f i) = B i ∈ s
  shows f i ≤ B
proof −
  have 0 ≤ (∑ i ∈ s − {i}. f i) and 0 ≤ f i
    using assms by (auto intro!: setsum-nonneg)
  moreover
  have (∑ i ∈ s − {i}. f i) + f i = B
    using assms by (simp add: setsum-diff1)
  ultimately show ?thesis by auto
qed

lemma setsum-nonneg-0:
  fixes f :: 'a ⇒ 'b::{ordered-ab-group-add}
  assumes finite s and pos: ⋀ i. i ∈ s ⟹ f i ≥ 0
  and (∑ i ∈ s. f i) = 0 and i: i ∈ s
  shows f i = 0
  using setsum-nonneg-leq-bound[OF assms] pos[OF i] by auto

lemma setsum-mono2:
fixes f :: 'a ⇒ 'b :: ordered-comm-monoid-add
assumes fin: finite B and sub: A ⊆ B and nn: ⋀b. b ∈ B−A ⟹ 0 ≤ f b
shows setsum f A ≤ setsum f B
proof −
  have setsum f A ≤ setsum f A + setsum f (B−A)
    by(simp add: add-increasing2[OF setsum-nonneg] nn Ball-def)
  also have ... = setsum f (A ∪ (B−A)) using fin finite-subset[OF sub fin]
    by (simp add:setsum-Un-disjoint del:Un-Diff-cancel)
  also have A ∪ (B−A) = B using sub by blast
  finally show ?thesis .
qed

lemma setsum-mono3: finite B ==> A <= B ==>
    ALL x: B − A.
      0 <= ((f x)::'a::{comm-monoid-add,ordered-ab-semigroup-add}) ==>
        setsum f A <= setsum f B
  apply (subgoal-tac setsum f B = setsum f A + setsum f (B − A))
```

```
  apply (erule ssubst)
  apply (subgoal-tac setsum f A + 0 <= setsum f A + setsum f (B - A))
  apply simp
  apply (rule add-left-mono)
  apply (erule setsum-nonneg)
  apply (subst setsum-Un-disjoint [THEN sym])
  apply (erule finite-subset, assumption)
  apply (rule finite-subset)
  prefer 2
  apply assumption
  apply (auto simp add: sup-absorb2)
done

lemma setsum-right-distrib:
  fixes f :: 'a => ('b::semiring-0)
  shows r * setsum f A = setsum (%n. r * f n) A
proof (cases finite A)
  case True
  thus ?thesis
  proof induct
    case empty thus ?case by simp
  next
    case (insert x A) thus ?case by (simp add: right-distrib)
  qed
next
  case False thus ?thesis by (simp add: setsum-def)
qed

lemma setsum-left-distrib:
  setsum f A * (r::'a::semiring-0) = (∑ n∈A. f n * r)
proof (cases finite A)
  case True
  then show ?thesis
  proof induct
    case empty thus ?case by simp
  next
    case (insert x A) thus ?case by (simp add: left-distrib)
  qed
next
  case False thus ?thesis by (simp add: setsum-def)
qed

lemma setsum-divide-distrib:
  setsum f A / (r::'a::field) = (∑ n∈A. f n / r)
proof (cases finite A)
  case True
  then show ?thesis
  proof induct
    case empty thus ?case by simp
```

```
  next
    case (insert x A) thus ?case by (simp add: add-divide-distrib)
  qed
next
  case False thus ?thesis by (simp add: setsum-def)
qed

lemma setsum-abs[iff]:
  fixes f :: 'a => ('b::ordered-ab-group-add-abs)
  shows abs (setsum f A) ≤ setsum (%i. abs(f i)) A
proof (cases finite A)
  case True
  thus ?thesis
  proof induct
    case empty thus ?case by simp
  next
    case (insert x A)
    thus ?case by (auto intro: abs-triangle-ineq order-trans)
  qed
next
  case False thus ?thesis by (simp add: setsum-def)
qed

lemma setsum-abs-ge-zero[iff]:
  fixes f :: 'a => ('b::ordered-ab-group-add-abs)
  shows 0 ≤ setsum (%i. abs(f i)) A
proof (cases finite A)
  case True
  thus ?thesis
  proof induct
    case empty thus ?case by simp
  next
    case (insert x A) thus ?case by auto
  qed
next
  case False thus ?thesis by (simp add: setsum-def)
qed

lemma abs-setsum-abs[simp]:
  fixes f :: 'a => ('b::ordered-ab-group-add-abs)
  shows abs (∑ a∈A. abs(f a)) = (∑ a∈A. abs(f a))
proof (cases finite A)
  case True
  thus ?thesis
  proof induct
    case empty thus ?case by simp
  next
    case (insert a A)
    hence |∑ a∈insert a A. |f a|| = ||f a| + (∑ a∈A. |f a|)| by simp
```

```
    also have ... = ||f a| + |∑ a∈A. |f a|||  using insert by simp
    also have ... = |f a| + |∑ a∈A. |f a||
      by (simp del: abs-of-nonneg)
    also have ... = (∑ a∈insert a A. |f a|) using insert by simp
    finally show ?case .
  qed
next
  case False thus ?thesis by (simp add: setsum-def)
qed

lemma setsum-Plus:
  fixes A :: 'a set and B :: 'b set
  assumes fin: finite A finite B
  shows setsum f (A <+> B) = setsum (f ∘ Inl) A + setsum (f ∘ Inr) B
proof −
  have A <+> B = Inl ` A ∪ Inr ` B by auto
  moreover from fin have finite (Inl ` A :: ('a + 'b) set) finite (Inr ` B :: ('a +
'b) set)
    by(auto intro: finite-imageI)
  moreover have Inl ` A ∩ Inr ` B = ({} :: ('a + 'b) set) by auto
  moreover have inj-on (Inl :: 'a ⇒ 'a + 'b) A inj-on (Inr :: 'b ⇒ 'a + 'b) B
by(auto intro: inj-onI)
  ultimately show ?thesis using fin by(simp add: setsum-Un-disjoint setsum-reindex)
qed
```

Commuting outer and inner summation

```
lemma setsum-commute:
  (∑ i∈A. ∑ j∈B. f i j) = (∑ j∈B. ∑ i∈A. f i j)
proof (simp add: setsum-cartesian-product)
  have (∑ (x,y) ∈ A <*> B. f x y) =
    (∑ (y,x) ∈ (%(i, j). (j, i)) ` (A × B). f x y)
    (is ?s = -)
    apply (simp add: setsum-reindex [where f = %(i, j). (j, i)] swap-inj-on)
    apply (simp add: split-def)
    done
  also have ... = (∑ (y,x)∈B × A. f x y)
    (is - = ?t)
    apply (simp add: swap-product)
    done
  finally show ?s = ?t .
qed

lemma setsum-product:
  fixes f :: 'a => ('b::semiring-0)
  shows setsum f A * setsum g B = (∑ i∈A. ∑ j∈B. f i * g j)
  by (simp add: setsum-right-distrib setsum-left-distrib) (rule setsum-commute)

lemma setsum-mult-setsum-if-inj:
fixes f :: 'a => ('b::semiring-0)
```

shows *inj-on* (%(*a*,*b*). *f a* ∗ *g b*) (*A* × *B*) ==>
setsum f A ∗ *setsum g B* = *setsum id* {*f a* ∗ *g b*|*a b*. *a*:*A* & *b*:*B*}
by(*auto simp*: *setsum-product setsum-cartesian-product*
intro!: *setsum-reindex-cong*[*symmetric*])

lemma *setsum-constant* [*simp*]: (∑ *x* ∈ *A*. *y*) = *of-nat*(*card A*) ∗ *y*
apply (*cases finite A*)
apply (*erule finite-induct*)
apply (*auto simp add*: *algebra-simps*)
done

lemma *setsum-bounded*:
assumes *le*: ⋀*i*. *i*∈*A* ⟹ *f i* ≤ (*K*::'*a*::{*semiring-1*, *ordered-ab-semigroup-add*})
shows *setsum f A* ≤ *of-nat*(*card A*) ∗ *K*
proof (*cases finite A*)
case *True*
thus *?thesis* **using** *le setsum-mono*[**where** *K*=*A* **and** *g* = %*x*. *K*] **by** *simp*
next
case *False* **thus** *?thesis* **by** (*simp add*: *setsum-def*)
qed

28.2.2 Cardinality as special case of *setsum*

lemma *card-eq-setsum*:
card A = *setsum* (λ*x*. *1*) *A*
by (*simp only*: *card-def setsum-def*)

lemma *card-UN-disjoint*:
finite I ==> (*ALL i*:*I*. *finite* (*A i*)) ==>
(*ALL i*:*I*. *ALL j*:*I*. *i* ≠ *j* --> *A i Int A j* = {}) ==> *card* (*UNION I A*) = (∑ *i*∈*I*. *card*(*A i*))
apply (*simp add*: *card-eq-setsum del*: *setsum-constant*)
apply (*subgoal-tac*
setsum (%*i*. *card* (*A i*)) *I* = *setsum* (%*i*. (*setsum* (%*x*. *1*) (*A i*))) *I*)
apply (*simp add*: *setsum-UN-disjoint del*: *setsum-constant*)
apply (*simp cong*: *setsum-cong*)
done

lemma *card-Union-disjoint*:
finite C ==> (*ALL A*:*C*. *finite A*) ==>
(*ALL A*:*C*. *ALL B*:*C*. *A* ≠ *B* --> *A Int B* = {}) ==> *card* (*Union C*) = *setsum card C*
apply (*frule card-UN-disjoint* [*of C id*])
apply (*unfold Union-def id-def*, *assumption*+)
done

The image of a finite set can be expressed using *fold-image*.

lemma *image-eq-fold-image*:
finite A ==> *f* ' *A* = *fold-image* (*op Un*) (%*x*. {*f x*}) {} *A*

proof (*induct rule*: *finite-induct*)
case *empty* **then show** *?case* **by** *simp*
next
interpret *ab-semigroup-mult op Un*
proof qed *auto*
case *insert*
then show *?case* **by** *simp*
qed

28.2.3 Cardinality of products

lemma *card-SigmaI* [*simp*]:
⟦ *finite A*; *ALL a*:*A*. *finite* (*B a*) ⟧
⟹ *card* (*SIGMA x*: *A*. *B x*) = (∑ *a*∈*A*. *card* (*B a*))
by(*simp add*: *card-eq-setsum setsum-Sigma del*:*setsum-constant*)

lemma *card-cartesian-product*: *card* (*A* <*> *B*) = *card*(*A*) ∗ *card*(*B*)
by (*cases finite A* ∧ *finite B*)
(*auto simp add*: *card-eq-0-iff dest*: *finite-cartesian-productD1 finite-cartesian-productD2*)

lemma *card-cartesian-product-singleton*: *card*({*x*} <*> *A*) = *card*(*A*)
by (*simp add*: *card-cartesian-product*)

28.3 Generalized product over a set

definition (**in** *comm-monoid-mult*) *setprod* :: ('*b* ⇒ '*a*) => '*b set* => '*a* **where**
setprod f A = (*if finite A then fold-image* (*op* ∗) *f 1 A else 1*)

sublocale *comm-monoid-mult* < *setprod*!: *comm-monoid-big op* ∗ *1 setprod* **proof**
qed (*fact setprod-def*)

abbreviation
Setprod (∏ - [*1000*] *999*) **where**
∏ *A* == *setprod* (%*x*. *x*) *A*

syntax
-*setprod* :: *pttrn* => '*a set* => '*b* => '*b*::*comm-monoid-mult* ((*3PROD* -:-. -) [*0*, *51*, *10*] *10*)
syntax (*xsymbols*)
-*setprod* :: *pttrn* => '*a set* => '*b* => '*b*::*comm-monoid-mult* ((*3*∏ -∈-. -) [*0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
-*setprod* :: *pttrn* => '*a set* => '*b* => '*b*::*comm-monoid-mult* ((*3*∏ -∈-. -) [*0*, *51*, *10*] *10*)

translations — Beware of argument permutation!
PROD i:*A*. *b* == *CONST setprod* (%*i*. *b*) *A*
∏ *i*∈*A*. *b* == *CONST setprod* (%*i*. *b*) *A*

Instead of $\prod x{\in}\{x.\ P\}.\ e$ we introduce the shorter $\prod x|P.\ e$.

syntax
-qsetprod :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3PROD* - |/ -./ -) [*0*,*0*,*10*] *10*)
syntax (*xsymbols*)
-qsetprod :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3*∏ - | (-)./ -) [*0*,*0*,*10*] *10*)
syntax (*HTML* **output**)
-qsetprod :: *pttrn* ⇒ *bool* ⇒ *'a* ⇒ *'a* ((*3*∏ - | (-)./ -) [*0*,*0*,*10*] *10*)

translations
PROD x|P. t => *CONST setprod* (%*x. t*) {*x. P*}
∏ *x|P. t* => *CONST setprod* (%*x. t*) {*x. P*}

lemma *setprod-empty*: *setprod f* {} = *1*
by (*fact setprod.empty*)

lemma *setprod-insert*: [| *finite A*; *a* ∉ *A* |] ==>
setprod f (*insert a A*) = *f a* ∗ *setprod f A*
by (*fact setprod.insert*)

lemma *setprod-infinite*: ~ *finite A* ==> *setprod f A* = *1*
by (*fact setprod.infinite*)

lemma *setprod-reindex*:
inj-on f B ==> *setprod h* (*f* ' *B*) = *setprod* (*h* ∘ *f*) *B*
by(*auto simp*: *setprod-def fold-image-reindex dest*!:*finite-imageD*)

lemma *setprod-reindex-id*: *inj-on f B* ==> *setprod f B* = *setprod id* (*f* ' *B*)
by (*auto simp add*: *setprod-reindex*)

lemma *setprod-cong*:
A = *B* ==> (!!*x. x*:*B* ==> *f x* = *g x*) ==> *setprod f A* = *setprod g B*
by(*fastsimp simp*: *setprod-def intro*: *fold-image-cong*)

lemma *strong-setprod-cong*[*cong*]:
A = *B* ==> (!!*x. x*:*B* =*simp*=> *f x* = *g x*) ==> *setprod f A* = *setprod g B*
by(*fastsimp simp*: *simp-implies-def setprod-def intro*: *fold-image-cong*)

lemma *setprod-reindex-cong*: *inj-on f A* ==>
B = *f* ' *A* ==> *g* = *h* ∘ *f* ==> *setprod h B* = *setprod g A*
by (*frule setprod-reindex*, *simp*)

lemma *strong-setprod-reindex-cong*: **assumes** *i*: *inj-on f A*
and *B*: *B* = *f* ' *A* **and** *eq*: ⋀*x. x* ∈ *A* ⟹ *g x* = (*h* ∘ *f*) *x*
shows *setprod h B* = *setprod g A*
proof−
have *setprod h B* = *setprod* (*h o f*) *A*
by (*simp add*: *B setprod-reindex*[*OF i*, *of h*])
then show *?thesis* **apply** *simp*
apply (*rule setprod-cong*)

apply *simp*
by (*simp add*: *eq*)
qed

lemma *setprod-Un-one*:
assumes *fS*: *finite S* **and** *fT*: *finite T*
and *I0*: $\forall x \in S \cap T.\ f\ x = 1$
shows *setprod f* $(S \cup T)$ = *setprod f S* $*$ *setprod f T*
using *fS fT*
apply (*simp add*: *setprod-def*)
apply (*rule fold-image-Un-one*)
using *I0* **by** *auto*

lemma *setprod-1*: *setprod* (%*i*. *1*) *A* = *1*
apply (*case-tac finite A*)
apply (*erule finite-induct*, *auto simp add*: *mult-ac*)
done

lemma *setprod-1'*: *ALL a:F*. *f a* = *1* ==> *setprod f F* = *1*
apply (*subgoal-tac setprod f F* = *setprod* (%*x*. *1*) *F*)
apply (*erule ssubst*, *rule setprod-1*)
apply (*rule setprod-cong*, *auto*)
done

lemma *setprod-Un-Int*: *finite A* ==> *finite B*
==> *setprod g* (*A Un B*) $*$ *setprod g* (*A Int B*) = *setprod g A* $*$ *setprod g B*
by(*simp add*: *setprod-def fold-image-Un-Int*[*symmetric*])

lemma *setprod-Un-disjoint*: *finite A* ==> *finite B*
==> *A Int B* = {} ==> *setprod g* (*A Un B*) = *setprod g A* $*$ *setprod g B*
by (*subst setprod-Un-Int* [*symmetric*], *auto*)

lemma *setprod-mono-one-left*:
assumes *fT*: *finite T* **and** *ST*: $S \subseteq T$
and *z*: $\forall i \in T - S.\ f\ i = 1$
shows *setprod f S* = *setprod f T*
proof−
have *eq*: $T = S \cup (T - S)$ **using** *ST* **by** *blast*
have *d*: $S \cap (T - S) = \{\}$ **using** *ST* **by** *blast*
from *fT ST* **have** *f*: *finite S finite* $(T - S)$ **by** (*auto intro*: *finite-subset*)
show *?thesis*
by (*simp add*: *setprod-Un-disjoint*[*OF f d*, *unfolded eq*[*symmetric*]] *setprod-1'*[*OF z*])
qed

lemmas *setprod-mono-one-right* = *setprod-mono-one-left* [*THEN sym*]

lemma *setprod-delta*:

```
  assumes fS: finite S
  shows setprod (λk. if k=a then b k else 1) S = (if a ∈ S then b a else 1)
proof−
  let ?f = (λk. if k=a then b k else 1)
  {assume a: a ∉ S
   hence ∀ k∈ S. ?f k = 1 by simp
   hence ?thesis  using a by (simp add: setprod-1 cong add: setprod-cong) }
  moreover
  {assume a: a ∈ S
   let ?A = S − {a}
   let ?B = {a}
   have eq: S = ?A ∪ ?B using a by blast
   have dj: ?A ∩ ?B = {} by simp
   from fS have fAB: finite ?A finite ?B by auto
   have fA1: setprod ?f ?A = 1 apply (rule setprod-1′) by auto
   have setprod ?f ?A * setprod ?f ?B = setprod ?f S
     using setprod-Un-disjoint[OF fAB dj, of ?f, unfolded eq[symmetric]]
     by simp
    then have ?thesis  using a by (simp add: fA1 cong add: setprod-cong cong
del: if-weak-cong)}
  ultimately show ?thesis by blast
qed

lemma setprod-delta′:
  assumes fS: finite S shows
  setprod (λk. if a = k then b k else 1) S =
    (if a∈ S then b a else 1)
  using setprod-delta[OF fS, of a b, symmetric]
  by (auto intro: setprod-cong)


lemma setprod-UN-disjoint:
    finite I ==> (ALL i:I. finite (A i)) ==>
       (ALL i:I. ALL j:I. i ≠ j −−> A i Int A j = {}) ==>
      setprod f (UNION I A) = setprod (%i. setprod f (A i)) I
by(simp add: setprod-def fold-image-UN-disjoint cong: setprod-cong)

lemma setprod-Union-disjoint:
  [| (ALL A:C. finite A);
     (ALL A:C. ALL B:C. A ≠ B −−> A Int B = {}) |]
  ==> setprod f (Union C) = setprod (setprod f) C
apply (cases finite C)
 prefer 2 apply (force dest: finite-UnionD simp add: setprod-def)
  apply (frule setprod-UN-disjoint [of C id f])
 apply (unfold Union-def id-def, assumption+)
done

lemma setprod-Sigma: finite A ==> ALL x:A. finite (B x) ==>
    (∏x∈A. (∏y∈ B x. f x y)) =
```

$(\prod (x,y)\in(SIGMA\ x{:}A.\ B\ x).\ f\ x\ y)$
by(*simp add*:*setprod-def fold-image-Sigma split-def cong*:*setprod-cong*)

Here we can eliminate the finiteness assumptions, by cases.

lemma *setprod-cartesian-product*:
$(\prod x\in A.\ (\prod y\in B.\ f\ x\ y)) = (\prod (x,y)\in(A <*> B).\ f\ x\ y)$
apply (*cases finite A*)
apply (*cases finite B*)
apply (*simp add*: *setprod-Sigma*)
apply (*cases A={}*, *simp*)
apply (*simp add*: *setprod-1*)
apply (*auto simp add*: *setprod-def*
dest: *finite-cartesian-productD1 finite-cartesian-productD2*)
done

lemma *setprod-timesf*:
setprod (%*x*. *f x* ∗ *g x*) *A* = (*setprod f A* ∗ *setprod g A*)
by(*simp add*:*setprod-def fold-image-distrib*)

28.3.1 Properties in more restricted classes of structures

lemma *setprod-eq-1-iff* [*simp*]:
finite F ==> (*setprod f F* = *1*) = (*ALL a*:*F*. *f a* = (*1*::*nat*))
by (*induct set*: *finite*) *auto*

lemma *setprod-zero*:
finite A ==> *EX x*: *A*. *f x* = (*0*::'*a*::*comm-semiring-1*) ==> *setprod f A* = *0*
apply (*induct set*: *finite*, *force*, *clarsimp*)
apply (*erule disjE*, *auto*)
done

lemma *setprod-nonneg* [*rule-format*]:
(*ALL x*: *A*. (*0*::'*a*::*linordered-semidom*) ≤ *f x*) −−> *0* ≤ *setprod f A*
by (*cases finite A*, *induct set*: *finite*, *simp-all add*: *mult-nonneg-nonneg*)

lemma *setprod-pos* [*rule-format*]: (*ALL x*: *A*. (*0*::'*a*::*linordered-semidom*) < *f x*)
−−> *0* < *setprod f A*
by (*cases finite A*, *induct set*: *finite*, *simp-all add*: *mult-pos-pos*)

lemma *setprod-zero-iff*[*simp*]: *finite A* ==>
(*setprod f A* = (*0*::'*a*::{*comm-semiring-1*,*no-zero-divisors*})) =
(*EX x*: *A*. *f x* = *0*)
by (*erule finite-induct*, *auto simp*:*no-zero-divisors*)

lemma *setprod-pos-nat*:
finite S ==> (*ALL x* : *S*. *f x* > (*0*::*nat*)) ==> *setprod f S* > *0*
using *setprod-zero-iff* **by**(*simp del*:*neq0-conv add*:*neq0-conv*[*symmetric*])

lemma *setprod-pos-nat-iff*[*simp*]:

finite S ==> (setprod f S > 0) = (ALL x : S. f x > (0::nat))
using *setprod-zero-iff* **by**(*simp del:neq0-conv add:neq0-conv*[*symmetric*])

lemma *setprod-Un*: *finite A ==> finite B ==> (ALL x: A Int B. f x ≠ 0) ==>*
(setprod f (A Un B) :: 'a ::{*field*})
*= setprod f A * setprod f B / setprod f (A Int B)*
by (*subst setprod-Un-Int* [*symmetric*], *auto*)

lemma *setprod-diff1*: *finite A ==> f a ≠ 0 ==>*
(setprod f (A − {a}) :: 'a :: {field}) =
(if a:A then setprod f A / f a else setprod f A)
by (*erule finite-induct*) (*auto simp add*: *insert-Diff-if*)

lemma *setprod-inversef*:
fixes *f* :: *'b ⇒ 'a*::*field-inverse-zero*
shows *finite A ==> setprod* (*inverse ∘ f*) *A = inverse* (*setprod f A*)
by (*erule finite-induct*) *auto*

lemma *setprod-dividef*:
fixes *f* :: *'b ⇒ 'a*::*field-inverse-zero*
shows *finite A*
==> setprod (%*x. f x / g x*) *A = setprod f A / setprod g A*
apply (*subgoal-tac*
setprod (%*x. f x / g x*) *A = setprod* (%*x. f x * (inverse ∘ g) x*) *A*)
apply (*erule ssubst*)
apply (*subst divide-inverse*)
apply (*subst setprod-timesf*)
apply (*subst setprod-inversef*, *assumption+*, *rule refl*)
apply (*rule setprod-cong*, *rule refl*)
apply (*subst divide-inverse*, *auto*)
done

lemma *setprod-dvd-setprod* [*rule-format*]:
(*ALL x : A. f x dvd g x*) ⟶ *setprod f A dvd setprod g A*
apply (*cases finite A*)
apply (*induct set*: *finite*)
apply (*auto simp add*: *dvd-def*)
apply (*rule-tac x = k * ka* **in** *exI*)
apply (*simp add*: *algebra-simps*)
done

lemma *setprod-dvd-setprod-subset*:
finite B ⟹ A <= B ⟹ setprod f A dvd setprod f B
apply (*subgoal-tac setprod f B = setprod f A * setprod f* (*B − A*))
apply (*unfold dvd-def*, *blast*)
apply (*subst setprod-Un-disjoint* [*symmetric*])
apply (*auto elim*: *finite-subset intro*: *setprod-cong*)
done

lemma *setprod-dvd-setprod-subset2*:
finite B $\Longrightarrow$ *A <= B* $\Longrightarrow$ *ALL x : A. (f x::'a::comm-semiring-1) dvd g x* $\Longrightarrow$
setprod f A dvd setprod g B
apply (*rule dvd-trans*)
apply (*rule setprod-dvd-setprod*, *erule* (*1*) *bspec*)
apply (*erule* (*1*) *setprod-dvd-setprod-subset*)
done

lemma *dvd-setprod*: *finite A* $\Longrightarrow$ *i:A* $\Longrightarrow$
(*f i ::'a::comm-semiring-1*) *dvd setprod f A*
by (*induct set*: *finite*) (*auto intro*: *dvd-mult*)

lemma *dvd-setsum* [*rule-format*]: (*ALL i : A. d dvd f i*) $\longrightarrow$
(*d::'a::comm-semiring-1*) *dvd* (*SUM x : A. f x*)
apply (*cases finite A*)
apply (*induct set*: *finite*)
apply *auto*
done

lemma *setprod-mono*:
fixes *f* :: *'a* $\Rightarrow$ *'b::linordered-semidom*
assumes $\forall$ *i*$\in$*A. 0* $\leq$ *f i* $\wedge$ *f i* $\leq$ *g i*
shows *setprod f A* $\leq$ *setprod g A*
proof (*cases finite A*)
case *True*
hence *?thesis setprod f A* $\geq$ *0* **using** *subset-refl*[*of A*]
proof (*induct A rule*: *finite-subset-induct*)
case (*insert a F*)
thus *setprod f* (*insert a F*) $\leq$ *setprod g* (*insert a F*) *0* $\leq$ *setprod f* (*insert a F*)
unfolding *setprod-insert*[*OF insert*(*1*,*3*)]
using *assms*[*rule-format*,*OF insert*(*2*)] *insert*
by (*auto intro*: *mult-mono mult-nonneg-nonneg*)
qed *auto*
thus *?thesis* **by** *simp*
qed *auto*

lemma *abs-setprod*:
fixes *f* :: *'a* $\Rightarrow$ *'b::*{*linordered-field*,*abs*}
shows *abs* (*setprod f A*) = *setprod* (λ*x. abs* (*f x*)) *A*
proof (*cases finite A*)
case *True* **thus** *?thesis*
by *induct* (*auto simp add*: *field-simps abs-mult*)
qed *auto*

lemma *setprod-constant*: *finite A* ==> ($\prod$ *x*$\in$ *A. (y::'a::*{*comm-monoid-mult*}))
= *y*^(*card A*)
apply (*erule finite-induct*)
apply *auto*
done

lemma *setprod-gen-delta*:
assumes *fS*: *finite S*
shows *setprod* (λ*k*. *if k=a then b k else c*) *S* = (*if a* ∈ *S then* (*b a* ::'*a*::{*comm-monoid-mult*})
∗ *c* ^ (*card S* − *1*) *else c* ^ *card S*)
proof−
let *?f* = (λ*k*. *if k=a then b k else c*)
{assume *a*: *a* ∉ *S*
hence ∀ *k*∈ *S*. *?f k* = *c* **by** *simp*
hence *?thesis* **using** *a setprod-constant*[*OF fS*, *of c*] **by** (*simp add*: *setprod-1*
cong add: *setprod-cong*) **}**
moreover
{assume *a*: *a* ∈ *S*
let *?A* = *S* − {*a*}
let *?B* = {*a*}
have *eq*: *S* = *?A* ∪ *?B* **using** *a* **by** *blast*
have *dj*: *?A* ∩ *?B* = {} **by** *simp*
from *fS* **have** *fAB*: *finite ?A finite ?B* **by** *auto*
have *fA0*:*setprod ?f ?A* = *setprod* (λ*i*. *c*) *?A*
apply (*rule setprod-cong*) **by** *auto*
have *cA*: *card ?A* = *card S* − *1* **using** *fS a* **by** *auto*
have *fA1*: *setprod ?f ?A* = *c* ^ *card ?A* **unfolding** *fA0* **apply** (*rule setprod-constant*)
using *fS* **by** *auto*
have *setprod ?f ?A* ∗ *setprod ?f ?B* = *setprod ?f S*
using *setprod-Un-disjoint*[*OF fAB dj*, *of ?f*, *unfolded eq*[*symmetric*]]
by *simp*
then have *?thesis* **using** *a cA*
by (*simp add*: *fA1 field-simps cong add*: *setprod-cong cong del*: *if-weak-cong*)**}**
ultimately show *?thesis* **by** *blast*
qed

28.4 Versions of *inf* and *sup* on non-empty sets

no-notation *times* (**infixl** ∗ *70*)
no-notation *Groups.one* (*1*)

locale *semilattice-big* = *semilattice* +
fixes *F* :: '*a set* ⇒ '*a*
assumes *F-eq*: *finite A* ⟹ *F A* = *fold1* (*op* ∗) *A*

sublocale *semilattice-big* < *folding-one-idem* **proof**
qed (*simp-all add*: *F-eq*)

notation *times* (**infixl** ∗ *70*)
notation *Groups.one* (*1*)

context *lattice*
begin

definition *Inf-fin* :: *'a set* ⇒ *'a* (⨅$_{fin}$- [*900*] *900*) **where**
Inf-fin = *fold1 inf*

definition *Sup-fin* :: *'a set* ⇒ *'a* (⨆$_{fin}$- [*900*] *900*) **where**
Sup-fin = *fold1 sup*

end

sublocale *lattice* < *Inf-fin*!: *semilattice-big inf Inf-fin* **proof**
qed (*simp add*: *Inf-fin-def*)

sublocale *lattice* < *Sup-fin*!: *semilattice-big sup Sup-fin* **proof**
qed (*simp add*: *Sup-fin-def*)

context *semilattice-inf*
begin

lemma *ab-semigroup-idem-mult-inf*:
class.ab-semigroup-idem-mult inf
proof qed (*rule inf-assoc inf-commute inf-idem*)+

lemma *fold-inf-insert*[*simp*]: *finite A* ⟹ *fold inf b* (*insert a A*) = *inf a* (*fold inf b A*)
by(*rule fun-left-comm-idem.fold-insert-idem*[*OF ab-semigroup-idem-mult.fun-left-comm-idem*[*OF ab-semigroup-idem-mult-inf*]])

lemma *inf-le-fold-inf*: *finite A* ⟹ *ALL a*:*A*. *b* ≤ *a* ⟹ *inf b c* ≤ *fold inf c A*
by (*induct pred*: *finite*) (*auto intro*: *le-infI1*)

lemma *fold-inf-le-inf*: *finite A* ⟹ *a* ∈ *A* ⟹ *fold inf b A* ≤ *inf a b*
proof(*induct arbitrary*: *a pred*:*finite*)
case *empty* **thus** *?case* **by** *simp*
next
case (*insert x A*)
show *?case*
proof *cases*
assume *A* = {} **thus** *?thesis* **using** *insert* **by** *simp*
next
assume *A* ≠ {} **thus** *?thesis* **using** *insert* **by** (*auto intro*: *le-infI2*)
qed
qed

lemma *below-fold1-iff*:
assumes *finite A A* ≠ {}
shows *x* ≤ *fold1 inf A* ⟷ (∀ *a*∈*A*. *x* ≤ *a*)
proof −
interpret *ab-semigroup-idem-mult inf*
by (*rule ab-semigroup-idem-mult-inf*)
show *?thesis* **using** *assms* **by** (*induct rule*: *finite-ne-induct*) *simp-all*

qed

lemma *fold1-belowI*:
assumes *finite A*
and $a \in A$
shows *fold1 inf A* $\leq a$
proof −
from *assms* **have** $A \neq \{\}$ **by** *auto*
from ‹*finite A*› ‹$A \neq \{\}$› ‹$a \in A$› **show** *?thesis*
proof (*induct rule*: *finite-ne-induct*)
case *singleton* **thus** *?case* **by** *simp*
next
interpret *ab-semigroup-idem-mult inf*
by (*rule ab-semigroup-idem-mult-inf*)
case (*insert x F*)
from *insert*(*5*) **have** $a = x \vee a \in F$ **by** *simp*
thus *?case*
proof
assume $a = x$ **thus** *?thesis* **using** *insert*
by (*simp add*: *mult-ac*)
next
assume $a \in F$
hence *bel*: *fold1 inf F* $\leq a$ **by** (*rule insert*)
have *inf* (*fold1 inf* (*insert x F*)) *a* = *inf x* (*inf* (*fold1 inf F*) *a*)
using *insert* **by** (*simp add*: *mult-ac*)
also have *inf* (*fold1 inf F*) *a* = *fold1 inf F*
using *bel* **by** (*auto intro*: *antisym*)
also have *inf x* ... = *fold1 inf* (*insert x F*)
using *insert* **by** (*simp add*: *mult-ac*)
finally have *aux*: *inf* (*fold1 inf* (*insert x F*)) *a* = *fold1 inf* (*insert x F*) **.**
moreover have *inf* (*fold1 inf* (*insert x F*)) *a* $\leq a$ **by** *simp*
ultimately show *?thesis* **by** *simp*
qed
qed
qed

end

context *semilattice-sup*
begin

lemma *ab-semigroup-idem-mult-sup*: *class.ab-semigroup-idem-mult sup*
by (*rule semilattice-inf.ab-semigroup-idem-mult-inf*)(*rule dual-semilattice*)

lemma *fold-sup-insert*[*simp*]: *finite A* $\Longrightarrow$ *fold sup b* (*insert a A*) = *sup a* (*fold sup b A*)
by(*rule semilattice-inf.fold-inf-insert*)(*rule dual-semilattice*)

lemma *fold-sup-le-sup*: *finite A* $\Longrightarrow$ *ALL a*:*A*. $a \leq b \Longrightarrow$ *fold sup c A* $\leq$ *sup b c*

by(*rule semilattice-inf.inf-le-fold-inf*)(*rule dual-semilattice*)

lemma *sup-le-fold-sup*: *finite A* $\Longrightarrow$ *a* $\in$ *A* $\Longrightarrow$ *sup a b* $\leq$ *fold sup b A*
by(*rule semilattice-inf.fold-inf-le-inf*)(*rule dual-semilattice*)

end

context *lattice*
begin

lemma *Inf-le-Sup* [*simp*]: $[\![$ *finite A*; *A* $\neq$ {} $]\!] \Longrightarrow \bigsqcap_{fin} A \leq \bigsqcup_{fin} A$
apply(*unfold Sup-fin-def Inf-fin-def*)
apply(*subgoal-tac EX a. a*:*A*)
prefer *2* **apply** *blast*
apply(*erule exE*)
apply(*rule order-trans*)
apply(*erule* (*1*) *fold1-belowI*)
apply(*erule* (*1*) *semilattice-inf.fold1-belowI* [*OF dual-semilattice*])
done

lemma *sup-Inf-absorb* [*simp*]:
finite A $\Longrightarrow$ *a* $\in$ *A* $\Longrightarrow$ *sup a* $(\bigsqcap_{fin} A)$ = *a*
apply(*subst sup-commute*)
apply(*simp add*: *Inf-fin-def sup-absorb2 fold1-belowI*)
done

lemma *inf-Sup-absorb* [*simp*]:
finite A $\Longrightarrow$ *a* $\in$ *A* $\Longrightarrow$ *inf a* $(\bigsqcup_{fin} A)$ = *a*
by (*simp add*: *Sup-fin-def inf-absorb1*
semilattice-inf.fold1-belowI [*OF dual-semilattice*])

end

context *distrib-lattice*
begin

lemma *sup-Inf1-distrib*:
assumes *finite A*
and *A* $\neq$ {}
shows *sup x* $(\bigsqcap_{fin} A) = \bigsqcap_{fin}\{sup\ x\ a | a.\ a \in A\}$
proof −
interpret *ab-semigroup-idem-mult inf*
by (*rule ab-semigroup-idem-mult-inf*)
from *assms* **show** *?thesis*
by (*simp add*: *Inf-fin-def image-def*
hom-fold1-commute [**where** *h*=*sup x*, *OF sup-inf-distrib1*])
(*rule arg-cong* [**where** *f*=*fold1 inf*], *blast*)
qed

lemma *sup-Inf2-distrib*:
assumes *A*: *finite A* $A \neq \{\}$ **and** *B*: *finite B* $B \neq \{\}$
shows *sup* $(\bigsqcap_{fin} A)$ $(\bigsqcap_{fin} B) = \bigsqcap_{fin}\{sup\ a\ b | a\ b.\ a \in A \wedge b \in B\}$
using *A* **proof** (*induct rule*: *finite-ne-induct*)
case *singleton* **thus** *?case*
by (*simp add*: *sup-Inf1-distrib* [*OF B*] *fold1-singleton-def* [*OF Inf-fin-def*])
next
interpret *ab-semigroup-idem-mult inf*
by (*rule ab-semigroup-idem-mult-inf*)
case (*insert x A*)
have *finB*: *finite* $\{sup\ x\ b\ |b.\ b \in B\}$
by(*rule finite-surj*[**where** $f = sup\ x$, *OF B*(*1*)], *auto*)
have *finAB*: *finite* $\{sup\ a\ b\ |a\ b.\ a \in A \wedge b \in B\}$
proof −
have $\{sup\ a\ b\ |a\ b.\ a \in A \wedge b \in B\} = (UN\ a{:}A.\ UN\ b{:}B.\ \{sup\ a\ b\})$
by *blast*
thus *?thesis* **by**(*simp add*: *insert*(*1*) *B*(*1*))
qed
have *ne*: $\{sup\ a\ b\ |a\ b.\ a \in A \wedge b \in B\} \neq \{\}$ **using** *insert B* **by** *blast*
have *sup* $(\bigsqcap_{fin}(insert\ x\ A))$ $(\bigsqcap_{fin} B) = sup\ (inf\ x\ (\bigsqcap_{fin} A))\ (\bigsqcap_{fin} B)$
using *insert* **by** (*simp add*: *fold1-insert-idem-def* [*OF Inf-fin-def*])
also have $\ldots = inf\ (sup\ x\ (\bigsqcap_{fin} B))\ (sup\ (\bigsqcap_{fin} A)\ (\bigsqcap_{fin} B))$ **by**(*rule sup-inf-distrib2*)
also have $\ldots = inf\ (\bigsqcap_{fin}\{sup\ x\ b | b.\ b \in B\})\ (\bigsqcap_{fin}\{sup\ a\ b | a\ b.\ a \in A \wedge b \in B\})$
using *insert* **by**(*simp add*:*sup-Inf1-distrib*[*OF B*])
also have $\ldots = \bigsqcap_{fin}(\{sup\ x\ b\ |b.\ b \in B\} \cup \{sup\ a\ b\ |a\ b.\ a \in A \wedge b \in B\})$
(**is** - $= \bigsqcap_{fin} ?M$)
using *B insert*
by (*simp add*: *Inf-fin-def fold1-Un2* [*OF finB - finAB ne*])
also have $?M = \{sup\ a\ b\ |a\ b.\ a \in insert\ x\ A \wedge b \in B\}$
by *blast*
finally show *?case* **.**
qed

lemma *inf-Sup1-distrib*:
assumes *finite A* **and** $A \neq \{\}$
shows *inf x* $(\bigsqcup_{fin} A) = \bigsqcup_{fin}\{inf\ x\ a | a.\ a \in A\}$
proof −
interpret *ab-semigroup-idem-mult sup*
by (*rule ab-semigroup-idem-mult-sup*)
from *assms* **show** *?thesis*
by (*simp add*: *Sup-fin-def image-def hom-fold1-commute* [**where** *h*=*inf x*, *OF inf-sup-distrib1*])
(*rule arg-cong* [**where** *f*=*fold1 sup*], *blast*)
qed

lemma *inf-Sup2-distrib*:
assumes *A*: *finite A* $A \neq \{\}$ **and** *B*: *finite B* $B \neq \{\}$
shows *inf* $(\bigsqcup_{fin} A)$ $(\bigsqcup_{fin} B) = \bigsqcup_{fin}\{inf\ a\ b | a\ b.\ a \in A \wedge b \in B\}$

using *A* **proof** (*induct rule*: *finite-ne-induct*)
case *singleton* **thus** *?case*
by(*simp add*: *inf-Sup1-distrib* [*OF B*] *fold1-singleton-def* [*OF Sup-fin-def*])
next
case (*insert x A*)
have *finB*: *finite* {*inf x b* |*b*. *b* ∈ *B*}
by(*rule finite-surj*[**where** *f* = %*b*. *inf x b*, *OF B*(*1*)], *auto*)
have *finAB*: *finite* {*inf a b* |*a b*. *a* ∈ *A* ∧ *b* ∈ *B*}
proof −
have {*inf a b* |*a b*. *a* ∈ *A* ∧ *b* ∈ *B*} = (*UN a*:*A*. *UN b*:*B*. {*inf a b*})
by *blast*
thus *?thesis* **by**(*simp add*: *insert*(*1*) *B*(*1*))
qed
have *ne*: {*inf a b* |*a b*. *a* ∈ *A* ∧ *b* ∈ *B*} ≠ {} **using** *insert B* **by** *blast*
interpret *ab-semigroup-idem-mult sup*
by (*rule ab-semigroup-idem-mult-sup*)
have *inf* (⨆*fin*(*insert x A*)) (⨆*fin B*) = *inf* (*sup x* (⨆*fin A*)) (⨆*fin B*)
using *insert* **by** (*simp add*: *fold1-insert-idem-def* [*OF Sup-fin-def*])
also have . . . = *sup* (*inf x* (⨆*fin B*)) (*inf* (⨆*fin A*) (⨆*fin B*)) **by**(*rule inf-sup-distrib2*)
also have . . . = *sup* (⨆*fin*{*inf x b*|*b*. *b* ∈ *B*}) (⨆*fin*{*inf a b*|*a b*. *a* ∈ *A* ∧ *b* ∈ *B*})
using *insert* **by**(*simp add*:*inf-Sup1-distrib*[*OF B*])
also have . . . = ⨆*fin*({*inf x b* |*b*. *b* ∈ *B*} ∪ {*inf a b* |*a b*. *a* ∈ *A* ∧ *b* ∈ *B*})
(**is** - = ⨆*fin ?M*)
using *B insert*
by (*simp add*: *Sup-fin-def fold1-Un2* [*OF finB* - *finAB ne*])
also have *?M* = {*inf a b* |*a b*. *a* ∈ *insert x A* ∧ *b* ∈ *B*}
by *blast*
finally show *?case* **.**
qed

end

context *complete-lattice*
begin

lemma *Inf-fin-Inf*:
assumes *finite A* **and** *A* ≠ {}
shows ⨅*fin A* = *Inf A*
proof −
interpret *ab-semigroup-idem-mult inf*
by (*rule ab-semigroup-idem-mult-inf*)
from ‹*A* ≠ {}› **obtain** *b B* **where** *A* = *insert b B* **by** *auto*
moreover with ‹*finite A*› **have** *finite B* **by** *simp*
ultimately show *?thesis*
by (*simp add*: *Inf-fin-def fold1-eq-fold-idem inf-Inf-fold-inf* [*symmetric*])
(*simp add*: *Inf-fold-inf*)
qed

lemma *Sup-fin-Sup*:
assumes *finite A* **and** $A \neq \{\}$
shows $\bigsqcup_{fin} A = Sup\ A$
proof −
interpret *ab-semigroup-idem-mult sup*
by (*rule ab-semigroup-idem-mult-sup*)
from ‹$A \neq \{\}$› **obtain** *b B* **where** $A = insert\ b\ B$ **by** *auto*
moreover with ‹*finite A*› **have** *finite B* **by** *simp*
ultimately show *?thesis*
by (*simp add*: *Sup-fin-def fold1-eq-fold-idem sup-Sup-fold-sup* [*symmetric*])
(*simp add*: *Sup-fold-sup*)
qed

end

28.5 Versions of *min* and *max* on non-empty sets

definition (**in** *linorder*) *Min* :: *'a set* ⇒ *'a* **where**
Min = *fold1 min*

definition (**in** *linorder*) *Max* :: *'a set* ⇒ *'a* **where**
Max = *fold1 max*

sublocale *linorder* < *Min*!: *semilattice-big min Min* **proof**
qed (*simp add*: *Min-def*)

sublocale *linorder* < *Max*!: *semilattice-big max Max* **proof**
qed (*simp add*: *Max-def*)

context *linorder*
begin

lemmas *Min-singleton* = *Min.singleton*
lemmas *Max-singleton* = *Max.singleton*

lemma *Min-insert*:
assumes *finite A* **and** $A \neq \{\}$
shows $Min\ (insert\ x\ A) = min\ x\ (Min\ A)$
using *assms* **by** *simp*

lemma *Max-insert*:
assumes *finite A* **and** $A \neq \{\}$
shows $Max\ (insert\ x\ A) = max\ x\ (Max\ A)$
using *assms* **by** *simp*

lemma *Min-Un*:
assumes *finite A* **and** $A \neq \{\}$ **and** *finite B* **and** $B \neq \{\}$
shows $Min\ (A \cup B) = min\ (Min\ A)\ (Min\ B)$
using *assms* **by** (*rule Min.union-idem*)

lemma *Max-Un*:
assumes *finite A* **and** $A \neq \{\}$ **and** *finite B* **and** $B \neq \{\}$
shows $Max\ (A \cup B) = max\ (Max\ A)\ (Max\ B)$
using *assms* **by** (*rule Max.union-idem*)

lemma *hom-Min-commute*:
assumes $\bigwedge x\ y.\ h\ (min\ x\ y) = min\ (h\ x)\ (h\ y)$
and *finite N* **and** $N \neq \{\}$
shows $h\ (Min\ N) = Min\ (h\ `\ N)$
using *assms* **by** (*rule Min.hom-commute*)

lemma *hom-Max-commute*:
assumes $\bigwedge x\ y.\ h\ (max\ x\ y) = max\ (h\ x)\ (h\ y)$
and *finite N* **and** $N \neq \{\}$
shows $h\ (Max\ N) = Max\ (h\ `\ N)$
using *assms* **by** (*rule Max.hom-commute*)

lemma *ab-semigroup-idem-mult-min*:
class.ab-semigroup-idem-mult min
proof qed (*auto simp add*: *min-def*)

lemma *ab-semigroup-idem-mult-max*:
class.ab-semigroup-idem-mult max
proof qed (*auto simp add*: *max-def*)

lemma *max-lattice*:
class.semilattice-inf $(op \geq)$ $(op >)$ *max*
by (*fact min-max.dual-semilattice*)

lemma *dual-max*:
ord.max $(op \geq) = min$
by (*auto simp add*: *ord.max-def-raw min-def expand-fun-eq*)

lemma *dual-min*:
ord.min $(op \geq) = max$
by (*auto simp add*: *ord.min-def-raw max-def expand-fun-eq*)

lemma *strict-below-fold1-iff*:
assumes *finite A* **and** $A \neq \{\}$
shows $x < fold1\ min\ A \longleftrightarrow (\forall\ a{\in}A.\ x < a)$
proof −
interpret *ab-semigroup-idem-mult min*
by (*rule ab-semigroup-idem-mult-min*)
from *assms* **show** *?thesis*
by (*induct rule*: *finite-ne-induct*)
(*simp-all add*: *fold1-insert*)
qed

lemma *fold1-below-iff*:
assumes *finite A* **and** $A \neq \{\}$
shows *fold1 min A* $\leq x \longleftrightarrow (\exists\, a{\in}A.\ a \leq x)$
proof −
interpret *ab-semigroup-idem-mult min*
by (*rule ab-semigroup-idem-mult-min*)
from *assms* **show** *?thesis*
by (*induct rule*: *finite-ne-induct*)
(*simp-all add*: *fold1-insert min-le-iff-disj*)
qed

lemma *fold1-strict-below-iff*:
assumes *finite A* **and** $A \neq \{\}$
shows *fold1 min A* $< x \longleftrightarrow (\exists\, a{\in}A.\ a < x)$
proof −
interpret *ab-semigroup-idem-mult min*
by (*rule ab-semigroup-idem-mult-min*)
from *assms* **show** *?thesis*
by (*induct rule*: *finite-ne-induct*)
(*simp-all add*: *fold1-insert min-less-iff-disj*)
qed

lemma *fold1-antimono*:
assumes $A \neq \{\}$ **and** $A \subseteq B$ **and** *finite B*
shows *fold1 min B* $\leq$ *fold1 min A*
proof *cases*
assume $A = B$ **thus** *?thesis* **by** *simp*
next
interpret *ab-semigroup-idem-mult min*
by (*rule ab-semigroup-idem-mult-min*)
assume $A \neq B$
have *B*: $B = A \cup (B{-}A)$ **using** ‹$A \subseteq B$› **by** *blast*
have *fold1 min B* = *fold1 min* $(A \cup (B{-}A))$ **by**(*subst B*)(*rule refl*)
also have $\ldots$ = *min* (*fold1 min A*) (*fold1 min* $(B{-}A)$)
proof −
have *finite A* **by**(*rule finite-subset*[*OF* ‹$A \subseteq B$› ‹*finite B*›])
moreover have *finite*$(B{-}A)$ **by**(*rule finite-Diff*[*OF* ‹*finite B*›])
moreover have $(B{-}A) \neq \{\}$ **using** *prems* **by** *blast*
moreover have *A Int* $(B{-}A) = \{\}$ **using** *prems* **by** *blast*
ultimately show *?thesis* **using** ‹$A \neq \{\}$› **by** (*rule-tac fold1-Un*)
qed
also have $\ldots \leq$ *fold1 min A* **by** (*simp add*: *min-le-iff-disj*)
finally show *?thesis* **.**
qed

lemma *Min-in* [*simp*]:
assumes *finite A* **and** $A \neq \{\}$
shows *Min A* $\in A$
proof −

interpret *ab-semigroup-idem-mult min*
by (*rule ab-semigroup-idem-mult-min*)
from *assms fold1-in* **show** *?thesis* **by** (*fastsimp simp*: *Min-def min-def*)
qed

lemma *Max-in* [*simp*]:
assumes *finite A* **and** $A \neq \{\}$
shows $Max\ A \in A$
proof −
interpret *ab-semigroup-idem-mult max*
by (*rule ab-semigroup-idem-mult-max*)
from *assms fold1-in* [*of A*] **show** *?thesis* **by** (*fastsimp simp*: *Max-def max-def*)
qed

lemma *Min-le* [*simp*]:
assumes *finite A* **and** $x \in A$
shows $Min\ A \leq x$
using *assms* **by** (*simp add*: *Min-def min-max.fold1-belowI*)

lemma *Max-ge* [*simp*]:
assumes *finite A* **and** $x \in A$
shows $x \leq Max\ A$
proof −
interpret *semilattice-inf op* $\geq$ *op* $>$ *max*
by (*rule max-lattice*)
from *assms* **show** *?thesis* **by** (*simp add*: *Max-def fold1-belowI*)
qed

lemma *Min-ge-iff* [*simp*, *no-atp*]:
assumes *finite A* **and** $A \neq \{\}$
shows $x \leq Min\ A \longleftrightarrow (\forall\, a{\in}A.\ x \leq a)$
using *assms* **by** (*simp add*: *Min-def min-max.below-fold1-iff*)

lemma *Max-le-iff* [*simp*, *no-atp*]:
assumes *finite A* **and** $A \neq \{\}$
shows $Max\ A \leq x \longleftrightarrow (\forall\, a{\in}A.\ a \leq x)$
proof −
interpret *semilattice-inf op* $\geq$ *op* $>$ *max*
by (*rule max-lattice*)
from *assms* **show** *?thesis* **by** (*simp add*: *Max-def below-fold1-iff*)
qed

lemma *Min-gr-iff* [*simp*, *no-atp*]:
assumes *finite A* **and** $A \neq \{\}$
shows $x < Min\ A \longleftrightarrow (\forall\, a{\in}A.\ x < a)$
using *assms* **by** (*simp add*: *Min-def strict-below-fold1-iff*)

lemma *Max-less-iff* [*simp*, *no-atp*]:
assumes *finite A* **and** $A \neq \{\}$

shows $Max\ A < x \longleftrightarrow (\forall\, a{\in}A.\ a < x)$
proof −
interpret *dual*: *linorder op* $\geq$ *op* $>$
by (*rule dual-linorder*)
from *assms* **show** *?thesis*
by (*simp add*: *Max-def dual.strict-below-fold1-iff* [*folded dual.dual-max*])
qed

lemma *Min-le-iff* [*no-atp*]:
assumes *finite A* **and** $A \neq \{\}$
shows $Min\ A \leq x \longleftrightarrow (\exists\, a{\in}A.\ a \leq x)$
using *assms* **by** (*simp add*: *Min-def fold1-below-iff*)

lemma *Max-ge-iff* [*no-atp*]:
assumes *finite A* **and** $A \neq \{\}$
shows $x \leq Max\ A \longleftrightarrow (\exists\, a{\in}A.\ x \leq a)$
proof −
interpret *dual*: *linorder op* $\geq$ *op* $>$
by (*rule dual-linorder*)
from *assms* **show** *?thesis*
by (*simp add*: *Max-def dual.fold1-below-iff* [*folded dual.dual-max*])
qed

lemma *Min-less-iff* [*no-atp*]:
assumes *finite A* **and** $A \neq \{\}$
shows $Min\ A < x \longleftrightarrow (\exists\, a{\in}A.\ a < x)$
using *assms* **by** (*simp add*: *Min-def fold1-strict-below-iff*)

lemma *Max-gr-iff* [*no-atp*]:
assumes *finite A* **and** $A \neq \{\}$
shows $x < Max\ A \longleftrightarrow (\exists\, a{\in}A.\ x < a)$
proof −
interpret *dual*: *linorder op* $\geq$ *op* $>$
by (*rule dual-linorder*)
from *assms* **show** *?thesis*
by (*simp add*: *Max-def dual.fold1-strict-below-iff* [*folded dual.dual-max*])
qed

lemma *Min-eqI*:
assumes *finite A*
assumes $\bigwedge y.\ y \in A \Longrightarrow y \geq x$
and $x \in A$
shows $Min\ A = x$
proof (*rule antisym*)
from ‹$x \in A$› **have** $A \neq \{\}$ **by** *auto*
with *assms* **show** $Min\ A \geq x$ **by** *simp*
next
from *assms* **show** $x \geq Min\ A$ **by** *simp*
qed

lemma *Max-eqI*:
assumes *finite A*
assumes $\bigwedge y.\ y \in A \Longrightarrow y \leq x$
and $x \in A$
shows $Max\ A = x$
proof (*rule antisym*)
from ‹$x \in A$› **have** $A \neq \{\}$ **by** *auto*
with *assms* **show** $Max\ A \leq x$ **by** *simp*
next
from *assms* **show** $x \leq Max\ A$ **by** *simp*
qed

lemma *Min-antimono*:
assumes $M \subseteq N$ **and** $M \neq \{\}$ **and** *finite N*
shows $Min\ N \leq Min\ M$
using *assms* **by** (*simp add*: *Min-def fold1-antimono*)

lemma *Max-mono*:
assumes $M \subseteq N$ **and** $M \neq \{\}$ **and** *finite N*
shows $Max\ M \leq Max\ N$
proof −
interpret *dual*: *linorder op* $\geq$ *op* $>$
by (*rule dual-linorder*)
from *assms* **show** *?thesis*
by (*simp add*: *Max-def dual.fold1-antimono* [*folded dual.dual-max*])
qed

lemma *finite-linorder-max-induct*[*consumes 1*, *case-names empty insert*]:
assumes *fin*: *finite A*
and *empty*: $P\ \{\}$
and *insert*: $(!!b\ A.\ finite\ A \Longrightarrow ALL\ a{:}A.\ a < b \Longrightarrow P\ A \Longrightarrow P(insert\ b\ A))$
shows $P\ A$
using *fin empty insert*
proof (*induct rule*: *finite-psubset-induct*)
case (*psubset A*)
have *IH*: $\bigwedge B.\ [\![B < A;\ P\ \{\};\ (\bigwedge A\ b.\ [\![finite\ A;\ \forall a{\in}A.\ a{<}b;\ P\ A]\!] \Longrightarrow P\ (insert\ b\ A))]\!] \Longrightarrow P\ B$ **by** *fact*
have *fin*: *finite A* **by** *fact*
have *empty*: $P\ \{\}$ **by** *fact*
have *step*: $\bigwedge b\ A.\ [\![finite\ A;\ \forall a{\in}A.\ a < b;\ P\ A]\!] \Longrightarrow P\ (insert\ b\ A)$ **by** *fact*
show $P\ A$
proof (*cases* $A = \{\}$)
assume $A = \{\}$
then show $P\ A$ **using** ‹$P\ \{\}$› **by** *simp*
next
let *?B* $= A - \{Max\ A\}$
let *?A* $= insert\ (Max\ A)$ *?B*
have *finite ?B* **using** ‹*finite A*› **by** *simp*

assume $A \neq \{\}$
with ‹*finite A*› **have** *Max A* : A **by** *auto*
then have A: *?A* = A **using** *insert-Diff-single insert-absorb* **by** *auto*
then have *P ?B* **using** ‹*P* {}› *step IH*[*of ?B*] **by** *blast*
moreover
have $\forall a{\in}$*?B*. $a <$ *Max A* **using** *Max-ge* [*OF* ‹*finite A*›] **by** *fastsimp*
ultimately show *P A* **using** *A insert-Diff-single step*[*OF* ‹*finite ?B*›] **by** *fastsimp*
qed
qed

lemma *finite-linorder-min-induct*[*consumes 1*, *case-names empty insert*]:
⟦*finite A*; *P* {}; $\bigwedge b\ A$. ⟦*finite A*; $\forall a{\in}A.\ b < a$; *P A*⟧ $\Longrightarrow$ *P* (*insert b A*)⟧ $\Longrightarrow$ *P A*
by(*rule linorder.finite-linorder-max-induct*[*OF dual-linorder*])

end

context *linordered-ab-semigroup-add*
begin

lemma *add-Min-commute*:
fixes k
assumes *finite N* **and** $N \neq \{\}$
shows k + *Min N* = *Min* {$k + m \mid m.\ m \in N$}
proof −
have $\bigwedge x\ y.\ k + min\ x\ y = min\ (k + x)\ (k + y)$
by (*simp add*: *min-def not-le*)
(*blast intro*: *antisym less-imp-le add-left-mono*)
with *assms* **show** *?thesis*
using *hom-Min-commute* [*of plus k N*]
by *simp* (*blast intro*: *arg-cong* [**where** f = *Min*])
qed

lemma *add-Max-commute*:
fixes k
assumes *finite N* **and** $N \neq \{\}$
shows k + *Max N* = *Max* {$k + m \mid m.\ m \in N$}
proof −
have $\bigwedge x\ y.\ k + max\ x\ y = max\ (k + x)\ (k + y)$
by (*simp add*: *max-def not-le*)
(*blast intro*: *antisym less-imp-le add-left-mono*)
with *assms* **show** *?thesis*
using *hom-Max-commute* [*of plus k N*]
by *simp* (*blast intro*: *arg-cong* [**where** f = *Max*])
qed

end

context *linordered-ab-group-add*

begin

lemma *minus-Max-eq-Min* [*simp*]:
finite S $\Longrightarrow$ *S* $\neq$ {} $\Longrightarrow$ − (*Max S*) = *Min* (*uminus* ' *S*)
by (*induct S rule*: *finite-ne-induct*) (*simp-all add*: *minus-max-eq-min*)

lemma *minus-Min-eq-Max* [*simp*]:
finite S $\Longrightarrow$ *S* $\neq$ {} $\Longrightarrow$ − (*Min S*) = *Max* (*uminus* ' *S*)
by (*induct S rule*: *finite-ne-induct*) (*simp-all add*: *minus-min-eq-max*)

end

end

29 Equiv-Relations: Equivalence Relations in Higher-Order Set Theory

theory *Equiv-Relations*
imports *Big-Operators Relation Plain*
begin

29.1 Equivalence relations

locale *equiv* =
fixes *A* **and** *r*
assumes *refl-on*: *refl-on A r*
and *sym*: *sym r*
and *trans*: *trans r*

Suppes, Theorem 70: *r* is an equiv relation iff r^{-1} *O r* = *r*.
First half: *equiv A r* ==> r^{-1} *O r* = *r*.

lemma *sym-trans-comp-subset*:
sym r ==> *trans r* ==> r^{-1} *O r* $\subseteq$ *r*
by (*unfold trans-def sym-def converse-def*) *blast*

lemma *refl-on-comp-subset*: *refl-on A r* ==> *r* $\subseteq$ r^{-1} *O r*
by (*unfold refl-on-def*) *blast*

lemma *equiv-comp-eq*: *equiv A r* ==> r^{-1} *O r* = *r*
apply (*unfold equiv-def*)
apply *clarify*
apply (*rule equalityI*)
apply (*iprover intro*: *sym-trans-comp-subset refl-on-comp-subset*)+
done

Second half.

lemma *comp-equivI*:

r^{-1} *O* $r = r$ ==> *Domain* $r = A$ ==> *equiv* A r
apply (*unfold equiv-def refl-on-def sym-def trans-def*)
apply (*erule equalityE*)
apply (*subgoal-tac* $\forall x\ y.\ (x, y) \in r$ --> $(y, x) \in r$)
apply *fast*
apply *fast*
done

29.2 Equivalence classes

lemma *equiv-class-subset*:
equiv A r ==> $(a, b) \in r$ ==> $r``\{a\} \subseteq r``\{b\}$
— lemma for the next result
by (*unfold equiv-def trans-def sym-def*) *blast*

theorem *equiv-class-eq*: *equiv* A r ==> $(a, b) \in r$ ==> $r``\{a\} = r``\{b\}$
apply (*assumption* | *rule equalityI equiv-class-subset*)+
apply (*unfold equiv-def sym-def*)
apply *blast*
done

lemma *equiv-class-self*: *equiv* A r ==> $a \in A$ ==> $a \in r``\{a\}$
by (*unfold equiv-def refl-on-def*) *blast*

lemma *subset-equiv-class*:
equiv A r ==> $r``\{b\} \subseteq r``\{a\}$ ==> $b \in A$ ==> $(a,b) \in r$
— lemma for the next result
by (*unfold equiv-def refl-on-def*) *blast*

lemma *eq-equiv-class*:
$r``\{a\} = r``\{b\}$ ==> *equiv* A r ==> $b \in A$ ==> $(a, b) \in r$
by (*iprover intro*: *equalityD2 subset-equiv-class*)

lemma *equiv-class-nondisjoint*:
equiv A r ==> $x \in (r``\{a\} \cap r``\{b\})$ ==> $(a, b) \in r$
by (*unfold equiv-def trans-def sym-def*) *blast*

lemma *equiv-type*: *equiv* A r ==> $r \subseteq A \times A$
by (*unfold equiv-def refl-on-def*) *blast*

theorem *equiv-class-eq-iff*:
equiv A r ==> $((x, y) \in r) = (r``\{x\} = r``\{y\}\ \&\ x \in A\ \&\ y \in A)$
by (*blast intro*!: *equiv-class-eq dest*: *eq-equiv-class equiv-type*)

theorem *eq-equiv-class-iff*:
equiv A r ==> $x \in A$ ==> $y \in A$ ==> $(r``\{x\} = r``\{y\}) = ((x, y) \in r)$
by (*blast intro*!: *equiv-class-eq dest*: *eq-equiv-class equiv-type*)

29.3 Quotients

definition *quotient* :: *'a set* ⇒ (*'a* × *'a*) *set* ⇒ *'a set set* (**infixl** *'//'* *90*) **where**
[*code del*]: $A//r = (\bigcup x \in A.\ \{r``\{x\}\})$ — set of equiv classes

lemma *quotientI*: $x \in A$ ==> $r``\{x\} \in A//r$
by (*unfold quotient-def*) *blast*

lemma *quotientE*:
$X \in A//r$ ==> (!!x. $X = r``\{x\}$ ==> $x \in A$ ==> P) ==> P
by (*unfold quotient-def*) *blast*

lemma *Union-quotient*: *equiv A r* ==> *Union* $(A//r) = A$
by (*unfold equiv-def refl-on-def quotient-def*) *blast*

lemma *quotient-disj*:
equiv A r ==> $X \in A//r$ ==> $Y \in A//r$ ==> $X = Y \mid (X \cap Y = \{\})$
apply (*unfold quotient-def*)
apply *clarify*
apply (*rule equiv-class-eq*)
apply *assumption*
apply (*unfold equiv-def trans-def sym-def*)
apply *blast*
done

lemma *quotient-eqI*:
[|*equiv A r*; $X \in A//r$; $Y \in A//r$; $x \in X$; $y \in Y$; $(x,y) \in r$|] ==> $X = Y$
apply (*clarify elim*!: *quotientE*)
apply (*rule equiv-class-eq*, *assumption*)
apply (*unfold equiv-def sym-def trans-def*, *blast*)
done

lemma *quotient-eq-iff*:
[|*equiv A r*; $X \in A//r$; $Y \in A//r$; $x \in X$; $y \in Y$|] ==> $(X = Y) = ((x,y) \in r)$
apply (*rule iffI*)
prefer *2* **apply** (*blast del*: *equalityI intro*: *quotient-eqI*)
apply (*clarify elim*!: *quotientE*)
apply (*unfold equiv-def sym-def trans-def*, *blast*)
done

lemma *eq-equiv-class-iff2*:
⟦ *equiv A r*; $x \in A$; $y \in A$ ⟧ ⟹ $(\{x\}//r = \{y\}//r) = ((x,y) : r)$
by(*simp add*:*quotient-def eq-equiv-class-iff*)

lemma *quotient-empty* [*simp*]: $\{\}//r = \{\}$
by(*simp add*: *quotient-def*)

lemma *quotient-is-empty* [*iff*]: $(A//r = \{\}) = (A = \{\})$

by(*simp add*: *quotient-def*)

lemma *quotient-is-empty2* [*iff*]: $(\{\} = A//r) = (A = \{\})$
by(*simp add*: *quotient-def*)

lemma *singleton-quotient*: $\{x\}//r = \{r `` \{x\}\}$
by(*simp add*:*quotient-def*)

lemma *quotient-diff1*:
$[\![$ *inj-on* (%*a*. $\{a\}//r$) A; $a \in A$ $]\!] \Longrightarrow (A - \{a\})//r = A//r - \{a\}//r$
apply(*simp add*:*quotient-def inj-on-def*)
apply *blast*
done

29.4 Defining unary operations upon equivalence classes

A congruence-preserving function

locale *congruent* =
fixes r **and** f
assumes *congruent*: $(y,z) \in r$ ==> $f\ y = f\ z$

abbreviation
RESPECTS :: $('a$ => $'b)$ => $('a * 'a)$ *set* => *bool*
(**infixr** *respects 80*) **where**
f respects r == *congruent r f*

lemma *UN-constant-eq*: $a \in A$ ==> $\forall y \in A.\ f\ y = c$ ==> $(\bigcup y \in A.\ f(y))=c$
— lemma required to prove *UN-equiv-class*
by *auto*

lemma *UN-equiv-class*:
equiv A r ==> *f respects r* ==> $a \in A$
==> $(\bigcup x \in r``\{a\}.\ f\ x) = f\ a$
— Conversion rule
apply (*rule equiv-class-self* [*THEN UN-constant-eq*], *assumption*+)
apply (*unfold equiv-def congruent-def sym-def*)
apply (*blast del*: *equalityI*)
done

lemma *UN-equiv-class-type*:
equiv A r ==> *f respects r* ==> $X \in A//r$ ==>
(!!x. $x \in A$ ==> $f\ x \in B$) ==> $(\bigcup x \in X.\ f\ x) \in B$
apply (*unfold quotient-def*)
apply *clarify*
apply (*subst UN-equiv-class*)
apply *auto*
done

Sufficient conditions for injectiveness. Could weaken premises! major premise could be an inclusion; bcong could be $!!y.\ y \in A \Longrightarrow f\ y \in B$.

lemma *UN-equiv-class-inject*:
equiv A r ==> f respects r ==>
(⋃x ∈ X. f x) = (⋃y ∈ Y. f y) ==> X ∈ A//r ==> Y ∈ A//r
==> (!!x y. x ∈ A ==> y ∈ A ==> f x = f y ==> (x, y) ∈ r)
==> X = Y
apply (*unfold quotient-def*)
apply *clarify*
apply (*rule equiv-class-eq*)
apply *assumption*
apply (*subgoal-tac f x = f xa*)
apply *blast*
apply (*erule box-equals*)
apply (*assumption | rule UN-equiv-class*)+
done

29.5 Defining binary operations upon equivalence classes

A congruence-preserving function of two arguments

locale *congruent2* =
fixes *r1* **and** *r2* **and** *f*
assumes *congruent2*:
(y1,z1) ∈ r1 ==> (y2,z2) ∈ r2 ==> f y1 y2 = f z1 z2

Abbreviation for the common case where the relations are identical

abbreviation
RESPECTS2:: [*'a => 'a => 'b*, (*'a * 'a*) *set*] => *bool*
(**infixr** *respects2 80*) **where**
f respects2 r == congruent2 r r f

lemma *congruent2-implies-congruent*:
equiv A r1 ==> congruent2 r1 r2 f ==> a ∈ A ==> congruent r2 (f a)
by (*unfold congruent-def congruent2-def equiv-def refl-on-def*) *blast*

lemma *congruent2-implies-congruent-UN*:
equiv A1 r1 ==> equiv A2 r2 ==> congruent2 r1 r2 f ==> a ∈ A2 ==>
congruent r1 (λx1. ⋃x2 ∈ r2``{a}. f x1 x2)
apply (*unfold congruent-def*)
apply *clarify*
apply (*rule equiv-type* [*THEN subsetD*, *THEN SigmaE2*], *assumption+*)
apply (*simp add*: *UN-equiv-class congruent2-implies-congruent*)
apply (*unfold congruent2-def equiv-def refl-on-def*)
apply (*blast del*: *equalityI*)
done

lemma *UN-equiv-class2*:

equiv A1 r1 ==> equiv A2 r2 ==> congruent2 r1 r2 f ==> a1 ∈ A1 ==> a2 ∈ A2
==> (⋃x1 ∈ r1``{a1}. ⋃x2 ∈ r2``{a2}. f x1 x2) = f a1 a2
by (*simp add*: *UN-equiv-class congruent2-implies-congruent congruent2-implies-congruent-UN*)

lemma *UN-equiv-class-type2*:
equiv A1 r1 ==> equiv A2 r2 ==> congruent2 r1 r2 f
==> X1 ∈ A1//r1 ==> X2 ∈ A2//r2
==> (!!x1 x2. x1 ∈ A1 ==> x2 ∈ A2 ==> f x1 x2 ∈ B)
==> (⋃x1 ∈ X1. ⋃x2 ∈ X2. f x1 x2) ∈ B
apply (*unfold quotient-def*)
apply *clarify*
apply (*blast intro*: *UN-equiv-class-type congruent2-implies-congruent-UN congruent2-implies-congruent quotientI*)
done

lemma *UN-UN-split-split-eq*:
(⋃(*x1*, *x2*) ∈ *X*. ⋃(*y1*, *y2*) ∈ *Y*. *A x1 x2 y1 y2*) =
(⋃*x* ∈ *X*. ⋃*y* ∈ *Y*. (λ(*x1*, *x2*). (λ(*y1*, *y2*). *A x1 x2 y1 y2*) *y*) *x*)
— Allows a natural expression of binary operators,
— without explicit calls to *split*
by *auto*

lemma *congruent2I*:
equiv A1 r1 ==> equiv A2 r2
==> (!!y z w. w ∈ A2 ==> (y,z) ∈ r1 ==> f y w = f z w)
==> (!!y z w. w ∈ A1 ==> (y,z) ∈ r2 ==> f w y = f w z)
==> congruent2 r1 r2 f
— Suggested by John Harrison – the two subproofs may be
— *much* simpler than the direct proof.
apply (*unfold congruent2-def equiv-def refl-on-def*)
apply *clarify*
apply (*blast intro*: *trans*)
done

lemma *congruent2-commuteI*:
assumes *equivA*: *equiv A r*
and *commute*: !!*y z*. *y ∈ A ==> z ∈ A ==> f y z = f z y*
and *congt*: !!*y z w*. *w ∈ A ==> (y,z) ∈ r ==> f w y = f w z*
shows *f respects2 r*
apply (*rule congruent2I* [*OF equivA equivA*])
apply (*rule commute* [*THEN trans*])
apply (*rule-tac* [*3*] *commute* [*THEN trans*, *symmetric*])
apply (*rule-tac* [*5*] *sym*)
apply (*rule congt* | *assumption* |
erule equivA [*THEN equiv-type*, *THEN subsetD*, *THEN SigmaE2*])+
done

29.6 Quotients and finiteness

Suggested by Florian Kammüller

lemma *finite-quotient*: *finite A ==> r ⊆ A × A ==> finite (A//r)*
— recall *equiv ?A ?r ⟹ ?r ⊆ ?A × ?A*
apply (*rule finite-subset*)
apply (*erule-tac* [*2*] *finite-Pow-iff* [*THEN iffD2*])
apply (*unfold quotient-def*)
apply *blast*
done

lemma *finite-equiv-class*:
finite A ==> r ⊆ A × A ==> X ∈ A//r ==> finite X
apply (*unfold quotient-def*)
apply (*rule finite-subset*)
prefer *2* **apply** *assumption*
apply *blast*
done

lemma *equiv-imp-dvd-card*:
finite A ==> equiv A r ==> ∀ X ∈ A//r. k dvd card X
==> k dvd card A
apply (*rule Union-quotient* [*THEN subst* [**where** *P=λA. k dvd card A*]])
apply *assumption*
apply (*rule dvd-partition*)
prefer *3* **apply** (*blast dest*: *quotient-disj*)
apply (*simp-all add*: *Union-quotient equiv-type*)
done

lemma *card-quotient-disjoint*:
⟦ *finite A*; *inj-on* (*λx*. {*x*} // *r*) *A* ⟧ ⟹ *card*(*A*//*r*) = *card A*
apply(*simp add*:*quotient-def*)
apply(*subst card-UN-disjoint*)
apply *assumption*
apply *simp*
apply(*fastsimp simp add*:*inj-on-def*)
apply *simp*
done

end

30 Int: The Integers as Equivalence Classes over Pairs of Natural Numbers

theory *Int*
imports *Equiv-Relations Nat Wellfounded*
uses

(*Tools/numeral.ML*)
(*Tools/numeral-syntax.ML*)
(*Tools/int-arith.ML*)
begin

30.1 The equivalence relation underlying the integers

definition *intrel* :: $((nat \times nat) \times (nat \times nat))$ *set* **where**
[*code del*]: $intrel = \{((x, y), (u, v)) \mid x\ y\ u\ v.\ x + v = u + y\ \}$

typedef (*Integ*)
$int = UNIV//intrel$
by (*auto simp add*: *quotient-def*)

instantiation *int* :: {*zero*, *one*, *plus*, *minus*, *uminus*, *times*, *ord*, *abs*, *sgn*}
begin

definition
Zero-int-def [*code del*]: $0 = Abs\text{-}Integ\ (intrel\ ``\ \{(0, 0)\})$

definition
One-int-def [*code del*]: $1 = Abs\text{-}Integ\ (intrel\ ``\ \{(1, 0)\})$

definition
add-int-def [*code del*]: $z + w = Abs\text{-}Integ$
$(\bigcup(x, y) \in Rep\text{-}Integ\ z.\ \bigcup(u, v) \in Rep\text{-}Integ\ w.$
$intrel\ ``\ \{(x + u, y + v)\})$

definition
minus-int-def [*code del*]:
$-z = Abs\text{-}Integ\ (\bigcup(x, y) \in Rep\text{-}Integ\ z.\ intrel\ ``\ \{(y, x)\})$

definition
diff-int-def [*code del*]: $z - w = z + (-w :: int)$

definition
mult-int-def [*code del*]: $z * w = Abs\text{-}Integ$
$(\bigcup(x, y) \in Rep\text{-}Integ\ z.\ \bigcup(u,v\ ) \in Rep\text{-}Integ\ w.$
$intrel\ ``\ \{(x*u + y*v, x*v + y*u)\})$

definition
le-int-def [*code del*]:
$z \leq w \longleftrightarrow (\exists x\ y\ u\ v.\ x+v \leq u+y \wedge (x, y) \in Rep\text{-}Integ\ z \wedge (u, v) \in Rep\text{-}Integ\ w)$

definition
less-int-def [*code del*]: $(z::int) < w \longleftrightarrow z \leq w \wedge z \neq w$

definition

zabs-def: $|i::int| = (if\ i < 0\ then\ -\ i\ else\ i)$

definition
zsgn-def: $sgn\ (i::int) = (if\ i=0\ then\ 0\ else\ if\ 0<i\ then\ 1\ else\ -\ 1)$

instance ..

end

30.2 Construction of the Integers

lemma *intrel-iff* [*simp*]: $(((x,y),(u,v)) \in intrel) = (x+v = u+y)$
by (*simp add*: *intrel-def*)

lemma *equiv-intrel*: *equiv UNIV intrel*
by (*simp add*: *intrel-def equiv-def refl-on-def sym-def trans-def*)

Reduces equality of equivalence classes to the *intrel* relation: $(intrel\ ``\ \{x\} = intrel\ ``\ \{y\}) = ((x,\ y) \in intrel)$

lemmas *equiv-intrel-iff* [*simp*] = *eq-equiv-class-iff* [*OF equiv-intrel UNIV-I UNIV-I*]

All equivalence classes belong to set of representatives

lemma [*simp*]: $intrel``\{(x,y)\} \in Integ$
by (*auto simp add*: *Integ-def intrel-def quotient-def*)

Reduces equality on abstractions to equality on representatives: $[\![x \in Integ; y \in Integ]\!] \Longrightarrow (Abs\text{-}Integ\ x = Abs\text{-}Integ\ y) = (x = y)$

declare *Abs-Integ-inject* [*simp*,*no-atp*] *Abs-Integ-inverse* [*simp*,*no-atp*]

Case analysis on the representation of an integer as an equivalence class of pairs of naturals.

lemma *eq-Abs-Integ* [*case-names Abs-Integ*, *cases type*: *int*]:
(!!$x\ y.\ z = Abs\text{-}Integ(intrel``\{(x,y)\}) ==> P) ==> P$
apply (*rule Abs-Integ-cases* [*of z*])
apply (*auto simp add*: *Integ-def quotient-def*)
done

30.3 Arithmetic Operations

lemma *minus*: $-\ Abs\text{-}Integ(intrel``\{(x,y)\}) = Abs\text{-}Integ(intrel\ ``\ \{(y,x)\})$
proof −
have $(\lambda(x,y).\ intrel``\{(y,x)\})\ respects\ intrel$
by (*simp add*: *congruent-def*)
thus *?thesis*
by (*simp add*: *minus-int-def UN-equiv-class* [*OF equiv-intrel*])
qed

lemma *add*:

```
    Abs-Integ (intrel``{(x,y)}) + Abs-Integ (intrel``{(u,v)}) =
     Abs-Integ (intrel``{(x+u, y+v)})
proof −
  have (λz w. (λ(x,y). (λ(u,v). intrel `` {(x+u, y+v)}) w) z)
       respects2 intrel
    by (simp add: congruent2-def)
  thus ?thesis
    by (simp add: add-int-def UN-UN-split-split-eq
                  UN-equiv-class2 [OF equiv-intrel equiv-intrel])
qed
```

Congruence property for multiplication

```
lemma mult-congruent2:
    (%p1 p2. (%(x,y). (%(u,v). intrel``{(x*u + y*v, x*v + y*u)}) p2) p1)
     respects2 intrel
apply (rule equiv-intrel [THEN congruent2-commuteI])
 apply (force simp add: mult-ac, clarify)
apply (simp add: congruent-def mult-ac)
apply (rename-tac u v w x y z)
apply (subgoal-tac u*y + x*y = w*y + v*y  &  u*z + x*z = w*z + v*z)
apply (simp add: mult-ac)
apply (simp add: add-mult-distrib [symmetric])
done

lemma mult:
    Abs-Integ((intrel``{(x,y)})) * Abs-Integ((intrel``{(u,v)})) =
     Abs-Integ(intrel `` {(x*u + y*v, x*v + y*u)})
by (simp add: mult-int-def UN-UN-split-split-eq mult-congruent2
              UN-equiv-class2 [OF equiv-intrel equiv-intrel])
```

The integers form a *comm-ring-1*

```
instance int :: comm-ring-1
proof
  fix i j k :: int
  show (i + j) + k = i + (j + k)
    by (cases i, cases j, cases k) (simp add: add add-assoc)
  show i + j = j + i
    by (cases i, cases j) (simp add: add-ac add)
  show 0 + i = i
    by (cases i) (simp add: Zero-int-def add)
  show − i + i = 0
    by (cases i) (simp add: Zero-int-def minus add)
  show i − j = i + − j
    by (simp add: diff-int-def)
  show (i * j) * k = i * (j * k)
    by (cases i, cases j, cases k) (simp add: mult algebra-simps)
  show i * j = j * i
    by (cases i, cases j) (simp add: mult algebra-simps)
  show 1 * i = i
```

by (*cases i*) (*simp add*: *One-int-def mult*)
show $(i + j) * k = i * k + j * k$
by (*cases i*, *cases j*, *cases k*) (*simp add*: *add mult algebra-simps*)
show $0 \neq (1::int)$
by (*simp add*: *Zero-int-def One-int-def*)
qed

lemma *int-def*: *of-nat m* = *Abs-Integ* (*intrel* `` {(*m*, *0*)})
by (*induct m*, *simp-all add*: *Zero-int-def One-int-def add*)

30.4 The $\leq$ Ordering

lemma *le*:
(*Abs-Integ*(*intrel*``{(*x*,*y*)}) $\leq$ *Abs-Integ*(*intrel*``{(*u*,*v*)})) = ($x+v \leq u+y$)
by (*force simp add*: *le-int-def*)

lemma *less*:
(*Abs-Integ*(*intrel*``{(*x*,*y*)}) < *Abs-Integ*(*intrel*``{(*u*,*v*)})) = ($x+v < u+y$)
by (*simp add*: *less-int-def le order-less-le*)

instance *int* :: *linorder*
proof
fix *i j k* :: *int*
show *antisym*: $i \leq j \Longrightarrow j \leq i \Longrightarrow i = j$
by (*cases i*, *cases j*) (*simp add*: *le*)
show $(i < j) = (i \leq j \wedge \neg\, j \leq i)$
by (*auto simp add*: *less-int-def dest*: *antisym*)
show $i \leq i$
by (*cases i*) (*simp add*: *le*)
show $i \leq j \Longrightarrow j \leq k \Longrightarrow i \leq k$
by (*cases i*, *cases j*, *cases k*) (*simp add*: *le*)
show $i \leq j \vee j \leq i$
by (*cases i*, *cases j*) (*simp add*: *le linorder-linear*)
qed

instantiation *int* :: *distrib-lattice*
begin

definition
(*inf* :: *int* $\Rightarrow$ *int* $\Rightarrow$ *int*) = *min*

definition
(*sup* :: *int* $\Rightarrow$ *int* $\Rightarrow$ *int*) = *max*

instance
by *intro-classes*
(*auto simp add*: *inf-int-def sup-int-def min-max.sup-inf-distrib1*)

end

instance *int* :: *ordered-cancel-ab-semigroup-add*
proof
  fix *i j k* :: *int*
  show $i \leq j \Longrightarrow k + i \leq k + j$
    by (*cases i*, *cases j*, *cases k*) (*simp add*: *le add*)
qed

Strict Monotonicity of Multiplication

strict, in 1st argument; proof is by induction on k¿0

lemma *zmult-zless-mono2-lemma*:
    (*i*::*int*)<*j* ==> *0*<*k* ==> *of-nat k* ∗ *i* < *of-nat k* ∗ *j*
apply (*induct k*, *simp*)
apply (*simp add*: *left-distrib*)
apply (*case-tac k=0*)
apply (*simp-all add*: *add-strict-mono*)
done

lemma *zero-le-imp-eq-int*: (*0*::*int*) ≤ *k* ==> ∃ *n*. *k* = *of-nat n*
apply (*cases k*)
apply (*auto simp add*: *le add int-def Zero-int-def*)
apply (*rule-tac x=x−y* **in** *exI*, *simp*)
done

lemma *zero-less-imp-eq-int*: (*0*::*int*) < *k* ==> ∃ *n*>*0*. *k* = *of-nat n*
apply (*cases k*)
apply (*simp add*: *less int-def Zero-int-def*)
apply (*rule-tac x=x−y* **in** *exI*, *simp*)
done

lemma *zmult-zless-mono2*: [| *i*<*j*; (*0*::*int*) < *k* |] ==> *k*∗*i* < *k*∗*j*
apply (*drule zero-less-imp-eq-int*)
apply (*auto simp add*: *zmult-zless-mono2-lemma*)
done

The integers form an ordered integral domain

instance *int* :: *linordered-idom*
proof
  fix *i j k* :: *int*
  show $i < j \Longrightarrow 0 < k \Longrightarrow k * i < k * j$
    by (*rule zmult-zless-mono2*)
  show $|i| = (\mathit{if}\ i < 0\ \mathit{then}\ -i\ \mathit{else}\ i)$
    by (*simp only*: *zabs-def*)
  show *sgn* (*i*::*int*) = (*if i=0 then 0 else if 0*<*i then 1 else* − *1*)
    by (*simp only*: *zsgn-def*)
qed

lemma *zless-imp-add1-zle*: $w < z \Longrightarrow w + (1::int) \leq z$

apply (*cases w*, *cases z*)
apply (*simp add*: *less le add One-int-def*)
done

lemma *zless-iff-Suc-zadd*:
(*w* :: *int*) < *z* ⟷ (∃ *n*. *z* = *w* + *of-nat* (*Suc n*))
apply (*cases z*, *cases w*)
apply (*auto simp add*: *less add int-def*)
apply (*rename-tac a b c d*)
apply (*rule-tac x*=*a*+*d* − *Suc*(*c*+*b*) **in** *exI*)
apply *arith*
done

lemmas *int-distrib* =
left-distrib [*of z1*::*int z2 w*, *standard*]
right-distrib [*of w*::*int z1 z2*, *standard*]
left-diff-distrib [*of z1*::*int z2 w*, *standard*]
right-diff-distrib [*of w*::*int z1 z2*, *standard*]

30.5 Embedding of the Integers into any *ring-1*: *of-int*

context *ring-1*
begin

definition *of-int* :: *int* ⇒ *'a* **where**
[*code del*]: *of-int z* = *contents* (⋃(*i*, *j*) ∈ *Rep-Integ z*. { *of-nat i* − *of-nat j* })

lemma *of-int*: *of-int* (*Abs-Integ* (*intrel* `` {(*i*,*j*)})) = *of-nat i* − *of-nat j*
proof −
have (λ(*i*,*j*). { *of-nat i* − (*of-nat j* :: *'a*) }) *respects intrel*
by (*simp add*: *congruent-def algebra-simps of-nat-add* [*symmetric*]
del: *of-nat-add*)
thus *?thesis*
by (*simp add*: *of-int-def UN-equiv-class* [*OF equiv-intrel*])
qed

lemma *of-int-0* [*simp*]: *of-int 0* = *0*
by (*simp add*: *of-int Zero-int-def*)

lemma *of-int-1* [*simp*]: *of-int 1* = *1*
by (*simp add*: *of-int One-int-def*)

lemma *of-int-add* [*simp*]: *of-int* (*w*+*z*) = *of-int w* + *of-int z*
by (*cases w*, *cases z*, *simp add*: *algebra-simps of-int add*)

lemma *of-int-minus* [*simp*]: *of-int* (−*z*) = − (*of-int z*)
by (*cases z*, *simp add*: *algebra-simps of-int minus*)

lemma *of-int-diff* [*simp*]: *of-int* (*w* − *z*) = *of-int w* − *of-int z*

by (*simp add*: *diff-minus Groups.diff-minus*)

lemma *of-int-mult* [*simp*]: *of-int* (*w*∗*z*) = *of-int w* ∗ *of-int z*
apply (*cases w*, *cases z*)
apply (*simp add*: *algebra-simps of-int mult of-nat-mult*)
done

Collapse nested embeddings

lemma *of-int-of-nat-eq* [*simp*]: *of-int* (*of-nat n*) = *of-nat n*
by (*induct n*) *auto*

lemma *of-int-power*:
of-int (*z* ^ *n*) = *of-int z* ^ *n*
by (*induct n*) *simp-all*

end

Class for unital rings with characteristic zero. Includes non-ordered rings like the complex numbers.

class *ring-char-0* = *ring-1* + *semiring-char-0*
begin

lemma *of-int-eq-iff* [*simp*]:
of-int w = *of-int z* ⟷ *w* = *z*
apply (*cases w*, *cases z*, *simp add*: *of-int*)
apply (*simp only*: *diff-eq-eq diff-add-eq eq-diff-eq*)
apply (*simp only*: *of-nat-add* [*symmetric*] *of-nat-eq-iff*)
done

Special cases where either operand is zero

lemma *of-int-eq-0-iff* [*simp*]:
of-int z = 0 ⟷ *z* = 0
using *of-int-eq-iff* [*of z 0*] **by** *simp*

lemma *of-int-0-eq-iff* [*simp*]:
0 = *of-int z* ⟷ *z* = 0
using *of-int-eq-iff* [*of 0 z*] **by** *simp*

end

context *linordered-idom*
begin

Every *linordered-idom* has characteristic zero.

subclass *ring-char-0* **..**

lemma *of-int-le-iff* [*simp*]:
of-int w ≤ *of-int z* ⟷ *w* ≤ *z*

by (*cases w*, *cases z*, *simp add*: *of-int le minus algebra-simps of-nat-add* [*symmetric*] *del*: *of-nat-add*)

lemma *of-int-less-iff* [*simp*]:
of-int w < *of-int z* ⟷ *w* < *z*
by (*simp add*: *less-le order-less-le*)

lemma *of-int-0-le-iff* [*simp*]:
0 ≤ *of-int z* ⟷ *0* ≤ *z*
using *of-int-le-iff* [*of 0 z*] **by** *simp*

lemma *of-int-le-0-iff* [*simp*]:
of-int z ≤ *0* ⟷ *z* ≤ *0*
using *of-int-le-iff* [*of z 0*] **by** *simp*

lemma *of-int-0-less-iff* [*simp*]:
0 < *of-int z* ⟷ *0* < *z*
using *of-int-less-iff* [*of 0 z*] **by** *simp*

lemma *of-int-less-0-iff* [*simp*]:
of-int z < *0* ⟷ *z* < *0*
using *of-int-less-iff* [*of z 0*] **by** *simp*

end

lemma *of-int-eq-id* [*simp*]: *of-int* = *id*
proof
fix *z* **show** *of-int z* = *id z*
by (*cases z*) (*simp add*: *of-int add minus int-def diff-minus*)
qed

30.6 Magnitude of an Integer, as a Natural Number: *nat*

definition
nat :: *int* ⇒ *nat*
where
[*code del*]: *nat z* = *contents* (⋃(*x*, *y*) ∈ *Rep-Integ z*. {*x*−*y*})

lemma *nat*: *nat* (*Abs-Integ* (*intrel*``{(*x*,*y*)})) = *x*−*y*
proof −
have (λ(*x*,*y*). {*x*−*y*}) *respects intrel*
by (*simp add*: *congruent-def*) *arith*
thus *?thesis*
by (*simp add*: *nat-def UN-equiv-class* [*OF equiv-intrel*])
qed

lemma *nat-int* [*simp*]: *nat* (*of-nat n*) = *n*
by (*simp add*: *nat int-def*)

lemma *nat-zero* [*simp*]: *nat 0 = 0*
by (*simp add*: *Zero-int-def nat*)

lemma *int-nat-eq* [*simp*]: *of-nat* (*nat z*) = (*if 0 ≤ z then z else 0*)
by (*cases z*, *simp add*: *nat le int-def Zero-int-def*)

corollary *nat-0-le*: *0 ≤ z ==> of-nat* (*nat z*) = *z*
by *simp*

lemma *nat-le-0* [*simp*]: *z ≤ 0 ==> nat z = 0*
by (*cases z*, *simp add*: *nat le Zero-int-def*)

lemma *nat-le-eq-zle*: *0 < w | 0 ≤ z ==> (nat w ≤ nat z) = (w≤z)*
apply (*cases w*, *cases z*)
apply (*simp add*: *nat le linorder-not-le* [*symmetric*] *Zero-int-def*, *arith*)
done

An alternative condition is $(0::'a) \leq w$

corollary *nat-mono-iff*: *0 < z ==> (nat w < nat z) = (w < z)*
by (*simp add*: *nat-le-eq-zle linorder-not-le* [*symmetric*])

corollary *nat-less-eq-zless*: *0 ≤ w ==> (nat w < nat z) = (w<z)*
by (*simp add*: *nat-le-eq-zle linorder-not-le* [*symmetric*])

lemma *zless-nat-conj* [*simp*]: (*nat w < nat z*) = (*0 < z & w < z*)
apply (*cases w*, *cases z*)
apply (*simp add*: *nat le Zero-int-def linorder-not-le* [*symmetric*], *arith*)
done

lemma *nonneg-eq-int*:
fixes *z* :: *int*
assumes *0 ≤ z* **and** $\bigwedge m.$ *z = of-nat m* $\Longrightarrow$ *P*
shows *P*
using *assms* **by** (*blast dest*: *nat-0-le sym*)

lemma *nat-eq-iff*: (*nat w = m*) = (*if 0 ≤ w then w = of-nat m else m=0*)
by (*cases w*, *simp add*: *nat le int-def Zero-int-def*, *arith*)

corollary *nat-eq-iff2*: (*m = nat w*) = (*if 0 ≤ w then w = of-nat m else m=0*)
by (*simp only*: *eq-commute* [*of m*] *nat-eq-iff*)

lemma *nat-less-iff*: *0 ≤ w ==> (nat w < m) = (w < of-nat m)*
apply (*cases w*)
apply (*simp add*: *nat le int-def Zero-int-def linorder-not-le*[*symmetric*], *arith*)
done

lemma *nat-0-iff*[*simp*]: *nat*(*i*::*int*) = *0* $\longleftrightarrow$ *i≤0*
by(*simp add*: *nat-eq-iff*) *arith*

lemma *int-eq-iff*: $(of\text{-}nat\ m = z) = (m = nat\ z\ \&\ 0 \leq z)$
by (*auto simp add*: *nat-eq-iff2*)

lemma *zero-less-nat-eq* [*simp*]: $(0 < nat\ z) = (0 < z)$
by (*insert zless-nat-conj* [*of 0*], *auto*)

lemma *nat-add-distrib*:
$[|\ (0::int) \leq z;\ \ 0 \leq z'\ |] ==> nat\ (z+z') = nat\ z + nat\ z'$
by (*cases z*, *cases z'*, *simp add*: *nat add le Zero-int-def*)

lemma *nat-diff-distrib*:
$[|\ (0::int) \leq z';\ \ z' \leq z\ |] ==> nat\ (z-z') = nat\ z - nat\ z'$
by (*cases z*, *cases z'*,
simp add: *nat add minus diff-minus le Zero-int-def*)

lemma *nat-zminus-int* [*simp*]: $nat\ (-\ (of\text{-}nat\ n)) = 0$
by (*simp add*: *int-def minus nat Zero-int-def*)

lemma *zless-nat-eq-int-zless*: $(m < nat\ z) = (of\text{-}nat\ m < z)$
by (*cases z*, *simp add*: *nat less int-def*, *arith*)

context *ring-1*
begin

lemma *of-nat-nat*: $0 \leq z \Longrightarrow of\text{-}nat\ (nat\ z) = of\text{-}int\ z$
by (*cases z rule*: *eq-Abs-Integ*)
(*simp add*: *nat le of-int Zero-int-def of-nat-diff*)

end

For termination proofs:

lemma *measure-function-int*[*measure-function*]: *is-measure* (*nat o abs*) **..**

30.7 Lemmas about the Function *of-nat* and Orderings

lemma *negative-zless-0*: $-\ (of\text{-}nat\ (Suc\ n)) < (0 :: int)$
by (*simp add*: *order-less-le del*: *of-nat-Suc*)

lemma *negative-zless* [*iff*]: $-\ (of\text{-}nat\ (Suc\ n)) < (of\text{-}nat\ m :: int)$
by (*rule negative-zless-0* [*THEN order-less-le-trans*], *simp*)

lemma *negative-zle-0*: $-\ of\text{-}nat\ n \leq (0 :: int)$
by (*simp add*: *minus-le-iff*)

lemma *negative-zle* [*iff*]: $-\ of\text{-}nat\ n \leq (of\text{-}nat\ m :: int)$
by (*rule order-trans* [*OF negative-zle-0 of-nat-0-le-iff*])

lemma *not-zle-0-negative* [*simp*]: $\sim\ (0 \leq -\ (of\text{-}nat\ (Suc\ n) :: int))$

by (*subst le-minus-iff*, *simp del*: *of-nat-Suc*)

lemma *int-zle-neg*: ((*of-nat n* :: *int*) ≤ − *of-nat m*) = (*n* = *0* & *m* = *0*)
by (*simp add*: *int-def le minus Zero-int-def*)

lemma *not-int-zless-negative* [*simp*]: ~ ((*of-nat n* :: *int*) < − *of-nat m*)
by (*simp add*: *linorder-not-less*)

lemma *negative-eq-positive* [*simp*]: ((− *of-nat n* :: *int*) = *of-nat m*) = (*n* = *0* & *m* = *0*)
by (*force simp add*: *order-eq-iff* [*of* − *of-nat n*] *int-zle-neg*)

lemma *zle-iff-zadd*: (*w*::*int*) ≤ *z* ⟷ (∃ *n*. *z* = *w* + *of-nat n*)
proof −
  have (*w* ≤ *z*) = (*0* ≤ *z* − *w*)
    by (*simp only*: *le-diff-eq add-0-left*)
  also have ... = (∃ *n*. *z* − *w* = *of-nat n*)
    by (*auto elim*: *zero-le-imp-eq-int*)
  also have ... = (∃ *n*. *z* = *w* + *of-nat n*)
    by (*simp only*: *algebra-simps*)
  finally show *?thesis* **.**
qed

lemma *zadd-int-left*: *of-nat m* + (*of-nat n* + *z*) = *of-nat* (*m* + *n*) + (*z*::*int*)
by *simp*

lemma *int-Suc0-eq-1*: *of-nat* (*Suc 0*) = (*1*::*int*)
by *simp*

This version is proved for all ordered rings, not just integers! It is proved here because attribute *arith-split* is not available in theory *Rings*. But is it really better than just rewriting with *abs-if*?

lemma *abs-split* [*arith-split*,*no-atp*]:
    P(*abs*(*a*::'*a*::*linordered-idom*)) = ((*0* ≤ *a* −−> *P a*) & (*a* < *0* −−> *P*(−*a*)))
by (*force dest*: *order-less-le-trans simp add*: *abs-if linorder-not-less*)

lemma *negD*: (*x* :: *int*) < *0* ⟹ ∃ *n*. *x* = − (*of-nat* (*Suc n*))
apply (*cases x*)
apply (*auto simp add*: *le minus Zero-int-def int-def order-less-le*)
apply (*rule-tac x*=*y* − *Suc x* **in** *exI*, *arith*)
done

30.8 Cases and induction

Now we replace the case analysis rule by a more conventional one: whether an integer is negative or not.

theorem *int-cases* [*cases type*: *int*, *case-names nonneg neg*]:

[|!! *n*. (*z* :: *int*) = *of-nat n* ==> *P*; !! *n*. *z* = − (*of-nat* (*Suc n*)) ==> *P* |] ==> *P*
apply (*cases z* < *0*, *blast dest*!: *negD*)
apply (*simp add*: *linorder-not-less del*: *of-nat-Suc*)
apply *auto*
apply (*blast dest*: *nat-0-le* [*THEN sym*])
done

theorem *int-of-nat-induct* [*induct type*: *int*, *case-names nonneg neg*]:
[|!! *n*. *P* (*of-nat n* :: *int*); !!*n*. *P* (− (*of-nat* (*Suc n*))) |] ==> *P z*
by (*cases z rule*: *int-cases*) *auto*

Contributed by Brian Huffman

theorem *int-diff-cases*:
obtains (*diff*) *m n* **where** (*z*::*int*) = *of-nat m* − *of-nat n*
apply (*cases z rule*: *eq-Abs-Integ*)
apply (*rule-tac m*=*x* **and** *n*=*y* **in** *diff*)
apply (*simp add*: *int-def diff-def minus add*)
done

30.9 Binary representation

This formalization defines binary arithmetic in terms of the integers rather than using a datatype. This avoids multiple representations (leading zeroes, etc.) See *ZF/Tools/twos−compl.ML*, function *int-of-binary*, for the numerical interpretation.

The representation expects that (*m mod 2*) is 0 or 1, even if m is negative; For instance, *−5 div 2* = *−3* and *−5 mod 2* = *1*; thus *−5* = (*−3*)∗*2* + *1*.

This two's complement binary representation derives from the paper "An Efficient Representation of Arithmetic for Term Rewriting" by Dave Cohen and Phil Watson, Rewriting Techniques and Applications, Springer LNCS 488 (240-251), 1991.

30.9.1 The constructors *Bit0*, *Bit1*, *Pls* and *Min*

definition
Pls :: *int* **where**
[*code del*]: *Pls* = *0*

definition
Min :: *int* **where**
[*code del*]: *Min* = − *1*

definition
Bit0 :: *int* ⇒ *int* **where**
[*code del*]: *Bit0 k* = *k* + *k*

definition
Bit1 :: *int* ⇒ *int* **where**
[*code del*]: *Bit1 k* = *1* + *k* + *k*

class *number* = — for numeric types: nat, int, real, ...
fixes *number-of* :: *int* ⇒ *'a*

use *Tools/numeral.ML*

syntax
-Numeral :: *num-const* ⇒ *'a* (-)

use *Tools/numeral-syntax.ML*
setup *Numeral-Syntax.setup*

abbreviation
Numeral0 ≡ *number-of Pls*

abbreviation
Numeral1 ≡ *number-of* (*Bit1 Pls*)

lemma *Let-number-of* [*simp*]: *Let* (*number-of v*) *f* = *f* (*number-of v*)
— Unfold all *let*s involving constants
unfolding *Let-def* **..**

definition
succ :: *int* ⇒ *int* **where**
[*code del*]: *succ k* = *k* + *1*

definition
pred :: *int* ⇒ *int* **where**
[*code del*]: *pred k* = *k* − *1*

lemmas
max-number-of [*simp*] = *max-def*
[*of number-of u number-of v*, *standard*]
and
min-number-of [*simp*] = *min-def*
[*of number-of u number-of v*, *standard*]
— unfolding *minx* and *max* on numerals

lemmas *numeral-simps* =
succ-def pred-def Pls-def Min-def Bit0-def Bit1-def

Removal of leading zeroes

lemma *Bit0-Pls* [*simp*, *code-post*]:
Bit0 Pls = *Pls*
unfolding *numeral-simps* **by** *simp*

lemma *Bit1-Min* [*simp*, *code-post*]:
Bit1 Min = Min
unfolding *numeral-simps* **by** *simp*

lemmas *normalize-bin-simps* =
Bit0-Pls Bit1-Min

30.9.2 Successor and predecessor functions

Successor

lemma *succ-Pls*:
succ Pls = Bit1 Pls
unfolding *numeral-simps* **by** *simp*

lemma *succ-Min*:
succ Min = Pls
unfolding *numeral-simps* **by** *simp*

lemma *succ-Bit0*:
succ (Bit0 k) = Bit1 k
unfolding *numeral-simps* **by** *simp*

lemma *succ-Bit1*:
succ (Bit1 k) = Bit0 (succ k)
unfolding *numeral-simps* **by** *simp*

lemmas *succ-bin-simps* [*simp*] =
succ-Pls succ-Min succ-Bit0 succ-Bit1

Predecessor

lemma *pred-Pls*:
pred Pls = Min
unfolding *numeral-simps* **by** *simp*

lemma *pred-Min*:
pred Min = Bit0 Min
unfolding *numeral-simps* **by** *simp*

lemma *pred-Bit0*:
pred (Bit0 k) = Bit1 (pred k)
unfolding *numeral-simps* **by** *simp*

lemma *pred-Bit1*:
pred (Bit1 k) = Bit0 k
unfolding *numeral-simps* **by** *simp*

lemmas *pred-bin-simps* [*simp*] =
pred-Pls pred-Min pred-Bit0 pred-Bit1

30.9.3 Binary arithmetic

Addition

lemma *add-Pls*:
Pls + *k* = *k*
unfolding *numeral-simps* **by** *simp*

lemma *add-Min*:
Min + *k* = *pred k*
unfolding *numeral-simps* **by** *simp*

lemma *add-Bit0-Bit0*:
(*Bit0 k*) + (*Bit0 l*) = *Bit0* (*k* + *l*)
unfolding *numeral-simps* **by** *simp*

lemma *add-Bit0-Bit1*:
(*Bit0 k*) + (*Bit1 l*) = *Bit1* (*k* + *l*)
unfolding *numeral-simps* **by** *simp*

lemma *add-Bit1-Bit0*:
(*Bit1 k*) + (*Bit0 l*) = *Bit1* (*k* + *l*)
unfolding *numeral-simps* **by** *simp*

lemma *add-Bit1-Bit1*:
(*Bit1 k*) + (*Bit1 l*) = *Bit0* (*k* + *succ l*)
unfolding *numeral-simps* **by** *simp*

lemma *add-Pls-right*:
k + *Pls* = *k*
unfolding *numeral-simps* **by** *simp*

lemma *add-Min-right*:
k + *Min* = *pred k*
unfolding *numeral-simps* **by** *simp*

lemmas *add-bin-simps* [*simp*] =
add-Pls add-Min add-Pls-right add-Min-right
add-Bit0-Bit0 add-Bit0-Bit1 add-Bit1-Bit0 add-Bit1-Bit1

Negation

lemma *minus-Pls*:
− *Pls* = *Pls*
unfolding *numeral-simps* **by** *simp*

lemma *minus-Min*:
− *Min* = *Bit1 Pls*
unfolding *numeral-simps* **by** *simp*

lemma *minus-Bit0*:

$- (Bit0\ k) = Bit0\ (- k)$
unfolding *numeral-simps* **by** *simp*

lemma *minus-Bit1*:
$- (Bit1\ k) = Bit1\ (pred\ (- k))$
unfolding *numeral-simps* **by** *simp*

lemmas *minus-bin-simps* [*simp*] =
minus-Pls minus-Min minus-Bit0 minus-Bit1

Subtraction

lemma *diff-bin-simps* [*simp*]:
$k - Pls = k$
$k - Min = succ\ k$
$Pls - (Bit0\ l) = Bit0\ (Pls - l)$
$Pls - (Bit1\ l) = Bit1\ (Min - l)$
$Min - (Bit0\ l) = Bit1\ (Min - l)$
$Min - (Bit1\ l) = Bit0\ (Min - l)$
$(Bit0\ k) - (Bit0\ l) = Bit0\ (k - l)$
$(Bit0\ k) - (Bit1\ l) = Bit1\ (pred\ k - l)$
$(Bit1\ k) - (Bit0\ l) = Bit1\ (k - l)$
$(Bit1\ k) - (Bit1\ l) = Bit0\ (k - l)$
unfolding *numeral-simps* **by** *simp-all*

Multiplication

lemma *mult-Pls*:
$Pls * w = Pls$
unfolding *numeral-simps* **by** *simp*

lemma *mult-Min*:
$Min * k = - k$
unfolding *numeral-simps* **by** *simp*

lemma *mult-Bit0*:
$(Bit0\ k) * l = Bit0\ (k * l)$
unfolding *numeral-simps int-distrib* **by** *simp*

lemma *mult-Bit1*:
$(Bit1\ k) * l = (Bit0\ (k * l)) + l$
unfolding *numeral-simps int-distrib* **by** *simp*

lemmas *mult-bin-simps* [*simp*] =
mult-Pls mult-Min mult-Bit0 mult-Bit1

30.9.4 Binary comparisons

Preliminaries

lemma *even-less-0-iff*:

a + a < 0 ⟷ a < (0::'a::linordered-idom)
proof −
have *a + a < 0 ⟷ (1+1)*a < 0* **by** (*simp add*: *left-distrib*)
also have *(1+1)*a < 0 ⟷ a < 0*
by (*simp add*: *mult-less-0-iff zero-less-two*
order-less-not-sym [*OF zero-less-two*])
finally show *?thesis* **.**
qed

lemma *le-imp-0-less*:
assumes *le*: *0 ≤ z*
shows *(0::int) < 1 + z*
proof −
have *0 ≤ z* **by** *fact*
also have *... < z + 1* **by** (*rule less-add-one*)
also have *... = 1 + z* **by** (*simp add*: *add-ac*)
finally show *0 < 1 + z* **.**
qed

lemma *odd-less-0-iff*:
(1 + z + z < 0) = (z < (0::int))
proof (*cases z rule*: *int-cases*)
case (*nonneg n*)
thus *?thesis* **by** (*simp add*: *linorder-not-less add-assoc add-increasing*
le-imp-0-less [*THEN order-less-imp-le*])
next
case (*neg n*)
thus *?thesis* **by** (*simp del*: *of-nat-Suc of-nat-add of-nat-1*
add: *algebra-simps of-nat-1* [**where** *'a=int*, *symmetric*] *of-nat-add* [*symmetric*])
qed

lemma *bin-less-0-simps*:
Pls < 0 ⟷ False
Min < 0 ⟷ True
Bit0 w < 0 ⟷ w < 0
Bit1 w < 0 ⟷ w < 0
unfolding *numeral-simps*
by (*simp-all add*: *even-less-0-iff odd-less-0-iff*)

lemma *less-bin-lemma*: *k < l ⟷ k − l < (0::int)*
by *simp*

lemma *le-iff-pred-less*: *k ≤ l ⟷ pred k < l*
unfolding *numeral-simps*
proof
have *k − 1 < k* **by** *simp*
also assume *k ≤ l*
finally show *k − 1 < l* **.**
next

assume $k - 1 < l$
hence $(k - 1) + 1 \leq l$ **by** (*rule zless-imp-add1-zle*)
thus $k \leq l$ **by** *simp*
qed

lemma *succ-pred*: $succ\ (pred\ x) = x$
unfolding *numeral-simps* **by** *simp*

Less-than

lemma *less-bin-simps* [*simp*]:
$Pls < Pls \longleftrightarrow False$
$Pls < Min \longleftrightarrow False$
$Pls < Bit0\ k \longleftrightarrow Pls < k$
$Pls < Bit1\ k \longleftrightarrow Pls \leq k$
$Min < Pls \longleftrightarrow True$
$Min < Min \longleftrightarrow False$
$Min < Bit0\ k \longleftrightarrow Min < k$
$Min < Bit1\ k \longleftrightarrow Min < k$
$Bit0\ k < Pls \longleftrightarrow k < Pls$
$Bit0\ k < Min \longleftrightarrow k \leq Min$
$Bit1\ k < Pls \longleftrightarrow k < Pls$
$Bit1\ k < Min \longleftrightarrow k < Min$
$Bit0\ k < Bit0\ l \longleftrightarrow k < l$
$Bit0\ k < Bit1\ l \longleftrightarrow k \leq l$
$Bit1\ k < Bit0\ l \longleftrightarrow k < l$
$Bit1\ k < Bit1\ l \longleftrightarrow k < l$
unfolding *le-iff-pred-less*
less-bin-lemma [*of Pls*]
less-bin-lemma [*of Min*]
less-bin-lemma [*of k*]
less-bin-lemma [*of Bit0 k*]
less-bin-lemma [*of Bit1 k*]
less-bin-lemma [*of pred Pls*]
less-bin-lemma [*of pred k*]
by (*simp-all add*: *bin-less-0-simps succ-pred*)

Less-than-or-equal

lemma *le-bin-simps* [*simp*]:
$Pls \leq Pls \longleftrightarrow True$
$Pls \leq Min \longleftrightarrow False$
$Pls \leq Bit0\ k \longleftrightarrow Pls \leq k$
$Pls \leq Bit1\ k \longleftrightarrow Pls \leq k$
$Min \leq Pls \longleftrightarrow True$
$Min \leq Min \longleftrightarrow True$
$Min \leq Bit0\ k \longleftrightarrow Min < k$
$Min \leq Bit1\ k \longleftrightarrow Min \leq k$
$Bit0\ k \leq Pls \longleftrightarrow k \leq Pls$
$Bit0\ k \leq Min \longleftrightarrow k \leq Min$
$Bit1\ k \leq Pls \longleftrightarrow k < Pls$

Bit1 k ≤ *Min* ⟷ *k* ≤ *Min*
Bit0 k ≤ *Bit0 l* ⟷ *k* ≤ *l*
Bit0 k ≤ *Bit1 l* ⟷ *k* ≤ *l*
Bit1 k ≤ *Bit0 l* ⟷ *k* < *l*
Bit1 k ≤ *Bit1 l* ⟷ *k* ≤ *l*
unfolding *not-less* [*symmetric*]
by (*simp-all add*: *not-le*)

Equality

lemma *eq-bin-simps* [*simp*]:
Pls = *Pls* ⟷ *True*
Pls = *Min* ⟷ *False*
Pls = *Bit0 l* ⟷ *Pls* = *l*
Pls = *Bit1 l* ⟷ *False*
Min = *Pls* ⟷ *False*
Min = *Min* ⟷ *True*
Min = *Bit0 l* ⟷ *False*
Min = *Bit1 l* ⟷ *Min* = *l*
Bit0 k = *Pls* ⟷ *k* = *Pls*
Bit0 k = *Min* ⟷ *False*
Bit1 k = *Pls* ⟷ *False*
Bit1 k = *Min* ⟷ *k* = *Min*
Bit0 k = *Bit0 l* ⟷ *k* = *l*
Bit0 k = *Bit1 l* ⟷ *False*
Bit1 k = *Bit0 l* ⟷ *False*
Bit1 k = *Bit1 l* ⟷ *k* = *l*
unfolding *order-eq-iff* [**where** *'a*=*int*]
by (*simp-all add*: *not-less*)

30.10 Converting Numerals to Rings: *number-of*

class *number-ring* = *number* + *comm-ring-1* +
assumes *number-of-eq*: *number-of k* = *of-int k*

self-embedding of the integers

instantiation *int* :: *number-ring*
begin

definition *int-number-of-def* [*code del*]:
number-of w = (*of-int w* :: *int*)

instance proof
qed (*simp only*: *int-number-of-def*)

end

lemma *number-of-is-id*:
number-of (*k*::*int*) = *k*
unfolding *int-number-of-def* **by** *simp*

lemma *number-of-succ*:
number-of (*succ k*) = (*1* + *number-of k* ::'*a*::*number-ring*)
unfolding *number-of-eq numeral-simps* **by** *simp*

lemma *number-of-pred*:
number-of (*pred w*) = (− *1* + *number-of w* ::'*a*::*number-ring*)
unfolding *number-of-eq numeral-simps* **by** *simp*

lemma *number-of-minus*:
number-of (*uminus w*) = (− (*number-of w*)::'*a*::*number-ring*)
unfolding *number-of-eq* **by** (*rule of-int-minus*)

lemma *number-of-add*:
number-of (*v* + *w*) = (*number-of v* + *number-of w*::'*a*::*number-ring*)
unfolding *number-of-eq* **by** (*rule of-int-add*)

lemma *number-of-diff*:
number-of (*v* − *w*) = (*number-of v* − *number-of w*::'*a*::*number-ring*)
unfolding *number-of-eq* **by** (*rule of-int-diff*)

lemma *number-of-mult*:
number-of (*v* ∗ *w*) = (*number-of v* ∗ *number-of w*::'*a*::*number-ring*)
unfolding *number-of-eq* **by** (*rule of-int-mult*)

The correctness of shifting. But it doesn't seem to give a measurable speed-up.

lemma *double-number-of-Bit0*:
(*1* + *1*) ∗ *number-of w* = (*number-of* (*Bit0 w*) ::'*a*::*number-ring*)
unfolding *number-of-eq numeral-simps left-distrib* **by** *simp*

Converting numerals 0 and 1 to their abstract versions.

lemma *numeral-0-eq-0* [*simp*, *code-post*]:
Numeral0 = (*0*::'*a*::*number-ring*)
unfolding *number-of-eq numeral-simps* **by** *simp*

lemma *numeral-1-eq-1* [*simp*, *code-post*]:
Numeral1 = (*1*::'*a*::*number-ring*)
unfolding *number-of-eq numeral-simps* **by** *simp*

Special-case simplification for small constants.

Unary minus for the abstract constant 1. Cannot be inserted as a simprule until later: it is *number-of-Min* re-oriented!

lemma *numeral-m1-eq-minus-1*:
(−*1*::'*a*::*number-ring*) = − *1*
unfolding *number-of-eq numeral-simps* **by** *simp*

lemma *mult-minus1* [*simp*]:

$-1 * z = -(z::'a::number\text{-}ring)$
unfolding *number-of-eq numeral-simps* **by** *simp*

lemma *mult-minus1-right* [*simp*]:
$z * -1 = -(z::'a::number\text{-}ring)$
unfolding *number-of-eq numeral-simps* **by** *simp*

lemma *minus-number-of-mult* [*simp*]:
$-(number\text{-}of\ w) * z = number\text{-}of\ (uminus\ w) * (z::'a::number\text{-}ring)$
unfolding *number-of-eq* **by** *simp*

Subtraction

lemma *diff-number-of-eq*:
$number\text{-}of\ v - number\text{-}of\ w =$
$(number\text{-}of\ (v + uminus\ w)::'a::number\text{-}ring)$
unfolding *number-of-eq* **by** *simp*

lemma *number-of-Pls*:
$number\text{-}of\ Pls = (0::'a::number\text{-}ring)$
unfolding *number-of-eq numeral-simps* **by** *simp*

lemma *number-of-Min*:
$number\text{-}of\ Min = (-\ 1::'a::number\text{-}ring)$
unfolding *number-of-eq numeral-simps* **by** *simp*

lemma *number-of-Bit0*:
$number\text{-}of\ (Bit0\ w) = (0::'a::number\text{-}ring) + (number\text{-}of\ w) + (number\text{-}of\ w)$
unfolding *number-of-eq numeral-simps* **by** *simp*

lemma *number-of-Bit1*:
$number\text{-}of\ (Bit1\ w) = (1::'a::number\text{-}ring) + (number\text{-}of\ w) + (number\text{-}of\ w)$
unfolding *number-of-eq numeral-simps* **by** *simp*

30.10.1 Equality of Binary Numbers

First version by Norbert Voelker

definition $iszero :: 'a::semiring\text{-}1 \Rightarrow bool$ **where**
$iszero\ z \longleftrightarrow z = 0$

lemma *iszero-0*: *iszero 0*
by (*simp add*: *iszero-def*)

lemma *iszero-Numeral0*: $iszero\ (Numeral0 :: 'a::number\text{-}ring)$
by (*simp add*: *iszero-0*)

lemma *not-iszero-1*: $\neg$ *iszero 1*
by (*simp add*: *iszero-def*)

lemma *not-iszero-Numeral1*: ¬ *iszero* (*Numeral1* :: *'a*::*number-ring*)
by (*simp add*: *not-iszero-1*)

lemma *eq-number-of-eq* [*simp*]:
((*number-of x*::*'a*::*number-ring*) = *number-of y*) =
iszero (*number-of* (*x* + *uminus y*) :: *'a*)
unfolding *iszero-def number-of-add number-of-minus*
by (*simp add*: *algebra-simps*)

lemma *iszero-number-of-Pls*:
iszero ((*number-of Pls*)::*'a*::*number-ring*)
unfolding *iszero-def numeral-0-eq-0* **..**

lemma *nonzero-number-of-Min*:
~ *iszero* ((*number-of Min*)::*'a*::*number-ring*)
unfolding *iszero-def numeral-m1-eq-minus-1* **by** *simp*

30.10.2 Comparisons, for Ordered Rings

lemmas *double-eq-0-iff* = *double-zero*

lemma *odd-nonzero*:
1 + *z* + *z* ≠ (*0*::*int*)
proof (*cases z rule*: *int-cases*)
case (*nonneg n*)
have *le*: *0* ≤ *z*+*z* **by** (*simp add*: *nonneg add-increasing*)
thus *?thesis* **using** *le-imp-0-less* [*OF le*]
by (*auto simp add*: *add-assoc*)
next
case (*neg n*)
show *?thesis*
proof
assume *eq*: *1* + *z* + *z* = *0*
have (*0*::*int*) < *1* + (*of-nat n* + *of-nat n*)
by (*simp add*: *le-imp-0-less add-increasing*)
also have ... = − (*1* + *z* + *z*)
by (*simp add*: *neg add-assoc* [*symmetric*])
also have ... = *0* **by** (*simp add*: *eq*)
finally have *0*<*0* **..**
thus *False* **by** *blast*
qed
qed

lemma *iszero-number-of-Bit0*:
iszero (*number-of* (*Bit0 w*)::*'a*) =
iszero (*number-of w*::*'a*::{*ring-char-0*,*number-ring*})
proof −
have (*of-int w* + *of-int w* = (*0*::*'a*)) ⟹ (*w* = *0*)
proof −

```
    assume eq: of-int w + of-int w = (0::'a)
    then have of-int (w + w) = (of-int 0 :: 'a) by simp
    then have w + w = 0 by (simp only: of-int-eq-iff)
    then show w = 0 by (simp only: double-eq-0-iff)
  qed
  thus ?thesis
    by (auto simp add: iszero-def number-of-eq numeral-simps)
qed

lemma iszero-number-of-Bit1:
  ~ iszero (number-of (Bit1 w)::'a::{ring-char-0,number-ring})
proof -
  have 1 + of-int w + of-int w ≠ (0::'a)
  proof
    assume eq: 1 + of-int w + of-int w = (0::'a)
    hence of-int (1 + w + w) = (of-int 0 :: 'a) by simp
    hence 1 + w + w = 0 by (simp only: of-int-eq-iff)
    with odd-nonzero show False by blast
  qed
  thus ?thesis
    by (auto simp add: iszero-def number-of-eq numeral-simps)
qed

lemmas iszero-simps [simp] =
  iszero-0 not-iszero-1
  iszero-number-of-Pls nonzero-number-of-Min
  iszero-number-of-Bit0 iszero-number-of-Bit1
```

30.10.3 The Less-Than Relation

```
lemma double-less-0-iff:
  (a + a < 0) = (a < (0::'a::linordered-idom))
proof -
  have (a + a < 0) = ((1+1)*a < 0) by (simp add: left-distrib)
  also have ... = (a < 0)
    by (simp add: mult-less-0-iff zero-less-two
                  order-less-not-sym [OF zero-less-two])
  finally show ?thesis .
qed

lemma odd-less-0:
  (1 + z + z < 0) = (z < (0::int))
proof (cases z rule: int-cases)
  case (nonneg n)
  thus ?thesis by (simp add: linorder-not-less add-assoc add-increasing
                             le-imp-0-less [THEN order-less-imp-le])
next
  case (neg n)
  thus ?thesis by (simp del: of-nat-Suc of-nat-add of-nat-1
```

add: *algebra-simps of-nat-1* [**where** *'a=int*, *symmetric*] *of-nat-add* [*symmetric*])
qed

Less-Than or Equals

Reduces $a \leq b$ to $\neg\ b < a$ for ALL numerals.

lemmas *le-number-of-eq-not-less* =
linorder-not-less [*of number-of w number-of v*, *symmetric*,
standard]

Absolute value (*abs*)

lemma *abs-number-of*:
abs(*number-of x::'a::*{*linordered-idom*,*number-ring*}) =
(*if number-of x < (0::'a) then −number-of x else number-of x*)
by (*simp add*: *abs-if*)

Re-orientation of the equation nnn=x

lemma *number-of-reorient*:
(*number-of w* = *x*) = (*x* = *number-of w*)
by *auto*

30.10.4 Simplification of arithmetic operations on integer constants.

lemmas *arith-extra-simps* [*standard*, *simp*] =
number-of-add [*symmetric*]
number-of-minus [*symmetric*]
numeral-m1-eq-minus-1 [*symmetric*]
number-of-mult [*symmetric*]
diff-number-of-eq abs-number-of

For making a minimal simpset, one must include these default simprules. Also include *simp-thms*.

lemmas *arith-simps* =
normalize-bin-simps pred-bin-simps succ-bin-simps
add-bin-simps minus-bin-simps mult-bin-simps
abs-zero abs-one arith-extra-simps

Simplification of relational operations

lemma *less-number-of* [*simp*]:
(*number-of x::'a::*{*linordered-idom*,*number-ring*}) < *number-of y* ⟷ *x* < *y*
unfolding *number-of-eq* **by** (*rule of-int-less-iff*)

lemma *le-number-of* [*simp*]:
(*number-of x::'a::*{*linordered-idom*,*number-ring*}) ≤ *number-of y* ⟷ *x* ≤ *y*
unfolding *number-of-eq* **by** (*rule of-int-le-iff*)

lemma *eq-number-of* [*simp*]:

(*number-of* x::$'a$::{*ring-char-0*,*number-ring*}) = *number-of* $y \longleftrightarrow x = y$
unfolding *number-of-eq* **by** (*rule of-int-eq-iff*)

lemmas *rel-simps* =
less-number-of less-bin-simps
le-number-of le-bin-simps
eq-number-of-eq eq-bin-simps
iszero-simps

30.10.5 Simplification of arithmetic when nested to the right.

lemma *add-number-of-left* [*simp*]:
number-of v + (*number-of* w + z) =
(*number-of*(v + w) + z::$'a$::*number-ring*)
by (*simp add*: *add-assoc* [*symmetric*])

lemma *mult-number-of-left* [*simp*]:
number-of v * (*number-of* w * z) =
(*number-of*(v * w) * z::$'a$::*number-ring*)
by (*simp add*: *mult-assoc* [*symmetric*])

lemma *add-number-of-diff1*:
number-of v + (*number-of* w − c) =
number-of(v + w) − (c::$'a$::*number-ring*)
by (*simp add*: *diff-minus*)

lemma *add-number-of-diff2* [*simp*]:
number-of v + (c − *number-of* w) =
number-of (v + *uminus* w) + (c::$'a$::*number-ring*)
by (*simp add*: *algebra-simps diff-number-of-eq* [*symmetric*])

30.11 The Set of Integers

context *ring-1*
begin

definition *Ints* :: *'a set* **where**
[*code del*]: *Ints* = *range of-int*

notation (*xsymbols*)
Ints (ℤ)

lemma *Ints-of-int* [*simp*]: *of-int* z ∈ ℤ
by (*simp add*: *Ints-def*)

lemma *Ints-of-nat* [*simp*]: *of-nat* n ∈ ℤ
apply (*simp add*: *Ints-def*)
apply (*rule range-eqI*)
apply (*rule of-int-of-nat-eq* [*symmetric*])
done

lemma *Ints-0* [*simp*]: $0 \in \mathbb{Z}$
apply (*simp add*: *Ints-def*)
apply (*rule range-eqI*)
apply (*rule of-int-0* [*symmetric*])
done

lemma *Ints-1* [*simp*]: $1 \in \mathbb{Z}$
apply (*simp add*: *Ints-def*)
apply (*rule range-eqI*)
apply (*rule of-int-1* [*symmetric*])
done

lemma *Ints-add* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a + b \in \mathbb{Z}$
apply (*auto simp add*: *Ints-def*)
apply (*rule range-eqI*)
apply (*rule of-int-add* [*symmetric*])
done

lemma *Ints-minus* [*simp*]: $a \in \mathbb{Z} \Longrightarrow -a \in \mathbb{Z}$
apply (*auto simp add*: *Ints-def*)
apply (*rule range-eqI*)
apply (*rule of-int-minus* [*symmetric*])
done

lemma *Ints-diff* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a - b \in \mathbb{Z}$
apply (*auto simp add*: *Ints-def*)
apply (*rule range-eqI*)
apply (*rule of-int-diff* [*symmetric*])
done

lemma *Ints-mult* [*simp*]: $a \in \mathbb{Z} \Longrightarrow b \in \mathbb{Z} \Longrightarrow a * b \in \mathbb{Z}$
apply (*auto simp add*: *Ints-def*)
apply (*rule range-eqI*)
apply (*rule of-int-mult* [*symmetric*])
done

lemma *Ints-power* [*simp*]: $a \in \mathbb{Z} \Longrightarrow a \hat{} n \in \mathbb{Z}$
by (*induct n*) *simp-all*

lemma *Ints-cases* [*cases set*: *Ints*]:
  assumes $q \in \mathbb{Z}$
  obtains (*of-int*) z **where** $q = of\text{-}int\ z$
  unfolding *Ints-def*
proof −
  from ‹$q \in \mathbb{Z}$› **have** $q \in range\ of\text{-}int$ **unfolding** *Ints-def* **.**
  then obtain z **where** $q = of\text{-}int\ z$ **..**
  then show *thesis* **..**
qed

lemma *Ints-induct* [*case-names of-int*, *induct set*: *Ints*]:
q ∈ ℤ ⟹ (⋀*z*. *P* (*of-int z*)) ⟹ *P q*
by (*rule Ints-cases*) *auto*

end

The premise involving ℤ prevents $a = (1::'a) \,/\, (2::'a)$.

lemma *Ints-double-eq-0-iff*:
assumes *in-Ints*: *a* ∈ *Ints*
shows (*a* + *a* = *0*) = (*a* = (*0*::'*a*::*ring-char-0*))
proof −
from *in-Ints* **have** *a* ∈ *range of-int* **unfolding** *Ints-def* [*symmetric*] **.**
then obtain *z* **where** *a*: *a* = *of-int z* **..**
show *?thesis*
proof
assume *a* = *0*
thus *a* + *a* = *0* **by** *simp*
next
assume *eq*: *a* + *a* = *0*
hence *of-int* (*z* + *z*) = (*of-int 0* :: '*a*) **by** (*simp add*: *a*)
hence *z* + *z* = *0* **by** (*simp only*: *of-int-eq-iff*)
hence *z* = *0* **by** (*simp only*: *double-eq-0-iff*)
thus *a* = *0* **by** (*simp add*: *a*)
qed
qed

lemma *Ints-odd-nonzero*:
assumes *in-Ints*: *a* ∈ *Ints*
shows *1* + *a* + *a* ≠ (*0*::'*a*::*ring-char-0*)
proof −
from *in-Ints* **have** *a* ∈ *range of-int* **unfolding** *Ints-def* [*symmetric*] **.**
then obtain *z* **where** *a*: *a* = *of-int z* **..**
show *?thesis*
proof
assume *eq*: *1* + *a* + *a* = *0*
hence *of-int* (*1* + *z* + *z*) = (*of-int 0* :: '*a*) **by** (*simp add*: *a*)
hence *1* + *z* + *z* = *0* **by** (*simp only*: *of-int-eq-iff*)
with *odd-nonzero* **show** *False* **by** *blast*
qed
qed

lemma *Ints-number-of* [*simp*]:
(*number-of w* :: '*a*::*number-ring*) ∈ *Ints*
unfolding *number-of-eq Ints-def* **by** *simp*

lemma *Nats-number-of* [*simp*]:
Int.Pls ≤ *w* ⟹ (*number-of w* :: '*a*::*number-ring*) ∈ *Nats*
unfolding *Int.Pls-def number-of-eq*

by (*simp only*: *of-nat-nat* [*symmetric*] *of-nat-in-Nats*)

lemma *Ints-odd-less-0*:
assumes *in-Ints*: $a \in Ints$
shows $(1 + a + a < 0) = (a < (0::'a::linordered\text{-}idom))$
proof −
from *in-Ints* **have** $a \in range\ of\text{-}int$ **unfolding** *Ints-def* [*symmetric*] **.**
then obtain z **where** a: $a = of\text{-}int\ z$ **..**
hence $((1::'a) + a + a < 0) = (of\text{-}int\ (1 + z + z) < (of\text{-}int\ 0 :: 'a))$
by (*simp add*: a)
also have ... $= (z < 0)$ **by** (*simp only*: *of-int-less-iff odd-less-0*)
also have ... $= (a < 0)$ **by** (*simp add*: a)
finally show *?thesis* **.**
qed

30.12 *setsum* and *setprod*

lemma *of-nat-setsum*: *of-nat* $(setsum\ f\ A) = (\sum x{\in}A.\ of\text{-}nat(f\ x))$
apply (*cases finite A*)
apply (*erule finite-induct*, *auto*)
done

lemma *of-int-setsum*: *of-int* $(setsum\ f\ A) = (\sum x{\in}A.\ of\text{-}int(f\ x))$
apply (*cases finite A*)
apply (*erule finite-induct*, *auto*)
done

lemma *of-nat-setprod*: *of-nat* $(setprod\ f\ A) = (\prod x{\in}A.\ of\text{-}nat(f\ x))$
apply (*cases finite A*)
apply (*erule finite-induct*, *auto simp add*: *of-nat-mult*)
done

lemma *of-int-setprod*: *of-int* $(setprod\ f\ A) = (\prod x{\in}A.\ of\text{-}int(f\ x))$
apply (*cases finite A*)
apply (*erule finite-induct*, *auto*)
done

lemmas *int-setsum* = *of-nat-setsum* [**where** *'a=int*]
lemmas *int-setprod* = *of-nat-setprod* [**where** *'a=int*]

30.13 Inequality Reasoning for the Arithmetic Simproc

lemma *add-numeral-0*: $Numeral0 + a = (a::'a::number\text{-}ring)$
by *simp*

lemma *add-numeral-0-right*: $a + Numeral0 = (a::'a::number\text{-}ring)$
by *simp*

lemma *mult-numeral-1*: $Numeral1 * a = (a::'a::number\text{-}ring)$
by *simp*

lemma *mult-numeral-1-right*: $a * Numeral1 = (a::'a::number\text{-}ring)$
by *simp*

lemma *divide-numeral-1*: $a / Numeral1 = (a::'a::\{number\text{-}ring,field\})$
by *simp*

lemma *inverse-numeral-1*:
$inverse\ Numeral1 = (Numeral1::'a::\{number\text{-}ring,field\})$
by *simp*

Theorem lists for the cancellation simprocs. The use of binary numerals for 0 and 1 reduces the number of special cases.

lemmas *add-0s = add-numeral-0 add-numeral-0-right*
lemmas *mult-1s = mult-numeral-1 mult-numeral-1-right*
mult-minus1 mult-minus1-right

30.14 Special Arithmetic Rules for Abstract 0 and 1

Arithmetic computations are defined for binary literals, which leaves 0 and 1 as special cases. Addition already has rules for 0, but not 1. Multiplication and unary minus already have rules for both 0 and 1.

lemma *binop-eq*: $[|f\ x\ y = g\ x\ y;\ x = x';\ y = y'|] ==> f\ x'\ y' = g\ x'\ y'$
by *simp*

lemmas *add-number-of-eq = number-of-add* [*symmetric*]

Allow 1 on either or both sides

lemma *one-add-one-is-two*: $1 + 1 = (2::'a::number\text{-}ring)$
by (*simp del*: *numeral-1-eq-1 add*: *numeral-1-eq-1* [*symmetric*])

lemmas *add-special =*
one-add-one-is-two
binop-eq [*of op +*, *OF add-number-of-eq numeral-1-eq-1 refl*, *standard*]
binop-eq [*of op +*, *OF add-number-of-eq refl numeral-1-eq-1*, *standard*]

Allow 1 on either or both sides (1-1 already simplifies to 0)

lemmas *diff-special =*
binop-eq [*of op −*, *OF diff-number-of-eq numeral-1-eq-1 refl*, *standard*]
binop-eq [*of op −*, *OF diff-number-of-eq refl numeral-1-eq-1*, *standard*]

Allow 0 or 1 on either side with a binary numeral on the other

lemmas *eq-special =*
binop-eq [*of op =*, *OF eq-number-of-eq numeral-0-eq-0 refl*, *standard*]
binop-eq [*of op =*, *OF eq-number-of-eq numeral-1-eq-1 refl*, *standard*]
binop-eq [*of op =*, *OF eq-number-of-eq refl numeral-0-eq-0*, *standard*]

binop-eq [*of op* =, *OF eq-number-of-eq refl numeral-1-eq-1*, *standard*]

Allow 0 or 1 on either side with a binary numeral on the other

lemmas *less-special* =
binop-eq [*of op* <, *OF less-number-of numeral-0-eq-0 refl*, *standard*]
binop-eq [*of op* <, *OF less-number-of numeral-1-eq-1 refl*, *standard*]
binop-eq [*of op* <, *OF less-number-of refl numeral-0-eq-0*, *standard*]
binop-eq [*of op* <, *OF less-number-of refl numeral-1-eq-1*, *standard*]

Allow 0 or 1 on either side with a binary numeral on the other

lemmas *le-special* =
binop-eq [*of op* ≤, *OF le-number-of numeral-0-eq-0 refl*, *standard*]
binop-eq [*of op* ≤, *OF le-number-of numeral-1-eq-1 refl*, *standard*]
binop-eq [*of op* ≤, *OF le-number-of refl numeral-0-eq-0*, *standard*]
binop-eq [*of op* ≤, *OF le-number-of refl numeral-1-eq-1*, *standard*]

lemmas *arith-special*[*simp*] =
add-special diff-special eq-special less-special le-special

Legacy theorems

lemmas *zle-int* = *of-nat-le-iff* [**where** *'a*=*int*]
lemmas *int-int-eq* = *of-nat-eq-iff* [**where** *'a*=*int*]

30.15 Setting up simplification procedures

lemmas *int-arith-rules* =
neg-le-iff-le numeral-0-eq-0 numeral-1-eq-1
minus-zero diff-minus left-minus right-minus
mult-zero-left mult-zero-right mult-Bit1 mult-1-left mult-1-right
mult-minus-left mult-minus-right
minus-add-distrib minus-minus mult-assoc
of-nat-0 of-nat-1 of-nat-Suc of-nat-add of-nat-mult
of-int-0 of-int-1 of-int-add of-int-mult

use *Tools/int-arith.ML*
setup ⟪ *Int-Arith.global-setup* ⟫
declaration ⟪ *K Int-Arith.setup* ⟫

setup ⟪
Reorient-Proc.add
(*fn Const* (@{*const-name number-of*}, -) $ - => *true* | - => *false*)
⟫

simproc-setup *reorient-numeral* (*number-of w* = *x*) = *Reorient-Proc.proc*

30.16 Lemmas About Small Numerals

lemma *of-int-m1* [*simp*]: *of-int* −*1* = (−*1* :: *'a* :: *number-ring*)
proof −

have (*of-int −1* :: *'a*) = *of-int* (*− 1*) **by** *simp*
also have ... = *−* *of-int 1* **by** (*simp only*: *of-int-minus*)
also have ... = *−1* **by** *simp*
finally show *?thesis* **.**
qed

lemma *abs-minus-one* [*simp*]: *abs* (*−1*) = (*1*::*'a*::{*linordered-idom*,*number-ring*})
by (*simp add*: *abs-if*)

lemma *abs-power-minus-one* [*simp*]:
abs(*−1* ^ *n*) = (*1*::*'a*::{*linordered-idom*,*number-ring*})
by (*simp add*: *power-abs*)

lemma *of-int-number-of-eq* [*simp*]:
of-int (*number-of v*) = (*number-of v* :: *'a* :: *number-ring*)
by (*simp add*: *number-of-eq*)

Lemmas for specialist use, NOT as default simprules

lemma *mult-2*: *2* ∗ *z* = (*z*+*z*::*'a*::*number-ring*)
unfolding *one-add-one-is-two* [*symmetric*] *left-distrib* **by** *simp*

lemma *mult-2-right*: *z* ∗ *2* = (*z*+*z*::*'a*::*number-ring*)
by (*subst mult-commute*, *rule mult-2*)

30.17 More Inequality Reasoning

lemma *zless-add1-eq*: (*w* < *z* + (*1*::*int*)) = (*w*<*z* | *w*=*z*)
by *arith*

lemma *add1-zle-eq*: (*w* + (*1*::*int*) ≤ *z*) = (*w*<*z*)
by *arith*

lemma *zle-diff1-eq* [*simp*]: (*w* ≤ *z* − (*1*::*int*)) = (*w*<*z*)
by *arith*

lemma *zle-add1-eq-le* [*simp*]: (*w* < *z* + (*1*::*int*)) = (*w*≤*z*)
by *arith*

lemma *int-one-le-iff-zero-less*: ((*1*::*int*) ≤ *z*) = (*0* < *z*)
by *arith*

30.18 The functions *nat* and *int*

Simplify the terms *int* (*0*::*'a*), *int* (*Suc 0*) and *w* + *−* *z*

declare *Zero-int-def* [*symmetric*, *simp*]
declare *One-int-def* [*symmetric*, *simp*]

lemmas *diff-int-def-symmetric* = *diff-int-def* [*symmetric*, *simp*]

lemma *nat-0*: *nat 0 = 0*
by (*simp add*: *nat-eq-iff*)

lemma *nat-1*: *nat 1 = Suc 0*
by (*subst nat-eq-iff*, *simp*)

lemma *nat-2*: *nat 2 = Suc (Suc 0)*
by (*subst nat-eq-iff*, *simp*)

lemma *one-less-nat-eq* [*simp*]: *(Suc 0 < nat z) = (1 < z)*
apply (*insert zless-nat-conj* [*of 1 z*])
apply (*auto simp add*: *nat-1*)
done

This simplifies expressions of the form *int n = z* where z is an integer literal.

lemmas *int-eq-iff-number-of* [*simp*] = *int-eq-iff* [*of - number-of v*, *standard*]

lemma *split-nat* [*arith-split*]:
  P(nat(i::int)) = ((∀ n. i = of-nat n ⟶ P n) & (i < 0 ⟶ P 0))
  (**is** *?P = (?L & ?R)*)
proof (*cases i < 0*)
  case *True* **thus** *?thesis* **by** *auto*
next
  case *False*
  have *?P = ?L*
  proof
    assume *?P* **thus** *?L* **using** *False* **by** *clarsimp*
  next
    assume *?L* **thus** *?P* **using** *False* **by** *simp*
  qed
  with *False* **show** *?thesis* **by** *simp*
qed

context *ring-1*
begin

lemma *of-int-of-nat* [*nitpick-simp*]:
  of-int k = (if k < 0 then - of-nat (nat (- k)) else of-nat (nat k))
proof (*cases k < 0*)
  case *True* **then have** *0 ≤ - k* **by** *simp*
  then have *of-nat (nat (- k)) = of-int (- k)* **by** (*rule of-nat-nat*)
  with *True* **show** *?thesis* **by** *simp*
next
  case *False* **then show** *?thesis* **by** (*simp add*: *not-less of-nat-nat*)
qed

end

lemma *nat-mult-distrib*:
fixes $z\ z'$ :: *int*
assumes $0 \leq z$
shows *nat* $(z * z') =$ *nat* $z *$ *nat* z'
proof (*cases* $0 \leq z'$)
case *False* **with** *assms* **have** $z * z' \leq 0$
by (*simp add*: *not-le mult-le-0-iff*)
then have *nat* $(z * z') = 0$ **by** *simp*
moreover from *False* **have** *nat* $z' = 0$ **by** *simp*
ultimately show *?thesis* **by** *simp*
next
case *True* **with** *assms* **have** *ge-0*: $z * z' \geq 0$ **by** (*simp add*: *zero-le-mult-iff*)
show *?thesis*
by (*rule injD* [*of of-nat* :: *nat* ⇒ *int*, *OF inj-of-nat*])
(*simp only*: *of-nat-mult of-nat-nat* [*OF True*]
of-nat-nat [*OF assms*] *of-nat-nat* [*OF ge-0*], *simp*)
qed

lemma *nat-mult-distrib-neg*: $z \leq (0::int) ==> nat(z*z') = nat(-z) * nat(-z')$
apply (*rule trans*)
apply (*rule-tac* [*2*] *nat-mult-distrib*, *auto*)
done

lemma *nat-abs-mult-distrib*: *nat* $(abs\ (w * z)) =$ *nat* $(abs\ w) *$ *nat* $(abs\ z)$
apply (*cases z=0* | *w=0*)
apply (*auto simp add*: *abs-if nat-mult-distrib* [*symmetric*]
nat-mult-distrib-neg [*symmetric*] *mult-less-0-iff*)
done

30.19 Induction principles for int

Well-founded segments of the integers

definition
int-ge-less-than :: *int* => (*int* * *int*) *set*
where
int-ge-less-than d = {(*z'*,*z*). $d \leq z' \ \&\ z' < z$}

theorem *wf-int-ge-less-than*: *wf* (*int-ge-less-than d*)
proof −
have *int-ge-less-than d* ⊆ *measure* (%*z*. *nat* $(z-d)$)
by (*auto simp add*: *int-ge-less-than-def*)
thus *?thesis*
by (*rule wf-subset* [*OF wf-measure*])
qed

This variant looks odd, but is typical of the relations suggested by Rank-Finder.

definition

int-ge-less-than2 :: *int* => (*int* ∗ *int*) *set*
where
int-ge-less-than2 d = {(*z'*,*z*). *d* ≤ *z* & *z'* < *z*}

theorem *wf-int-ge-less-than2*: *wf* (*int-ge-less-than2 d*)
proof −
have *int-ge-less-than2 d* ⊆ *measure* (%*z*. *nat* (*1*+*z*−*d*))
by (*auto simp add*: *int-ge-less-than2-def*)
thus *?thesis*
by (*rule wf-subset* [*OF wf-measure*])
qed

abbreviation
int :: *nat* ⇒ *int*
where
int ≡ *of-nat*

theorem *int-ge-induct* [*case-names base step*, *induct set*: *int*]:
fixes *i* :: *int*
assumes *ge*: *k* ≤ *i* **and**
base: *P k* **and**
step: ⋀*i*. *k* ≤ *i* ⟹ *P i* ⟹ *P* (*i* + *1*)
shows *P i*
proof −
{ **fix** *n* **have** ⋀*i*::*int*. *n* = *nat*(*i*−*k*) ⟹ *k* ≤ *i* ⟹ *P i*
proof (*induct n*)
case *0*
hence *i* = *k* **by** *arith*
thus *P i* **using** *base* **by** *simp*
next
case (*Suc n*)
then have *n* = *nat*((*i* − *1*) − *k*) **by** *arith*
moreover
have *ki1*: *k* ≤ *i* − *1* **using** *Suc.prems* **by** *arith*
ultimately
have *P*(*i* − *1*) **by**(*rule Suc.hyps*)
from *step*[*OF ki1 this*] **show** *?case* **by** *simp*
qed
}
with *ge* **show** *?thesis* **by** *fast*
qed

theorem *int-gr-induct* [*case-names base step*, *induct set*: *int*]:
assumes *gr*: *k* < (*i*::*int*) **and**
base: *P*(*k*+*1*) **and**
step: ⋀*i*. ⟦*k* < *i*; *P i*⟧ ⟹ *P*(*i*+*1*)
shows *P i*

apply(*rule int-ge-induct*[*of* $k + 1$])
using *gr* **apply** *arith*
apply(*rule base*)
apply (*rule step*, *simp*+)
done

theorem *int-le-induct*[*consumes 1*,*case-names base step*]:
assumes *le*: $i \leq (k::int)$ **and**
base: $P(k)$ **and**
step: $\bigwedge i.\ [\![i \leq k;\ P\ i]\!] \Longrightarrow P(i - 1)$
shows $P\ i$
proof −
{ **fix** n **have** $\bigwedge i::int.\ n = nat(k-i) \Longrightarrow i \leq k \Longrightarrow P\ i$
proof (*induct n*)
case *0*
hence $i = k$ **by** *arith*
thus $P\ i$ **using** *base* **by** *simp*
next
case (*Suc n*)
hence $n = nat(k - (i+1))$ **by** *arith*
moreover
have *ki1*: $i + 1 \leq k$ **using** *Suc.prems* **by** *arith*
ultimately
have $P(i+1)$ **by**(*rule Suc.hyps*)
from *step*[*OF ki1 this*] **show** *?case* **by** *simp*
qed
}
with *le* **show** *?thesis* **by** *fast*
qed

theorem *int-less-induct* [*consumes 1*,*case-names base step*]:
assumes *less*: $(i::int) < k$ **and**
base: $P(k - 1)$ **and**
step: $\bigwedge i.\ [\![i < k;\ P\ i]\!] \Longrightarrow P(i - 1)$
shows $P\ i$
apply(*rule int-le-induct*[*of* - $k - 1$])
using *less* **apply** *arith*
apply(*rule base*)
apply (*rule step*, *simp*+)
done

theorem *int-induct* [*case-names base step1 step2*]:
fixes $k :: int$
assumes *base*: $P\ k$
and *step1*: $\bigwedge i.\ k \leq i \Longrightarrow P\ i \Longrightarrow P\ (i + 1)$
and *step2*: $\bigwedge i.\ k \geq i \Longrightarrow P\ i \Longrightarrow P\ (i - 1)$
shows $P\ i$
proof −
have $i \leq k \vee i \geq k$ **by** *arith*

```
  then show ?thesis proof
    assume i ≥ k then show ?thesis using base
      by (rule int-ge-induct) (fact step1)
  next
    assume i ≤ k then show ?thesis using base
      by (rule int-le-induct) (fact step2)
  qed
qed
```

30.20 Intermediate value theorems

```
lemma int-val-lemma:
     (∀ i<n::nat. abs(f(i+1) - f i) ≤ 1) -->
      f 0 ≤ k --> k ≤ f n --> (∃ i ≤ n. f i = (k::int))
unfolding One-nat-def
apply (induct n, simp)
apply (intro strip)
apply (erule impE, simp)
apply (erule-tac x = n in allE, simp)
apply (case-tac k = f (Suc n))
apply force
apply (erule impE)
 apply (simp add: abs-if split add: split-if-asm)
apply (blast intro: le-SucI)
done

lemmas nat0-intermed-int-val = int-val-lemma [rule-format (no-asm)]

lemma nat-intermed-int-val:
     [| ∀ i. m ≤ i & i < n --> abs(f(i + 1::nat) - f i) ≤ 1; m < n;
        f m ≤ k; k ≤ f n |] ==> ? i. m ≤ i & i ≤ n & f i = (k::int)
apply (cut-tac n = n-m and f = %i. f (i+m)  and k = k
       in int-val-lemma)
unfolding One-nat-def
apply simp
apply (erule exE)
apply (rule-tac x = i+m in exI, arith)
done
```

30.21 Products and 1, by T. M. Rasmussen

```
lemma zabs-less-one-iff [simp]: (|z| < 1) = (z = (0::int))
by arith

lemma abs-zmult-eq-1:
  assumes mn: |m * n| = 1
  shows |m| = (1::int)
proof -
  have 0: m ≠ 0 & n ≠ 0 using mn
    by auto
```

```
  have ~ (2 ≤ |m|)
  proof
    assume 2 ≤ |m|
    hence 2*|n| ≤ |m|*|n|
      by (simp add: mult-mono 0)
    also have ... = |m*n|
      by (simp add: abs-mult)
    also have ... = 1
      by (simp add: mn)
    finally have 2*|n| ≤ 1 .
    thus False using 0
      by auto
  qed
  thus ?thesis using 0
    by auto
qed

lemma pos-zmult-eq-1-iff-lemma: (m * n = 1) ==> m = (1::int) | m = -1
by (insert abs-zmult-eq-1 [of m n], arith)

lemma pos-zmult-eq-1-iff:
  assumes 0 < (m::int) shows (m * n = 1) = (m = 1 & n = 1)
proof -
 from assms have m * n = 1 ==> m = 1 by (auto dest: pos-zmult-eq-1-iff-lemma)
  thus ?thesis by (auto dest: pos-zmult-eq-1-iff-lemma)
qed

lemma zmult-eq-1-iff: (m*n = (1::int)) = ((m = 1 & n = 1) | (m = -1 & n =
-1))
apply (rule iffI)
 apply (frule pos-zmult-eq-1-iff-lemma)
 apply (simp add: mult-commute [of m])
 apply (frule pos-zmult-eq-1-iff-lemma, auto)
done

lemma infinite-UNIV-int: ¬ finite (UNIV::int set)
proof
  assume finite (UNIV::int set)
  moreover have inj (λi::int. 2 * i)
    by (rule injI) simp
  ultimately have surj (λi::int. 2 * i)
    by (rule finite-UNIV-inj-surj)
  then obtain i :: int where 1 = 2 * i by (rule surjE)
  then show False by (simp add: pos-zmult-eq-1-iff)
qed
```

30.22 Further theorems on numerals

30.22.1 Special Simplification for Constants

These distributive laws move literals inside sums and differences.

lemmas *left-distrib-number-of* [*simp*] =
left-distrib [*of - - number-of v*, *standard*]

lemmas *right-distrib-number-of* [*simp*] =
right-distrib [*of number-of v*, *standard*]

lemmas *left-diff-distrib-number-of* [*simp*] =
left-diff-distrib [*of - - number-of v*, *standard*]

lemmas *right-diff-distrib-number-of* [*simp*] =
right-diff-distrib [*of number-of v*, *standard*]

These are actually for fields, like real: but where else to put them?

lemmas *zero-less-divide-iff-number-of* [*simp*, *no-atp*] =
zero-less-divide-iff [*of number-of w*, *standard*]

lemmas *divide-less-0-iff-number-of* [*simp*, *no-atp*] =
divide-less-0-iff [*of number-of w*, *standard*]

lemmas *zero-le-divide-iff-number-of* [*simp*, *no-atp*] =
zero-le-divide-iff [*of number-of w*, *standard*]

lemmas *divide-le-0-iff-number-of* [*simp*, *no-atp*] =
divide-le-0-iff [*of number-of w*, *standard*]

Replaces *inverse #nn* by *1/#nn*. It looks strange, but then other simprocs simplify the quotient.

lemmas *inverse-eq-divide-number-of* [*simp*] =
inverse-eq-divide [*of number-of w*, *standard*]

These laws simplify inequalities, moving unary minus from a term into the literal.

lemmas *less-minus-iff-number-of* [*simp*, *no-atp*] =
less-minus-iff [*of number-of v*, *standard*]

lemmas *le-minus-iff-number-of* [*simp*, *no-atp*] =
le-minus-iff [*of number-of v*, *standard*]

lemmas *equation-minus-iff-number-of* [*simp*, *no-atp*] =
equation-minus-iff [*of number-of v*, *standard*]

lemmas *minus-less-iff-number-of* [*simp*, *no-atp*] =
minus-less-iff [*of - number-of v*, *standard*]

lemmas *minus-le-iff-number-of* [*simp*, *no-atp*] =
minus-le-iff [*of* - *number-of v*, *standard*]

lemmas *minus-equation-iff-number-of* [*simp*, *no-atp*] =
minus-equation-iff [*of* - *number-of v*, *standard*]

To Simplify Inequalities Where One Side is the Constant 1

lemma *less-minus-iff-1* [*simp*,*no-atp*]:
fixes *b*::'*b*::{*linordered-idom*,*number-ring*}
shows (*1* < − *b*) = (*b* < −*1*)
by *auto*

lemma *le-minus-iff-1* [*simp*,*no-atp*]:
fixes *b*::'*b*::{*linordered-idom*,*number-ring*}
shows (*1* ≤ − *b*) = (*b* ≤ −*1*)
by *auto*

lemma *equation-minus-iff-1* [*simp*,*no-atp*]:
fixes *b*::'*b*::*number-ring*
shows (*1* = − *b*) = (*b* = −*1*)
by (*subst equation-minus-iff*, *auto*)

lemma *minus-less-iff-1* [*simp*,*no-atp*]:
fixes *a*::'*b*::{*linordered-idom*,*number-ring*}
shows (− *a* < *1*) = (−*1* < *a*)
by *auto*

lemma *minus-le-iff-1* [*simp*,*no-atp*]:
fixes *a*::'*b*::{*linordered-idom*,*number-ring*}
shows (− *a* ≤ *1*) = (−*1* ≤ *a*)
by *auto*

lemma *minus-equation-iff-1* [*simp*,*no-atp*]:
fixes *a*::'*b*::*number-ring*
shows (− *a* = *1*) = (*a* = −*1*)
by (*subst minus-equation-iff*, *auto*)

Cancellation of constant factors in comparisons (< and ≤)

lemmas *mult-less-cancel-left-number-of* [*simp*, *no-atp*] =
mult-less-cancel-left [*of number-of v*, *standard*]

lemmas *mult-less-cancel-right-number-of* [*simp*, *no-atp*] =
mult-less-cancel-right [*of* - *number-of v*, *standard*]

lemmas *mult-le-cancel-left-number-of* [*simp*, *no-atp*] =
mult-le-cancel-left [*of number-of v*, *standard*]

lemmas *mult-le-cancel-right-number-of* [*simp*, *no-atp*] =

mult-le-cancel-right [*of* - *number-of v*, *standard*]

Multiplying out constant divisors in comparisons ($<$, $\leq$ and $=$)

lemmas *le-divide-eq-number-of1* [*simp*] = *le-divide-eq* [*of* - - *number-of w*, *standard*]
lemmas *divide-le-eq-number-of1* [*simp*] = *divide-le-eq* [*of* - *number-of w*, *standard*]
lemmas *less-divide-eq-number-of1* [*simp*] = *less-divide-eq* [*of* - - *number-of w*, *standard*]
lemmas *divide-less-eq-number-of1* [*simp*] = *divide-less-eq* [*of* - *number-of w*, *standard*]
lemmas *eq-divide-eq-number-of1* [*simp*] = *eq-divide-eq* [*of* - - *number-of w*, *standard*]
lemmas *divide-eq-eq-number-of1* [*simp*] = *divide-eq-eq* [*of* - *number-of w*, *standard*]

30.22.2 Optional Simplification Rules Involving Constants

Simplify quotients that are compared with a literal constant.

lemmas *le-divide-eq-number-of* = *le-divide-eq* [*of number-of w*, *standard*]
lemmas *divide-le-eq-number-of* = *divide-le-eq* [*of* - - *number-of w*, *standard*]
lemmas *less-divide-eq-number-of* = *less-divide-eq* [*of number-of w*, *standard*]
lemmas *divide-less-eq-number-of* = *divide-less-eq* [*of* - - *number-of w*, *standard*]
lemmas *eq-divide-eq-number-of* = *eq-divide-eq* [*of number-of w*, *standard*]
lemmas *divide-eq-eq-number-of* = *divide-eq-eq* [*of* - - *number-of w*, *standard*]

Not good as automatic simprules because they cause case splits.

lemmas *divide-const-simps* =
le-divide-eq-number-of divide-le-eq-number-of less-divide-eq-number-of
divide-less-eq-number-of eq-divide-eq-number-of divide-eq-eq-number-of
le-divide-eq-1 divide-le-eq-1 less-divide-eq-1 divide-less-eq-1

Division By -1

lemma *divide-minus1* [*simp*]:
$x/-1 = -(x::'a::\{$*field-inverse-zero*, *number-ring*$\})$
by *simp*

lemma *minus1-divide* [*simp*]:
$-1\ /\ (x::'a::\{$*field-inverse-zero*, *number-ring*$\}) = -\ (1/x)$
by (*simp add*: *divide-inverse*)

lemma *half-gt-zero-iff*:
$(0 < r/2) = (0 < (r::'a::\{$*linordered-field-inverse-zero*,*number-ring*$\}))$
by *auto*

lemmas *half-gt-zero* [*simp*] = *half-gt-zero-iff* [*THEN iffD2*, *standard*]

lemma *divide-Numeral1*:
$(x::'a::\{$*field*, *number-ring*$\})$ / *Numeral1* = x
by *simp*

lemma *divide-Numeral0*:
(*x*::'*a*::{*field-inverse-zero*, *number-ring*}) / *Numeral0* = *0*
by *simp*

30.23 The divides relation

lemma *zdvd-antisym-nonneg*:
0 <= *m* ==> *0* <= *n* ==> *m dvd n* ==> *n dvd m* ==> *m* = (*n*::*int*)
apply (*simp add*: *dvd-def*, *auto*)
apply (*auto simp add*: *mult-assoc zero-le-mult-iff zmult-eq-1-iff*)
done

lemma *zdvd-antisym-abs*: **assumes** (*a*::*int*) *dvd b* **and** *b dvd a*
shows |*a*| = |*b*|
proof *cases*
assume *a* = *0* **with** *assms* **show** *?thesis* **by** *simp*
next
assume *a* ≠ *0*
from ‹*a dvd b*› **obtain** *k* **where** *k*:*b* = *a*∗*k* **unfolding** *dvd-def* **by** *blast*
from ‹*b dvd a*› **obtain** *k'* **where** *k'*:*a* = *b*∗*k'* **unfolding** *dvd-def* **by** *blast*
from *k k'* **have** *a* = *a*∗*k*∗*k'* **by** *simp*
with *mult-cancel-left1*[**where** *c*=*a* **and** *b*=*k*∗*k'*]
have *kk'*:*k*∗*k'* = *1* **using** ‹*a*≠*0*› **by** (*simp add*: *mult-assoc*)
hence *k* = *1* ∧ *k'* = *1* ∨ *k* = −*1* ∧ *k'* = −*1* **by** (*simp add*: *zmult-eq-1-iff*)
thus *?thesis* **using** *k k'* **by** *auto*
qed

lemma *zdvd-zdiffD*: *k dvd m* − *n* ==> *k dvd n* ==> *k dvd* (*m*::*int*)
apply (*subgoal-tac m* = *n* + (*m* − *n*))
apply (*erule ssubst*)
apply (*blast intro*: *dvd-add*, *simp*)
done

lemma *zdvd-reduce*: (*k dvd n* + *k* ∗ *m*) = (*k dvd* (*n*::*int*))
apply (*rule iffI*)
apply (*erule-tac* [*2*] *dvd-add*)
apply (*subgoal-tac n* = (*n* + *k* ∗ *m*) − *k* ∗ *m*)
apply (*erule ssubst*)
apply (*erule dvd-diff*)
apply(*simp-all*)
done

lemma *dvd-imp-le-int*:
fixes *d i* :: *int*
assumes *i* ≠ *0* **and** *d dvd i*
shows |*d*| ≤ |*i*|
proof −
from ‹*d dvd i*› **obtain** *k* **where** *i* = *d* ∗ *k* **..**

```
  with ‹i ≠ 0› have k ≠ 0 by auto
  then have 1 ≤ |k| and 0 ≤ |d| by auto
  then have |d| * 1 ≤ |d| * |k| by (rule mult-left-mono)
  with ‹i = d * k› show ?thesis by (simp add: abs-mult)
qed

lemma zdvd-not-zless:
  fixes m n :: int
  assumes 0 < m and m < n
  shows ¬ n dvd m
proof
  from assms have 0 < n by auto
  assume n dvd m then obtain k where k: m = n * k ..
  with ‹0 < m› have 0 < n * k by auto
  with ‹0 < n› have 0 < k by (simp add: zero-less-mult-iff)
  with k ‹0 < n› ‹m < n› have n * k < n * 1 by simp
  with ‹0 < n› ‹0 < k› show False unfolding mult-less-cancel-left by auto
qed

lemma zdvd-mult-cancel: assumes d:k * m dvd k * n and kz:k ≠ (0::int)
  shows m dvd n
proof−
  from d obtain h where h: k*n = k*m * h unfolding dvd-def by blast
  {assume n ≠ m*h hence k* n ≠ k* (m*h) using kz by simp
    with h have False by (simp add: mult-assoc)}
  hence n = m * h by blast
  thus ?thesis by simp
qed

theorem zdvd-int: (x dvd y) = (int x dvd int y)
proof −
  have ⋀k. int y = int x * k ⟹ x dvd y
  proof −
    fix k
    assume A: int y = int x * k
    then show x dvd y proof (cases k)
      case (1 n) with A have y = x * n by (simp add: of-nat-mult [symmetric])
      then show ?thesis ..
    next
      case (2 n) with A have int y = int x * (− int (Suc n)) by simp
      also have ... = − (int x * int (Suc n)) by (simp only: mult-minus-right)
      also have ... = − int (x * Suc n) by (simp only: of-nat-mult [symmetric])
      finally have − int (x * Suc n) = int y ..
      then show ?thesis by (simp only: negative-eq-positive) auto
    qed
  qed
  then show ?thesis by (auto elim!: dvdE simp only: dvd-triv-left of-nat-mult)
qed
```

lemma *zdvd1-eq*[*simp*]: (*x*::*int*) *dvd 1* = (|*x*| = *1*)
proof
  assume *d*: *x dvd 1* **hence** *int* (*nat* |*x*|) *dvd int* (*nat 1*) **by** *simp*
  hence *nat* |*x*| *dvd 1* **by** (*simp add*: *zdvd-int*)
  hence *nat* |*x*| = *1* **by** *simp*
  thus |*x*| = *1* **by** (*cases x* < *0*, *auto*)
next
  assume |*x*|=*1*
  then have *x* = *1* ∨ *x* = −*1* **by** *auto*
  then show *x dvd 1* **by** (*auto intro*: *dvdI*)
qed

lemma *zdvd-mult-cancel1*:
  assumes *mp*:*m* ≠(*0*::*int*) **shows** (*m* ∗ *n dvd m*) = (|*n*| = *1*)
proof
  assume *n1*: |*n*| = *1* **thus** *m* ∗ *n dvd m*
    by (*cases n* >*0*, *auto simp add*: *minus-equation-iff*)
next
  assume *H*: *m* ∗ *n dvd m* **hence** *H2*: *m* ∗ *n dvd m* ∗ *1* **by** *simp*
  from *zdvd-mult-cancel*[*OF H2 mp*] **show** |*n*| = *1* **by** (*simp only*: *zdvd1-eq*)
qed

lemma *int-dvd-iff*: (*int m dvd z*) = (*m dvd nat* (*abs z*))
  unfolding *zdvd-int* **by** (*cases z* ≥ *0*) *simp-all*

lemma *dvd-int-iff*: (*z dvd int m*) = (*nat* (*abs z*) *dvd m*)
  unfolding *zdvd-int* **by** (*cases z* ≥ *0*) *simp-all*

lemma *nat-dvd-iff*: (*nat z dvd m*) = (*if* *0* ≤ *z then* (*z dvd int m*) *else m* = *0*)
  by (*auto simp add*: *dvd-int-iff*)

lemma *eq-nat-nat-iff*:
  0 ≤ *z* ⟹ *0* ≤ *z'* ⟹ *nat z* = *nat z'* ⟷ *z* = *z'*
  by (*auto elim*!: *nonneg-eq-int*)

lemma *nat-power-eq*:
  0 ≤ *z* ⟹ *nat* (*z* ^ *n*) = *nat z* ^ *n*
  by (*induct n*) (*simp-all add*: *nat-mult-distrib*)

lemma *zdvd-imp-le*: [| *z dvd n*; *0* < *n* |] ==> *z* ≤ (*n*::*int*)
  apply (*rule-tac z*=*n* **in** *int-cases*)
  apply (*auto simp add*: *dvd-int-iff*)
  apply (*rule-tac z*=*z* **in** *int-cases*)
  apply (*auto simp add*: *dvd-imp-le*)
  done

lemma *zdvd-period*:
  fixes *a d* :: *int*
  assumes *a dvd d*

shows *a dvd* $(x + t) \longleftrightarrow$ *a dvd* $((x + c * d) + t)$
proof −
from *assms* **obtain** *k* **where** $d = a * k$ **by** (*rule dvdE*)
show *?thesis* **proof**
assume *a dvd* $(x + t)$
then obtain *l* **where** $x + t = a * l$ **by** (*rule dvdE*)
then have $x = a * l - t$ **by** *simp*
with ‹$d = a * k$› **show** *a dvd x* + *c* ∗ *d* + *t* **by** *simp*
next
assume *a dvd x* + *c* ∗ *d* + *t*
then obtain *l* **where** $x + c * d + t = a * l$ **by** (*rule dvdE*)
then have $x = a * l - c * d - t$ **by** *simp*
with ‹$d = a * k$› **show** *a dvd* $(x + t)$ **by** *simp*
qed
qed

30.24 Configuration of the code generator

code-datatype *Pls Min Bit0 Bit1 number-of* :: *int* ⇒ *int*

lemmas *pred-succ-numeral-code* [*code*] =
pred-bin-simps succ-bin-simps

lemmas *plus-numeral-code* [*code*] =
add-bin-simps
arith-extra-simps(*1*) [**where** *'a* = *int*]

lemmas *minus-numeral-code* [*code*] =
minus-bin-simps
arith-extra-simps(*2*) [**where** *'a* = *int*]
arith-extra-simps(*5*) [**where** *'a* = *int*]

lemmas *times-numeral-code* [*code*] =
mult-bin-simps
arith-extra-simps(*4*) [**where** *'a* = *int*]

instantiation *int* :: *eq*
begin

definition [*code del*]: *eq-class.eq k l* ⟷ *k* − *l* = (*0*::*int*)

instance by *default* (*simp add*: *eq-int-def*)

end

lemma *eq-number-of-int-code* [*code*]:
eq-class.eq (*number-of k* :: *int*) (*number-of l*) ⟷ *eq-class.eq k l*
unfolding *eq-int-def number-of-is-id* **..**

lemma *eq-int-code* [*code*]:
eq-class.eq Int.Pls Int.Pls ⟷ *True*
eq-class.eq Int.Pls Int.Min ⟷ *False*
eq-class.eq Int.Pls (*Int.Bit0 k2*) ⟷ *eq-class.eq Int.Pls k2*
eq-class.eq Int.Pls (*Int.Bit1 k2*) ⟷ *False*
eq-class.eq Int.Min Int.Pls ⟷ *False*
eq-class.eq Int.Min Int.Min ⟷ *True*
eq-class.eq Int.Min (*Int.Bit0 k2*) ⟷ *False*
eq-class.eq Int.Min (*Int.Bit1 k2*) ⟷ *eq-class.eq Int.Min k2*
eq-class.eq (*Int.Bit0 k1*) *Int.Pls* ⟷ *eq-class.eq k1 Int.Pls*
eq-class.eq (*Int.Bit1 k1*) *Int.Pls* ⟷ *False*
eq-class.eq (*Int.Bit0 k1*) *Int.Min* ⟷ *False*
eq-class.eq (*Int.Bit1 k1*) *Int.Min* ⟷ *eq-class.eq k1 Int.Min*
eq-class.eq (*Int.Bit0 k1*) (*Int.Bit0 k2*) ⟷ *eq-class.eq k1 k2*
eq-class.eq (*Int.Bit0 k1*) (*Int.Bit1 k2*) ⟷ *False*
eq-class.eq (*Int.Bit1 k1*) (*Int.Bit0 k2*) ⟷ *False*
eq-class.eq (*Int.Bit1 k1*) (*Int.Bit1 k2*) ⟷ *eq-class.eq k1 k2*
unfolding *eq-equals* **by** *simp-all*

lemma *eq-int-refl* [*code nbe*]:
eq-class.eq (*k*::*int*) *k* ⟷ *True*
by (*rule HOL.eq-refl*)

lemma *less-eq-number-of-int-code* [*code*]:
(*number-of k* :: *int*) ≤ *number-of l* ⟷ *k* ≤ *l*
unfolding *number-of-is-id* **..**

lemma *less-eq-int-code* [*code*]:
Int.Pls ≤ *Int.Pls* ⟷ *True*
Int.Pls ≤ *Int.Min* ⟷ *False*
Int.Pls ≤ *Int.Bit0 k* ⟷ *Int.Pls* ≤ *k*
Int.Pls ≤ *Int.Bit1 k* ⟷ *Int.Pls* ≤ *k*
Int.Min ≤ *Int.Pls* ⟷ *True*
Int.Min ≤ *Int.Min* ⟷ *True*
Int.Min ≤ *Int.Bit0 k* ⟷ *Int.Min* < *k*
Int.Min ≤ *Int.Bit1 k* ⟷ *Int.Min* ≤ *k*
Int.Bit0 k ≤ *Int.Pls* ⟷ *k* ≤ *Int.Pls*
Int.Bit1 k ≤ *Int.Pls* ⟷ *k* < *Int.Pls*
Int.Bit0 k ≤ *Int.Min* ⟷ *k* ≤ *Int.Min*
Int.Bit1 k ≤ *Int.Min* ⟷ *k* ≤ *Int.Min*
Int.Bit0 k1 ≤ *Int.Bit0 k2* ⟷ *k1* ≤ *k2*
Int.Bit0 k1 ≤ *Int.Bit1 k2* ⟷ *k1* ≤ *k2*
Int.Bit1 k1 ≤ *Int.Bit0 k2* ⟷ *k1* < *k2*
Int.Bit1 k1 ≤ *Int.Bit1 k2* ⟷ *k1* ≤ *k2*
by *simp-all*

lemma *less-number-of-int-code* [*code*]:
(*number-of k* :: *int*) < *number-of l* ⟷ *k* < *l*
unfolding *number-of-is-id* **..**

lemma *less-int-code* [*code*]:
Int.Pls < *Int.Pls* ⟷ *False*
Int.Pls < *Int.Min* ⟷ *False*
Int.Pls < *Int.Bit0 k* ⟷ *Int.Pls* < *k*
Int.Pls < *Int.Bit1 k* ⟷ *Int.Pls* ≤ *k*
Int.Min < *Int.Pls* ⟷ *True*
Int.Min < *Int.Min* ⟷ *False*
Int.Min < *Int.Bit0 k* ⟷ *Int.Min* < *k*
Int.Min < *Int.Bit1 k* ⟷ *Int.Min* < *k*
Int.Bit0 k < *Int.Pls* ⟷ *k* < *Int.Pls*
Int.Bit1 k < *Int.Pls* ⟷ *k* < *Int.Pls*
Int.Bit0 k < *Int.Min* ⟷ *k* ≤ *Int.Min*
Int.Bit1 k < *Int.Min* ⟷ *k* < *Int.Min*
Int.Bit0 k1 < *Int.Bit0 k2* ⟷ *k1* < *k2*
Int.Bit0 k1 < *Int.Bit1 k2* ⟷ *k1* ≤ *k2*
Int.Bit1 k1 < *Int.Bit0 k2* ⟷ *k1* < *k2*
Int.Bit1 k1 < *Int.Bit1 k2* ⟷ *k1* < *k2*
by *simp-all*

definition
nat-aux :: *int* ⇒ *nat* ⇒ *nat* **where**
nat-aux i n = *nat i* + *n*

lemma [*code*]:
nat-aux i n = (*if i* ≤ *0 then n else nat-aux* (*i* − *1*) (*Suc n*)) — tail recursive
by (*auto simp add*: *nat-aux-def nat-eq-iff linorder-not-le order-less-imp-le*
dest: *zless-imp-add1-zle*)

lemma [*code*]: *nat i* = *nat-aux i 0*
by (*simp add*: *nat-aux-def*)

hide-const (**open**) *nat-aux*

lemma *zero-is-num-zero* [*code*, *code-unfold-post*]:
(*0*::*int*) = *Numeral0*
by *simp*

lemma *one-is-num-one* [*code*, *code-unfold-post*]:
(*1*::*int*) = *Numeral1*
by *simp*

code-modulename *SML*
Int Arith

code-modulename *OCaml*
Int Arith

code-modulename *Haskell*

Int Arith

types-code
int (*int*)
attach (*term-of*) ⟪
val term-of-int = HOLogic.mk-number HOLogic.intT;
⟫
attach (*test*) ⟪
fun gen-int i =
let val j = one-of [~*1*, *1*] ∗ *random-range 0 i*
in (*j*, *fn* () => *term-of-int j*) *end*;
⟫

setup ⟪
let

fun strip-number-of (@{*term Int.number-of* :: *int* => *int*} $ *t*) = *t*
| *strip-number-of t* = *t*;

fun numeral-codegen thy defs dep module b t gr =
let val i = HOLogic.dest-numeral (*strip-number-of t*)
in
SOME (*Codegen.str* (*string-of-int i*),
snd (*Codegen.invoke-tycodegen thy defs dep module false HOLogic.intT gr*))
end handle TERM - => NONE;

in

Codegen.add-codegen numeral-codegen numeral-codegen

end
⟫

consts-code
number-of :: *int* ⇒ *int* ((-))
0 :: *int* (*0*)
1 :: *int* (*1*)
uminus :: *int* => *int* (~)
op + :: *int* => *int* => *int* ((- +/ -))
op ∗ :: *int* => *int* => *int* ((- */ -))
op ≤ :: *int* => *int* => *bool* ((- <=/ -))
op < :: *int* => *int* => *bool* ((- </ -))

quickcheck-params [*default-type* = *int*]

hide-const (**open**) *Pls Min Bit0 Bit1 succ pred*

30.25 Legacy theorems

lemmas *zminus-zminus = minus-minus* [*of z*::*int*, *standard*]
lemmas *zminus-0 = minus-zero* [**where** *'a=int*]
lemmas *zminus-zadd-distrib = minus-add-distrib* [*of z*::*int w*, *standard*]
lemmas *zadd-commute = add-commute* [*of z*::*int w*, *standard*]
lemmas *zadd-assoc = add-assoc* [*of z1*::*int z2 z3*, *standard*]
lemmas *zadd-left-commute = add-left-commute* [*of x*::*int y z*, *standard*]
lemmas *zadd-ac = zadd-assoc zadd-commute zadd-left-commute*
lemmas *zmult-ac = mult-ac*
lemmas *zadd-0 = add-0-left* [*of z*::*int*, *standard*]
lemmas *zadd-0-right = add-0-right* [*of z*::*int*, *standard*]
lemmas *zadd-zminus-inverse2 = left-minus* [*of z*::*int*, *standard*]
lemmas *zmult-zminus = mult-minus-left* [*of z*::*int w*, *standard*]
lemmas *zmult-commute = mult-commute* [*of z*::*int w*, *standard*]
lemmas *zmult-assoc = mult-assoc* [*of z1*::*int z2 z3*, *standard*]
lemmas *zadd-zmult-distrib = left-distrib* [*of z1*::*int z2 w*, *standard*]
lemmas *zadd-zmult-distrib2 = right-distrib* [*of w*::*int z1 z2*, *standard*]
lemmas *zdiff-zmult-distrib = left-diff-distrib* [*of z1*::*int z2 w*, *standard*]
lemmas *zdiff-zmult-distrib2 = right-diff-distrib* [*of w*::*int z1 z2*, *standard*]

lemmas *zmult-1 = mult-1-left* [*of z*::*int*, *standard*]
lemmas *zmult-1-right = mult-1-right* [*of z*::*int*, *standard*]

lemmas *zle-refl = order-refl* [*of w*::*int*, *standard*]
lemmas *zle-trans = order-trans* [**where** *'a=int* **and** *x=i* **and** *y=j* **and** *z=k*, *standard*]
lemmas *zle-antisym = order-antisym* [*of z*::*int w*, *standard*]
lemmas *zle-linear = linorder-linear* [*of z*::*int w*, *standard*]
lemmas *zless-linear = linorder-less-linear* [**where** *'a = int*]

lemmas *zadd-left-mono = add-left-mono* [*of i*::*int j k*, *standard*]
lemmas *zadd-strict-right-mono = add-strict-right-mono* [*of i*::*int j k*, *standard*]
lemmas *zadd-zless-mono = add-less-le-mono* [*of w'*::*int w z' z*, *standard*]

lemmas *int-0-less-1 = zero-less-one* [**where** *'a=int*]
lemmas *int-0-neq-1 = zero-neq-one* [**where** *'a=int*]

lemmas *inj-int = inj-of-nat* [**where** *'a=int*]
lemmas *zadd-int = of-nat-add* [**where** *'a=int*, *symmetric*]
lemmas *int-mult = of-nat-mult* [**where** *'a=int*]
lemmas *zmult-int = of-nat-mult* [**where** *'a=int*, *symmetric*]
lemmas *int-eq-0-conv = of-nat-eq-0-iff* [**where** *'a=int* **and** *m=n*, *standard*]
lemmas *zless-int = of-nat-less-iff* [**where** *'a=int*]
lemmas *int-less-0-conv = of-nat-less-0-iff* [**where** *'a=int* **and** *m=k*, *standard*]
lemmas *zero-less-int-conv = of-nat-0-less-iff* [**where** *'a=int*]
lemmas *zero-zle-int = of-nat-0-le-iff* [**where** *'a=int*]
lemmas *int-le-0-conv = of-nat-le-0-iff* [**where** *'a=int* **and** *m=n*, *standard*]
lemmas *int-0 = of-nat-0* [**where** *'a=int*]
lemmas *int-1 = of-nat-1* [**where** *'a=int*]

lemmas *int-Suc* = *of-nat-Suc* [**where** *'a=int*]
lemmas *abs-int-eq* = *abs-of-nat* [**where** *'a=int* **and** *n=m*, *standard*]
lemmas *of-int-int-eq* = *of-int-of-nat-eq* [**where** *'a=int*]
lemmas *zdiff-int* = *of-nat-diff* [**where** *'a=int*, *symmetric*]
lemmas *zless-le* = *less-int-def*
lemmas *int-eq-of-nat* = *TrueI*

lemma *zpower-zadd-distrib*:
*x ^ (y + z) = ((x ^ y) * (x ^ z)::int)*
by (*rule power-add*)

lemma *zero-less-zpower-abs-iff*:
(0 < abs x ^ n) ⟷ (x ≠ (0::int) | n = 0)
by (*rule zero-less-power-abs-iff*)

lemma *zero-le-zpower-abs*: *(0::int) ≤ abs x ^ n*
by (*rule zero-le-power-abs*)

lemma *zpower-zpower*:
*(x ^ y) ^ z = (x ^ (y * z))::int)*
by (*rule power-mult* [*symmetric*])

lemma *int-power*:
int (m ^ n) = int m ^ n
by (*rule of-nat-power*)

lemmas *zpower-int* = *int-power* [*symmetric*]

end

31 Nat-Numeral: Binary numerals for the natural numbers

theory *Nat-Numeral*
imports *Int*
begin

31.1 Numerals for natural numbers

Arithmetic for naturals is reduced to that for the non-negative integers.

instantiation *nat* :: *number*
begin

definition
nat-number-of-def [*code-unfold*, *code del*]: *number-of v = nat (number-of v)*

instance ..

end

lemma [*code-post*]:
nat (*number-of v*) = *number-of v*
unfolding *nat-number-of-def* **..**

31.2 Special case: squares and cubes

lemma *numeral-2-eq-2*: *2* = *Suc* (*Suc 0*)
by (*simp add*: *nat-number-of-def*)

lemma *numeral-3-eq-3*: *3* = *Suc* (*Suc* (*Suc 0*))
by (*simp add*: *nat-number-of-def*)

context *power*
begin

abbreviation (*xsymbols*)
power2 :: *'a* ⇒ *'a* ((-^{2}) [*1000*] *999*) **where**
$x^2 \equiv x$ ^ *2*

notation (*latex* **output**)
power2 ((-^{2}) [*1000*] *999*)

notation (*HTML* **output**)
power2 ((-^{2}) [*1000*] *999*)

end

context *monoid-mult*
begin

lemma *power2-eq-square*: $a^2 = a * a$
by (*simp add*: *numeral-2-eq-2*)

lemma *power3-eq-cube*: *a* ^ *3* = *a* * *a* * *a*
by (*simp add*: *numeral-3-eq-3 mult-assoc*)

lemma *power-even-eq*:
a ^ (*2***n*) = (*a* ^ *n*) ^ *2*
by (*subst mult-commute*) (*simp add*: *power-mult*)

lemma *power-odd-eq*:
a ^ *Suc* (*2***n*) = *a* * (*a* ^ *n*) ^ *2*
by (*simp add*: *power-even-eq*)

end

context *semiring-1*
begin

lemma *zero-power2* [*simp*]: $0^2 = 0$
by (*simp add*: *power2-eq-square*)

lemma *one-power2* [*simp*]: $1^2 = 1$
by (*simp add*: *power2-eq-square*)

end

context *ring-1*
begin

lemma *power2-minus* [*simp*]:
$(- a)^2 = a^2$
by (*simp add*: *power2-eq-square*)

We cannot prove general results about the numeral $-1::'b$, so we have to use $-(1::'a)$ instead.

lemma *power-minus1-even* [*simp*]:
$(- 1) \hat{} (2*n) = 1$
proof (*induct n*)
case *0* **show** *?case* **by** *simp*
next
case (*Suc n*) **then show** *?case* **by** (*simp add*: *power-add*)
qed

lemma *power-minus1-odd*:
$(- 1) \hat{} Suc\ (2*n) = - 1$
by *simp*

lemma *power-minus-even* [*simp*]:
$(-a) \hat{} (2*n) = a \hat{} (2*n)$
by (*simp add*: *power-minus* [*of a*])

end

context *ring-1-no-zero-divisors*
begin

lemma *zero-eq-power2* [*simp*]:
$a^2 = 0 \longleftrightarrow a = 0$
unfolding *power2-eq-square* **by** *simp*

lemma *power2-eq-1-iff*:
$a^2 = 1 \longleftrightarrow a = 1 \vee a = - 1$
unfolding *power2-eq-square* **by** (*rule square-eq-1-iff*)

end

context *linordered-ring*
begin

lemma *sum-squares-ge-zero*:
$0 \leq x * x + y * y$
by (*intro add-nonneg-nonneg zero-le-square*)

lemma *not-sum-squares-lt-zero*:
$\neg\ x * x + y * y < 0$
by (*simp add*: *not-less sum-squares-ge-zero*)

end

context *linordered-ring-strict*
begin

lemma *sum-squares-eq-zero-iff*:
$x * x + y * y = 0 \longleftrightarrow x = 0 \wedge y = 0$
by (*simp add*: *add-nonneg-eq-0-iff*)

lemma *sum-squares-le-zero-iff*:
$x * x + y * y \leq 0 \longleftrightarrow x = 0 \wedge y = 0$
by (*simp add*: *le-less not-sum-squares-lt-zero sum-squares-eq-zero-iff*)

lemma *sum-squares-gt-zero-iff*:
$0 < x * x + y * y \longleftrightarrow x \neq 0 \vee y \neq 0$
by (*simp add*: *not-le* [*symmetric*] *sum-squares-le-zero-iff*)

end

context *linordered-semidom*
begin

lemma *power2-le-imp-le*:
$x^2 \leq y^2 \Longrightarrow 0 \leq y \Longrightarrow x \leq y$
unfolding *numeral-2-eq-2* **by** (*rule power-le-imp-le-base*)

lemma *power2-less-imp-less*:
$x^2 < y^2 \Longrightarrow 0 \leq y \Longrightarrow x < y$
by (*rule power-less-imp-less-base*)

lemma *power2-eq-imp-eq*:
$x^2 = y^2 \Longrightarrow 0 \leq x \Longrightarrow 0 \leq y \Longrightarrow x = y$
unfolding *numeral-2-eq-2* **by** (*erule* (*2*) *power-eq-imp-eq-base*) *simp*

end

context *linordered-idom*
begin

lemma *zero-le-power2* [*simp*]:
  $0 \leq a^2$
  by (*simp add*: *power2-eq-square*)

lemma *zero-less-power2* [*simp*]:
  $0 < a^2 \longleftrightarrow a \neq 0$
  by (*force simp add*: *power2-eq-square zero-less-mult-iff linorder-neq-iff*)

lemma *power2-less-0* [*simp*]:
  $\neg\ a^2 < 0$
  by (*force simp add*: *power2-eq-square mult-less-0-iff*)

lemma *abs-power2* [*simp*]:
  $abs\ (a^2) = a^2$
  by (*simp add*: *power2-eq-square abs-mult abs-mult-self*)

lemma *power2-abs* [*simp*]:
  $(abs\ a)^2 = a^2$
  by (*simp add*: *power2-eq-square abs-mult-self*)

lemma *odd-power-less-zero*:
  $a < 0 \Longrightarrow a\ \hat{}\ Suc\ (2{*}n) < 0$
proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  have $a\ \hat{}\ Suc\ (2 * Suc\ n) = (a{*}a) * a\ \hat{}\ Suc(2{*}n)$
    by (*simp add*: *mult-ac power-add power2-eq-square*)
  thus *?case*
    by (*simp del*: *power-Suc add*: *Suc mult-less-0-iff mult-neg-neg*)
qed

lemma *odd-0-le-power-imp-0-le*:
  $0 \leq a\ \hat{}\ Suc\ (2{*}n) \Longrightarrow 0 \leq a$
  using *odd-power-less-zero* [*of a n*]
    by (*force simp add*: *linorder-not-less* [*symmetric*])

lemma *zero-le-even-power'*[*simp*]:
  $0 \leq a\ \hat{}\ (2{*}n)$
proof (*induct n*)
  case *0*
    show *?case* **by** *simp*
next
  case (*Suc n*)
    have $a\ \hat{}\ (2 * Suc\ n) = (a{*}a) * a\ \hat{}\ (2{*}n)$

by (*simp add*: *mult-ac power-add power2-eq-square*)
thus *?case*
by (*simp add*: *Suc zero-le-mult-iff*)
qed

lemma *sum-power2-ge-zero*:
$0 \leq x^2 + y^2$
unfolding *power2-eq-square* **by** (*rule sum-squares-ge-zero*)

lemma *not-sum-power2-lt-zero*:
$\neg\ x^2 + y^2 < 0$
unfolding *power2-eq-square* **by** (*rule not-sum-squares-lt-zero*)

lemma *sum-power2-eq-zero-iff*:
$x^2 + y^2 = 0 \longleftrightarrow x = 0 \wedge y = 0$
unfolding *power2-eq-square* **by** (*rule sum-squares-eq-zero-iff*)

lemma *sum-power2-le-zero-iff*:
$x^2 + y^2 \leq 0 \longleftrightarrow x = 0 \wedge y = 0$
unfolding *power2-eq-square* **by** (*rule sum-squares-le-zero-iff*)

lemma *sum-power2-gt-zero-iff*:
$0 < x^2 + y^2 \longleftrightarrow x \neq 0 \vee y \neq 0$
unfolding *power2-eq-square* **by** (*rule sum-squares-gt-zero-iff*)

end

lemma *power2-sum*:
fixes $x\ y :: 'a$::*number-ring*
shows $(x + y)^2 = x^2 + y^2 + 2 * x * y$
by (*simp add*: *ring-distribs power2-eq-square mult-2*) (*rule mult-commute*)

lemma *power2-diff*:
fixes $x\ y :: 'a$::*number-ring*
shows $(x - y)^2 = x^2 + y^2 - 2 * x * y$
by (*simp add*: *ring-distribs power2-eq-square mult-2*) (*rule mult-commute*)

31.3 Predicate for negative binary numbers

definition *neg* :: *int* ⇒ *bool* **where**
neg Z ⟷ *Z* < *0*

lemma *not-neg-int* [*simp*]: ~ *neg* (*of-nat n*)
by (*simp add*: *neg-def*)

lemma *neg-zminus-int* [*simp*]: *neg* (− (*of-nat* (*Suc n*)))
by (*simp add*: *neg-def del*: *of-nat-Suc*)

lemmas *neg-eq-less-0* = *neg-def*

lemma *not-neg-eq-ge-0*: $(\sim neg\ x) = (0 \leq x)$
by (*simp add*: *neg-def linorder-not-less*)

To simplify inequalities when Numeral1 can get simplified to 1

lemma *not-neg-0*: $\sim\ neg\ 0$
by (*simp add*: *One-int-def neg-def*)

lemma *not-neg-1*: $\sim\ neg\ 1$
by (*simp add*: *neg-def linorder-not-less*)

lemma *neg-nat*: $neg\ z \Longrightarrow nat\ z = 0$
by (*simp add*: *neg-def order-less-imp-le*)

lemma *not-neg-nat*: $\sim\ neg\ z \Longrightarrow of\text{-}nat\ (nat\ z) = z$
by (*simp add*: *linorder-not-less neg-def*)

If *Numeral0* is rewritten to 0 then this rule can't be applied: *Numeral0* IS *number-of Pls*

lemma *not-neg-number-of-Pls*: $\sim\ neg\ (number\text{-}of\ Int.Pls)$
by (*simp add*: *neg-def*)

lemma *neg-number-of-Min*: $neg\ (number\text{-}of\ Int.Min)$
by (*simp add*: *neg-def*)

lemma *neg-number-of-Bit0*:
$neg\ (number\text{-}of\ (Int.Bit0\ w)) = neg\ (number\text{-}of\ w)$
by (*simp add*: *neg-def*)

lemma *neg-number-of-Bit1*:
$neg\ (number\text{-}of\ (Int.Bit1\ w)) = neg\ (number\text{-}of\ w)$
by (*simp add*: *neg-def*)

lemmas *neg-simps* [*simp*] =
not-neg-0 not-neg-1
not-neg-number-of-Pls neg-number-of-Min
neg-number-of-Bit0 neg-number-of-Bit1

31.4 Function *nat*: Coercion from Type *int* to *nat*

declare *nat-1* [*simp*]

lemma *nat-number-of* [*simp*]: $nat\ (number\text{-}of\ w) = number\text{-}of\ w$
by (*simp add*: *nat-number-of-def*)

lemma *nat-numeral-0-eq-0* [*simp*, *code-post*]: $Numeral0 = (0::nat)$
by (*simp add*: *nat-number-of-def*)

lemma *nat-numeral-1-eq-1* [*simp*]: $Numeral1 = (1::nat)$

by (*simp add*: *nat-number-of-def*)

lemma *Numeral1-eq1-nat*:
(*1*::*nat*) = *Numeral1*
by *simp*

lemma *numeral-1-eq-Suc-0* [*code-post*]: *Numeral1* = *Suc 0*
by (*simp only*: *nat-numeral-1-eq-1 One-nat-def*)

31.5 Function *int*: Coercion from Type *nat* to *int*

lemma *int-nat-number-of* [*simp*]:
int (*number-of v*) =
(*if neg* (*number-of v* :: *int*) *then 0*
else (*number-of v* :: *int*))
unfolding *nat-number-of-def number-of-is-id neg-def*
by *simp*

31.5.1 Successor

lemma *Suc-nat-eq-nat-zadd1*: (*0*::*int*) <= *z* ==> *Suc* (*nat z*) = *nat* (*1* + *z*)
apply (*rule sym*)
apply (*simp add*: *nat-eq-iff int-Suc*)
done

lemma *Suc-nat-number-of-add*:
Suc (*number-of v* + *n*) =
(*if neg* (*number-of v* :: *int*) *then 1*+*n else number-of* (*Int.succ v*) + *n*)
unfolding *nat-number-of-def number-of-is-id neg-def numeral-simps*
by (*simp add*: *Suc-nat-eq-nat-zadd1 add-ac*)

lemma *Suc-nat-number-of* [*simp*]:
Suc (*number-of v*) =
(*if neg* (*number-of v* :: *int*) *then 1 else number-of* (*Int.succ v*))
apply (*cut-tac n* = *0* **in** *Suc-nat-number-of-add*)
apply (*simp cong del*: *if-weak-cong*)
done

31.5.2 Addition

lemma *add-nat-number-of* [*simp*]:
(*number-of v* :: *nat*) + *number-of v'* =
(*if v* < *Int.Pls then number-of v'*
else if v' < *Int.Pls then number-of v*
else number-of (*v* + *v'*))
unfolding *nat-number-of-def number-of-is-id numeral-simps*
by (*simp add*: *nat-add-distrib*)

lemma *nat-number-of-add-1* [*simp*]:
number-of v + (*1*::*nat*) =

(*if* $v < Int.Pls$ *then* 1 *else number-of* (*Int.succ* v))
unfolding *nat-number-of-def number-of-is-id numeral-simps*
by (*simp add*: *nat-add-distrib*)

lemma *nat-1-add-number-of* [*simp*]:
(1::*nat*) + *number-of* v =
(*if* $v < Int.Pls$ *then* 1 *else number-of* (*Int.succ* v))
unfolding *nat-number-of-def number-of-is-id numeral-simps*
by (*simp add*: *nat-add-distrib*)

lemma *nat-1-add-1* [*simp*]: 1 + 1 = (2::*nat*)
by (*rule int-int-eq* [*THEN iffD1*]) *simp*

31.5.3 Subtraction

lemma *diff-nat-eq-if*:
nat z − *nat* z' =
(*if neg* z' *then nat* z
else let d = $z - z'$ *in*
if neg d *then* 0 *else nat* d)
by (*simp add*: *Let-def nat-diff-distrib* [*symmetric*] *neg-eq-less-0 not-neg-eq-ge-0*)

lemma *diff-nat-number-of* [*simp*]:
(*number-of* v :: *nat*) − *number-of* v' =
(*if* $v' < Int.Pls$ *then number-of* v
else let d = *number-of* (v + *uminus* v') *in*
if neg d *then* 0 *else nat* d)
unfolding *nat-number-of-def number-of-is-id numeral-simps neg-def*
by *auto*

lemma *nat-number-of-diff-1* [*simp*]:
number-of v − (1::*nat*) =
(*if* $v \leq Int.Pls$ *then* 0 *else number-of* (*Int.pred* v))
unfolding *nat-number-of-def number-of-is-id numeral-simps*
by *auto*

31.5.4 Multiplication

lemma *mult-nat-number-of* [*simp*]:
(*number-of* v :: *nat*) ∗ *number-of* v' =
(*if* $v < Int.Pls$ *then* 0 *else number-of* (v ∗ v'))
unfolding *nat-number-of-def number-of-is-id numeral-simps*
by (*simp add*: *nat-mult-distrib*)

31.6 Comparisons

31.6.1 Equals (=)

lemma *eq-nat-number-of* [*simp*]:

((*number-of* v :: *nat*) = *number-of* v') =
(*if neg* (*number-of* v :: *int*) *then* (*number-of* v' :: *int*) $\leq$ *0*
else if neg (*number-of* v' :: *int*) *then* (*number-of* v :: *int*) = *0*
else $v = v'$)
unfolding *nat-number-of-def number-of-is-id neg-def*
by *auto*

31.6.2 Less-than (¡)

lemma *less-nat-number-of* [*simp*]:
(*number-of* v :: *nat*) < *number-of* $v' \longleftrightarrow$
(*if* $v < v'$ *then Int.Pls* < v' *else False*)
unfolding *nat-number-of-def number-of-is-id numeral-simps*
by *auto*

31.6.3 Less-than-or-equal

lemma *le-nat-number-of* [*simp*]:
(*number-of* v :: *nat*) $\leq$ *number-of* $v' \longleftrightarrow$
(*if* $v \leq v'$ *then True else* $v \leq$ *Int.Pls*)
unfolding *nat-number-of-def number-of-is-id numeral-simps*
by *auto*

lemmas *numerals* = *nat-numeral-0-eq-0 nat-numeral-1-eq-1 numeral-2-eq-2*

31.7 Powers with Numeric Exponents

Squares of literal numerals will be evaluated.

lemmas *power2-eq-square-number-of* [*simp*] =
power2-eq-square [*of number-of w*, *standard*]

Simprules for comparisons where common factors can be cancelled.

lemmas *zero-compare-simps* =
add-strict-increasing add-strict-increasing2 add-increasing
zero-le-mult-iff zero-le-divide-iff
zero-less-mult-iff zero-less-divide-iff
mult-le-0-iff divide-le-0-iff
mult-less-0-iff divide-less-0-iff
zero-le-power2 power2-less-0

31.7.1 Nat

lemma *Suc-pred'*: *0* < *n* ==> *n* = *Suc*(*n* − *1*)
by *simp*

lemmas *expand-Suc* = *Suc-pred'* [*of number-of v*, *standard*]

31.7.2 Arith

lemma *Suc-eq-plus1*: *Suc n = n + 1*
unfolding *One-nat-def* **by** *simp*

lemma *Suc-eq-plus1-left*: *Suc n = 1 + n*
unfolding *One-nat-def* **by** *simp*

lemma *add-eq-if*: *(m::nat) + n = (if m=0 then n else Suc ((m − 1) + n))*
unfolding *One-nat-def* **by** *(cases m) simp-all*

lemma *mult-eq-if*: *(m::nat) * n = (if m=0 then 0 else n + ((m − 1) * n))*
unfolding *One-nat-def* **by** *(cases m) simp-all*

lemma *power-eq-if*: *(p ^ m :: nat) = (if m=0 then 1 else p * (p ^ (m − 1)))*
unfolding *One-nat-def* **by** *(cases m) simp-all*

31.8 Comparisons involving (0::nat)

Simplification already does $n < (0::'a)$, $n \leq (0::'a)$ and $(0::'a) \leq n$.

lemma *eq-number-of-0* [*simp*]:
number-of v = (0::nat) ⟷ v ≤ Int.Pls
unfolding *nat-number-of-def number-of-is-id numeral-simps*
by *auto*

lemma *eq-0-number-of* [*simp*]:
(0::nat) = number-of v ⟷ v ≤ Int.Pls
by *(rule trans [OF eq-sym-conv eq-number-of-0])*

lemma *less-0-number-of* [*simp*]:
(0::nat) < number-of v ⟷ Int.Pls < v
unfolding *nat-number-of-def number-of-is-id numeral-simps*
by *simp*

lemma *neg-imp-number-of-eq-0*: *neg (number-of v :: int) ==> number-of v = (0::nat)*
by *(simp del: nat-numeral-0-eq-0 add: nat-numeral-0-eq-0 [symmetric])*

31.9 Comparisons involving *Suc*

lemma *eq-number-of-Suc* [*simp*]:
(number-of v = Suc n) =
(let pv = number-of (Int.pred v) in
if neg pv then False else nat pv = n)
apply *(simp only: simp-thms Let-def neg-eq-less-0 linorder-not-less*
number-of-pred nat-number-of-def
split add: split-if)

apply (*rule-tac x = number-of v* **in** *spec*)
apply (*auto simp add*: *nat-eq-iff*)
done

lemma *Suc-eq-number-of* [*simp*]:
(*Suc n = number-of v*) =
(*let pv = number-of* (*Int.pred v*) *in*
if neg pv then False else nat pv = n)
by (*rule trans* [*OF eq-sym-conv eq-number-of-Suc*])

lemma *less-number-of-Suc* [*simp*]:
(*number-of v < Suc n*) =
(*let pv = number-of* (*Int.pred v*) *in*
if neg pv then True else nat pv < n)
apply (*simp only*: *simp-thms Let-def neg-eq-less-0 linorder-not-less*
number-of-pred nat-number-of-def
split add: *split-if*)
apply (*rule-tac x = number-of v* **in** *spec*)
apply (*auto simp add*: *nat-less-iff*)
done

lemma *less-Suc-number-of* [*simp*]:
(*Suc n < number-of v*) =
(*let pv = number-of* (*Int.pred v*) *in*
if neg pv then False else n < nat pv)
apply (*simp only*: *simp-thms Let-def neg-eq-less-0 linorder-not-less*
number-of-pred nat-number-of-def
split add: *split-if*)
apply (*rule-tac x = number-of v* **in** *spec*)
apply (*auto simp add*: *zless-nat-eq-int-zless*)
done

lemma *le-number-of-Suc* [*simp*]:
(*number-of v <= Suc n*) =
(*let pv = number-of* (*Int.pred v*) *in*
if neg pv then True else nat pv <= n)
by (*simp add*: *Let-def linorder-not-less* [*symmetric*])

lemma *le-Suc-number-of* [*simp*]:
(*Suc n <= number-of v*) =
(*let pv = number-of* (*Int.pred v*) *in*
if neg pv then False else n <= nat pv)
by (*simp add*: *Let-def linorder-not-less* [*symmetric*])

lemma *eq-number-of-Pls-Min*: (*Numeral0* ::*int*) ~= *number-of Int.Min*
by *auto*

31.10 Max and Min Combined with *Suc*

lemma *max-number-of-Suc* [*simp*]:
  max (*Suc n*) (*number-of v*) =
    (*let pv* = *number-of* (*Int.pred v*) *in*
     if neg pv then Suc n else Suc(*max n* (*nat pv*)))
apply (*simp only*: *Let-def neg-eq-less-0 number-of-pred nat-number-of-def*
      split add: *split-if nat.split*)
apply (*rule-tac x* = *number-of v* **in** *spec*)
apply *auto*
done

lemma *max-Suc-number-of* [*simp*]:
  max (*number-of v*) (*Suc n*) =
    (*let pv* = *number-of* (*Int.pred v*) *in*
     if neg pv then Suc n else Suc(*max* (*nat pv*) *n*))
apply (*simp only*: *Let-def neg-eq-less-0 number-of-pred nat-number-of-def*
      split add: *split-if nat.split*)
apply (*rule-tac x* = *number-of v* **in** *spec*)
apply *auto*
done

lemma *min-number-of-Suc* [*simp*]:
  min (*Suc n*) (*number-of v*) =
    (*let pv* = *number-of* (*Int.pred v*) *in*
     if neg pv then 0 else Suc(*min n* (*nat pv*)))
apply (*simp only*: *Let-def neg-eq-less-0 number-of-pred nat-number-of-def*
      split add: *split-if nat.split*)
apply (*rule-tac x* = *number-of v* **in** *spec*)
apply *auto*
done

lemma *min-Suc-number-of* [*simp*]:
  min (*number-of v*) (*Suc n*) =
    (*let pv* = *number-of* (*Int.pred v*) *in*
     if neg pv then 0 else Suc(*min* (*nat pv*) *n*))
apply (*simp only*: *Let-def neg-eq-less-0 number-of-pred nat-number-of-def*
      split add: *split-if nat.split*)
apply (*rule-tac x* = *number-of v* **in** *spec*)
apply *auto*
done

31.11 Literal arithmetic involving powers

lemma *power-nat-number-of*:
  (*number-of v* :: *nat*) ^ *n* =
    (*if neg* (*number-of v* :: *int*) *then* 0^*n else nat* ((*number-of v* :: *int*) ^ *n*))
by (*simp only*: *simp-thms neg-nat not-neg-eq-ge-0 nat-number-of-def nat-power-eq*
      split add: *split-if cong*: *imp-cong*)

lemmas *power-nat-number-of-number-of* = *power-nat-number-of* [*of* - *number-of w*, *standard*]
declare *power-nat-number-of-number-of* [*simp*]

For arbitrary rings

lemma *power-number-of-even*:
  fixes *z* :: *'a*::*number-ring*
  shows *z* ^ *number-of* (*Int.Bit0 w*) = (*let w* = *z* ^ (*number-of w*) *in w* * *w*)
by (*cases w* ≥ *0*) (*auto simp add*: *Let-def Bit0-def nat-number-of-def number-of-is-id nat-add-distrib power-add simp del*: *nat-number-of*)

lemma *power-number-of-odd*:
  fixes *z* :: *'a*::*number-ring*
  shows *z* ^ *number-of* (*Int.Bit1 w*) = (*if* (*0*::*int*) <= *number-of w*
    then (*let w* = *z* ^ (*number-of w*) *in z* * *w* * *w*) *else 1*)
unfolding *Let-def Bit1-def nat-number-of-def number-of-is-id*
apply (*cases 0* <= *w*)
apply (*simp only*: *mult-assoc nat-add-distrib power-add*, *simp*)
apply (*simp add*: *not-le mult-2* [*symmetric*] *add-assoc*)
done

lemmas *zpower-number-of-even* = *power-number-of-even* [**where** *'a*=*int*]
lemmas *zpower-number-of-odd* = *power-number-of-odd* [**where** *'a*=*int*]

lemmas *power-number-of-even-number-of* [*simp*] =
  power-number-of-even [*of number-of v*, *standard*]

lemmas *power-number-of-odd-number-of* [*simp*] =
  power-number-of-odd [*of number-of v*, *standard*]

lemma *nat-number-of-Pls*: *Numeral0* = (*0*::*nat*)
  by (*simp add*: *nat-number-of-def*)

lemma *nat-number-of-Min*: *number-of Int.Min* = (*0*::*nat*)
  apply (*simp only*: *number-of-Min nat-number-of-def nat-zminus-int*)
  done

lemma *nat-number-of-Bit0*:
  number-of (*Int.Bit0 w*) = (*let n*::*nat* = *number-of w in n* + *n*)
by (*cases w* ≥ *0*) (*auto simp add*: *Let-def Bit0-def nat-number-of-def number-of-is-id nat-add-distrib simp del*: *nat-number-of*)

lemma *nat-number-of-Bit1*:
  number-of (*Int.Bit1 w*) =
    (*if neg* (*number-of w* :: *int*) *then 0*
     else let n = *number-of w in Suc* (*n* + *n*))
unfolding *Let-def Bit1-def nat-number-of-def number-of-is-id neg-def*
apply (*cases w* < *0*)

apply (*simp add*: *mult-2* [*symmetric*] *add-assoc*)
apply (*simp only*: *nat-add-distrib*, *simp*)
done

lemmas *nat-number* =
nat-number-of-Pls nat-number-of-Min
nat-number-of-Bit0 nat-number-of-Bit1

lemmas *nat-number'* =
nat-number-of-Bit0 nat-number-of-Bit1

lemmas *nat-arith* =
add-nat-number-of
diff-nat-number-of
mult-nat-number-of
eq-nat-number-of
less-nat-number-of

lemmas *semiring-norm* =
Let-def arith-simps nat-arith rel-simps neg-simps if-False
if-True add-0 add-Suc add-number-of-left mult-number-of-left
numeral-1-eq-1 [*symmetric*] *Suc-eq-plus1*
numeral-0-eq-0 [*symmetric*] *numerals* [*symmetric*]
not-iszero-Numeral1

lemma *Let-Suc* [*simp*]: *Let* (*Suc n*) *f* == *f* (*Suc n*)
by (*fact Let-def*)

lemma *power-m1-even*: (−1) ^ (2∗*n*) = (1::'*a*::{*number-ring*})
by (*simp only*: *number-of-Min power-minus1-even*)

lemma *power-m1-odd*: (−1) ^ *Suc*(2∗*n*) = (−1::'*a*::{*number-ring*})
by (*simp only*: *number-of-Min power-minus1-odd*)

lemma *nat-number-of-add-left*:
number-of v + (*number-of v'* + (*k*::*nat*)) =
(*if neg* (*number-of v* :: *int*) *then number-of v'* + *k*
else if neg (*number-of v'* :: *int*) *then number-of v* + *k*
else number-of (*v* + *v'*) + *k*)
by (*auto simp add*: *neg-def*)

lemma *nat-number-of-mult-left*:
number-of v ∗ (*number-of v'* ∗ (*k*::*nat*)) =
(*if v* < *Int.Pls then* 0
else number-of (*v* ∗ *v'*) ∗ *k*)
by (*auto simp add*: *not-less Pls-def nat-number-of-def number-of-is-id*
nat-mult-distrib simp del: *nat-number-of*)

31.12 Literal arithmetic and *of-nat*

lemma *of-nat-double*:
 $0 \leq x \Longrightarrow$ *of-nat* (*nat* (2 * *x*)) = *of-nat* (*nat x*) + *of-nat* (*nat x*)
by (*simp only*: *mult-2 nat-add-distrib of-nat-add*)

lemma *nat-numeral-m1-eq-0*: $-1 = (0::nat)$
by (*simp only*: *nat-number-of-def*)

lemma *of-nat-number-of-lemma*:
 of-nat (*number-of v* :: *nat*) =
 (*if* 0 ≤ (*number-of v* :: *int*)
 then (*number-of v* :: 'a :: *number-ring*)
 else 0)
by (*simp add*: *int-number-of-def nat-number-of-def number-of-eq of-nat-nat*)

lemma *of-nat-number-of-eq* [*simp*]:
 of-nat (*number-of v* :: *nat*) =
 (*if neg* (*number-of v* :: *int*) *then* 0
 else (*number-of v* :: 'a :: *number-ring*))
by (*simp only*: *of-nat-number-of-lemma neg-def*, *simp*)

31.12.1 For simplifying *Suc m − K* and *K − Suc m*

Where K above is a literal

lemma *Suc-diff-eq-diff-pred*: *Numeral0* < *n* ==> *Suc m* − *n* = *m* − (*n* − *Numeral1*)
by (*simp split*: *nat-diff-split*)

Now just instantiating *n* to *number-of v* does the right simplification, but with some redundant inequality tests.

lemma *neg-number-of-pred-iff-0*:
 neg (*number-of* (*Int.pred v*)::*int*) = (*number-of v* = (0::*nat*))
apply (*subgoal-tac neg* (*number-of* (*Int.pred v*)) = (*number-of v* < *Suc* 0))
apply (*simp only*: *less-Suc-eq-le le-0-eq*)
apply (*subst less-number-of-Suc*, *simp*)
done

No longer required as a simprule because of the *inverse-fold* simproc

lemma *Suc-diff-number-of*:
 Int.Pls < *v* ==>
 Suc m − (*number-of v*) = *m* − (*number-of* (*Int.pred v*))
apply (*subst Suc-diff-eq-diff-pred*)
apply *simp*
apply (*simp del*: *nat-numeral-1-eq-1*)
apply (*auto simp only*: *diff-nat-number-of less-0-number-of* [*symmetric*]
 neg-number-of-pred-iff-0)
done

lemma *diff-Suc-eq-diff-pred*: $m - Suc\ n = (m - 1) - n$
by (*simp split*: *nat-diff-split*)

31.12.2 For *nat-case* and *nat-rec*

lemma *nat-case-number-of* [*simp*]:
  nat-case a f (*number-of v*) =
    (*let pv* = *number-of* (*Int.pred v*) *in*
     if neg pv then a else f (*nat pv*))
by (*simp split add*: *nat.split add*: *Let-def neg-number-of-pred-iff-0*)

lemma *nat-case-add-eq-if* [*simp*]:
  nat-case a f ((*number-of v*) + *n*) =
    (*let pv* = *number-of* (*Int.pred v*) *in*
     if neg pv then nat-case a f n else f (*nat pv* + *n*))
apply (*subst add-eq-if*)
apply (*simp split add*: *nat.split*
        del: *nat-numeral-1-eq-1*
        add: *nat-numeral-1-eq-1* [*symmetric*]
          numeral-1-eq-Suc-0 [*symmetric*]
          neg-number-of-pred-iff-0)
done

lemma *nat-rec-number-of* [*simp*]:
  nat-rec a f (*number-of v*) =
    (*let pv* = *number-of* (*Int.pred v*) *in*
     if neg pv then a else f (*nat pv*) (*nat-rec a f* (*nat pv*)))
apply (*case-tac* (*number-of v*) ::*nat*)
apply (*simp-all* (*no-asm-simp*) *add*: *Let-def neg-number-of-pred-iff-0*)
apply (*simp split add*: *split-if-asm*)
done

lemma *nat-rec-add-eq-if* [*simp*]:
  nat-rec a f (*number-of v* + *n*) =
    (*let pv* = *number-of* (*Int.pred v*) *in*
     if neg pv then nat-rec a f n
          else f (*nat pv* + *n*) (*nat-rec a f* (*nat pv* + *n*)))
apply (*subst add-eq-if*)
apply (*simp split add*: *nat.split*
        del: *nat-numeral-1-eq-1*
        add: *nat-numeral-1-eq-1* [*symmetric*]
          numeral-1-eq-Suc-0 [*symmetric*]
          neg-number-of-pred-iff-0)
done

31.12.3 Various Other Lemmas

lemma *card-UNIV-bool*[*simp*]: *card* (*UNIV* :: *bool set*) = *2*
by(*simp add*: *UNIV-bool*)

Evens and Odds, for Mutilated Chess Board

Lemmas for specialist use, NOT as default simprules

lemma *nat-mult-2*: $2 * z = (z+z::nat)$
unfolding *nat-1-add-1* [*symmetric*] *left-distrib* **by** *simp*

lemma *nat-mult-2-right*: $z * 2 = (z+z::nat)$
by (*subst mult-commute*, *rule nat-mult-2*)

Case analysis on $n < (2::'a)$

lemma *less-2-cases*: $(n::nat) < 2 ==> n = 0 \mid n = Suc\ 0$
by (*auto simp add*: *nat-1-add-1* [*symmetric*])

Removal of Small Numerals: 0, 1 and (in additive positions) 2

lemma *add-2-eq-Suc* [*simp*]: $2 + n = Suc\ (Suc\ n)$
by *simp*

lemma *add-2-eq-Suc'* [*simp*]: $n + 2 = Suc\ (Suc\ n)$
by *simp*

Can be used to eliminate long strings of Sucs, but not by default

lemma *Suc3-eq-add-3*: $Suc\ (Suc\ (Suc\ n)) = 3 + n$
by *simp*

end

32 Nat-Transfer: Generic transfer machinery; specific transfer from nats to ints and back.

theory *Nat-Transfer*
imports *Nat-Numeral*
uses (*Tools/transfer.ML*)
begin

32.1 Generic transfer machinery

definition *transfer-morphism*:: $('b \Rightarrow 'a) \Rightarrow ('b \Rightarrow bool) \Rightarrow bool$
  where $transfer\text{-}morphism\ f\ A \longleftrightarrow (\forall P.\ (\forall x.\ P\ x) \longrightarrow (\forall y.\ A\ y \longrightarrow P\ (f\ y)))$

lemma *transfer-morphismI*:
  assumes $\bigwedge P\ y.\ (\bigwedge x.\ P\ x) \Longrightarrow A\ y \Longrightarrow P\ (f\ y)$
  shows *transfer-morphism f A*
  using *assms* **by** (*auto simp add*: *transfer-morphism-def*)

use *Tools/transfer.ML*

setup *Transfer.setup*

32.2 Set up transfer from nat to int

set up transfer direction

lemma *transfer-morphism-nat-int*: *transfer-morphism nat* (*op* <= (*0*::*int*))
  by (*rule transfer-morphismI*) *simp*

declare *transfer-morphism-nat-int* [*transfer add*
  mode: *manual*
  return: *nat-0-le*
  labels: *nat-int*
]

basic functions and relations

lemma *transfer-nat-int-numerals* [*transfer key*: *transfer-morphism-nat-int*]:
    (*0*::*nat*) = *nat 0*
    (*1*::*nat*) = *nat 1*
    (*2*::*nat*) = *nat 2*
    (*3*::*nat*) = *nat 3*
  by *auto*

definition
  tsub :: *int* ⇒ *int* ⇒ *int*
where
  tsub x y = (*if x* >= *y then x* − *y else 0*)

lemma *tsub-eq*: *x* >= *y* ⟹ *tsub x y* = *x* − *y*
  by (*simp add*: *tsub-def*)

lemma *transfer-nat-int-functions* [*transfer key*: *transfer-morphism-nat-int*]:
    (*x*::*int*) >= *0* ⟹ *y* >= *0* ⟹ (*nat x*) + (*nat y*) = *nat* (*x* + *y*)
    (*x*::*int*) >= *0* ⟹ *y* >= *0* ⟹ (*nat x*) ∗ (*nat y*) = *nat* (*x* ∗ *y*)
    (*x*::*int*) >= *0* ⟹ *y* >= *0* ⟹ (*nat x*) − (*nat y*) = *nat* (*tsub x y*)
    (*x*::*int*) >= *0* ⟹ (*nat x*) ^*n* = *nat* (*x*^*n*)
  by (*auto simp add*: *eq-nat-nat-iff nat-mult-distrib*
    nat-power-eq tsub-def)

lemma *transfer-nat-int-function-closures* [*transfer key*: *transfer-morphism-nat-int*]:
    (*x*::*int*) >= *0* ⟹ *y* >= *0* ⟹ *x* + *y* >= *0*
    (*x*::*int*) >= *0* ⟹ *y* >= *0* ⟹ *x* ∗ *y* >= *0*
    (*x*::*int*) >= *0* ⟹ *y* >= *0* ⟹ *tsub x y* >= *0*
    (*x*::*int*) >= *0* ⟹ *x*^*n* >= *0*
    (*0*::*int*) >= *0*
    (*1*::*int*) >= *0*
    (*2*::*int*) >= *0*
    (*3*::*int*) >= *0*
    int z >= *0*
  by (*auto simp add*: *zero-le-mult-iff tsub-def*)

lemma *transfer-nat-int-relations* [*transfer key*: *transfer-morphism-nat-int*]:

$x >= 0 \Longrightarrow y >= 0 \Longrightarrow$
$(nat\ (x::int) = nat\ y) = (x = y)$
$x >= 0 \Longrightarrow y >= 0 \Longrightarrow$
$(nat\ (x::int) < nat\ y) = (x < y)$
$x >= 0 \Longrightarrow y >= 0 \Longrightarrow$
$(nat\ (x::int) <= nat\ y) = (x <= y)$
$x >= 0 \Longrightarrow y >= 0 \Longrightarrow$
$(nat\ (x::int)\ dvd\ nat\ y) = (x\ dvd\ y)$
by (*auto simp add*: *zdvd-int*)

first-order quantifiers

lemma *all-nat*: $(\forall x.\ P\ x) \longleftrightarrow (\forall x{\geq}0.\ P\ (nat\ x))$
by (*simp split add*: *split-nat*)

lemma *ex-nat*: $(\exists x.\ P\ x) \longleftrightarrow (\exists x.\ 0 \leq x \wedge P\ (nat\ x))$
proof
assume $\exists x.\ P\ x$
then obtain x **where** $P\ x$ **..**
then have $int\ x \geq 0 \wedge P\ (nat\ (int\ x))$ **by** *simp*
then show $\exists x{\geq}0.\ P\ (nat\ x)$ **..**
next
assume $\exists x{\geq}0.\ P\ (nat\ x)$
then show $\exists x.\ P\ x$ **by** *auto*
qed

lemma *transfer-nat-int-quantifiers* [*transfer key*: *transfer-morphism-nat-int*]:
$(ALL\ (x::nat).\ P\ x) = (ALL\ (x::int).\ x >= 0 \longrightarrow P\ (nat\ x))$
$(EX\ (x::nat).\ P\ x) = (EX\ (x::int).\ x >= 0\ \&\ P\ (nat\ x))$
by (*rule all-nat*, *rule ex-nat*)

lemma *all-cong*: $(\bigwedge x.\ Q\ x \Longrightarrow P\ x = P'\ x) \Longrightarrow$
$(ALL\ x.\ Q\ x \longrightarrow P\ x) = (ALL\ x.\ Q\ x \longrightarrow P'\ x)$
by *auto*

lemma *ex-cong*: $(\bigwedge x.\ Q\ x \Longrightarrow P\ x = P'\ x) \Longrightarrow$
$(EX\ x.\ Q\ x \wedge P\ x) = (EX\ x.\ Q\ x \wedge P'\ x)$
by *auto*

declare *transfer-morphism-nat-int* [*transfer add*
cong: *all-cong ex-cong*]

if

lemma *nat-if-cong* [*transfer key*: *transfer-morphism-nat-int*]:
$(if\ P\ then\ (nat\ x)\ else\ (nat\ y)) = nat\ (if\ P\ then\ x\ else\ y)$
by *auto*

operations with sets

definition

```
  nat-set :: int set ⇒ bool
where
  nat-set S = (ALL x:S. x >= 0)

lemma transfer-nat-int-set-functions:
    card A = card (int ‘ A)
    {} = nat ‘ ({}::int set)
    A Un B = nat ‘ (int ‘ A Un int ‘ B)
    A Int B = nat ‘ (int ‘ A Int int ‘ B)
    {x. P x} = nat ‘ {x. x >= 0 & P(nat x)}
  apply (rule card-image [symmetric])
  apply (auto simp add: inj-on-def image-def)
  apply (rule-tac x = int x in bexI)
  apply auto
  apply (rule-tac x = int x in bexI)
  apply auto
  apply (rule-tac x = int x in bexI)
  apply auto
  apply (rule-tac x = int x in exI)
  apply auto
done

lemma transfer-nat-int-set-function-closures:
    nat-set {}
    nat-set A ⟹ nat-set B ⟹ nat-set (A Un B)
    nat-set A ⟹ nat-set B ⟹ nat-set (A Int B)
    nat-set {x. x >= 0 & P x}
    nat-set (int ‘ C)
    nat-set A ⟹ x : A ⟹ x >= 0
  unfolding nat-set-def apply auto
done

lemma transfer-nat-int-set-relations:
    (finite A) = (finite (int ‘ A))
    (x : A) = (int x : int ‘ A)
    (A = B) = (int ‘ A = int ‘ B)
    (A < B) = (int ‘ A < int ‘ B)
    (A <= B) = (int ‘ A <= int ‘ B)
  apply (rule iffI)
  apply (erule finite-imageI)
  apply (erule finite-imageD)
  apply (auto simp add: image-def expand-set-eq inj-on-def)
  apply (drule-tac x = int x in spec, auto)
  apply (drule-tac x = int x in spec, auto)
  apply (drule-tac x = int x in spec, auto)
done

lemma transfer-nat-int-set-return-embed: nat-set A ⟹
    (int ‘ nat ‘ A = A)
```

by (*auto simp add*: *nat-set-def image-def*)

lemma *transfer-nat-int-set-cong*: (!!*x*. *x* >= *0* ⟹ *P x* = *P' x*) ⟹
{(*x*::*int*). *x* >= *0* & *P x*} = {*x*. *x* >= *0* & *P' x*}
by *auto*

declare *transfer-morphism-nat-int* [*transfer add*
return: *transfer-nat-int-set-functions*
transfer-nat-int-set-function-closures
transfer-nat-int-set-relations
transfer-nat-int-set-return-embed
cong: *transfer-nat-int-set-cong*
]

setsum and setprod

lemma *transfer-nat-int-sum-prod*:
setsum f A = *setsum* (%*x*. *f* (*nat x*)) (*int ' A*)
setprod f A = *setprod* (%*x*. *f* (*nat x*)) (*int ' A*)
apply (*subst setsum-reindex*)
apply (*unfold inj-on-def*, *auto*)
apply (*subst setprod-reindex*)
apply (*unfold inj-on-def o-def*, *auto*)
done

lemma *transfer-nat-int-sum-prod2*:
setsum f A = *nat*(*setsum* (%*x*. *int* (*f x*)) *A*)
setprod f A = *nat*(*setprod* (%*x*. *int* (*f x*)) *A*)
apply (*subst int-setsum* [*symmetric*])
apply *auto*
apply (*subst int-setprod* [*symmetric*])
apply *auto*
done

lemma *transfer-nat-int-sum-prod-closure*:
nat-set A ⟹ (!!*x*. *x* >= *0* ⟹ *f x* >= (*0*::*int*)) ⟹ *setsum f A* >= *0*
nat-set A ⟹ (!!*x*. *x* >= *0* ⟹ *f x* >= (*0*::*int*)) ⟹ *setprod f A* >= *0*
unfolding *nat-set-def*
apply (*rule setsum-nonneg*)
apply *auto*
apply (*rule setprod-nonneg*)
apply *auto*
done

lemma *transfer-nat-int-sum-prod-cong*:

$A = B \Longrightarrow$ *nat-set* $B \Longrightarrow (!!x.\ x >= 0 \Longrightarrow f\ x = g\ x) \Longrightarrow$
setsum $f\ A =$ *setsum* $g\ B$
$A = B \Longrightarrow$ *nat-set* $B \Longrightarrow (!!x.\ x >= 0 \Longrightarrow f\ x = g\ x) \Longrightarrow$
setprod $f\ A =$ *setprod* $g\ B$
unfolding *nat-set-def*
apply (*subst setsum-cong*, *assumption*)
apply *auto* [*2*]
apply (*subst setprod-cong*, *assumption*, *auto*)
done

declare *transfer-morphism-nat-int* [*transfer add*
return: *transfer-nat-int-sum-prod transfer-nat-int-sum-prod2*
transfer-nat-int-sum-prod-closure
cong: *transfer-nat-int-sum-prod-cong*]

32.3 Set up transfer from int to nat

set up transfer direction

lemma *transfer-morphism-int-nat*: *transfer-morphism int* ($\lambda n.$ *True*)
by (*rule transfer-morphismI*) *simp*

declare *transfer-morphism-int-nat* [*transfer add*
mode: *manual*
return: *nat-int*
labels: *int-nat*
]

basic functions and relations

definition
is-nat :: *int* ⇒ *bool*
where
is-nat $x = (x >= 0)$

lemma *transfer-int-nat-numerals*:
$0 =$ *int* *0*
$1 =$ *int* *1*
$2 =$ *int* *2*
$3 =$ *int* *3*
by *auto*

lemma *transfer-int-nat-functions*:
(*int* x) + (*int* y) = *int* $(x + y)$
(*int* x) * (*int* y) = *int* $(x * y)$
tsub (*int* x) (*int* y) = *int* $(x - y)$
(*int* x) ^ n = *int* $(x \hat{} n)$
by (*auto simp add*: *int-mult tsub-def int-power*)

lemma *transfer-int-nat-function-closures*:
is-nat $x \Longrightarrow$ *is-nat* $y \Longrightarrow$ *is-nat* $(x + y)$

is-nat x $\Longrightarrow$ *is-nat y* $\Longrightarrow$ *is-nat* (*x* * *y*)
is-nat x $\Longrightarrow$ *is-nat y* $\Longrightarrow$ *is-nat* (*tsub x y*)
is-nat x $\Longrightarrow$ *is-nat* (*x^n*)
is-nat 0
is-nat 1
is-nat 2
is-nat 3
is-nat (*int z*)
by (*simp-all only*: *is-nat-def transfer-nat-int-function-closures*)

lemma *transfer-int-nat-relations*:
(*int x* = *int y*) = (*x* = *y*)
(*int x* < *int y*) = (*x* < *y*)
(*int x* <= *int y*) = (*x* <= *y*)
(*int x dvd int y*) = (*x dvd y*)
by (*auto simp add*: *zdvd-int*)

declare *transfer-morphism-int-nat* [*transfer add return*:
transfer-int-nat-numerals
transfer-int-nat-functions
transfer-int-nat-function-closures
transfer-int-nat-relations
]

first-order quantifiers

lemma *transfer-int-nat-quantifiers*:
(*ALL* (*x*::*int*) >= *0*. *P x*) = (*ALL* (*x*::*nat*). *P* (*int x*))
(*EX* (*x*::*int*) >= *0*. *P x*) = (*EX* (*x*::*nat*). *P* (*int x*))
apply (*subst all-nat*)
apply *auto* [*1*]
apply (*subst ex-nat*)
apply *auto*
done

declare *transfer-morphism-int-nat* [*transfer add*
return: *transfer-int-nat-quantifiers*]

if

lemma *int-if-cong*: (*if P then* (*int x*) *else* (*int y*)) =
int (*if P then x else y*)
by *auto*

declare *transfer-morphism-int-nat* [*transfer add return*: *int-if-cong*]

operations with sets

lemma *transfer-int-nat-set-functions*:
nat-set A $\Longrightarrow$ *card A* = *card* (*nat* ‘ *A*)
{} = *int* ‘ ({}::*nat set*)
nat-set A $\Longrightarrow$ *nat-set B* $\Longrightarrow$ *A Un B* = *int* ‘ (*nat* ‘ *A Un nat* ‘ *B*)

nat-set A $\Longrightarrow$ *nat-set B* $\Longrightarrow$ *A Int B = int ' (nat ' A Int nat ' B)*
{x. x >= 0 & P x} = int ' {x. P(int x)}

by (*simp-all only*: *is-nat-def transfer-nat-int-set-functions*
transfer-nat-int-set-function-closures
transfer-nat-int-set-return-embed nat-0-le
cong: *transfer-nat-int-set-cong*)

lemma *transfer-int-nat-set-function-closures*:
nat-set {}
nat-set A $\Longrightarrow$ *nat-set B* $\Longrightarrow$ *nat-set (A Un B)*
nat-set A $\Longrightarrow$ *nat-set B* $\Longrightarrow$ *nat-set (A Int B)*
nat-set {x. x >= 0 & P x}
nat-set (int ' C)
nat-set A $\Longrightarrow$ *x : A* $\Longrightarrow$ *is-nat x*
by (*simp-all only*: *transfer-nat-int-set-function-closures is-nat-def*)

lemma *transfer-int-nat-set-relations*:
nat-set A $\Longrightarrow$ *finite A = finite (nat ' A)*
is-nat x $\Longrightarrow$ *nat-set A* $\Longrightarrow$ *(x : A) = (nat x : nat ' A)*
nat-set A $\Longrightarrow$ *nat-set B* $\Longrightarrow$ *(A = B) = (nat ' A = nat ' B)*
nat-set A $\Longrightarrow$ *nat-set B* $\Longrightarrow$ *(A < B) = (nat ' A < nat ' B)*
nat-set A $\Longrightarrow$ *nat-set B* $\Longrightarrow$ *(A <= B) = (nat ' A <= nat ' B)*
by (*simp-all only*: *is-nat-def transfer-nat-int-set-relations*
transfer-nat-int-set-return-embed nat-0-le)

lemma *transfer-int-nat-set-return-embed*: *nat ' int ' A = A*
by (*simp only*: *transfer-nat-int-set-relations*
transfer-nat-int-set-function-closures
transfer-nat-int-set-return-embed nat-0-le)

lemma *transfer-int-nat-set-cong*: (!!*x*. *P x = P' x*) $\Longrightarrow$
{(x::nat). P x} = {x. P' x}
by *auto*

declare *transfer-morphism-int-nat* [*transfer add*
return: *transfer-int-nat-set-functions*
transfer-int-nat-set-function-closures
transfer-int-nat-set-relations
transfer-int-nat-set-return-embed
cong: *transfer-int-nat-set-cong*
]

setsum and setprod

lemma *transfer-int-nat-sum-prod*:
nat-set A $\Longrightarrow$ *setsum f A = setsum (%x. f (int x)) (nat ' A)*
nat-set A $\Longrightarrow$ *setprod f A = setprod (%x. f (int x)) (nat ' A)*
apply (*subst setsum-reindex*)
apply (*unfold inj-on-def nat-set-def*, *auto simp add*: *eq-nat-nat-iff*)

apply (*subst setprod-reindex*)
apply (*unfold inj-on-def nat-set-def o-def*, *auto simp add*: *eq-nat-nat-iff*
cong: *setprod-cong*)
done

lemma *transfer-int-nat-sum-prod2*:
(!!*x*. *x*:*A* ⟹ *is-nat* (*f x*)) ⟹ *setsum f A* = *int*(*setsum* (%*x*. *nat* (*f x*)) *A*)
(!!*x*. *x*:*A* ⟹ *is-nat* (*f x*)) ⟹
setprod f A = *int*(*setprod* (%*x*. *nat* (*f x*)) *A*)
unfolding *is-nat-def*
apply (*subst int-setsum*, *auto*)
apply (*subst int-setprod*, *auto simp add*: *cong*: *setprod-cong*)
done

declare *transfer-morphism-int-nat* [*transfer add*
return: *transfer-int-nat-sum-prod transfer-int-nat-sum-prod2*
cong: *setsum-cong setprod-cong*]

end

33 Divides: The division operators div and mod

theory *Divides*
imports *Nat-Numeral Nat-Transfer*
uses ~~/*src*/*Provers*/*Arith*/*cancel-div-mod.ML*
begin

33.1 Syntactic division operations

class *div* = *dvd* +
fixes *div* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** *div 70*)
and *mod* :: *'a* ⇒ *'a* ⇒ *'a* (**infixl** *mod 70*)

33.2 Abstract division in commutative semirings.

class *semiring-div* = *comm-semiring-1-cancel* + *no-zero-divisors* + *div* +
assumes *mod-div-equality*: *a div b* ∗ *b* + *a mod b* = *a*
and *div-by-0* [*simp*]: *a div 0* = *0*
and *div-0* [*simp*]: *0 div a* = *0*
and *div-mult-self1* [*simp*]: *b* ≠ *0* ⟹ (*a* + *c* ∗ *b*) *div b* = *c* + *a div b*
and *div-mult-mult1* [*simp*]: *c* ≠ *0* ⟹ (*c* ∗ *a*) *div* (*c* ∗ *b*) = *a div b*
begin

op div and *op mod*

lemma *mod-div-equality2*: *b* ∗ (*a div b*) + *a mod b* = *a*
unfolding *mult-commute* [*of b*]
by (*rule mod-div-equality*)

lemma *mod-div-equality'*: *a mod b + a div b * b = a*
using *mod-div-equality* [*of a b*]
by (*simp only*: *add-ac*)

lemma *div-mod-equality*: *((a div b) * b + a mod b) + c = a + c*
by (*simp add*: *mod-div-equality*)

lemma *div-mod-equality2*: *(b * (a div b) + a mod b) + c = a + c*
by (*simp add*: *mod-div-equality2*)

lemma *mod-by-0* [*simp*]: *a mod 0 = a*
using *mod-div-equality* [*of a zero*] **by** *simp*

lemma *mod-0* [*simp*]: *0 mod a = 0*
using *mod-div-equality* [*of zero a*] *div-0* **by** *simp*

lemma *div-mult-self2* [*simp*]:
assumes *b ≠ 0*
shows *(a + b * c) div b = c + a div b*
using *assms div-mult-self1* [*of b a c*] **by** (*simp add*: *mult-commute*)

lemma *mod-mult-self1* [*simp*]: *(a + c * b) mod b = a mod b*
proof (*cases b = 0*)
case *True* **then show** *?thesis* **by** *simp*
next
case *False*
have *a + c * b = (a + c * b) div b * b + (a + c * b) mod b*
by (*simp add*: *mod-div-equality*)
also from *False div-mult-self1* [*of b a c*] **have**
*... = (c + a div b) * b + (a + c * b) mod b*
by (*simp add*: *algebra-simps*)
finally have *a = a div b * b + (a + c * b) mod b*
by (*simp add*: *add-commute* [*of a*] *add-assoc left-distrib*)
then have *a div b * b + (a + c * b) mod b = a div b * b + a mod b*
by (*simp add*: *mod-div-equality*)
then show *?thesis* **by** *simp*
qed

lemma *mod-mult-self2* [*simp*]: *(a + b * c) mod b = a mod b*
by (*simp add*: *mult-commute* [*of b*])

lemma *div-mult-self1-is-id* [*simp*]: *b ≠ 0 ⟹ b * a div b = a*
using *div-mult-self2* [*of b 0 a*] **by** *simp*

lemma *div-mult-self2-is-id* [*simp*]: *b ≠ 0 ⟹ a * b div b = a*
using *div-mult-self1* [*of b 0 a*] **by** *simp*

lemma *mod-mult-self1-is-0* [*simp*]: *b * a mod b = 0*

using *mod-mult-self2* [*of 0 b a*] **by** *simp*

lemma *mod-mult-self2-is-0* [*simp*]: *a * b mod b = 0*
using *mod-mult-self1* [*of 0 a b*] **by** *simp*

lemma *div-by-1* [*simp*]: *a div 1 = a*
using *div-mult-self2-is-id* [*of 1 a*] *zero-neq-one* **by** *simp*

lemma *mod-by-1* [*simp*]: *a mod 1 = 0*
proof −
from *mod-div-equality* [*of a one*] *div-by-1* **have** *a + a mod 1 = a* **by** *simp*
then have *a + a mod 1 = a + 0* **by** *simp*
then show *?thesis* **by** (*rule add-left-imp-eq*)
qed

lemma *mod-self* [*simp*]: *a mod a = 0*
using *mod-mult-self2-is-0* [*of 1*] **by** *simp*

lemma *div-self* [*simp*]: *a ≠ 0 ⟹ a div a = 1*
using *div-mult-self2-is-id* [*of - 1*] **by** *simp*

lemma *div-add-self1* [*simp*]:
assumes *b ≠ 0*
shows *(b + a) div b = a div b + 1*
using *assms div-mult-self1* [*of b a 1*] **by** (*simp add*: *add-commute*)

lemma *div-add-self2* [*simp*]:
assumes *b ≠ 0*
shows *(a + b) div b = a div b + 1*
using *assms div-add-self1* [*of b a*] **by** (*simp add*: *add-commute*)

lemma *mod-add-self1* [*simp*]:
(b + a) mod b = a mod b
using *mod-mult-self1* [*of a 1 b*] **by** (*simp add*: *add-commute*)

lemma *mod-add-self2* [*simp*]:
(a + b) mod b = a mod b
using *mod-mult-self1* [*of a 1 b*] **by** *simp*

lemma *mod-div-decomp*:
fixes *a b*
obtains *q r* **where** *q = a div b* **and** *r = a mod b*
and *a = q * b + r*
proof −
from *mod-div-equality* **have** *a = a div b * b + a mod b* **by** *simp*
moreover have *a div b = a div b* **..**
moreover have *a mod b = a mod b* **..**
note *that* **ultimately show** *thesis* **by** *blast*
qed

lemma *dvd-eq-mod-eq-0* [*code*, *code-unfold*, *code-inline del*]: *a dvd b* ⟷ *b mod a = 0*
proof
  assume *b mod a = 0*
  with *mod-div-equality* [*of b a*] **have** *b div a * a = b* **by** *simp*
  then have *b = a * (b div a)* **unfolding** *mult-commute* **..**
  then have ∃ *c*. *b = a * c* **..**
  then show *a dvd b* **unfolding** *dvd-def* **.**
next
  assume *a dvd b*
  then have ∃ *c*. *b = a * c* **unfolding** *dvd-def* **.**
  then obtain *c* **where** *b = a * c* **..**
  then have *b mod a = a * c mod a* **by** *simp*
  then have *b mod a = c * a mod a* **by** (*simp add*: *mult-commute*)
  then show *b mod a = 0* **by** *simp*
qed

lemma *mod-div-trivial* [*simp*]: *a mod b div b = 0*
proof (*cases b = 0*)
  assume *b = 0*
  thus *?thesis* **by** *simp*
next
  assume *b ≠ 0*
  hence *a div b + a mod b div b = (a mod b + a div b * b) div b*
    by (*rule div-mult-self1* [*symmetric*])
  also have *... = a div b*
    by (*simp only*: *mod-div-equality'*)
  also have *... = a div b + 0*
    by *simp*
  finally show *?thesis*
    by (*rule add-left-imp-eq*)
qed

lemma *mod-mod-trivial* [*simp*]: *a mod b mod b = a mod b*
proof −
  have *a mod b mod b = (a mod b + a div b * b) mod b*
    by (*simp only*: *mod-mult-self1*)
  also have *... = a mod b*
    by (*simp only*: *mod-div-equality'*)
  finally show *?thesis* **.**
qed

lemma *dvd-imp-mod-0*: *a dvd b* ⟹ *b mod a = 0*
by (*rule dvd-eq-mod-eq-0*[*THEN iffD1*])

lemma *dvd-div-mult-self*: *a dvd b* ⟹ *(b div a) * a = b*
by (*subst* (*2*) *mod-div-equality* [*of b a*, *symmetric*]) (*simp add*:*dvd-imp-mod-0*)

lemma *dvd-mult-div-cancel*: *a dvd b* $\Longrightarrow$ *a* $*$ (*b div a*) = *b*
by (*drule dvd-div-mult-self*) (*simp add*: *mult-commute*)

lemma *dvd-div-mult*: *a dvd b* $\Longrightarrow$ (*b div a*) $*$ *c* = *b* $*$ *c div a*
apply (*cases a* = *0*)
apply *simp*
apply (*auto simp*: *dvd-def mult-assoc*)
done

lemma *div-dvd-div*[*simp*]:
a dvd b $\Longrightarrow$ *a dvd c* $\Longrightarrow$ (*b div a dvd c div a*) = (*b dvd c*)
apply (*cases a* = *0*)
apply *simp*
apply (*unfold dvd-def*)
apply *auto*
apply(*blast intro*:*mult-assoc*[*symmetric*])
apply(*fastsimp simp add*: *mult-assoc*)
done

lemma *dvd-mod-imp-dvd*: [| *k dvd m mod n*; *k dvd n* |] ==> *k dvd m*
apply (*subgoal-tac k dvd* (*m div n*) $*$*n* + *m mod n*)
apply (*simp add*: *mod-div-equality*)
apply (*simp only*: *dvd-add dvd-mult*)
done

Addition respects modular equivalence.

lemma *mod-add-left-eq*: (*a* + *b*) *mod c* = (*a mod c* + *b*) *mod c*
proof −
have (*a* + *b*) *mod c* = (*a div c* $*$ *c* + *a mod c* + *b*) *mod c*
by (*simp only*: *mod-div-equality*)
also have ... = (*a mod c* + *b* + *a div c* $*$ *c*) *mod c*
by (*simp only*: *add-ac*)
also have ... = (*a mod c* + *b*) *mod c*
by (*rule mod-mult-self1*)
finally show *?thesis* **.**
qed

lemma *mod-add-right-eq*: (*a* + *b*) *mod c* = (*a* + *b mod c*) *mod c*
proof −
have (*a* + *b*) *mod c* = (*a* + (*b div c* $*$ *c* + *b mod c*)) *mod c*
by (*simp only*: *mod-div-equality*)
also have ... = (*a* + *b mod c* + *b div c* $*$ *c*) *mod c*
by (*simp only*: *add-ac*)
also have ... = (*a* + *b mod c*) *mod c*
by (*rule mod-mult-self1*)
finally show *?thesis* **.**
qed

lemma *mod-add-eq*: (*a* + *b*) *mod c* = (*a mod c* + *b mod c*) *mod c*

by (*rule trans* [*OF mod-add-left-eq mod-add-right-eq*])

lemma *mod-add-cong*:
assumes $a \bmod c = a' \bmod c$
assumes $b \bmod c = b' \bmod c$
shows $(a + b) \bmod c = (a' + b') \bmod c$
proof −
have $(a \bmod c + b \bmod c) \bmod c = (a' \bmod c + b' \bmod c) \bmod c$
unfolding *assms* **..**
thus *?thesis*
by (*simp only*: *mod-add-eq* [*symmetric*])
qed

lemma *div-add* [*simp*]: *z dvd x* $\Longrightarrow$ *z dvd y*
$\Longrightarrow (x + y)$ *div z* $= x$ *div z* $+ y$ *div z*
by (*cases* $z = 0$, *simp*, *unfold dvd-def*, *auto simp add*: *algebra-simps*)

Multiplication respects modular equivalence.

lemma *mod-mult-left-eq*: $(a * b) \bmod c = ((a \bmod c) * b) \bmod c$
proof −
have $(a * b) \bmod c = ((a \text{ div } c * c + a \bmod c) * b) \bmod c$
by (*simp only*: *mod-div-equality*)
also have $\ldots = (a \bmod c * b + a \text{ div } c * b * c) \bmod c$
by (*simp only*: *algebra-simps*)
also have $\ldots = (a \bmod c * b) \bmod c$
by (*rule mod-mult-self1*)
finally show *?thesis* **.**
qed

lemma *mod-mult-right-eq*: $(a * b) \bmod c = (a * (b \bmod c)) \bmod c$
proof −
have $(a * b) \bmod c = (a * (b \text{ div } c * c + b \bmod c)) \bmod c$
by (*simp only*: *mod-div-equality*)
also have $\ldots = (a * (b \bmod c) + a * (b \text{ div } c) * c) \bmod c$
by (*simp only*: *algebra-simps*)
also have $\ldots = (a * (b \bmod c)) \bmod c$
by (*rule mod-mult-self1*)
finally show *?thesis* **.**
qed

lemma *mod-mult-eq*: $(a * b) \bmod c = ((a \bmod c) * (b \bmod c)) \bmod c$
by (*rule trans* [*OF mod-mult-left-eq mod-mult-right-eq*])

lemma *mod-mult-cong*:
assumes $a \bmod c = a' \bmod c$
assumes $b \bmod c = b' \bmod c$
shows $(a * b) \bmod c = (a' * b') \bmod c$
proof −
have $(a \bmod c * (b \bmod c)) \bmod c = (a' \bmod c * (b' \bmod c)) \bmod c$

unfolding *assms* **..**
thus *?thesis*
by (*simp only*: *mod-mult-eq* [*symmetric*])
qed

lemma *mod-mod-cancel*:
assumes *c dvd b*
shows *a mod b mod c = a mod c*
proof −
from ‹*c dvd b*› **obtain** *k* **where** *b = c * k*
by (*rule dvdE*)
have *a mod b mod c = a mod (c * k) mod c*
by (*simp only*: ‹*b = c * k*›)
also have *... = (a mod (c * k) + a div (c * k) * k * c) mod c*
by (*simp only*: *mod-mult-self1*)
also have *... = (a div (c * k) * (c * k) + a mod (c * k)) mod c*
by (*simp only*: *add-ac mult-ac*)
also have *... = a mod c*
by (*simp only*: *mod-div-equality*)
finally show *?thesis* **.**
qed

lemma *div-mult-div-if-dvd*:
*y dvd x ⟹ z dvd w ⟹ (x div y) * (w div z) = (x * w) div (y * z)*
apply (*cases y = 0*, *simp*)
apply (*cases z = 0*, *simp*)
apply (*auto elim*!: *dvdE simp add*: *algebra-simps*)
apply (*subst mult-assoc* [*symmetric*])
apply (*simp add*: *no-zero-divisors*)
done

lemma *div-mult-swap*:
assumes *c dvd b*
shows *a * (b div c) = (a * b) div c*
proof −
from *assms* **have** *b div c * (a div 1) = b * a div (c * 1)*
by (*simp only*: *div-mult-div-if-dvd one-dvd*)
then show *?thesis* **by** (*simp add*: *mult-commute*)
qed

lemma *div-mult-mult2* [*simp*]:
*c ≠ 0 ⟹ (a * c) div (b * c) = a div b*
by (*drule div-mult-mult1*) (*simp add*: *mult-commute*)

lemma *div-mult-mult1-if* [*simp*]:
*(c * a) div (c * b) = (if c = 0 then 0 else a div b)*
by *simp-all*

lemma *mod-mult-mult1*:

```
  (c * a) mod (c * b) = c * (a mod b)
proof (cases c = 0)
  case True then show ?thesis by simp
next
  case False
  from mod-div-equality
  have ((c * a) div (c * b)) * (c * b) + (c * a) mod (c * b) = c * a .
  with False have c * ((a div b) * b + a mod b) + (c * a) mod (c * b)
    = c * a + c * (a mod b) by (simp add: algebra-simps)
  with mod-div-equality show ?thesis by simp
qed

lemma mod-mult-mult2:
  (a * c) mod (b * c) = (a mod b) * c
  using mod-mult-mult1 [of c a b] by (simp add: mult-commute)

lemma dvd-mod: k dvd m ⟹ k dvd n ⟹ k dvd (m mod n)
  unfolding dvd-def by (auto simp add: mod-mult-mult1)

lemma dvd-mod-iff: k dvd n ⟹ k dvd (m mod n) ⟷ k dvd m
by (blast intro: dvd-mod-imp-dvd dvd-mod)

lemma div-power:
  y dvd x ⟹ (x div y) ^ n = x ^ n div y ^ n
apply (induct n)
 apply simp
apply(simp add: div-mult-div-if-dvd dvd-power-same)
done

lemma dvd-div-eq-mult:
  assumes a ≠ 0 and a dvd b
  shows b div a = c ⟷ b = c * a
proof
  assume b = c * a
  then show b div a = c by (simp add: assms)
next
  assume b div a = c
  then have b div a * a = c * a by simp
  moreover from ‹a dvd b› have b div a * a = b by (simp add: dvd-div-mult-self)
  ultimately show b = c * a by simp
qed

lemma dvd-div-div-eq-mult:
  assumes a ≠ 0 c ≠ 0 and a dvd b c dvd d
  shows b div a = d div c ⟷ b * c = a * d
 using assms by (auto simp add: mult-commute [of - a] dvd-div-mult-self dvd-div-eq-mult
div-mult-swap intro: sym)

end
```

class *ring-div* = *semiring-div* + *comm-ring-1*
begin

subclass *ring-1-no-zero-divisors* **..**

Negation respects modular equivalence.

lemma *mod-minus-eq*: $(-\ a)\ mod\ b = (-\ (a\ mod\ b))\ mod\ b$
proof −
  have $(-\ a)\ mod\ b = (-\ (a\ div\ b * b + a\ mod\ b))\ mod\ b$
    by (*simp only*: *mod-div-equality*)
  also have $\ldots = (-\ (a\ mod\ b) + -\ (a\ div\ b) * b)\ mod\ b$
    by (*simp only*: *minus-add-distrib minus-mult-left add-ac*)
  also have $\ldots = (-\ (a\ mod\ b))\ mod\ b$
    by (*rule mod-mult-self1*)
  finally show *?thesis* **.**
qed

lemma *mod-minus-cong*:
  assumes $a\ mod\ b = a'\ mod\ b$
  shows $(-\ a)\ mod\ b = (-\ a')\ mod\ b$
proof −
  have $(-\ (a\ mod\ b))\ mod\ b = (-\ (a'\ mod\ b))\ mod\ b$
    unfolding *assms* **..**
  thus *?thesis*
    by (*simp only*: *mod-minus-eq* [*symmetric*])
qed

Subtraction respects modular equivalence.

lemma *mod-diff-left-eq*: $(a - b)\ mod\ c = (a\ mod\ c - b)\ mod\ c$
  unfolding *diff-minus*
  by (*intro mod-add-cong mod-minus-cong*) *simp-all*

lemma *mod-diff-right-eq*: $(a - b)\ mod\ c = (a - b\ mod\ c)\ mod\ c$
  unfolding *diff-minus*
  by (*intro mod-add-cong mod-minus-cong*) *simp-all*

lemma *mod-diff-eq*: $(a - b)\ mod\ c = (a\ mod\ c - b\ mod\ c)\ mod\ c$
  unfolding *diff-minus*
  by (*intro mod-add-cong mod-minus-cong*) *simp-all*

lemma *mod-diff-cong*:
  assumes $a\ mod\ c = a'\ mod\ c$
  assumes $b\ mod\ c = b'\ mod\ c$
  shows $(a - b)\ mod\ c = (a' - b')\ mod\ c$
  unfolding *diff-minus* **using** *assms*
  by (*intro mod-add-cong mod-minus-cong*)

lemma *dvd-neg-div*: $y\ dvd\ x \Longrightarrow -x\ div\ y = -\ (x\ div\ y)$

apply (*case-tac* $y = 0$) **apply** *simp*
apply (*auto simp add*: *dvd-def*)
apply (*subgoal-tac* $-(y * k) = y * - k$)
apply (*erule ssubst*)
apply (*erule div-mult-self1-is-id*)
apply *simp*
done

lemma *dvd-div-neg*: *y dvd x* $\Longrightarrow$ *x div* $-y = -$ (*x div y*)
apply (*case-tac* $y = 0$) **apply** *simp*
apply (*auto simp add*: *dvd-def*)
apply (*subgoal-tac* $y * k = -y * -k$)
apply (*erule ssubst*)
apply (*rule div-mult-self1-is-id*)
apply *simp*
apply *simp*
done

end

33.3 Division on *nat*

We define *op div* and *op mod* on *nat* by means of a characteristic relation with two input arguments m, n and two output arguments q(uotient) and r(emainder).

definition *divmod-nat-rel* :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat* $\times$ *nat* $\Rightarrow$ *bool* **where**
divmod-nat-rel m n qr $\longleftrightarrow$
m = *fst qr* $*$ *n* + *snd qr* $\wedge$
(*if n* = *0 then fst qr* = *0 else if n* $>$ *0 then* 0 $\leq$ *snd qr* $\wedge$ *snd qr* $<$ *n else n* $<$ *snd qr* $\wedge$ *snd qr* $\leq$ *0*)

divmod-nat-rel is total:

lemma *divmod-nat-rel-ex*:
obtains *q r* **where** *divmod-nat-rel m n* (*q*, *r*)
proof (*cases* $n = 0$)
case *True* **with** *that* **show** *thesis*
by (*auto simp add*: *divmod-nat-rel-def*)
next
case *False*
have $\exists\, q\ r.\ m = q * n + r \wedge r < n$
proof (*induct m*)
case *0* **with** ‹$n \neq 0$›
have (*0*::*nat*) = *0* $*$ *n* + *0* $\wedge$ *0* $<$ *n* **by** *simp*
then show *?case* **by** *blast*
next
case (*Suc m*) **then obtain** $q'\ r'$
where *m*: $m = q' * n + r'$ **and** *n*: $r' < n$ **by** *auto*
then show *?case* **proof** (*cases Suc* $r' < n$)

```
      case True
      from m n have Suc m = q' * n + Suc r' by simp
      with True show ?thesis by blast
    next
      case False then have n ≤ Suc r' by auto
      moreover from n have Suc r' ≤ n by auto
      ultimately have n = Suc r' by auto
      with m have Suc m = Suc q' * n + 0 by simp
      with ‹n ≠ 0› show ?thesis by blast
    qed
  qed
  with that show thesis
    using ‹n ≠ 0› by (auto simp add: divmod-nat-rel-def)
qed
```

divmod-nat-rel is injective:

```
lemma divmod-nat-rel-unique:
  assumes divmod-nat-rel m n qr
    and divmod-nat-rel m n qr'
  shows qr = qr'
proof (cases n = 0)
  case True with assms show ?thesis
    by (cases qr, cases qr')
      (simp add: divmod-nat-rel-def)
next
  case False
  have aux: ⋀q r q' r'. q' * n + r' = q * n + r ⟹ r < n ⟹ q' ≤ (q::nat)
  apply (rule leI)
  apply (subst less-iff-Suc-add)
  apply (auto simp add: add-mult-distrib)
  done
  from ‹n ≠ 0› assms have fst qr = fst qr'
    by (auto simp add: divmod-nat-rel-def intro: order-antisym dest: aux sym)
  moreover from this assms have snd qr = snd qr'
    by (simp add: divmod-nat-rel-def)
  ultimately show ?thesis by (cases qr, cases qr') simp
qed
```

We instantiate divisibility on the natural numbers by means of *divmod-nat-rel*:

```
instantiation nat :: semiring-div
begin

definition divmod-nat :: nat ⇒ nat ⇒ nat × nat where
  [code del]: divmod-nat m n = (THE qr. divmod-nat-rel m n qr)

lemma divmod-nat-rel-divmod-nat:
  divmod-nat-rel m n (divmod-nat m n)
proof −
  from divmod-nat-rel-ex
```

```
    obtain qr where rel: divmod-nat-rel m n qr .
  then show ?thesis
  by (auto simp add: divmod-nat-def intro: theI elim: divmod-nat-rel-unique)
qed

lemma divmod-nat-eq:
  assumes divmod-nat-rel m n qr
  shows divmod-nat m n = qr
  using assms by (auto intro: divmod-nat-rel-unique divmod-nat-rel-divmod-nat)

definition div-nat where
  m div n = fst (divmod-nat m n)

definition mod-nat where
  m mod n = snd (divmod-nat m n)

lemma divmod-nat-div-mod:
  divmod-nat m n = (m div n, m mod n)
  unfolding div-nat-def mod-nat-def by simp

lemma div-eq:
  assumes divmod-nat-rel m n (q, r)
  shows m div n = q
  using assms by (auto dest: divmod-nat-eq simp add: divmod-nat-div-mod)

lemma mod-eq:
  assumes divmod-nat-rel m n (q, r)
  shows m mod n = r
  using assms by (auto dest: divmod-nat-eq simp add: divmod-nat-div-mod)

lemma divmod-nat-rel: divmod-nat-rel m n (m div n, m mod n)
  by (simp add: div-nat-def mod-nat-def divmod-nat-rel-divmod-nat)

lemma divmod-nat-zero:
  divmod-nat m 0 = (0, m)
proof −
  from divmod-nat-rel [of m 0] show ?thesis
    unfolding divmod-nat-div-mod divmod-nat-rel-def by simp
qed

lemma divmod-nat-base:
  assumes m < n
  shows divmod-nat m n = (0, m)
proof −
  from divmod-nat-rel [of m n] show ?thesis
    unfolding divmod-nat-div-mod divmod-nat-rel-def
    using assms by (cases m div n = 0)
      (auto simp add: gr0-conv-Suc [of m div n])
qed
```

lemma *divmod-nat-step*:
assumes $0 < n$ **and** $n \leq m$
shows *divmod-nat* $m\ n = (Suc\ ((m - n)\ div\ n), (m - n)\ mod\ n)$
proof −
from *divmod-nat-rel* **have** *divmod-nat-m-n*: *divmod-nat-rel* $m\ n\ (m\ div\ n, m\ mod\ n)$ **.**
with *assms* **have** *m-div-n*: $m\ div\ n \geq 1$
by (*cases* $m\ div\ n$) (*auto simp add*: *divmod-nat-rel-def*)
have *divmod-nat-rel* $(m - n)\ n\ (m\ div\ n - Suc\ 0, m\ mod\ n)$
proof −
from *assms* **have**
$n \neq 0$
$\bigwedge k.\ m = Suc\ k * n + m\ mod\ n \Longrightarrow m - n = (Suc\ k - Suc\ 0) * n + m\ mod\ n$
by *simp-all*
then show *?thesis* **using** *assms divmod-nat-m-n*
by (*cases* $m\ div\ n$)
(*simp-all only*: *divmod-nat-rel-def fst-conv snd-conv*, *simp-all*)
qed
with *divmod-nat-eq* **have** *divmod-nat* $(m - n)\ n = (m\ div\ n - Suc\ 0, m\ mod\ n)$ **by** *simp*
moreover from *divmod-nat-div-mod* **have** *divmod-nat* $(m - n)\ n = ((m - n)\ div\ n, (m - n)\ mod\ n)$ **.**
ultimately have $m\ div\ n = Suc\ ((m - n)\ div\ n)$
and $m\ mod\ n = (m - n)\ mod\ n$ **using** *m-div-n* **by** *simp-all*
then show *?thesis* **using** *divmod-nat-div-mod* **by** *simp*
qed

The "recursion" equations for *op div* and *op mod*

lemma *div-less* [*simp*]:
fixes $m\ n :: nat$
assumes $m < n$
shows $m\ div\ n = 0$
using *assms divmod-nat-base divmod-nat-div-mod* **by** *simp*

lemma *le-div-geq*:
fixes $m\ n :: nat$
assumes $0 < n$ **and** $n \leq m$
shows $m\ div\ n = Suc\ ((m - n)\ div\ n)$
using *assms divmod-nat-step divmod-nat-div-mod* **by** *simp*

lemma *mod-less* [*simp*]:
fixes $m\ n :: nat$
assumes $m < n$
shows $m\ mod\ n = m$
using *assms divmod-nat-base divmod-nat-div-mod* **by** *simp*

lemma *le-mod-geq*:

fixes $m\ n :: nat$
assumes $n \leq m$
shows $m\ mod\ n = (m - n)\ mod\ n$
using *assms divmod-nat-step divmod-nat-div-mod* **by** (*cases* $n = 0$) *simp-all*

instance proof −
have [*simp*]: $\bigwedge n$::*nat*. $n\ div\ 0 = 0$
by (*simp add*: *div-nat-def divmod-nat-zero*)
have [*simp*]: $\bigwedge n$::*nat*. $0\ div\ n = 0$
proof −
fix $n :: nat$
show $0\ div\ n = 0$
by (*cases* $n = 0$) *simp-all*
qed
show *OFCLASS*(*nat*, *semiring-div-class*) **proof**
fix $m\ n :: nat$
show $m\ div\ n * n + m\ mod\ n = m$
using *divmod-nat-rel* [*of m n*] **by** (*simp add*: *divmod-nat-rel-def*)
next
fix $m\ n\ q :: nat$
assume $n \neq 0$
then show $(q + m * n)\ div\ n = m + q\ div\ n$
by (*induct m*) (*simp-all add*: *le-div-geq*)
next
fix $m\ n\ q :: nat$
assume $m \neq 0$
then show $(m * n)\ div\ (m * q) = n\ div\ q$
proof (*cases* $n \neq 0 \wedge q \neq 0$)
case *False* **then show** *?thesis* **by** *auto*
next
case *True* **with** ‹$m \neq 0$›
have $m > 0$ **and** $n > 0$ **and** $q > 0$ **by** *auto*
then have $\bigwedge a\ b$. *divmod-nat-rel n q* $(a, b) \Longrightarrow$ *divmod-nat-rel* $(m * n)$ $(m * q)$ $(a, m * b)$
by (*auto simp add*: *divmod-nat-rel-def*) (*simp-all add*: *algebra-simps*)
moreover from *divmod-nat-rel* **have** *divmod-nat-rel n q* $(n\ div\ q, n\ mod\ q)$ **.**
ultimately have *divmod-nat-rel* $(m * n)$ $(m * q)$ $(n\ div\ q, m * (n\ mod\ q))$ **.**
then show *?thesis* **by** (*simp add*: *div-eq*)
qed
qed *simp-all*
qed

end

lemma *divmod-nat-if* [*code*]: *divmod-nat m n* = (*if* $n = 0 \vee m < n$ *then* $(0, m)$ *else*
let (q, r) = *divmod-nat* $(m - n)\ n$ *in* $(Suc\ q, r)$)
by (*simp add*: *divmod-nat-zero divmod-nat-base divmod-nat-step*)
(*simp add*: *divmod-nat-div-mod*)

Simproc for cancelling *op div* and *op mod*

ML ⟪
local

structure CancelDivMod = CancelDivModFun(struct

val div-name = @{const-name div};
val mod-name = @{const-name mod};
val mk-binop = HOLogic.mk-binop;
val mk-sum = Nat-Arith.mk-sum;
val dest-sum = Nat-Arith.dest-sum;

val div-mod-eqs = map mk-meta-eq [@{thm div-mod-equality}, @{thm div-mod-equality2}];

val trans = trans;

val prove-eq-sums = Arith-Data.prove-conv2 all-tac (Arith-Data.simp-all-tac
(@{thm add-0-left} :: @{thm add-0-right} :: @{thms add-ac}))

end)

in

val cancel-div-mod-nat-proc = Simplifier.simproc @{theory}
cancel-div-mod [(m::nat) + n] (K CancelDivMod.proc);

val - = Addsimprocs [cancel-div-mod-nat-proc];

end
⟫

33.3.1 Quotient

lemma *div-geq*: $0 < n \Longrightarrow \neg\ m < n \Longrightarrow m\ div\ n = Suc\ ((m - n)\ div\ n)$
by (*simp add*: *le-div-geq linorder-not-less*)

lemma *div-if*: $0 < n \Longrightarrow m\ div\ n = (if\ m < n\ then\ 0\ else\ Suc\ ((m - n)\ div\ n))$
by (*simp add*: *div-geq*)

lemma *div-mult-self-is-m* [*simp*]: $0<n ==> (m*n)\ div\ n = (m::nat)$
by *simp*

lemma *div-mult-self1-is-m* [*simp*]: $0<n ==> (n*m)\ div\ n = (m::nat)$
by *simp*

33.3.2 Remainder

lemma *mod-less-divisor* [*simp*]:
fixes $m\ n :: nat$

assumes $n > 0$
shows $m\ mod\ n < (n::nat)$
using *assms divmod-nat-rel* [*of m n*] **unfolding** *divmod-nat-rel-def* **by** *auto*

lemma *mod-less-eq-dividend* [*simp*]:
fixes $m\ n :: nat$
shows $m\ mod\ n \leq m$
proof (*rule add-leD2*)
from *mod-div-equality* **have** $m\ div\ n * n + m\ mod\ n = m$ **.**
then show $m\ div\ n * n + m\ mod\ n \leq m$ **by** *auto*
qed

lemma *mod-geq*: $\neg\ m < (n::nat) \Longrightarrow m\ mod\ n = (m - n)\ mod\ n$
by (*simp add*: *le-mod-geq linorder-not-less*)

lemma *mod-if*: $m\ mod\ (n::nat) = (if\ m < n\ then\ m\ else\ (m - n)\ mod\ n)$
by (*simp add*: *le-mod-geq*)

lemma *mod-1* [*simp*]: $m\ mod\ Suc\ 0 = 0$
by (*induct m*) (*simp-all add*: *mod-geq*)

lemma *mod-mult-distrib*: $(m\ mod\ n) * (k::nat) = (m * k)\ mod\ (n * k)$
apply (*cases* $n = 0$, *simp*)
apply (*cases* $k = 0$, *simp*)
apply (*induct m rule*: *nat-less-induct*)
apply (*subst mod-if*, *simp*)
apply (*simp add*: *mod-geq diff-mult-distrib*)
done

lemma *mod-mult-distrib2*: $(k::nat) * (m\ mod\ n) = (k*m)\ mod\ (k*n)$
by (*simp add*: *mult-commute* [*of k*] *mod-mult-distrib*)

lemma *mult-div-cancel*: $(n::nat) * (m\ div\ n) = m - (m\ mod\ n)$
by (*cut-tac* $a = m$ **and** $b = n$ **in** *mod-div-equality2*, *arith*)

lemma *mod-le-divisor*[*simp*]: $0 < n \Longrightarrow m\ mod\ n \leq (n::nat)$
apply (*drule mod-less-divisor* [**where** $m = m$])
apply *simp*
done

33.3.3 Quotient and Remainder

lemma *divmod-nat-rel-mult1-eq*:
$divmod\text{-}nat\text{-}rel\ b\ c\ (q, r) \Longrightarrow c > 0$
$\Longrightarrow divmod\text{-}nat\text{-}rel\ (a * b)\ c\ (a * q + a * r\ div\ c, a * r\ mod\ c)$
by (*auto simp add*: *split-ifs divmod-nat-rel-def algebra-simps*)

lemma *div-mult1-eq*:

$(a * b)$ *div* $c = a * (b$ *div* $c) + a * (b$ *mod* $c)$ *div* $(c::nat)$
apply (*cases* $c = 0$, *simp*)
apply (*blast intro*: *divmod-nat-rel* [*THEN divmod-nat-rel-mult1-eq*, *THEN div-eq*])
done

lemma *divmod-nat-rel-add1-eq*:
divmod-nat-rel $a\ c\ (aq, ar) \Longrightarrow$ *divmod-nat-rel* $b\ c\ (bq, br) \Longrightarrow\ c > 0$
$\Longrightarrow$ *divmod-nat-rel* $(a + b)\ c\ (aq + bq + (ar + br)$ *div* $c, (ar + br)$ *mod* $c)$
by (*auto simp add*: *split-ifs divmod-nat-rel-def algebra-simps*)

lemma *div-add1-eq*:
$(a+b)$ *div* $(c::nat) = a$ *div* $c + b$ *div* $c + ((a$ *mod* $c + b$ *mod* $c)$ *div* $c)$
apply (*cases* $c = 0$, *simp*)
apply (*blast intro*: *divmod-nat-rel-add1-eq* [*THEN div-eq*] *divmod-nat-rel*)
done

lemma *mod-lemma*: [| $(0::nat) < c$; $r < b$ |] ==> $b * (q$ *mod* $c) + r < b * c$
apply (*cut-tac* $m = q$ **and** $n = c$ **in** *mod-less-divisor*)
apply (*drule-tac* [2] $m = q$ *mod* c **in** *less-imp-Suc-add*, *auto*)
apply (*erule-tac* $P = \%x.$ *?lhs* $<$ *?rhs* x **in** *ssubst*)
apply (*simp add*: *add-mult-distrib2*)
done

lemma *divmod-nat-rel-mult2-eq*:
divmod-nat-rel $a\ b\ (q, r) \Longrightarrow 0 < b \Longrightarrow 0 < c$
$\Longrightarrow$ *divmod-nat-rel* $a\ (b * c)\ (q$ *div* $c, b *(q$ *mod* $c) + r)$
by (*auto simp add*: *mult-ac divmod-nat-rel-def add-mult-distrib2* [*symmetric*] *mod-lemma*)

lemma *div-mult2-eq*: a *div* $(b*c) = (a$ *div* $b)$ *div* $(c::nat)$
apply (*cases* $b = 0$, *simp*)
apply (*cases* $c = 0$, *simp*)
apply (*force simp add*: *divmod-nat-rel* [*THEN divmod-nat-rel-mult2-eq*, *THEN div-eq*])
done

lemma *mod-mult2-eq*: a *mod* $(b*c) = b*(a$ *div* b *mod* $c) + a$ *mod* $(b::nat)$
apply (*cases* $b = 0$, *simp*)
apply (*cases* $c = 0$, *simp*)
apply (*auto simp add*: *mult-commute divmod-nat-rel* [*THEN divmod-nat-rel-mult2-eq*, *THEN mod-eq*])
done

33.3.4 Further Facts about Quotient and Remainder

lemma *div-1* [*simp*]: m *div Suc* $0 = m$
by (*induct* m) (*simp-all add*: *div-geq*)

lemma *div-le-mono* [*rule-format* (*no-asm*)]:
  $\forall m{::}nat.\ m \leq n \longrightarrow (m\ div\ k) \leq (n\ div\ k)$
apply (*case-tac k=0*, *simp*)
apply (*induct n rule*: *nat-less-induct*, *clarify*)
apply (*case-tac n<k*)

apply *simp*

apply (*case-tac m<k*)

apply *simp*

apply (*simp add*: *div-geq diff-le-mono*)
done

lemma *div-le-mono2*: $!!m{::}nat.\ [|\ 0<m;\ m\leq n\ |] ==> (k\ div\ n) \leq (k\ div\ m)$
apply (*subgoal-tac 0<n*)
 prefer *2* **apply** *simp*
apply (*induct-tac k rule*: *nat-less-induct*)
apply (*rename-tac k*)
apply (*case-tac k<n*, *simp*)
apply (*subgoal-tac* ~ (*k<m*))
 prefer *2* **apply** *simp*
apply (*simp add*: *div-geq*)
apply (*subgoal-tac* (*k−n*) *div n* ≤ (*k−m*) *div n*)
 prefer *2*
 apply (*blast intro*: *div-le-mono diff-le-mono2*)
apply (*rule le-trans*, *simp*)
apply (*simp*)
done

lemma *div-le-dividend* [*simp*]: *m div n* ≤ (*m*::*nat*)
apply (*case-tac n=0*, *simp*)
apply (*subgoal-tac m div n* ≤ *m div 1*, *simp*)
apply (*rule div-le-mono2*)
apply (*simp-all* (*no-asm-simp*))
done

lemma *div-less-dividend* [*rule-format*]:
  $!!n{::}nat.\ 1<n ==> 0 < m \longrightarrow m\ div\ n < m$
apply (*induct-tac m rule*: *nat-less-induct*)
apply (*rename-tac m*)
apply (*case-tac m<n*, *simp*)
apply (*subgoal-tac 0<n*)
 prefer *2* **apply** *simp*
apply (*simp add*: *div-geq*)

apply (*case-tac n<m*)
apply (*subgoal-tac* (*m−n*) *div n* < (*m−n*))
apply (*rule impI less-trans-Suc*)+
apply *assumption*
apply (*simp-all*)
done

declare *div-less-dividend* [*simp*]

A fact for the mutilated chess board

lemma *mod-Suc*: *Suc*(*m*) *mod n* = (*if Suc*(*m mod n*) = *n then 0 else Suc*(*m mod n*))
apply (*case-tac n=0*, *simp*)
apply (*induct m rule*: *nat-less-induct*)
apply (*case-tac Suc* (*na*) <*n*)

apply (*frule lessI* [*THEN less-trans*], *simp add*: *less-not-refl3*)

apply (*simp add*: *linorder-not-less le-Suc-eq mod-geq*)
apply (*auto simp add*: *Suc-diff-le le-mod-geq*)
done

lemma *mod-eq-0-iff*: (*m mod d = 0*) = (∃ *q*::*nat*. *m* = *d*∗*q*)
by (*auto simp add*: *dvd-eq-mod-eq-0* [*symmetric*] *dvd-def*)

lemmas *mod-eq-0D* [*dest*!] = *mod-eq-0-iff* [*THEN iffD1*]

lemma *mod-eqD*: (*m mod d = r*) ==> ∃ *q*::*nat*. *m* = *r* + *q*∗*d*
apply (*cut-tac a* = *m* **in** *mod-div-equality*)
apply (*simp only*: *add-ac*)
apply (*blast intro*: *sym*)
done

lemma *split-div*:
P(*n div k* :: *nat*) =
((*k* = *0* ⟶ *P 0*) ∧ (*k* ≠ *0* ⟶ (!*i*. !*j*<*k*. *n* = *k*∗*i* + *j* ⟶ *P i*)))
(**is** *?P* = *?Q* **is** - = (- ∧ (- ⟶ *?R*)))
proof
assume *P*: *?P*
show *?Q*
proof (*cases*)
assume *k* = *0*
with *P* **show** *?Q* **by** *simp*
next
assume *not0*: *k* ≠ *0*
thus *?Q*
proof (*simp*, *intro allI impI*)
fix *i j*

```
      assume n: n = k*i + j and j: j < k
      show P i
      proof (cases)
        assume i = 0
        with n j P show P i by simp
      next
        assume i ≠ 0
        with not0 n j P show P i by(simp add:add-ac)
      qed
    qed
  qed
next
  assume Q: ?Q
  show ?P
  proof (cases)
    assume k = 0
    with Q show ?P by simp
  next
    assume not0: k ≠ 0
    with Q have R: ?R by simp
    from not0 R[THEN spec,of n div k,THEN spec, of n mod k]
    show ?P by simp
  qed
qed

lemma split-div-lemma:
  assumes 0 < n
  shows n * q ≤ m ∧ m < n * Suc q ⟷ q = ((m::nat) div n) (is ?lhs ⟷ ?rhs)
proof
  assume ?rhs
  with mult-div-cancel have nq: n * q = m - (m mod n) by simp
  then have A: n * q ≤ m by simp
  have n - (m mod n) > 0 using mod-less-divisor assms by auto
  then have m < m + (n - (m mod n)) by simp
  then have m < n + (m - (m mod n)) by simp
  with nq have m < n + n * q by simp
  then have B: m < n * Suc q by simp
  from A B show ?lhs ..
next
  assume P: ?lhs
  then have divmod-nat-rel m n (q, m - n * q)
    unfolding divmod-nat-rel-def by (auto simp add: mult-ac)
  with divmod-nat-rel-unique divmod-nat-rel [of m n]
  have (q, m - n * q) = (m div n, m mod n) by auto
  then show ?rhs by simp
qed

theorem split-div':
  P ((m::nat) div n) = ((n = 0 ∧ P 0) ∨
```

```
   (∃ q. (n * q ≤ m ∧ m < n * (Suc q)) ∧ P q))
  apply (case-tac 0 < n)
  apply (simp only: add: split-div-lemma)
  apply simp-all
  done

lemma split-mod:
 P(n mod k :: nat) =
 ((k = 0 ⟶ P n) ∧ (k ≠ 0 ⟶ (!i. !j<k. n = k*i + j ⟶ P j)))
 (is ?P = ?Q is - = (- ∧ (- ⟶ ?R)))
proof
  assume P: ?P
  show ?Q
  proof (cases)
    assume k = 0
    with P show ?Q by simp
  next
    assume not0: k ≠ 0
    thus ?Q
    proof (simp, intro allI impI)
      fix i j
      assume n = k*i + j j < k
      thus P j using not0 P by(simp add:add-ac mult-ac)
    qed
  qed
next
  assume Q: ?Q
  show ?P
  proof (cases)
    assume k = 0
    with Q show ?P by simp
  next
    assume not0: k ≠ 0
    with Q have R: ?R by simp
    from not0 R[THEN spec,of n div k,THEN spec, of n mod k]
    show ?P by simp
  qed
qed

theorem mod-div-equality': (m::nat) mod n = m - (m div n) * n
  apply (rule-tac P=%x. m mod n = x - (m div n) * n in
    subst [OF mod-div-equality [of - n]])
  apply arith
  done

lemma div-mod-equality':
  fixes m n :: nat
  shows m div n * n = m - m mod n
proof -
```

```
  have m mod n ≤ m mod n ..
  from div-mod-equality have
    m div n * n + m mod n - m mod n = m - m mod n by simp
  with diff-add-assoc [OF ‹m mod n ≤ m mod n›, of m div n * n] have
    m div n * n + (m mod n - m mod n) = m - m mod n
    by simp
  then show ?thesis by simp
qed
```

33.3.5 An "induction" law for modulus arithmetic.

```
lemma mod-induct-0:
  assumes step: ∀ i<p. P i ⟶ P ((Suc i) mod p)
  and base: P i and i: i<p
  shows P 0
proof (rule ccontr)
  assume contra: ¬(P 0)
  from i have p: 0<p by simp
  have ∀ k. 0<k ⟶ ¬ P (p-k) (is ∀ k. ?A k)
  proof
    fix k
    show ?A k
    proof (induct k)
      show ?A 0 by simp  — by contradiction
    next
      fix n
      assume ih: ?A n
      show ?A (Suc n)
      proof (clarsimp)
        assume y: P (p - Suc n)
        have n: Suc n < p
        proof (rule ccontr)
          assume ¬(Suc n < p)
          hence p - Suc n = 0
            by simp
          with y contra show False
            by simp
        qed
        hence n2: Suc (p - Suc n) = p-n by arith
        from p have p - Suc n < p by arith
        with y step have z: P ((Suc (p - Suc n)) mod p)
          by blast
        show False
        proof (cases n=0)
          case True
          with z n2 contra show ?thesis by simp
        next
          case False
          with p have p-n < p by arith
```

```
        with z n2 False ih show ?thesis by simp
       qed
      qed
    qed
  qed
  moreover
  from i obtain k where 0<k ∧ i+k=p
    by (blast dest: less-imp-add-positive)
  hence 0<k ∧ i=p−k by auto
  moreover
  note base
  ultimately
  show False by blast
qed

lemma mod-induct:
  assumes step: ∀ i<p. P i ⟶ P ((Suc i) mod p)
  and base: P i and i: i<p and j: j<p
  shows P j
proof −
  have ∀ j<p. P j
  proof
    fix j
    show j<p ⟶ P j (is ?A j)
    proof (induct j)
      from step base i show ?A 0
        by (auto elim: mod-induct-0)
    next
      fix k
      assume ih: ?A k
      show ?A (Suc k)
      proof
        assume suc: Suc k < p
        hence k: k<p by simp
        with ih have P k ..
        with step k have P (Suc k mod p)
          by blast
        moreover
        from suc have Suc k mod p = Suc k
          by simp
        ultimately
        show P (Suc k) by simp
      qed
    qed
  qed
  with j show ?thesis by blast
qed

lemma div2-Suc-Suc [simp]: Suc (Suc m) div 2 = Suc (m div 2)
```

by (*auto simp add*: *numeral-2-eq-2 le-div-geq*)

lemma *add-self-div-2* [*simp*]: (*m* + *m*) *div* 2 = (*m*::*nat*)
by (*simp add*: *nat-mult-2* [*symmetric*])

lemma *mod2-Suc-Suc* [*simp*]: *Suc*(*Suc*(*m*)) *mod* 2 = *m mod* 2
apply (*subgoal-tac m mod* 2 < 2)
apply (*erule less-2-cases* [*THEN disjE*])
apply (*simp-all* (*no-asm-simp*) *add*: *Let-def mod-Suc*)
done

lemma *mod2-gr-0* [*simp*]: 0 < (*m*::*nat*) *mod* 2 ⟷ *m mod* 2 = 1
proof −
 { fix *n* :: *nat* **have** (*n*::*nat*) < 2 ⟹ *n* = 0 ∨ *n* = 1 **by** (*cases n*) *simp-all* **}**
 moreover have *m mod* 2 < 2 **by** *simp*
 ultimately have *m mod* 2 = 0 ∨ *m mod* 2 = 1 **.**
 then show *?thesis* **by** *auto*
qed

These lemmas collapse some needless occurrences of Suc: at least three Sucs, since two and fewer are rewritten back to Suc again! We already have some rules to simplify operands smaller than 3.

lemma *div-Suc-eq-div-add3* [*simp*]: *m div* (*Suc* (*Suc* (*Suc n*))) = *m div* (3+*n*)
by (*simp add*: *Suc3-eq-add-3*)

lemma *mod-Suc-eq-mod-add3* [*simp*]: *m mod* (*Suc* (*Suc* (*Suc n*))) = *m mod* (3+*n*)
by (*simp add*: *Suc3-eq-add-3*)

lemma *Suc-div-eq-add3-div*: (*Suc* (*Suc* (*Suc m*))) *div n* = (3+*m*) *div n*
by (*simp add*: *Suc3-eq-add-3*)

lemma *Suc-mod-eq-add3-mod*: (*Suc* (*Suc* (*Suc m*))) *mod n* = (3+*m*) *mod n*
by (*simp add*: *Suc3-eq-add-3*)

lemmas *Suc-div-eq-add3-div-number-of* =
 Suc-div-eq-add3-div [*of* - *number-of v*, *standard*]
declare *Suc-div-eq-add3-div-number-of* [*simp*]

lemmas *Suc-mod-eq-add3-mod-number-of* =
 Suc-mod-eq-add3-mod [*of* - *number-of v*, *standard*]
declare *Suc-mod-eq-add3-mod-number-of* [*simp*]

lemma *Suc-times-mod-eq*: 1<*k* ==> *Suc* (*k* ∗ *m*) *mod k* = 1
apply (*induct m*)
apply (*simp-all add*: *mod-Suc*)
done

declare *Suc-times-mod-eq* [*of number-of w*, *standard*, *simp*]

lemma [*simp*]: *n div k* ≤ (*Suc n*) *div k*
by (*simp add*: *div-le-mono*)

lemma *Suc-n-div-2-gt-zero* [*simp*]: (*0*::*nat*) < *n* ==> *0* < (*n* + *1*) *div 2*
by (*cases n*) *simp-all*

lemma *div-2-gt-zero* [*simp*]: **assumes** *A*: (*1*::*nat*) < *n* **shows** *0* < *n div 2*
proof −
  from *A* **have** *B*: *0* < *n* − *1* **and** *C*: *n* − *1* + *1* = *n* **by** *simp-all*
  from *Suc-n-div-2-gt-zero* [*OF B*] *C* **show** *?thesis* **by** *simp*
qed

lemma *mod-mult-self3* [*simp*]: (*k*∗*n* + *m*) *mod n* = *m mod* (*n*::*nat*)
by (*simp add*: *mult-ac add-ac*)

lemma *mod-mult-self4* [*simp*]: *Suc* (*k*∗*n* + *m*) *mod n* = *Suc m mod n*
proof −
  have *Suc* (*k* ∗ *n* + *m*) *mod n* = (*k* ∗ *n* + *Suc m*) *mod n* **by** *simp*
  also have ... = *Suc m mod n* **by** (*rule mod-mult-self3*)
  finally show *?thesis* **.**
qed

lemma *mod-Suc-eq-Suc-mod*: *Suc m mod n* = *Suc* (*m mod n*) *mod n*
apply (*subst mod-Suc* [*of m*])
apply (*subst mod-Suc* [*of m mod n*], *simp*)
done

33.4 Division on *int*

definition *divmod-int-rel* :: *int* ⇒ *int* ⇒ *int* × *int* ⇒ *bool* **where**
  — definition of quotient and remainder
  [*code*]: *divmod-int-rel a b* = (λ(*q*, *r*). *a* = *b* ∗ *q* + *r* ∧
    (*if 0* < *b then 0* ≤ *r* ∧ *r* < *b else b* < *r* ∧ *r* ≤ *0*))

definition *adjust* :: *int* ⇒ *int* × *int* ⇒ *int* × *int* **where**
  — for the division algorithm
  [*code*]: *adjust b* = (λ(*q*, *r*). *if 0* ≤ *r* − *b then* (*2* ∗ *q* + *1*, *r* − *b*)
    else (*2* ∗ *q*, *r*))

algorithm for the case *a*≥*0*, *b*>*0*

function *posDivAlg* :: *int* ⇒ *int* ⇒ *int* × *int* **where**
  posDivAlg a b = (*if a* < *b* ∨ *b* ≤ *0 then* (*0*, *a*)
    else adjust b (*posDivAlg a* (*2* ∗ *b*)))
by *auto*
termination by (*relation measure* (λ(*a*, *b*). *nat* (*a* − *b* + *1*)))
  (*auto simp add*: *mult-2*)

algorithm for the case *a*<*0*, *b*>*0*

function *negDivAlg* :: *int* ⇒ *int* ⇒ *int* × *int* **where**
negDivAlg a b = (*if* 0 ≤*a* + *b* ∨ *b* ≤ 0 *then* (−1, *a* + *b*)
else adjust b (*negDivAlg a* (2 ∗ *b*)))
by *auto*
termination by (*relation measure* (λ(*a*, *b*). *nat* (− *a* − *b*)))
(*auto simp add*: *mult-2*)

algorithm for the general case $b \neq (0::'a)$

definition *negateSnd* :: *int* × *int* ⇒ *int* × *int* **where**
[*code-unfold*]: *negateSnd* = *apsnd uminus*

definition *divmod-int* :: *int* ⇒ *int* ⇒ *int* × *int* **where**
— The full division algorithm considers all possible signs for a, b including the special case *a*=0, *b*<0 because *negDivAlg* requires *a* < (0::'*a*).
divmod-int a b = (*if* 0 ≤ *a then if* 0 ≤ *b then posDivAlg a b*
else if a = 0 *then* (0, 0)
else negateSnd (*negDivAlg* (−*a*) (−*b*))
else
if 0 < *b then negDivAlg a b*
else negateSnd (*posDivAlg* (−*a*) (−*b*)))

instantiation *int* :: *Divides.div*
begin

definition
a div b = *fst* (*divmod-int a b*)

definition
a mod b = *snd* (*divmod-int a b*)

instance ..

end

lemma *divmod-int-mod-div*:
divmod-int p q = (*p div q*, *p mod q*)
by (*auto simp add*: *div-int-def mod-int-def*)

Here is the division algorithm in ML:

```
fun posDivAlg (a,b) =
  if a<b then (0,a)
  else let val (q,r) = posDivAlg(a, 2*b)
           in  if 0\<le>r-b then (2*q+1, r-b) else (2*q, r)
       end

fun negDivAlg (a,b) =
  if 0\<le>a+b then (~1,a+b)
```

```
      else let val (q,r) = negDivAlg(a, 2*b)
               in  if 0\<le>r-b then (2*q+1, r-b) else (2*q, r)
           end;

    fun negateSnd (q,r:int) = (q,~r);

    fun divmod (a,b) = if 0\<le>a then
                          if b>0 then posDivAlg (a,b)
                           else if a=0 then (0,0)
                                else negateSnd (negDivAlg (~a,~b))
                       else
                          if 0<b then negDivAlg (a,b)
                          else        negateSnd (posDivAlg (~a,~b));
```

33.4.1 Uniqueness and Monotonicity of Quotients and Remainders

lemma *unique-quotient-lemma*:
[| $b*q' + r' \leq b*q + r$; $0 \leq r'$; $r' < b$; $r < b$ |]
==> $q' \leq (q::int)$
apply (*subgoal-tac* $r' + b * (q'-q) \leq r$)
prefer *2* **apply** (*simp add*: *right-diff-distrib*)
apply (*subgoal-tac* $0 < b * (1 + q - q')$)
apply (*erule-tac* [*2*] *order-le-less-trans*)
prefer *2* **apply** (*simp add*: *right-diff-distrib right-distrib*)
apply (*subgoal-tac* $b * q' < b * (1 + q)$)
prefer *2* **apply** (*simp add*: *right-diff-distrib right-distrib*)
apply (*simp add*: *mult-less-cancel-left*)
done

lemma *unique-quotient-lemma-neg*:
[| $b*q' + r' \leq b*q + r$; $r \leq 0$; $b < r$; $b < r'$ |]
==> $q \leq (q'::int)$
by (*rule-tac* $b = -b$ **and** $r = -r'$ **and** $r' = -r$ **in** *unique-quotient-lemma*, *auto*)

lemma *unique-quotient*:
[| *divmod-int-rel* a b (q, r); *divmod-int-rel* a b (q', r'); $b \neq 0$ |]
==> $q = q'$
apply (*simp add*: *divmod-int-rel-def linorder-neq-iff split*: *split-if-asm*)
apply (*blast intro*: *order-antisym*
dest: *order-eq-refl* [*THEN unique-quotient-lemma*]
order-eq-refl [*THEN unique-quotient-lemma-neg*] *sym*)+
done

lemma *unique-remainder*:

[| *divmod-int-rel a b* (*q*, *r*); *divmod-int-rel a b* (*q'*, *r'*); $b \neq 0$ |]
==> $r = r'$
apply (*subgoal-tac q = q'*)
apply (*simp add*: *divmod-int-rel-def*)
apply (*blast intro*: *unique-quotient*)
done

33.4.2 Correctness of *posDivAlg*, the Algorithm for Non-Negative Dividends

And positive divisors

lemma *adjust-eq* [*simp*]:
adjust b (*q*,*r*) =
(*let diff* = *r*−*b in*
if 0 ≤ *diff then* (2∗*q* + 1, *diff*)
else (2∗*q*, *r*))
by (*simp add*: *Let-def adjust-def*)

declare *posDivAlg.simps* [*simp del*]

use with a simproc to avoid repeatedly proving the premise

lemma *posDivAlg-eqn*:
0 < *b* ==>
posDivAlg a b = (*if a*<*b then* (0,*a*) *else adjust b* (*posDivAlg a* (2∗*b*)))
by (*rule posDivAlg.simps* [*THEN trans*], *simp*)

Correctness of *posDivAlg*: it computes quotients correctly

theorem *posDivAlg-correct*:
assumes 0 ≤ *a* **and** 0 < *b*
shows *divmod-int-rel a b* (*posDivAlg a b*)
using *prems* **apply** (*induct a b rule*: *posDivAlg.induct*)
apply *auto*
apply (*simp add*: *divmod-int-rel-def*)
apply (*subst posDivAlg-eqn*, *simp add*: *right-distrib*)
apply (*case-tac a* < *b*)
apply *simp-all*
apply (*erule splitE*)
apply (*auto simp add*: *right-distrib Let-def mult-ac mult-2-right*)
done

33.4.3 Correctness of *negDivAlg*, the Algorithm for Negative Dividends

And positive divisors

declare *negDivAlg.simps* [*simp del*]

use with a simproc to avoid repeatedly proving the premise

lemma *negDivAlg-eqn*:
 0 < b ==>
 negDivAlg a b =
 (if 0≤a+b then (−1,a+b) else adjust b (negDivAlg a (2∗b)))
by (*rule negDivAlg.simps* [*THEN trans*], *simp*)

lemma *negDivAlg-correct*:
 assumes *a < 0* **and** *b > 0*
 shows *divmod-int-rel a b* (*negDivAlg a b*)
using *prems* **apply** (*induct a b rule*: *negDivAlg.induct*)
apply (*auto simp add*: *linorder-not-le*)
apply (*simp add*: *divmod-int-rel-def*)
apply (*subst negDivAlg-eqn*, *assumption*)
apply (*case-tac a + b < (0*::*int)*)
apply *simp-all*
apply (*erule splitE*)
apply (*auto simp add*: *right-distrib Let-def mult-ac mult-2-right*)
done

33.4.4 Existence Shown by Proving the Division Algorithm to be Correct

lemma *divmod-int-rel-0*: *b ≠ 0 ==> divmod-int-rel 0 b* (*0*, *0*)
by (*auto simp add*: *divmod-int-rel-def linorder-neq-iff*)

lemma *posDivAlg-0* [*simp*]: *posDivAlg 0 b* = (*0*, *0*)
by (*subst posDivAlg.simps*, *auto*)

lemma *negDivAlg-minus1* [*simp*]: *negDivAlg −1 b* = (*−1*, *b − 1*)
by (*subst negDivAlg.simps*, *auto*)

lemma *negateSnd-eq* [*simp*]: *negateSnd*(*q*,*r*) = (*q*,*−r*)
by (*simp add*: *negateSnd-def*)

lemma *divmod-int-rel-neg*: *divmod-int-rel* (*−a*) (*−b*) *qr ==> divmod-int-rel a b* (*negateSnd qr*)
by (*auto simp add*: *split-ifs divmod-int-rel-def*)

lemma *divmod-int-correct*: *b ≠ 0 ==> divmod-int-rel a b* (*divmod-int a b*)
by (*force simp add*: *linorder-neq-iff divmod-int-rel-0 divmod-int-def divmod-int-rel-neg*
 posDivAlg-correct negDivAlg-correct)

Arbitrary definitions for division by zero. Useful to simplify certain equations.

lemma *DIVISION-BY-ZERO* [*simp*]: *a div* (*0*::*int*) = *0* & *a mod* (*0*::*int*) = *a*
by (*simp add*: *div-int-def mod-int-def divmod-int-def posDivAlg.simps*)

Basic laws about division and remainder

lemma *zmod-zdiv-equality*: $(a::int) = b * (a\ div\ b) + (a\ mod\ b)$
apply (*case-tac* $b = 0$, *simp*)
apply (*cut-tac* $a = a$ **and** $b = b$ **in** *divmod-int-correct*)
apply (*auto simp add*: *divmod-int-rel-def div-int-def mod-int-def*)
done

lemma *zdiv-zmod-equality*: $(b * (a\ div\ b) + (a\ mod\ b)) + k = (a::int)+k$
by(*simp add*: *zmod-zdiv-equality*[*symmetric*])

lemma *zdiv-zmod-equality2*: $((a\ div\ b) * b + (a\ mod\ b)) + k = (a::int)+k$
by(*simp add*: *mult-commute zmod-zdiv-equality*[*symmetric*])

Tool setup

ML ⟪
local

structure CancelDivMod = CancelDivModFun(*struct*

val div-name = @{const-name div};
val mod-name = @{const-name mod};
val mk-binop = HOLogic.mk-binop;
val mk-sum = Arith-Data.mk-sum HOLogic.intT;
val dest-sum = Arith-Data.dest-sum;

val div-mod-eqs = map mk-meta-eq [@{thm zdiv-zmod-equality}, @{thm zdiv-zmod-equality2}];

val trans = trans;

val prove-eq-sums = Arith-Data.prove-conv2 all-tac (*Arith-Data.simp-all-tac*
(@{*thm diff-minus*} :: @{*thms add-0s*} @ @{*thms add-ac*}))

end)

in

val cancel-div-mod-int-proc = Simplifier.simproc @{theory}
cancel-zdiv-zmod [(*k*::*int*) + *l*] (*K CancelDivMod.proc*);

val - = Addsimprocs [*cancel-div-mod-int-proc*];

end
⟫

lemma *pos-mod-conj* : $(0::int) < b$ ==> $0 \le a\ mod\ b\ \&\ a\ mod\ b < b$
apply (*cut-tac* $a = a$ **and** $b = b$ **in** *divmod-int-correct*)
apply (*auto simp add*: *divmod-int-rel-def mod-int-def*)
done

lemmas *pos-mod-sign* [*simp*] = *pos-mod-conj* [*THEN conjunct1*, *standard*]

and *pos-mod-bound* [*simp*] = *pos-mod-conj* [*THEN conjunct2*, *standard*]

lemma *neg-mod-conj* : $b < (0::int) ==> a \ mod \ b \leq 0 \ \& \ b < a \ mod \ b$
apply (*cut-tac* $a = a$ **and** $b = b$ **in** *divmod-int-correct*)
apply (*auto simp add*: *divmod-int-rel-def div-int-def mod-int-def*)
done

lemmas *neg-mod-sign* [*simp*] = *neg-mod-conj* [*THEN conjunct1*, *standard*]
and *neg-mod-bound* [*simp*] = *neg-mod-conj* [*THEN conjunct2*, *standard*]

33.4.5 General Properties of div and mod

lemma *divmod-int-rel-div-mod*: $b \neq 0 ==>$ *divmod-int-rel* $a \ b \ (a \ div \ b, \ a \ mod \ b)$
apply (*cut-tac* $a = a$ **and** $b = b$ **in** *zmod-zdiv-equality*)
apply (*force simp add*: *divmod-int-rel-def linorder-neq-iff*)
done

lemma *divmod-int-rel-div*: [| *divmod-int-rel* $a \ b \ (q, \ r)$; $b \neq 0$ |] ==> $a \ div \ b = q$
by (*simp add*: *divmod-int-rel-div-mod* [*THEN unique-quotient*])

lemma *divmod-int-rel-mod*: [| *divmod-int-rel* $a \ b \ (q, \ r)$; $b \neq 0$ |] ==> $a \ mod \ b = r$
by (*simp add*: *divmod-int-rel-div-mod* [*THEN unique-remainder*])

lemma *div-pos-pos-trivial*: [| $(0::int) \leq a$; $a < b$ |] ==> $a \ div \ b = 0$
apply (*rule divmod-int-rel-div*)
apply (*auto simp add*: *divmod-int-rel-def*)
done

lemma *div-neg-neg-trivial*: [| $a \leq (0::int)$; $b < a$ |] ==> $a \ div \ b = 0$
apply (*rule divmod-int-rel-div*)
apply (*auto simp add*: *divmod-int-rel-def*)
done

lemma *div-pos-neg-trivial*: [| $(0::int) < a$; $a+b \leq 0$ |] ==> $a \ div \ b = -1$
apply (*rule divmod-int-rel-div*)
apply (*auto simp add*: *divmod-int-rel-def*)
done

lemma *mod-pos-pos-trivial*: [| $(0::int) \leq a$; $a < b$ |] ==> $a \ mod \ b = a$
apply (*rule-tac* $q = 0$ **in** *divmod-int-rel-mod*)
apply (*auto simp add*: *divmod-int-rel-def*)
done

lemma *mod-neg-neg-trivial*: [| $a \leq (0::int)$; $b < a$ |] ==> $a \ mod \ b = a$
apply (*rule-tac* $q = 0$ **in** *divmod-int-rel-mod*)
apply (*auto simp add*: *divmod-int-rel-def*)

done

lemma *mod-pos-neg-trivial*: [| (*0*::*int*) < *a*; *a*+*b* ≤ *0* |] ==> *a mod b* = *a*+*b*
apply (*rule-tac q* = −*1* **in** *divmod-int-rel-mod*)
apply (*auto simp add*: *divmod-int-rel-def*)
done

There is no *mod-neg-pos-trivial*.

lemma *zdiv-zminus-zminus* [*simp*]: (−*a*) *div* (−*b*) = *a div* (*b*::*int*)
apply (*case-tac b* = *0*, *simp*)
apply (*simp add*: *divmod-int-rel-div-mod* [*THEN divmod-int-rel-neg*, *simplified*,
THEN divmod-int-rel-div, *THEN sym*])

done

lemma *zmod-zminus-zminus* [*simp*]: (−*a*) *mod* (−*b*) = − (*a mod* (*b*::*int*))
apply (*case-tac b* = *0*, *simp*)
apply (*subst divmod-int-rel-div-mod* [*THEN divmod-int-rel-neg*, *simplified*, *THEN*
divmod-int-rel-mod],
auto)
done

33.4.6 Laws for div and mod with Unary Minus

lemma *zminus1-lemma*:
divmod-int-rel a b (*q*, *r*)
==> *divmod-int-rel* (−*a*) *b* (*if r*=*0 then* −*q else* −*q* − *1*,
if r=*0 then 0 else b*−*r*)
by (*force simp add*: *split-ifs divmod-int-rel-def linorder-neq-iff right-diff-distrib*)

lemma *zdiv-zminus1-eq-if*:
b ≠ (*0*::*int*)
==> (−*a*) *div b* =
(*if a mod b* = *0 then* − (*a div b*) *else* − (*a div b*) − *1*)
by (*blast intro*: *divmod-int-rel-div-mod* [*THEN zminus1-lemma*, *THEN divmod-int-rel-div*])

lemma *zmod-zminus1-eq-if*:
(−*a*::*int*) *mod b* = (*if a mod b* = *0 then 0 else b* − (*a mod b*))
apply (*case-tac b* = *0*, *simp*)
apply (*blast intro*: *divmod-int-rel-div-mod* [*THEN zminus1-lemma*, *THEN divmod-int-rel-mod*])
done

lemma *zmod-zminus1-not-zero*:
fixes *k l* :: *int*
shows − *k mod l* ≠ *0* ⟹ *k mod l* ≠ *0*
unfolding *zmod-zminus1-eq-if* **by** *auto*

lemma *zdiv-zminus2*: *a div (−b) = (−a::int) div b*
by (*cut-tac a = −a* **in** *zdiv-zminus-zminus*, *auto*)

lemma *zmod-zminus2*: *a mod (−b) = − ((−a::int) mod b)*
by (*cut-tac a = −a* **and** *b = b* **in** *zmod-zminus-zminus*, *auto*)

lemma *zdiv-zminus2-eq-if*:
b ≠ (0::int)
==> a div (−b) =
(if a mod b = 0 then − (a div b) else − (a div b) − 1)
by (*simp add*: *zdiv-zminus1-eq-if zdiv-zminus2*)

lemma *zmod-zminus2-eq-if*:
a mod (−b::int) = (if a mod b = 0 then 0 else (a mod b) − b)
by (*simp add*: *zmod-zminus1-eq-if zmod-zminus2*)

lemma *zmod-zminus2-not-zero*:
fixes *k l* :: *int*
shows *k mod − l ≠ 0 ⟹ k mod l ≠ 0*
unfolding *zmod-zminus2-eq-if* **by** *auto*

33.4.7 Division of a Number by Itself

lemma *self-quotient-aux1*: [| *(0::int) < a*; *a = r + a∗q*; *r < a* |] *==> 1 ≤ q*
apply (*subgoal-tac 0 < a∗q*)
apply (*simp add*: *zero-less-mult-iff*, *arith*)
done

lemma *self-quotient-aux2*: [| *(0::int) < a*; *a = r + a∗q*; *0 ≤ r* |] *==> q ≤ 1*
apply (*subgoal-tac 0 ≤ a∗ (1−q)*)
apply (*simp add*: *zero-le-mult-iff*)
apply (*simp add*: *right-diff-distrib*)
done

lemma *self-quotient*: [| *divmod-int-rel a a (q, r)*; *a ≠ (0::int)* |] *==> q = 1*
apply (*simp add*: *split-ifs divmod-int-rel-def linorder-neq-iff*)
apply (*rule order-antisym*, *safe*, *simp-all*)
apply (*rule-tac* [*3*] *a = −a* **and** *r = −r* **in** *self-quotient-aux1*)
apply (*rule-tac a = −a* **and** *r = −r* **in** *self-quotient-aux2*)
apply (*force intro*: *self-quotient-aux1 self-quotient-aux2 simp add*: *add-commute*)+
done

lemma *self-remainder*: [| *divmod-int-rel a a (q, r)*; *a ≠ (0::int)* |] *==> r = 0*
apply (*frule self-quotient*, *assumption*)
apply (*simp add*: *divmod-int-rel-def*)
done

lemma *zdiv-self* [*simp*]: *a ≠ 0 ==> a div a = (1::int)*
by (*simp add*: *divmod-int-rel-div-mod* [*THEN self-quotient*])

lemma *zmod-self* [*simp*]: *a mod a = (0::int)*
apply (*case-tac a = 0*, *simp*)
apply (*simp add*: *divmod-int-rel-div-mod* [*THEN self-remainder*])
done

33.4.8 Computation of Division and Remainder

lemma *zdiv-zero* [*simp*]: *(0::int) div b = 0*
by (*simp add*: *div-int-def divmod-int-def*)

lemma *div-eq-minus1*: *(0::int) < b ==> −1 div b = −1*
by (*simp add*: *div-int-def divmod-int-def*)

lemma *zmod-zero* [*simp*]: *(0::int) mod b = 0*
by (*simp add*: *mod-int-def divmod-int-def*)

lemma *zmod-minus1*: *(0::int) < b ==> −1 mod b = b − 1*
by (*simp add*: *mod-int-def divmod-int-def*)

a positive, b positive

lemma *div-pos-pos*: *[| 0 < a; 0 ≤ b |] ==> a div b = fst (posDivAlg a b)*
by (*simp add*: *div-int-def divmod-int-def*)

lemma *mod-pos-pos*: *[| 0 < a; 0 ≤ b |] ==> a mod b = snd (posDivAlg a b)*
by (*simp add*: *mod-int-def divmod-int-def*)

a negative, b positive

lemma *div-neg-pos*: *[| a < 0; 0 < b |] ==> a div b = fst (negDivAlg a b)*
by (*simp add*: *div-int-def divmod-int-def*)

lemma *mod-neg-pos*: *[| a < 0; 0 < b |] ==> a mod b = snd (negDivAlg a b)*
by (*simp add*: *mod-int-def divmod-int-def*)

a positive, b negative

lemma *div-pos-neg*:
[| 0 < a; b < 0 |] ==> a div b = fst (negateSnd (negDivAlg (−a) (−b)))
by (*simp add*: *div-int-def divmod-int-def*)

lemma *mod-pos-neg*:
[| 0 < a; b < 0 |] ==> a mod b = snd (negateSnd (negDivAlg (−a) (−b)))
by (*simp add*: *mod-int-def divmod-int-def*)

a negative, b negative

lemma *div-neg-neg*:
[| a < 0; b ≤ 0 |] ==> a div b = fst (negateSnd (posDivAlg (−a) (−b)))
by (*simp add*: *div-int-def divmod-int-def*)

lemma *mod-neg-neg*:
[| $a < 0$; $b \leq 0$ |] ==> a *mod* b = *snd* (*negateSnd* (*posDivAlg* ($-a$) ($-b$)))
by (*simp add*: *mod-int-def divmod-int-def*)

Simplify expresions in which div and mod combine numerical constants

lemma *divmod-int-relI*:
⟦a == b * q + r; *if* 0 < b *then* 0 ≤ r ∧ r < b *else* b < r ∧ r ≤ 0⟧
⟹ *divmod-int-rel* a b (q, r)
unfolding *divmod-int-rel-def* **by** *simp*

lemmas *divmod-int-rel-div-eq* = *divmod-int-relI* [*THEN divmod-int-rel-div*, *THEN eq-reflection*]
lemmas *divmod-int-rel-mod-eq* = *divmod-int-relI* [*THEN divmod-int-rel-mod*, *THEN eq-reflection*]
lemmas *arithmetic-simps* =
arith-simps
add-special
add-0-left
add-0-right
mult-zero-left
mult-zero-right
mult-1-left
mult-1-right

ML ⟪⟪
local
val mk-number = *HOLogic.mk-number HOLogic.intT*;
fun mk-cert u k l = @{*term plus* :: *int* ⇒ *int* ⇒ *int*} $
(@{*term times* :: *int* ⇒ *int* ⇒ *int*} $ *u* $ *mk-number k*) $
mk-number l;
fun prove ctxt prop = *Goal.prove ctxt* [] [] *prop*
(*K* (*ALLGOALS* (*full-simp-tac* (*HOL-basic-ss addsimps* @{*thms arithmetic-simps*}))));
fun binary-proc proc ss ct =
(*case Thm.term-of ct of*
- $ *t* $ *u* =>
(*case try* (*pairself* ('(*snd o HOLogic.dest-number*))) (*t*, *u*) *of*
SOME args => *proc* (*Simplifier.the-context ss*) *args*
| *NONE* => *NONE*)
| - => *NONE*);
in
fun divmod-proc rule = *binary-proc* (*fn ctxt* => *fn* ((*m*, *t*), (*n*, *u*)) =>
if n = 0 *then NONE*
else let val (*k*, *l*) = *Integer.div-mod m n*;
in SOME (*rule OF* [*prove ctxt* (*Logic.mk-equals* (*t*, *mk-cert u k l*))]) *end*);
end
⟫⟫

simproc-setup *binary-int-div* (*number-of m div number-of n* :: *int*) =

⟪ *K* (*divmod-proc* (@{*thm divmod-int-rel-div-eq*})) ⟫

simproc-setup *binary-int-mod* (*number-of m mod number-of n* :: *int*) =
⟪ *K* (*divmod-proc* (@{*thm divmod-int-rel-mod-eq*})) ⟫

lemmas *posDivAlg-eqn-number-of* [*simp*] =
posDivAlg-eqn [*of number-of v number-of w*, *standard*]

lemmas *negDivAlg-eqn-number-of* [*simp*] =
negDivAlg-eqn [*of number-of v number-of w*, *standard*]

Special-case simplification

lemma *zmod-minus1-right* [*simp*]: *a mod* (−*1*::*int*) = *0*
apply (*cut-tac a* = *a* **and** *b* = −*1* **in** *neg-mod-sign*)
apply (*cut-tac* [*2*] *a* = *a* **and** *b* = −*1* **in** *neg-mod-bound*)
apply (*auto simp del*: *neg-mod-sign neg-mod-bound*)
done

lemma *zdiv-minus1-right* [*simp*]: *a div* (−*1*::*int*) = −*a*
by (*cut-tac a* = *a* **and** *b* = −*1* **in** *zmod-zdiv-equality*, *auto*)

lemmas *div-pos-pos-1-number-of* [*simp*] =
div-pos-pos [*OF int-0-less-1*, *of number-of w*, *standard*]

lemmas *div-pos-neg-1-number-of* [*simp*] =
div-pos-neg [*OF int-0-less-1*, *of number-of w*, *standard*]

lemmas *mod-pos-pos-1-number-of* [*simp*] =
mod-pos-pos [*OF int-0-less-1*, *of number-of w*, *standard*]

lemmas *mod-pos-neg-1-number-of* [*simp*] =
mod-pos-neg [*OF int-0-less-1*, *of number-of w*, *standard*]

lemmas *posDivAlg-eqn-1-number-of* [*simp*] =
posDivAlg-eqn [*of concl*: *1 number-of w*, *standard*]

lemmas *negDivAlg-eqn-1-number-of* [*simp*] =
negDivAlg-eqn [*of concl*: *1 number-of w*, *standard*]

33.4.9 Monotonicity in the First Argument (Dividend)

lemma *zdiv-mono1*: [| *a* ≤ *a'*; *0* < (*b*::*int*) |] ==> *a div b* ≤ *a' div b*
apply (*cut-tac a* = *a* **and** *b* = *b* **in** *zmod-zdiv-equality*)
apply (*cut-tac a* = *a'* **and** *b* = *b* **in** *zmod-zdiv-equality*)
apply (*rule unique-quotient-lemma*)
apply (*erule subst*)

apply (*erule subst*, *simp-all*)
done

lemma *zdiv-mono1-neg*: [| $a \leq a'$; $(b::int) < 0$ |] ==> $a' \; div \; b \leq a \; div \; b$
apply (*cut-tac* $a = a$ **and** $b = b$ **in** *zmod-zdiv-equality*)
apply (*cut-tac* $a = a'$ **and** $b = b$ **in** *zmod-zdiv-equality*)
apply (*rule unique-quotient-lemma-neg*)
apply (*erule subst*)
apply (*erule subst*, *simp-all*)
done

33.4.10 Monotonicity in the Second Argument (Divisor)

lemma *q-pos-lemma*:
[| $0 \leq b'*q' + r'$; $r' < b'$; $0 < b'$ |] ==> $0 \leq (q'::int)$
apply (*subgoal-tac* $0 < b'* (q' + 1)$)
apply (*simp add*: *zero-less-mult-iff*)
apply (*simp add*: *right-distrib*)
done

lemma *zdiv-mono2-lemma*:
[| $b*q + r = b'*q' + r'$; $0 \leq b'*q' + r'$;
$r' < b'$; $0 \leq r$; $0 < b'$; $b' \leq b$ |]
==> $q \leq (q'::int)$
apply (*frule q-pos-lemma*, *assumption*+)
apply (*subgoal-tac* $b*q < b* (q' + 1)$)
apply (*simp add*: *mult-less-cancel-left*)
apply (*subgoal-tac* $b*q = r' - r + b'*q'$)
prefer *2* **apply** *simp*
apply (*simp* (*no-asm-simp*) *add*: *right-distrib*)
apply (*subst add-commute*, *rule zadd-zless-mono*, *arith*)
apply (*rule mult-right-mono*, *auto*)
done

lemma *zdiv-mono2*:
[| $(0::int) \leq a$; $0 < b'$; $b' \leq b$ |] ==> $a \; div \; b \leq a \; div \; b'$
apply (*subgoal-tac* $b \neq 0$)
prefer *2* **apply** *arith*
apply (*cut-tac* $a = a$ **and** $b = b$ **in** *zmod-zdiv-equality*)
apply (*cut-tac* $a = a$ **and** $b = b'$ **in** *zmod-zdiv-equality*)
apply (*rule zdiv-mono2-lemma*)
apply (*erule subst*)
apply (*erule subst*, *simp-all*)
done

lemma *q-neg-lemma*:
[| $b'*q' + r' < 0$; $0 \leq r'$; $0 < b'$ |] ==> $q' \leq (0::int)$
apply (*subgoal-tac* $b'*q' < 0$)
apply (*simp add*: *mult-less-0-iff*, *arith*)

done

lemma *zdiv-mono2-neg-lemma*:
[| $b*q + r = b'*q' + r'$; $b'*q' + r' < 0$;
$r < b$; $0 \leq r'$; $0 < b'$; $b' \leq b$ |]
==> $q' \leq (q::int)$
apply (*frule q-neg-lemma*, *assumption*+)
apply (*subgoal-tac* $b*q' < b*(q + 1)$)
apply (*simp add*: *mult-less-cancel-left*)
apply (*simp add*: *right-distrib*)
apply (*subgoal-tac* $b*q' \leq b'*q'$)
prefer *2* **apply** (*simp add*: *mult-right-mono-neg*, *arith*)
done

lemma *zdiv-mono2-neg*:
[| $a < (0::int)$; $0 < b'$; $b' \leq b$ |] ==> a *div* $b' \leq a$ *div* b
apply (*cut-tac* $a = a$ **and** $b = b$ **in** *zmod-zdiv-equality*)
apply (*cut-tac* $a = a$ **and** $b = b'$ **in** *zmod-zdiv-equality*)
apply (*rule zdiv-mono2-neg-lemma*)
apply (*erule subst*)
apply (*erule subst*, *simp-all*)
done

33.4.11 More Algebraic Laws for div and mod

proving (a*b) div c = a * (b div c) + a * (b mod c)

lemma *zmult1-lemma*:
[| *divmod-int-rel* b c (q, r); $c \neq 0$ |]
==> *divmod-int-rel* $(a * b)$ c $(a*q + a*r$ *div* c, $a*r$ *mod* $c)$
by (*auto simp add*: *split-ifs divmod-int-rel-def linorder-neq-iff right-distrib mult-ac*)

lemma *zdiv-zmult1-eq*: $(a*b)$ *div* $c = a*(b$ *div* $c) + a*(b$ *mod* $c)$ *div* $(c::int)$
apply (*case-tac* $c = 0$, *simp*)
apply (*blast intro*: *divmod-int-rel-div-mod* [*THEN zmult1-lemma*, *THEN divmod-int-rel-div*])
done

lemma *zmod-zmult1-eq*: $(a*b)$ *mod* $c = a*(b$ *mod* $c)$ *mod* $(c::int)$
apply (*case-tac* $c = 0$, *simp*)
apply (*blast intro*: *divmod-int-rel-div-mod* [*THEN zmult1-lemma*, *THEN divmod-int-rel-mod*])
done

lemma *zmod-zdiv-trivial*: $(a$ *mod* $b)$ *div* $b = (0::int)$
apply (*case-tac* $b = 0$, *simp*)
apply (*auto simp add*: *linorder-neq-iff div-pos-pos-trivial div-neg-neg-trivial*)
done

proving (a+b) div c = a div c + b div c + ((a mod c + b mod c) div c)

lemma *zadd1-lemma*:
[| *divmod-int-rel* a c (aq, ar); *divmod-int-rel* b c (bq, br); $c \neq 0$ |]

```
    ==> divmod-int-rel (a+b) c (aq + bq + (ar+br) div c, (ar+br) mod c)
by (force simp add: split-ifs divmod-int-rel-def linorder-neq-iff right-distrib)


lemma zdiv-zadd1-eq:
    (a+b) div (c::int) = a div c + b div c + ((a mod c + b mod c) div c)
apply (case-tac c = 0, simp)
apply (blast intro: zadd1-lemma [OF divmod-int-rel-div-mod divmod-int-rel-div-mod]
divmod-int-rel-div)
done

instance int :: ring-div
proof
  fix a b c :: int
  assume not0: b ≠ 0
  show (a + c * b) div b = c + a div b
    unfolding zdiv-zadd1-eq [of a c * b] using not0
      by (simp add: zmod-zmult1-eq zmod-zdiv-trivial zdiv-zmult1-eq)
next
  fix a b c :: int
  assume a ≠ 0
  then show (a * b) div (a * c) = b div c
  proof (cases b ≠ 0 ∧ c ≠ 0)
    case False then show ?thesis by auto
  next
    case True then have b ≠ 0 and c ≠ 0 by auto
    with ‹a ≠ 0›
    have ⋀q r. divmod-int-rel b c (q, r) ⟹ divmod-int-rel (a * b) (a * c) (q, a *
r)
      apply (auto simp add: divmod-int-rel-def)
      apply (auto simp add: algebra-simps)
     apply (auto simp add: zero-less-mult-iff zero-le-mult-iff mult-le-0-iff mult-commute
[of a] mult-less-cancel-right)
      done
    moreover with ‹c ≠ 0› divmod-int-rel-div-mod have divmod-int-rel b c (b div
c, b mod c) by auto
    ultimately have divmod-int-rel (a * b) (a * c) (b div c, a * (b mod c)) .
    moreover from ‹a ≠ 0› ‹c ≠ 0› have a * c ≠ 0 by simp
    ultimately show ?thesis by (rule divmod-int-rel-div)
  qed
qed auto

lemma posDivAlg-div-mod:
  assumes k ≥ 0
  and l ≥ 0
  shows posDivAlg k l = (k div l, k mod l)
proof (cases l = 0)
  case True then show ?thesis by (simp add: posDivAlg.simps)
next
```

```
  case False with assms posDivAlg-correct
    have divmod-int-rel k l (fst (posDivAlg k l), snd (posDivAlg k l))
    by simp
  from divmod-int-rel-div [OF this ‹l ≠ 0›] divmod-int-rel-mod [OF this ‹l ≠ 0›]
  show ?thesis by simp
qed

lemma negDivAlg-div-mod:
  assumes k < 0
  and l > 0
  shows negDivAlg k l = (k div l, k mod l)
proof −
  from assms have l ≠ 0 by simp
  from assms negDivAlg-correct
    have divmod-int-rel k l (fst (negDivAlg k l), snd (negDivAlg k l))
    by simp
  from divmod-int-rel-div [OF this ‹l ≠ 0›] divmod-int-rel-mod [OF this ‹l ≠ 0›]
  show ?thesis by simp
qed

lemma zmod-eq-0-iff: (m mod d = 0) = (EX q::int. m = d*q)
by (simp add: dvd-eq-mod-eq-0 [symmetric] dvd-def)


lemmas zmod-eq-0D [dest!] = zmod-eq-0-iff [THEN iffD1]
```

33.4.12 Proving $a\ div\ (b * c) = a\ div\ b\ div\ c$

first, four lemmas to bound the remainder for the cases b¡0 and b¿0

```
lemma zmult2-lemma-aux1: [| (0::int) < c;  b < r;  r ≤ 0 |] ==> b*c < b*(q
mod c) + r
apply (subgoal-tac b * (c − q mod c) < r * 1)
 apply (simp add: algebra-simps)
apply (rule order-le-less-trans)
 apply (erule-tac [2] mult-strict-right-mono)
 apply (rule mult-left-mono-neg)
  using add1-zle-eq[of q mod c]apply(simp add: algebra-simps)
 apply (simp)
apply (simp)
done

lemma zmult2-lemma-aux2:
    [| (0::int) < c;   b < r;  r ≤ 0 |] ==> b * (q mod c) + r ≤ 0
apply (subgoal-tac b * (q mod c) ≤ 0)
 apply arith
apply (simp add: mult-le-0-iff)
done

lemma zmult2-lemma-aux3: [| (0::int) < c;  0 ≤ r;  r < b |] ==> 0 ≤ b * (q
```

mod c) + *r*
apply (*subgoal-tac 0* ≤ *b* ∗ (*q mod c*))
apply *arith*
apply (*simp add*: *zero-le-mult-iff*)
done

lemma *zmult2-lemma-aux4*: [| (*0*::*int*) < *c*; *0* ≤ *r*; *r* < *b* |] ==> *b* ∗ (*q mod c*)
+ *r* < *b* ∗ *c*
apply (*subgoal-tac r* ∗ *1* < *b* ∗ (*c* − *q mod c*))
apply (*simp add*: *right-diff-distrib*)
apply (*rule order-less-le-trans*)
apply (*erule mult-strict-right-mono*)
apply (*rule-tac* [*2*] *mult-left-mono*)
apply *simp*
using *add1-zle-eq*[*of q mod c*] **apply** (*simp add*: *algebra-simps*)
apply *simp*
done

lemma *zmult2-lemma*: [| *divmod-int-rel a b* (*q*, *r*); *b* ≠ *0*; *0* < *c* |]
==> *divmod-int-rel a* (*b* ∗ *c*) (*q div c*, *b*∗(*q mod c*) + *r*)
by (*auto simp add*: *mult-ac divmod-int-rel-def linorder-neq-iff*
zero-less-mult-iff right-distrib [*symmetric*]
zmult2-lemma-aux1 zmult2-lemma-aux2 zmult2-lemma-aux3
zmult2-lemma-aux4)

lemma *zdiv-zmult2-eq*: (*0*::*int*) < *c* ==> *a div* (*b*∗*c*) = (*a div b*) *div c*
apply (*case-tac b* = *0*, *simp*)
apply (*force simp add*: *divmod-int-rel-div-mod* [*THEN zmult2-lemma*, *THEN divmod-int-rel-div*])
done

lemma *zmod-zmult2-eq*:
(*0*::*int*) < *c* ==> *a mod* (*b*∗*c*) = *b*∗(*a div b mod c*) + *a mod b*
apply (*case-tac b* = *0*, *simp*)
apply (*force simp add*: *divmod-int-rel-div-mod* [*THEN zmult2-lemma*, *THEN divmod-int-rel-mod*])
done

33.4.13 Splitting Rules for div and mod

The proofs of the two lemmas below are essentially identical

lemma *split-pos-lemma*:
0<*k* ==>
P(*n div k* :: *int*)(*n mod k*) = (∀ *i j*. *0*≤*j* & *j*<*k* & *n* = *k*∗*i* + *j* −−> *P i j*)
apply (*rule iffI*, *clarify*)
apply (*erule-tac P*=*P ?x ?y* **in** *rev-mp*)
apply (*subst mod-add-eq*)
apply (*subst zdiv-zadd1-eq*)
apply (*simp add*: *div-pos-pos-trivial mod-pos-pos-trivial*)

converse direction

apply (*drule-tac* $x = n\ div\ k$ **in** *spec*)
apply (*drule-tac* $x = n\ mod\ k$ **in** *spec*, *simp*)
done

lemma *split-neg-lemma*:
$k<0 \Longrightarrow$
$P(n\ div\ k :: int)(n\ mod\ k) = (\forall i\ j.\ k<j\ \&\ j\leq 0\ \&\ n = k*i + j \longrightarrow P\ i\ j)$
apply (*rule iffI*, *clarify*)
apply (*erule-tac* $P=P\ ?x\ ?y$ **in** *rev-mp*)
apply (*subst mod-add-eq*)
apply (*subst zdiv-zadd1-eq*)
apply (*simp add*: *div-neg-neg-trivial mod-neg-neg-trivial*)

converse direction

apply (*drule-tac* $x = n\ div\ k$ **in** *spec*)
apply (*drule-tac* $x = n\ mod\ k$ **in** *spec*, *simp*)
done

lemma *split-zdiv*:
$P(n\ div\ k :: int) =$
$((k = 0 \longrightarrow P\ 0)\ \&$
$(0<k \longrightarrow (\forall i\ j.\ 0\leq j\ \&\ j<k\ \&\ n = k*i + j \longrightarrow P\ i))\ \&$
$(k<0 \longrightarrow (\forall i\ j.\ k<j\ \&\ j\leq 0\ \&\ n = k*i + j \longrightarrow P\ i)))$
apply (*case-tac* $k=0$, *simp*)
apply (*simp only*: *linorder-neq-iff*)
apply (*erule disjE*)
apply (*simp-all add*: *split-pos-lemma* [*of concl*: $\%x\ y.\ P\ x$]
split-neg-lemma [*of concl*: $\%x\ y.\ P\ x$])
done

lemma *split-zmod*:
$P(n\ mod\ k :: int) =$
$((k = 0 \longrightarrow P\ n)\ \&$
$(0<k \longrightarrow (\forall i\ j.\ 0\leq j\ \&\ j<k\ \&\ n = k*i + j \longrightarrow P\ j))\ \&$
$(k<0 \longrightarrow (\forall i\ j.\ k<j\ \&\ j\leq 0\ \&\ n = k*i + j \longrightarrow P\ j)))$
apply (*case-tac* $k=0$, *simp*)
apply (*simp only*: *linorder-neq-iff*)
apply (*erule disjE*)
apply (*simp-all add*: *split-pos-lemma* [*of concl*: $\%x\ y.\ P\ y$]
split-neg-lemma [*of concl*: $\%x\ y.\ P\ y$])
done

Enable (lin)arith to deal with *op div* and *op mod* when these are applied to some constant that is of the form *number-of k*:

declare *split-zdiv* [*of* - - *number-of k*, *standard*, *arith-split*]
declare *split-zmod* [*of* - - *number-of k*, *standard*, *arith-split*]

33.4.14 Speeding up the Division Algorithm with Shifting

computing div by shifting

lemma *pos-zdiv-mult-2*: $(0::int) \leq a \Longrightarrow (1 + 2*b)\ div\ (2*a) = b\ div\ a$
proof *cases*
  assume *a=0*
    thus *?thesis* **by** *simp*
next
  assume $a \neq 0$ **and** *le-a*: $0 \leq a$
  hence *a-pos*: $1 \leq a$ **by** *arith*
  hence *one-less-a2*: $1 < 2 * a$ **by** *arith*
  hence *le-2a*: $2 * (1 + b\ mod\ a) \leq 2 * a$
    unfolding *mult-le-cancel-left*
    by (*simp add*: *add1-zle-eq add-commute* [*of 1*])
  with *a-pos* **have** $0 \leq b\ mod\ a$ **by** *simp*
  hence *le-addm*: $0 \leq 1\ mod\ (2*a) + 2*(b\ mod\ a)$
    by (*simp add*: *mod-pos-pos-trivial one-less-a2*)
  with *le-2a*
  have $(1\ mod\ (2*a) + 2*(b\ mod\ a))\ div\ (2*a) = 0$
    by (*simp add*: *div-pos-pos-trivial le-addm mod-pos-pos-trivial one-less-a2*
        right-distrib)
  thus *?thesis*
    by (*subst zdiv-zadd1-eq*,
      simp add: *mod-mult-mult1 one-less-a2*
        div-pos-pos-trivial)
qed

lemma *neg-zdiv-mult-2*:
  assumes *A*: $a \leq (0::int)$ **shows** $(1 + 2*b)\ div\ (2*a) = (b+1)\ div\ a$
proof −
  have *R*: $1 + -(2 * (b + 1)) = -(1 + 2 * b)$ **by** *simp*
  have $(1 + 2 * (-b - 1))\ div\ (2 * (-a)) = (-b - 1)\ div\ (-a)$
    by (*rule pos-zdiv-mult-2*, *simp add*: *A*)
  thus *?thesis*
    by (*simp only*: *R zdiv-zminus-zminus diff-minus*
      minus-add-distrib [*symmetric*] *mult-minus-right*)
qed

lemma *zdiv-number-of-Bit0* [*simp*]:
    number-of (*Int.Bit0 v*) *div number-of* (*Int.Bit0 w*) =
      number-of v div (*number-of w* :: *int*)
by (*simp only*: *number-of-eq numeral-simps*) (*simp add*: *mult-2* [*symmetric*])

lemma *zdiv-number-of-Bit1* [*simp*]:
    number-of (*Int.Bit1 v*) *div number-of* (*Int.Bit0 w*) =
      (*if* $(0::int) \leq$ *number-of w*
       then number-of v div (*number-of w*)
       else (*number-of v* + $(1::int)$) *div* (*number-of w*))
apply (*simp only*: *number-of-eq numeral-simps UNIV-I split*: *split-if*)

apply (*simp add*: *pos-zdiv-mult-2 neg-zdiv-mult-2 add-ac mult-2* [*symmetric*])
done

33.4.15 Computing mod by Shifting (proofs resemble those for div)

lemma *pos-zmod-mult-2*:
fixes *a b* :: *int*
assumes $0 \leq a$
shows $(1 + 2 * b)\ mod\ (2 * a) = 1 + 2 * (b\ mod\ a)$
proof (*cases* $0 < a$)
case *False* **with** *assms* **show** *?thesis* **by** *simp*
next
case *True*
then have $b\ mod\ a < a$ **by** (*rule pos-mod-bound*)
then have $1 + b\ mod\ a \leq a$ **by** *simp*
then have *A*: $2 * (1 + b\ mod\ a) \leq 2 * a$ **by** *simp*
from ‹$0 < a$› **have** $0 \leq b\ mod\ a$ **by** (*rule pos-mod-sign*)
then have *B*: $0 \leq 1 + 2 * (b\ mod\ a)$ **by** *simp*
have $((1::int)\ mod\ ((2::int) * a) + (2::int) * b\ mod\ ((2::int) * a))\ mod\ ((2::int) * a) = (1::int) + (2::int) * (b\ mod\ a)$
using ‹$0 < a$› **and** *A*
by (*auto simp add*: *mod-mult-mult1 mod-pos-pos-trivial ring-distribs intro*!: *mod-pos-pos-trivial B*)
then show *?thesis* **by** (*subst mod-add-eq*)
qed

lemma *neg-zmod-mult-2*:
fixes *a b* :: *int*
assumes $a \leq 0$
shows $(1 + 2 * b)\ mod\ (2 * a) = 2 * ((b + 1)\ mod\ a) - 1$
proof −
from *assms* **have** $0 \leq -a$ **by** *auto*
then have $(1 + 2 * (-b - 1))\ mod\ (2 * (-a)) = 1 + 2 * ((-b - 1)\ mod\ (-a))$
by (*rule pos-zmod-mult-2*)
then show *?thesis* **by** (*simp add*: *zmod-zminus2 algebra-simps*) (*simp add*: *diff-minus add-ac*)
qed

lemma *zmod-number-of-Bit0* [*simp*]:
number-of (*Int.Bit0 v*) *mod number-of* (*Int.Bit0 w*) =
(*2*::*int*) * (*number-of v mod number-of w*)
apply (*simp only*: *number-of-eq numeral-simps*)
apply (*simp add*: *mod-mult-mult1 pos-zmod-mult-2*
neg-zmod-mult-2 add-ac mult-2 [*symmetric*])
done

lemma *zmod-number-of-Bit1* [*simp*]:

```
    number-of (Int.Bit1 v) mod number-of (Int.Bit0 w) =
     (if (0::int) ≤ number-of w
          then 2 * (number-of v mod number-of w) + 1
          else 2 * ((number-of v + (1::int)) mod number-of w) − 1)
apply (simp only: number-of-eq numeral-simps)
apply (simp add: mod-mult-mult1 pos-zmod-mult-2
            neg-zmod-mult-2 add-ac mult-2 [symmetric])
done
```

33.4.16 Quotients of Signs

```
lemma div-neg-pos-less0: [| a < (0::int);  0 < b |] ==> a div b < 0
apply (subgoal-tac a div b ≤ −1, force)
apply (rule order-trans)
apply (rule-tac a' = −1 in zdiv-mono1)
apply (auto simp add: div-eq-minus1)
done

lemma div-nonneg-neg-le0: [| (0::int) ≤ a; b < 0 |] ==> a div b ≤ 0
by (drule zdiv-mono1-neg, auto)

lemma div-nonpos-pos-le0: [| (a::int) ≤ 0; b > 0 |] ==> a div b ≤ 0
by (drule zdiv-mono1, auto)
```

Now for some equivalences of the form $a \mathit{div} b >=< 0 \longleftrightarrow \ldots$ conditional upon the sign of a or b. There are many more. They should all be simp rules unless that causes too much search.

```
lemma pos-imp-zdiv-nonneg-iff: (0::int) < b ==> (0 ≤ a div b) = (0 ≤ a)
apply auto
apply (drule-tac [2] zdiv-mono1)
apply (auto simp add: linorder-neq-iff)
apply (simp (no-asm-use) add: linorder-not-less [symmetric])
apply (blast intro: div-neg-pos-less0)
done

lemma neg-imp-zdiv-nonneg-iff:
  b < (0::int) ==> (0 ≤ a div b) = (a ≤ (0::int))
apply (subst zdiv-zminus-zminus [symmetric])
apply (subst pos-imp-zdiv-nonneg-iff, auto)
done


lemma pos-imp-zdiv-neg-iff: (0::int) < b ==> (a div b < 0) = (a < 0)
by (simp add: linorder-not-le [symmetric] pos-imp-zdiv-nonneg-iff)


lemma neg-imp-zdiv-neg-iff: b < (0::int) ==> (a div b < 0) = (0 < a)
by (simp add: linorder-not-le [symmetric] neg-imp-zdiv-nonneg-iff)
```

lemma *nonneg1-imp-zdiv-pos-iff*:
(*0*::*int*) <= *a* ⟹ (*a div b* > *0*) = (*a* >= *b* & *b*>*0*)
apply *rule*
apply *rule*
using *div-pos-pos-trivial*[*of a b*]**apply** *arith*
apply(*cases b=0*)**apply** *simp*
using *div-nonneg-neg-le0*[*of a b*]**apply** *arith*
using *int-one-le-iff-zero-less*[*of a div b*] *zdiv-mono1*[*of b a b*]**apply** *simp*
done

33.4.17 The Divides Relation

lemmas *zdvd-iff-zmod-eq-0-number-of* [*simp*] =
dvd-eq-mod-eq-0 [*of number-of x*::*int number-of y*::*int*, *standard*]

lemma *zdvd-zmod*: *f dvd m* ==> *f dvd* (*n*::*int*) ==> *f dvd m mod n*
by (*rule dvd-mod*)

lemma *zdvd-zmod-imp-zdvd*: *k dvd m mod n* ==> *k dvd n* ==> *k dvd* (*m*::*int*)
by (*rule dvd-mod-imp-dvd*)

lemma *zmult-div-cancel*: (*n*::*int*) * (*m div n*) = *m* − (*m mod n*)
using *zmod-zdiv-equality*[**where** *a*=*m* **and** *b*=*n*]
by (*simp add*: *algebra-simps*)

lemma *zpower-zmod*: ((*x*::*int*) *mod m*) ^*y mod m* = *x* ^*y mod m*
apply (*induct y*, *auto*)
apply (*rule zmod-zmult1-eq* [*THEN trans*])
apply (*simp* (*no-asm-simp*))
apply (*rule mod-mult-eq* [*symmetric*])
done

lemma *zdiv-int*: *int* (*a div b*) = (*int a*) *div* (*int b*)
apply (*subst split-div*, *auto*)
apply (*subst split-zdiv*, *auto*)
apply (*rule-tac a*=*int* (*b* * *i*) + *int j* **and** *b*=*int b* **and** *r*=*int j* **and** *r′*=*ja* **in** *unique-quotient*)
apply (*auto simp add*: *divmod-int-rel-def of-nat-mult*)
done

lemma *zmod-int*: *int* (*a mod b*) = (*int a*) *mod* (*int b*)
apply (*subst split-mod*, *auto*)
apply (*subst split-zmod*, *auto*)
apply (*rule-tac a*=*int* (*b* * *i*) + *int j* **and** *b*=*int b* **and** *q*=*int i* **and** *q′*=*ia*
in *unique-remainder*)
apply (*auto simp add*: *divmod-int-rel-def of-nat-mult*)
done

lemma *abs-div*: (*y*::*int*) *dvd x* ⟹ *abs* (*x div y*) = *abs x div abs y*

by (*unfold dvd-def*, *cases y=0*, *auto simp add*: *abs-mult*)

lemma *zdvd-mult-div-cancel*:(*n*::*int*) *dvd m* $\Longrightarrow$ *n* $*$ (*m div n*) = *m*
apply (*subgoal-tac m mod n* = *0*)
apply (*simp add*: *zmult-div-cancel*)
apply (*simp only*: *dvd-eq-mod-eq-0*)
done

Suggested by Matthias Daum

lemma *int-power-div-base*:
$[\![0 < m; 0 < k]\!] \Longrightarrow k \hat{} m \text{ div } k = (k::int) \hat{} (m - Suc\ 0)$
apply (*subgoal-tac k* ^ *m* = *k* ^ ((*m* − *Suc 0*) + *Suc 0*))
apply (*erule ssubst*)
apply (*simp only*: *power-add*)
apply *simp-all*
done

by Brian Huffman

lemma *zminus-zmod*: − ((*x*::*int*) *mod m*) *mod m* = − *x mod m*
by (*rule mod-minus-eq* [*symmetric*])

lemma *zdiff-zmod-left*: (*x mod m* − *y*) *mod m* = (*x* − *y*) *mod* (*m*::*int*)
by (*rule mod-diff-left-eq* [*symmetric*])

lemma *zdiff-zmod-right*: (*x* − *y mod m*) *mod m* = (*x* − *y*) *mod* (*m*::*int*)
by (*rule mod-diff-right-eq* [*symmetric*])

lemmas *zmod-simps* =
mod-add-left-eq [*symmetric*]
mod-add-right-eq [*symmetric*]
zmod-zmult1-eq [*symmetric*]
mod-mult-left-eq [*symmetric*]
zpower-zmod
zminus-zmod zdiff-zmod-left zdiff-zmod-right

Distributive laws for function *nat*.

lemma *nat-div-distrib*: 0 ≤ *x* $\Longrightarrow$ *nat* (*x div y*) = *nat x div nat y*
apply (*rule linorder-cases* [*of y 0*])
apply (*simp add*: *div-nonneg-neg-le0*)
apply *simp*
apply (*simp add*: *nat-eq-iff pos-imp-zdiv-nonneg-iff zdiv-int*)
done

lemma *nat-mod-distrib*:
$[\![0 \le x; 0 \le y]\!] \Longrightarrow nat\ (x \text{ mod } y) = nat\ x \text{ mod } nat\ y$
apply (*case-tac y* = *0*, *simp*)
apply (*simp add*: *nat-eq-iff zmod-int*)
done

transfer setup

lemma *transfer-nat-int-functions*:
  $(x::int) >= 0 \Longrightarrow y >= 0 \Longrightarrow (nat\ x)\ div\ (nat\ y) = nat\ (x\ div\ y)$
  $(x::int) >= 0 \Longrightarrow y >= 0 \Longrightarrow (nat\ x)\ mod\ (nat\ y) = nat\ (x\ mod\ y)$
 by (*auto simp add*: *nat-div-distrib nat-mod-distrib*)

lemma *transfer-nat-int-function-closures*:
  $(x::int) >= 0 \Longrightarrow y >= 0 \Longrightarrow x\ div\ y >= 0$
  $(x::int) >= 0 \Longrightarrow y >= 0 \Longrightarrow x\ mod\ y >= 0$
 apply (*cases* $y = 0$)
 apply (*auto simp add*: *pos-imp-zdiv-nonneg-iff*)
 apply (*cases* $y = 0$)
 apply *auto*
done

declare *transfer-morphism-nat-int* [*transfer add return*:
 transfer-nat-int-functions
 transfer-nat-int-function-closures
]

lemma *transfer-int-nat-functions*:
  $(int\ x)\ div\ (int\ y) = int\ (x\ div\ y)$
  $(int\ x)\ mod\ (int\ y) = int\ (x\ mod\ y)$
 by (*auto simp add*: *zdiv-int zmod-int*)

lemma *transfer-int-nat-function-closures*:
  $is\text{-}nat\ x \Longrightarrow is\text{-}nat\ y \Longrightarrow is\text{-}nat\ (x\ div\ y)$
  $is\text{-}nat\ x \Longrightarrow is\text{-}nat\ y \Longrightarrow is\text{-}nat\ (x\ mod\ y)$
 by (*simp-all only*: *is-nat-def transfer-nat-int-function-closures*)

declare *transfer-morphism-int-nat* [*transfer add return*:
 transfer-int-nat-functions
 transfer-int-nat-function-closures
]

Suggested by Matthias Daum

lemma *int-div-less-self*: $[\![0 < x;\ 1 < k]\!] \Longrightarrow x\ div\ k < (x::int)$
apply (*subgoal-tac nat x div nat k < nat x*)
 apply (*simp add*: *nat-div-distrib* [*symmetric*])
apply (*rule Divides.div-less-dividend*, *simp-all*)
done

lemma *zmod-eq-dvd-iff*: $(x::int)\ mod\ n = y\ mod\ n \longleftrightarrow n\ dvd\ x - y$
proof
 assume *H*: $x\ mod\ n = y\ mod\ n$
 hence $x\ mod\ n - y\ mod\ n = 0$ **by** *simp*
 hence $(x\ mod\ n - y\ mod\ n)\ mod\ n = 0$ **by** *simp*
 hence $(x - y)\ mod\ n = 0$ **by** (*simp add*: *mod-diff-eq*[*symmetric*])
 thus $n\ dvd\ x - y$ **by** (*simp add*: *dvd-eq-mod-eq-0*)

next
assume *H*: *n dvd x − y*
then obtain *k* **where** *k*: *x−y = n∗k* **unfolding** *dvd-def* **by** *blast*
hence *x = n∗k + y* **by** *simp*
hence *x mod n = (n∗k + y) mod n* **by** *simp*
thus *x mod n = y mod n* **by** (*simp add*: *mod-add-left-eq*)
qed

lemma *nat-mod-eq-lemma*: **assumes** *xyn*: (*x*::*nat*) *mod n = y mod n* **and** *xy*:*y ≤ x*
shows ∃ *q*. *x = y + n ∗ q*
proof−
from *xy* **have** *th*: *int x − int y = int* (*x − y*) **by** *simp*
from *xyn* **have** *int x mod int n = int y mod int n*
by (*simp add*: *zmod-int*[*symmetric*])
hence *int n dvd int x − int y* **by** (*simp only*: *zmod-eq-dvd-iff*[*symmetric*])
hence *n dvd x − y* **by** (*simp add*: *th zdvd-int*)
then show *?thesis* **using** *xy* **unfolding** *dvd-def* **apply** *clarsimp* **apply** (*rule-tac x=k* **in** *exI*) **by** *arith*
qed

lemma *nat-mod-eq-iff*: (*x*::*nat*) *mod n = y mod n* ⟷ (∃ *q1 q2*. *x + n ∗ q1 = y + n ∗ q2*)
(**is** *?lhs = ?rhs*)
proof
assume *H*: *x mod n = y mod n*
{assume *xy*: *x ≤ y*
from *H* **have** *th*: *y mod n = x mod n* **by** *simp*
from *nat-mod-eq-lemma*[*OF th xy*] **have** *?rhs*
apply *clarify* **apply** (*rule-tac x=q* **in** *exI*) **by** (*rule exI*[**where** *x=0*], *simp*)**}**
moreover
{assume *xy*: *y ≤ x*
from *nat-mod-eq-lemma*[*OF H xy*] **have** *?rhs*
apply *clarify* **apply** (*rule-tac x=0* **in** *exI*) **by** (*rule-tac x=q* **in** *exI*, *simp*)**}**
ultimately show *?rhs* **using** *linear*[*of x y*] **by** *blast*
next
assume *?rhs* **then obtain** *q1 q2* **where** *q12*: *x + n ∗ q1 = y + n ∗ q2* **by** *blast*
hence (*x + n ∗ q1*) *mod n* = (*y + n ∗ q2*) *mod n* **by** *simp*
thus *?lhs* **by** *simp*
qed

lemma *div-nat-number-of* [*simp*]:
(*number-of v* :: *nat*) *div number-of v′* =
(*if neg* (*number-of v* :: *int*) *then 0*
else nat (*number-of v div number-of v′*))
unfolding *nat-number-of-def number-of-is-id neg-def*
by (*simp add*: *nat-div-distrib*)

lemma *one-div-nat-number-of* [*simp*]:

Suc 0 div number-of v′ = nat (1 div number-of v′)
by (*simp del*: *nat-numeral-1-eq-1 add*: *numeral-1-eq-Suc-0* [*symmetric*])

lemma *mod-nat-number-of* [*simp*]:
(number-of v :: nat) mod number-of v′ =
(if neg (number-of v :: int) then 0
else if neg (number-of v′ :: int) then number-of v
else nat (number-of v mod number-of v′))
unfolding *nat-number-of-def number-of-is-id neg-def*
by (*simp add*: *nat-mod-distrib*)

lemma *one-mod-nat-number-of* [*simp*]:
Suc 0 mod number-of v′ =
(if neg (number-of v′ :: int) then Suc 0
else nat (1 mod number-of v′))
by (*simp del*: *nat-numeral-1-eq-1 add*: *numeral-1-eq-Suc-0* [*symmetric*])

lemmas *dvd-eq-mod-eq-0-number-of =*
dvd-eq-mod-eq-0 [*of number-of x number-of y*, *standard*]

declare *dvd-eq-mod-eq-0-number-of* [*simp*]

33.4.18 Nitpick

lemma *zmod-zdiv-equality′*:
(m::int) mod n = m − (m div n) ∗ n
by (*rule-tac P*=%*x*. *m mod n = x − (m div n) ∗ n*
in *subst* [*OF mod-div-equality* [*of* - *n*]])
arith

lemmas [*nitpick-def*] = *dvd-eq-mod-eq-0* [*THEN eq-reflection*]
mod-div-equality′ [*THEN eq-reflection*]
zmod-zdiv-equality′ [*THEN eq-reflection*]

33.4.19 Code generation

definition *pdivmod* :: *int* ⇒ *int* ⇒ *int* × *int* **where**
pdivmod k l = (|k| div |l|, |k| mod |l|)

lemma *pdivmod-posDivAlg* [*code*]:
pdivmod k l = (if l = 0 then (0, |k|) else posDivAlg |k| |l|)
by (*subst posDivAlg-div-mod*) (*simp-all add*: *pdivmod-def*)

lemma *divmod-int-pdivmod*: *divmod-int k l = (if k = 0 then (0, 0) else if l = 0 then (0, k) else*
apsnd ((op ∗) (sgn l)) (if 0 < l ∧ 0 ≤ k ∨ l < 0 ∧ k < 0
then pdivmod k l
else (let (r, s) = pdivmod k l in
if s = 0 then (− r, 0) else (− r − 1, |l| − s))))
proof −

```
  have aux: ⋀q::int. − k = l * q ⟷ k = l * − q by auto
  show ?thesis
    by (simp add: divmod-int-mod-div pdivmod-def)
      (auto simp add: aux not-less not-le zdiv-zminus1-eq-if
      zmod-zminus1-eq-if zdiv-zminus2-eq-if zmod-zminus2-eq-if)
qed

lemma divmod-int-code [code]: divmod-int k l = (if k = 0 then (0, 0) else if l =
0 then (0, k) else
  apsnd ((op *) (sgn l)) (if sgn k = sgn l
    then pdivmod k l
    else (let (r, s) = pdivmod k l in
      if s = 0 then (− r, 0) else (− r − 1, |l| − s))))
proof −
  have k ≠ 0 ⟹ l ≠ 0 ⟹ 0 < l ∧ 0 ≤ k ∨ l < 0 ∧ k < 0 ⟷ sgn k = sgn l
    by (auto simp add: not-less sgn-if)
  then show ?thesis by (simp add: divmod-int-pdivmod)
qed

context ring-1
begin

lemma of-int-num [code]:
  of-int k = (if k = 0 then 0 else if k < 0 then
    − of-int (− k) else let
      (l, m) = divmod-int k 2;
      l' = of-int l
    in if m = 0 then l' + l' else l' + l' + 1)
proof −
  have aux1: k mod (2::int) ≠ (0::int) ⟹
    of-int k = of-int (k div 2 * 2 + 1)
  proof −
    have k mod 2 < 2 by (auto intro: pos-mod-bound)
    moreover have 0 ≤ k mod 2 by (auto intro: pos-mod-sign)
    moreover assume k mod 2 ≠ 0
    ultimately have k mod 2 = 1 by arith
    moreover have of-int k = of-int (k div 2 * 2 + k mod 2) by simp
    ultimately show ?thesis by auto
  qed
  have aux2: ⋀x. of-int 2 * x = x + x
  proof −
    fix x
    have int2: (2::int) = 1 + 1 by arith
    show of-int 2 * x = x + x
    unfolding int2 of-int-add left-distrib by simp
  qed
  have aux3: ⋀x. x * of-int 2 = x + x
  proof −
    fix x
```

have *int2*: (*2*::*int*) = *1* + *1* **by** *arith*
show *x* ∗ *of-int 2* = *x* + *x*
unfolding *int2 of-int-add right-distrib* **by** *simp*
qed
from *aux1* **show** *?thesis* **by** (*auto simp add*: *divmod-int-mod-div Let-def aux2 aux3*)
qed

end

code-modulename *SML*
Divides Arith

code-modulename *OCaml*
Divides Arith

code-modulename *Haskell*
Divides Arith

end

34 Code-Numeral: Type of target language numerals

theory *Code-Numeral*
imports *Nat-Numeral Nat-Transfer Divides*
begin

Code numerals are isomorphic to HOL *nat* but mapped to target-language builtin numerals.

34.1 Datatype of target language numerals

typedef (**open**) *code-numeral* = *UNIV* :: *nat set*
morphisms *nat-of of-nat* **by** *rule*

lemma *of-nat-nat-of* [*simp*]:
of-nat (*nat-of k*) = *k*
by (*rule nat-of-inverse*)

lemma *nat-of-of-nat* [*simp*]:
nat-of (*of-nat n*) = *n*
by (*rule of-nat-inverse*) (*rule UNIV-I*)

lemma [*measure-function*]:
is-measure nat-of **by** (*rule is-measure-trivial*)

lemma *code-numeral*:
(⋀*n*::*code-numeral*. *PROP P n*) ≡ (⋀*n*::*nat*. *PROP P* (*of-nat n*))
proof
fix *n* :: *nat*
assume ⋀*n*::*code-numeral*. *PROP P n*
then show *PROP P* (*of-nat n*) **.**
next
fix *n* :: *code-numeral*
assume ⋀*n*::*nat*. *PROP P* (*of-nat n*)
then have *PROP P* (*of-nat* (*nat-of n*)) **.**
then show *PROP P n* **by** *simp*
qed

lemma *code-numeral-case*:
assumes ⋀*n*. *k* = *of-nat n* ⟹ *P*
shows *P*
by (*rule assms* [*of nat-of k*]) *simp*

lemma *code-numeral-induct-raw*:
assumes ⋀*n*. *P* (*of-nat n*)
shows *P k*
proof −
from *assms* **have** *P* (*of-nat* (*nat-of k*)) **.**
then show *?thesis* **by** *simp*
qed

lemma *nat-of-inject* [*simp*]:
nat-of k = *nat-of l* ⟷ *k* = *l*
by (*rule nat-of-inject*)

lemma *of-nat-inject* [*simp*]:
of-nat n = *of-nat m* ⟷ *n* = *m*
by (*rule of-nat-inject*) (*rule UNIV-I*)+

instantiation *code-numeral* :: *zero*
begin

definition [*simp*, *code del*]:
0 = *of-nat 0*

instance ..

end

definition [*simp*]:
Suc-code-numeral k = *of-nat* (*Suc* (*nat-of k*))

rep-datatype *0* :: *code-numeral Suc-code-numeral*
proof −

fix P :: *code-numeral* $\Rightarrow$ *bool*
fix k :: *code-numeral*
assume $P\ 0$ **then have** *init*: P (*of-nat 0*) **by** *simp*
assume $\bigwedge k.\ P\ k \Longrightarrow P$ (*Suc-code-numeral k*)
then have $\bigwedge n.\ P$ (*of-nat n*) $\Longrightarrow P$ (*Suc-code-numeral* (*of-nat n*)) **.**
then have *step*: $\bigwedge n.\ P$ (*of-nat n*) $\Longrightarrow P$ (*of-nat* (*Suc n*)) **by** *simp*
from *init step* **have** P (*of-nat* (*nat-of k*))
by (*induct* (*nat-of k*)) *simp-all*
then show $P\ k$ **by** *simp*
qed *simp-all*

declare *code-numeral-case* [*case-names nat*, *cases type*: *code-numeral*]
declare *code-numeral.induct* [*case-names nat*, *induct type*: *code-numeral*]

lemma *code-numeral-decr* [*termination-simp*]:
$k \neq$ *of-nat 0* $\Longrightarrow$ *nat-of k* $-$ *Suc 0* $<$ *nat-of k*
by (*cases k*) *simp*

lemma [*simp*, *code*]:
code-numeral-size $=$ *nat-of*
proof (*rule ext*)
fix k
have *code-numeral-size k* $=$ *nat-size* (*nat-of k*)
by (*induct k rule*: *code-numeral.induct*) (*simp-all del*: *zero-code-numeral-def Suc-code-numeral-def*, *simp-all*)
also have *nat-size* (*nat-of k*) $=$ *nat-of k* **by** (*induct* (*nat-of k*)) *simp-all*
finally show *code-numeral-size k* $=$ *nat-of k* **.**
qed

lemma [*simp*, *code*]:
size $=$ *nat-of*
proof (*rule ext*)
fix k
show *size k* $=$ *nat-of k*
by (*induct k*) (*simp-all del*: *zero-code-numeral-def Suc-code-numeral-def*, *simp-all*)
qed

lemmas [*code del*] $=$ *code-numeral.recs code-numeral.cases*

lemma [*code*]:
eq-class.eq k l $\longleftrightarrow$ *eq-class.eq* (*nat-of k*) (*nat-of l*)
by (*cases k*, *cases l*) (*simp add*: *eq*)

lemma [*code nbe*]:
eq-class.eq (*k*::*code-numeral*) *k* $\longleftrightarrow$ *True*
by (*rule HOL.eq-refl*)

34.2 Indices as datatype of ints

instantiation *code-numeral* :: *number*
begin

definition
number-of = *of-nat o nat*

instance ..

end

lemma *nat-of-number* [*simp*]:
nat-of (*number-of k*) = *number-of k*
by (*simp add*: *number-of-code-numeral-def nat-number-of-def number-of-is-id*)

code-datatype *number-of* :: *int* ⇒ *code-numeral*

34.3 Basic arithmetic

instantiation *code-numeral* :: {*minus*, *linordered-semidom*, *semiring-div*, *linorder*}
begin

definition [*simp*, *code del*]:
(*1*::*code-numeral*) = *of-nat 1*

definition [*simp*, *code del*]:
n + *m* = *of-nat* (*nat-of n* + *nat-of m*)

definition [*simp*, *code del*]:
n − *m* = *of-nat* (*nat-of n* − *nat-of m*)

definition [*simp*, *code del*]:
n ∗ *m* = *of-nat* (*nat-of n* ∗ *nat-of m*)

definition [*simp*, *code del*]:
n div m = *of-nat* (*nat-of n div nat-of m*)

definition [*simp*, *code del*]:
n mod m = *of-nat* (*nat-of n mod nat-of m*)

definition [*simp*, *code del*]:
n ≤ *m* ⟷ *nat-of n* ≤ *nat-of m*

definition [*simp*, *code del*]:
n < *m* ⟷ *nat-of n* < *nat-of m*

instance proof
qed (*auto simp add*: *code-numeral left-distrib intro*: *mult-commute*)

end

lemma *zero-code-numeral-code* [*code*, *code-unfold*]:
(*0*::*code-numeral*) = *Numeral0*
by (*simp add*: *number-of-code-numeral-def Pls-def*)
lemma [*code-post*]: *Numeral0* = (*0*::*code-numeral*)
using *zero-code-numeral-code* **..**

lemma *one-code-numeral-code* [*code*, *code-unfold*]:
(*1*::*code-numeral*) = *Numeral1*
by (*simp add*: *number-of-code-numeral-def Pls-def Bit1-def*)
lemma [*code-post*]: *Numeral1* = (*1*::*code-numeral*)
using *one-code-numeral-code* **..**

lemma *plus-code-numeral-code* [*code nbe*]:
of-nat n + *of-nat m* = *of-nat* (*n* + *m*)
by *simp*

definition *subtract-code-numeral* :: *code-numeral* ⇒ *code-numeral* ⇒ *code-numeral*
where
[*simp*, *code del*]: *subtract-code-numeral* = *op* −

lemma *subtract-code-numeral-code* [*code nbe*]:
subtract-code-numeral (*of-nat n*) (*of-nat m*) = *of-nat* (*n* − *m*)
by *simp*

lemma *minus-code-numeral-code* [*code*]:
n − *m* = *subtract-code-numeral n m*
by *simp*

lemma *times-code-numeral-code* [*code nbe*]:
of-nat n ∗ *of-nat m* = *of-nat* (*n* ∗ *m*)
by *simp*

lemma *less-eq-code-numeral-code* [*code nbe*]:
of-nat n ≤ *of-nat m* ⟷ *n* ≤ *m*
by *simp*

lemma *less-code-numeral-code* [*code nbe*]:
of-nat n < *of-nat m* ⟷ *n* < *m*
by *simp*

lemma *code-numeral-zero-minus-one*:
(*0*::*code-numeral*) − *1* = *0*
by *simp*

lemma *Suc-code-numeral-minus-one*:
Suc-code-numeral n − *1* = *n*
by *simp*

lemma *of-nat-code* [*code*]:
of-nat = *Nat.of-nat*
proof
fix *n* :: *nat*
have *Nat.of-nat n* = *of-nat n*
by (*induct n*) *simp-all*
then show *of-nat n* = *Nat.of-nat n*
by (*rule sym*)
qed

lemma *code-numeral-not-eq-zero*: *i* ≠ *of-nat 0* ⟷ *i* ≥ *1*
by (*cases i*) *auto*

definition *nat-of-aux* :: *code-numeral* ⇒ *nat* ⇒ *nat* **where**
nat-of-aux i n = *nat-of i* + *n*

lemma *nat-of-aux-code* [*code*]:
nat-of-aux i n = (*if i* = *0 then n else nat-of-aux* (*i* − *1*) (*Suc n*))
by (*auto simp add*: *nat-of-aux-def code-numeral-not-eq-zero*)

lemma *nat-of-code* [*code*]:
nat-of i = *nat-of-aux i 0*
by (*simp add*: *nat-of-aux-def*)

definition *div-mod-code-numeral* :: *code-numeral* ⇒ *code-numeral* ⇒ *code-numeral*
× *code-numeral* **where**
[*code del*]: *div-mod-code-numeral n m* = (*n div m*, *n mod m*)

lemma [*code*]:
div-mod-code-numeral n m = (*if m* = *0 then* (*0*, *n*) *else* (*n div m*, *n mod m*))
unfolding *div-mod-code-numeral-def* **by** *auto*

lemma [*code*]:
n div m = *fst* (*div-mod-code-numeral n m*)
unfolding *div-mod-code-numeral-def* **by** *simp*

lemma [*code*]:
n mod m = *snd* (*div-mod-code-numeral n m*)
unfolding *div-mod-code-numeral-def* **by** *simp*

definition *int-of* :: *code-numeral* ⇒ *int* **where**
int-of = *Nat.of-nat o nat-of*

lemma *int-of-code* [*code*]:
int-of k = (*if k* = *0 then 0*
else (*if k mod 2* = *0 then 2* ∗ *int-of* (*k div 2*) *else 2* ∗ *int-of* (*k div 2*) + *1*))
proof −
have (*nat-of k div 2*) ∗ *2* + *nat-of k mod 2* = *nat-of k*

by (*rule mod-div-equality*)
then have *int* ((*nat-of k div 2*) * *2* + *nat-of k mod 2*) = *int* (*nat-of k*)
by *simp*
then have *int* (*nat-of k*) = *int* (*nat-of k div 2*) * *2* + *int* (*nat-of k mod 2*)
unfolding *int-mult zadd-int* [*symmetric*] **by** *simp*
then show *?thesis* **by** (*auto simp add*: *int-of-def mult-ac*)
qed

hide-const (**open**) *of-nat nat-of int-of*

34.3.1 Lazy Evaluation of an indexed function

function *iterate-upto* :: (*code-numeral* => *'a*) => *code-numeral* => *code-numeral* => *'a Predicate.pred*
where
iterate-upto f n m = *Predicate.Seq* (%*u*. *if n* > *m then Predicate.Empty else Predicate.Insert* (*f n*) (*iterate-upto f* (*n* + *1*) *m*))
by *pat-completeness auto*

termination by (*relation measure* (%(*f*, *n*, *m*). *Code-Numeral.nat-of* (*m* + *1* − *n*))) *auto*

hide-const (**open**) *iterate-upto*

34.4 Code generator setup

Implementation of indices by bounded integers

code-type *code-numeral*
(*SML int*)
(*OCaml Big'-int.big'-int*)
(*Haskell Int*)
(*Scala Int*)

code-instance *code-numeral* :: *eq*
(*Haskell* −)

setup ⟪
fold (*Numeral.add-code* @{*const-name number-code-numeral-inst.number-of-code-numeral*}
false Code-Printer.literal-naive-numeral) [*SML*, *Haskell*]
#> *Numeral.add-code* @{*const-name number-code-numeral-inst.number-of-code-numeral*}
false Code-Printer.literal-numeral OCaml
#> *Numeral.add-code* @{*const-name number-code-numeral-inst.number-of-code-numeral*}
false Code-Printer.literal-naive-numeral Scala
⟫

code-reserved *SML Int int*
code-reserved *Scala Int*

code-const *op* + :: *code-numeral* ⇒ *code-numeral* ⇒ *code-numeral*

```
  (SML Int.+/ ((-),/ (-)))
  (OCaml Big'-int.add'-big'-int)
  (Haskell infixl 6 +)
  (Scala infixl 7 +)

code-const subtract-code-numeral :: code-numeral ⇒ code-numeral ⇒ code-numeral
  (SML Int.max/ (-/ −/ -,/ 0 : int))
 (OCaml Big'-int.max'-big'-int/ (Big'-int.sub'-big'-int/ -/ -)/ Big'-int.zero'-big'-int)
  (Haskell max/ (-/ −/ -)/ (0 :: Int))
  (Scala !(-/ −/ -).max(0))

code-const op * :: code-numeral ⇒ code-numeral ⇒ code-numeral
  (SML Int.*/ ((-),/ (-)))
  (OCaml Big'-int.mult'-big'-int)
  (Haskell infixl 7 *)
  (Scala infixl 8 *)

code-const div-mod-code-numeral
  (SML !(fn n => fn m =>/ if m = 0/ then (0, n) else/ (n div m, n mod m)))
 (OCaml Big'-int.quomod'-big'-int/ (Big'-int.abs'-big'-int -)/ (Big'-int.abs'-big'-int
-))
  (Haskell divMod)
  (Scala !((n: Int) => (m: Int) =>/ if (m == 0)/ (0, n) else/ (n '/ m, n %
m)))

code-const eq-class.eq :: code-numeral ⇒ code-numeral ⇒ bool
  (SML !((- : Int.int) = -))
  (OCaml Big'-int.eq'-big'-int)
  (Haskell infixl 4 ==)
  (Scala infixl 5 ==)

code-const op ≤ :: code-numeral ⇒ code-numeral ⇒ bool
  (SML Int.<=/ ((-),/ (-)))
  (OCaml Big'-int.le'-big'-int)
  (Haskell infix 4 <=)
  (Scala infixl 4 <=)

code-const op < :: code-numeral ⇒ code-numeral ⇒ bool
  (SML Int.</ ((-),/ (-)))
  (OCaml Big'-int.lt'-big'-int)
  (Haskell infix 4 <)
  (Scala infixl 4 <)

end
```

35 Numeral-Simprocs: Combination and Cancellation Simprocs for Numeral Expressions

theory *Numeral-Simprocs*
imports *Divides*
uses
~~/*src*/*Provers*/*Arith*/*assoc-fold.ML*
~~/*src*/*Provers*/*Arith*/*cancel-numerals.ML*
~~/*src*/*Provers*/*Arith*/*combine-numerals.ML*
~~/*src*/*Provers*/*Arith*/*cancel-numeral-factor.ML*
~~/*src*/*Provers*/*Arith*/*extract-common-term.ML*
(*Tools*/*numeral-simprocs.ML*)
(*Tools*/*nat-numeral-simprocs.ML*)
begin

declare *split-div* [*of* - - *number-of k*, *standard*, *arith-split*]
declare *split-mod* [*of* - - *number-of k*, *standard*, *arith-split*]

For *combine-numerals*

lemma *left-add-mult-distrib*: $i*u + (j*u + k) = (i+j)*u + (k::nat)$
by (*simp add*: *add-mult-distrib*)

For *cancel-numerals*

lemma *nat-diff-add-eq1*:
$j <= (i::nat) ==> ((i*u + m) - (j*u + n)) = (((i-j)*u + m) - n)$
by (*simp split add*: *nat-diff-split add*: *add-mult-distrib*)

lemma *nat-diff-add-eq2*:
$i <= (j::nat) ==> ((i*u + m) - (j*u + n)) = (m - ((j-i)*u + n))$
by (*simp split add*: *nat-diff-split add*: *add-mult-distrib*)

lemma *nat-eq-add-iff1*:
$j <= (i::nat) ==> (i*u + m = j*u + n) = ((i-j)*u + m = n)$
by (*auto split add*: *nat-diff-split simp add*: *add-mult-distrib*)

lemma *nat-eq-add-iff2*:
$i <= (j::nat) ==> (i*u + m = j*u + n) = (m = (j-i)*u + n)$
by (*auto split add*: *nat-diff-split simp add*: *add-mult-distrib*)

lemma *nat-less-add-iff1*:
$j <= (i::nat) ==> (i*u + m < j*u + n) = ((i-j)*u + m < n)$
by (*auto split add*: *nat-diff-split simp add*: *add-mult-distrib*)

lemma *nat-less-add-iff2*:
$i <= (j::nat) ==> (i*u + m < j*u + n) = (m < (j-i)*u + n)$
by (*auto split add*: *nat-diff-split simp add*: *add-mult-distrib*)

lemma *nat-le-add-iff1*:
$j <= (i::nat) ==> (i*u + m <= j*u + n) = ((i-j)*u + m <= n)$

by (*auto split add*: *nat-diff-split simp add*: *add-mult-distrib*)

lemma *nat-le-add-iff2*:
  $i <= (j::nat) ==> (i*u + m <= j*u + n) = (m <= (j-i)*u + n)$
by (*auto split add*: *nat-diff-split simp add*: *add-mult-distrib*)

For *cancel-numeral-factors*

lemma *nat-mult-le-cancel1*: $(0::nat) < k ==> (k*m <= k*n) = (m<=n)$
by *auto*

lemma *nat-mult-less-cancel1*: $(0::nat) < k ==> (k*m < k*n) = (m<n)$
by *auto*

lemma *nat-mult-eq-cancel1*: $(0::nat) < k ==> (k*m = k*n) = (m=n)$
by *auto*

lemma *nat-mult-div-cancel1*: $(0::nat) < k ==> (k*m)\ div\ (k*n) = (m\ div\ n)$
by *auto*

lemma *nat-mult-dvd-cancel-disj*[*simp*]:
  $(k*m)\ dvd\ (k*n) = (k=0 \mid m\ dvd\ (n::nat))$
by(*auto simp*: *dvd-eq-mod-eq-0 mod-mult-distrib2*[*symmetric*])

lemma *nat-mult-dvd-cancel1*: $0 < k \Longrightarrow (k*m)\ dvd\ (k*n::nat) = (m\ dvd\ n)$
by(*auto*)

For *cancel-factor*

lemma *nat-mult-le-cancel-disj*: $(k*m <= k*n) = ((0::nat) < k --> m<=n)$
by *auto*

lemma *nat-mult-less-cancel-disj*: $(k*m < k*n) = ((0::nat) < k \ \&\ m<n)$
by *auto*

lemma *nat-mult-eq-cancel-disj*: $(k*m = k*n) = (k = (0::nat) \mid m=n)$
by *auto*

lemma *nat-mult-div-cancel-disj*[*simp*]:
  $(k*m)\ div\ (k*n) = (if\ k = (0::nat)\ then\ 0\ else\ m\ div\ n)$
by (*simp add*: *nat-mult-div-cancel1*)

use *Tools/numeral-simprocs.ML*

use *Tools/nat-numeral-simprocs.ML*

declaration ⟪
  K (*Lin-Arith.add-simps* (@{*thms neg-simps*} @ [@{*thm Suc-nat-number-of*}, @{*thm int-nat-number-of*}])
  #> *Lin-Arith.add-simps* (@{*thms ring-distribs*} @ [@{*thm Let-number-of*}, @{*thm*

Let-0}, @{*thm Let-1*},
@{*thm nat-0*}, @{*thm nat-1*},
@{*thm add-nat-number-of*}, @{*thm diff-nat-number-of*}, @{*thm mult-nat-number-of*},
@{*thm eq-nat-number-of*}, @{*thm less-nat-number-of*}, @{*thm le-number-of-eq-not-less*},
@{*thm le-Suc-number-of*}, @{*thm le-number-of-Suc*},
@{*thm less-Suc-number-of*}, @{*thm less-number-of-Suc*},
@{*thm Suc-eq-number-of*}, @{*thm eq-number-of-Suc*},
@{*thm mult-Suc*}, @{*thm mult-Suc-right*},
@{*thm add-Suc*}, @{*thm add-Suc-right*},
@{*thm eq-number-of-0*}, @{*thm eq-0-number-of*}, @{*thm less-0-number-of*},
@{*thm of-int-number-of-eq*}, @{*thm of-nat-number-of-eq*}, @{*thm nat-number-of*},
@{*thm if-True*}, @{*thm if-False*}])
#> *Lin-Arith.add-simprocs* (*Numeral-Simprocs.assoc-fold-simproc*
:: *Numeral-Simprocs.combine-numerals*
:: *Numeral-Simprocs.cancel-numerals*)
#> *Lin-Arith.add-simprocs* (*Nat-Numeral-Simprocs.combine-numerals* :: *Nat-Numeral-Simprocs.cancel-nume*
⟫

end

36 Semiring-Normalization: Semiring normalization

theory *Semiring-Normalization*
imports *Numeral-Simprocs Nat-Transfer*
uses
Tools/semiring-normalizer.ML
begin

Prelude

class *comm-semiring-1-cancel-crossproduct* = *comm-semiring-1-cancel* +
assumes *crossproduct-eq*: $w * y + x * z = w * z + x * y \longleftrightarrow w = x \vee y = z$
begin

lemma *crossproduct-noteq*:
$a \neq b \wedge c \neq d \longleftrightarrow a * c + b * d \neq a * d + b * c$
by (*simp add*: *crossproduct-eq*)

lemma *add-scale-eq-noteq*:
$r \neq 0 \Longrightarrow a = b \wedge c \neq d \Longrightarrow a + r * c \neq b + r * d$
proof (*rule notI*)
assume *nz*: $r \neq 0$ **and** *cnd*: $a = b \wedge c \neq d$
and *eq*: $a + (r * c) = b + (r * d)$
have $(0 * d) + (r * c) = (0 * c) + (r * d)$
using *add-imp-eq eq mult-zero-left* **by** (*simp add*: *cnd*)
then show *False* **using** *crossproduct-eq* [*of 0 d*] *nz cnd* **by** *simp*
qed

lemma *add-0-iff*:
  $b = b + a \longleftrightarrow a = 0$
  using *add-imp-eq* [*of b a 0*] **by** *auto*

end

sublocale *idom* < *comm-semiring-1-cancel-crossproduct*
proof
  fix *w x y z*
  show $w * y + x * z = w * z + x * y \longleftrightarrow w = x \vee y = z$
  proof
    assume $w * y + x * z = w * z + x * y$
    then have $w * y + x * z - w * z - x * y = 0$ **by** (*simp add*: *algebra-simps*)
    then have $w * (y - z) - x * (y - z) = 0$ **by** (*simp add*: *algebra-simps*)
    then have $(y - z) * (w - x) = 0$ **by** (*simp add*: *algebra-simps*)
    then have $y - z = 0 \vee w - x = 0$ **by** (*rule divisors-zero*)
    then show $w = x \vee y = z$ **by** *auto*
  qed (*auto simp add*: *add-ac*)
qed

instance *nat* :: *comm-semiring-1-cancel-crossproduct*
proof
  fix *w x y z* :: *nat*
  have *aux*: $\bigwedge y\ z.\ y < z \Longrightarrow w * y + x * z = w * z + x * y \Longrightarrow w = x$
  proof −
    fix *y z* :: *nat*
    assume $y < z$ **then have** $\exists k.\ z = y + k \wedge k \neq 0$ **by** (*intro exI* [*of* - *z* − *y*])
auto
    then obtain *k* **where** $z = y + k$ **and** $k \neq 0$ **by** *blast*
    assume $w * y + x * z = w * z + x * y$
    then have $(w * y + x * y) + x * k = (w * y + x * y) + w * k$ **by** (*simp*
add: ‹$z = y + k$› *algebra-simps*)
    then have $x * k = w * k$ **by** *simp*
    then show $w = x$ **using** ‹$k \neq 0$› **by** *simp*
  qed
  show $w * y + x * z = w * z + x * y \longleftrightarrow w = x \vee y = z$
    by (*auto simp add*: *neq-iff dest*!: *aux*)
qed

Semiring normalization proper

setup *Semiring-Normalizer.setup*

context *comm-semiring-1*
begin

lemma *normalizing-semiring-ops*:
  shows *TERM* $(x + y)$ **and** *TERM* $(x * y)$ **and** *TERM* $(x \hat{} n)$
    and *TERM 0* **and** *TERM 1* **.**

lemma *normalizing-semiring-rules*:
$(a * m) + (b * m) = (a + b) * m$
$(a * m) + m = (a + 1) * m$
$m + (a * m) = (a + 1) * m$
$m + m = (1 + 1) * m$
$0 + a = a$
$a + 0 = a$
$a * b = b * a$
$(a + b) * c = (a * c) + (b * c)$
$0 * a = 0$
$a * 0 = 0$
$1 * a = a$
$a * 1 = a$
$(lx * ly) * (rx * ry) = (lx * rx) * (ly * ry)$
$(lx * ly) * (rx * ry) = lx * (ly * (rx * ry))$
$(lx * ly) * (rx * ry) = rx * ((lx * ly) * ry)$
$(lx * ly) * rx = (lx * rx) * ly$
$(lx * ly) * rx = lx * (ly * rx)$
$lx * (rx * ry) = (lx * rx) * ry$
$lx * (rx * ry) = rx * (lx * ry)$
$(a + b) + (c + d) = (a + c) + (b + d)$
$(a + b) + c = a + (b + c)$
$a + (c + d) = c + (a + d)$
$(a + b) + c = (a + c) + b$
$a + c = c + a$
$a + (c + d) = (a + c) + d$
$(x \hat{} p) * (x \hat{} q) = x \hat{} (p + q)$
$x * (x \hat{} q) = x \hat{} (Suc\ q)$
$(x \hat{} q) * x = x \hat{} (Suc\ q)$
$x * x = x \hat{} 2$
$(x * y) \hat{} q = (x \hat{} q) * (y \hat{} q)$
$(x \hat{} p) \hat{} q = x \hat{} (p * q)$
$x \hat{} 0 = 1$
$x \hat{} 1 = x$
$x * (y + z) = (x * y) + (x * z)$
$x \hat{} (Suc\ q) = x * (x \hat{} q)$
$x \hat{} (2*n) = (x \hat{} n) * (x \hat{} n)$
$x \hat{} (Suc\ (2*n)) = x * ((x \hat{} n) * (x \hat{} n))$
by (*simp-all add*: *algebra-simps power-add power2-eq-square power-mult-distrib power-mult*)

lemmas *normalizing-comm-semiring-1-axioms* =
comm-semiring-1-axioms [*normalizer*
semiring ops: *normalizing-semiring-ops*
semiring rules: *normalizing-semiring-rules*]

declaration
⟪ *Semiring-Normalizer.semiring-funs* @{*thm normalizing-comm-semiring-1-axioms*}
⟫

end

context *comm-ring-1*
begin

lemma *normalizing-ring-ops*: **shows** *TERM* $(x- y)$ **and** *TERM* $(- x)$ **.**

lemma *normalizing-ring-rules*:
$- x = (- 1) * x$
$x - y = x + (- y)$
by (*simp-all add*: *diff-minus*)

lemmas *normalizing-comm-ring-1-axioms* =
comm-ring-1-axioms [*normalizer*
semiring ops: *normalizing-semiring-ops*
semiring rules: *normalizing-semiring-rules*
ring ops: *normalizing-ring-ops*
ring rules: *normalizing-ring-rules*]

declaration
⟪ *Semiring-Normalizer.semiring-funs* @{*thm normalizing-comm-ring-1-axioms*}
⟫

end

context *comm-semiring-1-cancel-crossproduct*
begin

declare
normalizing-comm-semiring-1-axioms [*normalizer del*]

lemmas
normalizing-comm-semiring-1-cancel-crossproduct-axioms =
comm-semiring-1-cancel-crossproduct-axioms [*normalizer*
semiring ops: *normalizing-semiring-ops*
semiring rules: *normalizing-semiring-rules*
idom rules: *crossproduct-noteq add-scale-eq-noteq*]

declaration
⟪ *Semiring-Normalizer.semiring-funs* @{*thm normalizing-comm-semiring-1-cancel-crossproduct-axioms*}
⟫

end

context *idom*
begin

declare *normalizing-comm-ring-1-axioms* [*normalizer del*]

lemmas *normalizing-idom-axioms = idom-axioms* [*normalizer*
semiring ops: *normalizing-semiring-ops*
semiring rules: *normalizing-semiring-rules*
ring ops: *normalizing-ring-ops*
ring rules: *normalizing-ring-rules*
idom rules: *crossproduct-noteq add-scale-eq-noteq*
ideal rules: *right-minus-eq add-0-iff*]

declaration
⟪ *Semiring-Normalizer.semiring-funs* @{*thm normalizing-idom-axioms*} ⟫

end

context *field*
begin

lemma *normalizing-field-ops*:
shows *TERM* (*x* / *y*) **and** *TERM* (*inverse x*) **.**

lemmas *normalizing-field-rules = divide-inverse inverse-eq-divide*

lemmas *normalizing-field-axioms =*
field-axioms [*normalizer*
semiring ops: *normalizing-semiring-ops*
semiring rules: *normalizing-semiring-rules*
ring ops: *normalizing-ring-ops*
ring rules: *normalizing-ring-rules*
field ops: *normalizing-field-ops*
field rules: *normalizing-field-rules*
idom rules: *crossproduct-noteq add-scale-eq-noteq*
ideal rules: *right-minus-eq add-0-iff*]

declaration
⟪ *Semiring-Normalizer.field-funs* @{*thm normalizing-field-axioms*} ⟫

end

hide-fact (**open**) *normalizing-comm-semiring-1-axioms*
normalizing-comm-semiring-1-cancel-crossproduct-axioms normalizing-semiring-ops
normalizing-semiring-rules

hide-fact (**open**) *normalizing-comm-ring-1-axioms*
normalizing-idom-axioms normalizing-ring-ops normalizing-ring-rules

hide-fact (**open**) *normalizing-field-axioms normalizing-field-ops normalizing-field-rules*

end

37 Groebner-Basis: Groebner bases

theory *Groebner-Basis*
imports *Semiring-Normalization*
uses
(*Tools/groebner.ML*)
begin

37.1 Groebner Bases

lemmas *bool-simps* = *simp-thms*(*1−34*)

lemma *dnf*:
$(P \,\&\, (Q \mid R)) = ((P\&Q) \mid (P\&R))$ $((Q \mid R) \,\&\, P) = ((Q\&P) \mid (R\&P))$
$(P \wedge Q) = (Q \wedge P)$ $(P \vee Q) = (Q \vee P)$
by *blast+*

lemmas *weak-dnf-simps* = *dnf bool-simps*

lemma *nnf-simps*:
$(\neg(P \wedge Q)) = (\neg P \vee \neg Q)$ $(\neg(P \vee Q)) = (\neg P \wedge \neg Q)$ $(P \longrightarrow Q) = (\neg P \vee Q)$
$(P = Q) = ((P \wedge Q) \vee (\neg P \wedge \neg\, Q))$ $(\neg\, \neg(P)) = P$
by *blast+*

lemma *PFalse*:
$P \equiv \mathit{False} \Longrightarrow \neg\, P$
$\neg\, P \Longrightarrow (P \equiv \mathit{False})$
by *auto*

ML ⟪
structure Algebra-Simplification = Named-Thms(
val name = algebra
val description = pre−simplification rules for algebraic methods
)
⟫

setup *Algebra-Simplification.setup*

use *Tools/groebner.ML*

method-setup *algebra = Groebner.algebra-method*
solve polynomial equations over (semi)rings and ideal membership problems using Groebner bases

declare *dvd-def*[*algebra*]
declare *dvd-eq-mod-eq-0*[*symmetric*, *algebra*]
declare *mod-div-trivial*[*algebra*]
declare *mod-mod-trivial*[*algebra*]
declare *conjunct1*[*OF DIVISION-BY-ZERO*, *algebra*]
declare *conjunct2*[*OF DIVISION-BY-ZERO*, *algebra*]

declare *zmod-zdiv-equality*[*symmetric*,*algebra*]
declare *zdiv-zmod-equality*[*symmetric*, *algebra*]
declare *zdiv-zminus-zminus*[*algebra*]
declare *zmod-zminus-zminus*[*algebra*]
declare *zdiv-zminus2*[*algebra*]
declare *zmod-zminus2*[*algebra*]
declare *zdiv-zero*[*algebra*]
declare *zmod-zero*[*algebra*]
declare *mod-by-1*[*algebra*]
declare *div-by-1*[*algebra*]
declare *zmod-minus1-right*[*algebra*]
declare *zdiv-minus1-right*[*algebra*]
declare *mod-div-trivial*[*algebra*]
declare *mod-mod-trivial*[*algebra*]
declare *mod-mult-self2-is-0*[*algebra*]
declare *mod-mult-self1-is-0*[*algebra*]
declare *zmod-eq-0-iff*[*algebra*]
declare *dvd-0-left-iff*[*algebra*]
declare *zdvd1-eq*[*algebra*]
declare *zmod-eq-dvd-iff*[*algebra*]
declare *nat-mod-eq-iff*[*algebra*]

end

38 SetInterval: Set intervals

theory *SetInterval*
imports *Int Nat-Transfer*
begin

context *ord*
begin
definition
  lessThan :: *'a* => *'a set* ((*1*{..<-})) **where**
  {..<*u*} == {*x*. *x* < *u*}

definition
  atMost :: *'a* => *'a set* ((*1*{..-})) **where**
  {..*u*} == {*x*. *x* ≤ *u*}

definition
  greaterThan :: *'a* => *'a set* ((*1*{-<..})) **where**
  {*l*<..} == {*x*. *l*<*x*}

definition
  atLeast :: *'a* => *'a set* ((*1*{-..})) **where**
  {*l*..} == {*x*. *l*≤*x*}

definition
greaterThanLessThan :: *'a => 'a => 'a set* ((*1*{-<..<-})) **where**
{*l*<..<*u*} == {*l*<..} *Int* {..<*u*}

definition
atLeastLessThan :: *'a => 'a => 'a set* ((*1*{-..<-})) **where**
{*l*..<*u*} == {*l*..} *Int* {..<*u*}

definition
greaterThanAtMost :: *'a => 'a => 'a set* ((*1*{-<..-})) **where**
{*l*<..*u*} == {*l*<..} *Int* {..*u*}

definition
atLeastAtMost :: *'a => 'a => 'a set* ((*1*{-..-})) **where**
{*l*..*u*} == {*l*..} *Int* {..*u*}

end

A note of warning when using {..<*n*} on type *nat*: it is equivalent to {*0*..<*n*} but some lemmas involving {*m*..<*n*} may not exist in {..<*n*}-form as well.

syntax
-UNION-le :: *'a => 'a => 'b set => 'b set* ((*3UN -<=-./ -*) [*0*, *0*, *10*] *10*)
-UNION-less :: *'a => 'a => 'b set => 'b set* ((*3UN -<-./ -*) [*0*, *0*, *10*] *10*)
-INTER-le :: *'a => 'a => 'b set => 'b set* ((*3INT -<=-./ -*) [*0*, *0*, *10*] *10*)
-INTER-less :: *'a => 'a => 'b set => 'b set* ((*3INT -<-./ -*) [*0*, *0*, *10*] *10*)

syntax (*xsymbols*)
-UNION-le :: *'a => 'a => 'b set => 'b set* ((*3⋃ -≤-./ -*) [*0*, *0*, *10*] *10*)
-UNION-less :: *'a => 'a => 'b set => 'b set* ((*3⋃ -<-./ -*) [*0*, *0*, *10*] *10*)
-INTER-le :: *'a => 'a => 'b set => 'b set* ((*3⋂ -≤-./ -*) [*0*, *0*, *10*] *10*)
-INTER-less :: *'a => 'a => 'b set => 'b set* ((*3⋂ -<-./ -*) [*0*, *0*, *10*] *10*)

syntax (*latex* **output**)
-UNION-le :: *'a ⇒ 'a => 'b set => 'b set* ((*3⋃(00- ≤ -)/ -*) [*0*, *0*, *10*] *10*)
-UNION-less :: *'a ⇒ 'a => 'b set => 'b set* ((*3⋃(00- < -)/ -*) [*0*, *0*, *10*] *10*)
-INTER-le :: *'a ⇒ 'a => 'b set => 'b set* ((*3⋂(00- ≤ -)/ -*) [*0*, *0*, *10*] *10*)
-INTER-less :: *'a ⇒ 'a => 'b set => 'b set* ((*3⋂(00- < -)/ -*) [*0*, *0*, *10*] *10*)

translations
UN i<=n. A == *UN i:{..n}. A*
UN i<n. A == *UN i:{..<n}. A*
INT i<=n. A == *INT i:{..n}. A*
INT i<n. A == *INT i:{..<n}. A*

38.1 Various equivalences

lemma (**in** *ord*) *lessThan-iff* [*iff*]: (*i*: *lessThan k*) = (*i*<*k*)
by (*simp add*: *lessThan-def*)

lemma *Compl-lessThan* [*simp*]:
!!*k*:: '*a*::*linorder*. −*lessThan k* = *atLeast k*
apply (*auto simp add*: *lessThan-def atLeast-def*)
done

lemma *single-Diff-lessThan* [*simp*]: !!*k*:: '*a*::*order*. {*k*} − *lessThan k* = {*k*}
by *auto*

lemma (**in** *ord*) *greaterThan-iff* [*iff*]: (*i*: *greaterThan k*) = (*k*<*i*)
by (*simp add*: *greaterThan-def*)

lemma *Compl-greaterThan* [*simp*]:
!!*k*:: '*a*::*linorder*. −*greaterThan k* = *atMost k*
by (*auto simp add*: *greaterThan-def atMost-def*)

lemma *Compl-atMost* [*simp*]: !!*k*:: '*a*::*linorder*. −*atMost k* = *greaterThan k*
apply (*subst Compl-greaterThan* [*symmetric*])
apply (*rule double-complement*)
done

lemma (**in** *ord*) *atLeast-iff* [*iff*]: (*i*: *atLeast k*) = (*k*<=*i*)
by (*simp add*: *atLeast-def*)

lemma *Compl-atLeast* [*simp*]:
!!*k*:: '*a*::*linorder*. −*atLeast k* = *lessThan k*
by (*auto simp add*: *lessThan-def atLeast-def*)

lemma (**in** *ord*) *atMost-iff* [*iff*]: (*i*: *atMost k*) = (*i*<=*k*)
by (*simp add*: *atMost-def*)

lemma *atMost-Int-atLeast*: !!*n*:: '*a*::*order*. *atMost n Int atLeast n* = {*n*}
by (*blast intro*: *order-antisym*)

38.2 Logical Equivalences for Set Inclusion and Equality

lemma *atLeast-subset-iff* [*iff*]:
(*atLeast x* ⊆ *atLeast y*) = (*y* ≤ (*x*::'*a*::*order*))
by (*blast intro*: *order-trans*)

lemma *atLeast-eq-iff* [*iff*]:
(*atLeast x* = *atLeast y*) = (*x* = (*y*::'*a*::*linorder*))
by (*blast intro*: *order-antisym order-trans*)

lemma *greaterThan-subset-iff* [*iff*]:
(*greaterThan x* ⊆ *greaterThan y*) = (*y* ≤ (*x*::'*a*::*linorder*))

apply (*auto simp add*: *greaterThan-def*)
apply (*subst linorder-not-less* [*symmetric*], *blast*)
done

lemma *greaterThan-eq-iff* [*iff*]:
(*greaterThan x* = *greaterThan y*) = (*x* = (*y*::'*a*::*linorder*))
apply (*rule iffI*)
apply (*erule equalityE*)
apply *simp-all*
done

lemma *atMost-subset-iff* [*iff*]: (*atMost x* ⊆ *atMost y*) = (*x* ≤ (*y*::'*a*::*order*))
by (*blast intro*: *order-trans*)

lemma *atMost-eq-iff* [*iff*]: (*atMost x* = *atMost y*) = (*x* = (*y*::'*a*::*linorder*))
by (*blast intro*: *order-antisym order-trans*)

lemma *lessThan-subset-iff* [*iff*]:
(*lessThan x* ⊆ *lessThan y*) = (*x* ≤ (*y*::'*a*::*linorder*))
apply (*auto simp add*: *lessThan-def*)
apply (*subst linorder-not-less* [*symmetric*], *blast*)
done

lemma *lessThan-eq-iff* [*iff*]:
(*lessThan x* = *lessThan y*) = (*x* = (*y*::'*a*::*linorder*))
apply (*rule iffI*)
apply (*erule equalityE*)
apply *simp-all*
done

38.3 Two-sided intervals

context *ord*
begin

lemma *greaterThanLessThan-iff* [*simp*,*no-atp*]:
(*i* : {*l*<..<*u*}) = (*l* < *i* & *i* < *u*)
by (*simp add*: *greaterThanLessThan-def*)

lemma *atLeastLessThan-iff* [*simp*,*no-atp*]:
(*i* : {*l*..<*u*}) = (*l* <= *i* & *i* < *u*)
by (*simp add*: *atLeastLessThan-def*)

lemma *greaterThanAtMost-iff* [*simp*,*no-atp*]:
(*i* : {*l*<..*u*}) = (*l* < *i* & *i* <= *u*)
by (*simp add*: *greaterThanAtMost-def*)

lemma *atLeastAtMost-iff* [*simp*,*no-atp*]:
(*i* : {*l*..*u*}) = (*l* <= *i* & *i* <= *u*)

by (*simp add*: *atLeastAtMost-def*)

The above four lemmas could be declared as iffs. Unfortunately this breaks many proofs. Since it only helps blast, it is better to leave well alone

end

38.3.1 Emptyness, singletons, subset

context *order*
begin

lemma *atLeastatMost-empty*[*simp*]:
$b < a \Longrightarrow \{a..b\} = \{\}$
by(*auto simp*: *atLeastAtMost-def atLeast-def atMost-def*)

lemma *atLeastatMost-empty-iff*[*simp*]:
$\{a..b\} = \{\} \longleftrightarrow (\sim\ a <= b)$
by *auto* (*blast intro*: *order-trans*)

lemma *atLeastatMost-empty-iff2*[*simp*]:
$\{\} = \{a..b\} \longleftrightarrow (\sim\ a <= b)$
by *auto* (*blast intro*: *order-trans*)

lemma *atLeastLessThan-empty*[*simp*]:
$b <= a \Longrightarrow \{a..<b\} = \{\}$
by(*auto simp*: *atLeastLessThan-def*)

lemma *atLeastLessThan-empty-iff*[*simp*]:
$\{a..<b\} = \{\} \longleftrightarrow (\sim\ a < b)$
by *auto* (*blast intro*: *le-less-trans*)

lemma *atLeastLessThan-empty-iff2*[*simp*]:
$\{\} = \{a..<b\} \longleftrightarrow (\sim\ a < b)$
by *auto* (*blast intro*: *le-less-trans*)

lemma *greaterThanAtMost-empty*[*simp*]: $l \leq k ==> \{k<..l\} = \{\}$
by(*auto simp*:*greaterThanAtMost-def greaterThan-def atMost-def*)

lemma *greaterThanAtMost-empty-iff*[*simp*]: $\{k<..l\} = \{\} \longleftrightarrow \sim\ k < l$
by *auto* (*blast intro*: *less-le-trans*)

lemma *greaterThanAtMost-empty-iff2*[*simp*]: $\{\} = \{k<..l\} \longleftrightarrow \sim\ k < l$
by *auto* (*blast intro*: *less-le-trans*)

lemma *greaterThanLessThan-empty*[*simp*]:$l \leq k ==> \{k<..<l\} = \{\}$
by(*auto simp*:*greaterThanLessThan-def greaterThan-def lessThan-def*)

lemma *atLeastAtMost-singleton* [*simp*]: $\{a..a\} = \{a\}$
by (*auto simp add*: *atLeastAtMost-def atMost-def atLeast-def*)

lemma *atLeastAtMost-singleton'*: $a = b \Longrightarrow \{a\ ..\ b\} = \{a\}$ **by** *simp*

lemma *atLeastatMost-subset-iff*[*simp*]:
$\{a..b\} <= \{c..d\} \longleftrightarrow (\sim\ a <= b) \mid c <= a\ \&\ b <= d$
unfolding *atLeastAtMost-def atLeast-def atMost-def*
by (*blast intro*: *order-trans*)

lemma *atLeastatMost-psubset-iff*:
$\{a..b\} < \{c..d\} \longleftrightarrow$
$((\sim\ a <= b) \mid c <= a\ \&\ b <= d\ \&\ (c < a \mid b < d))\ \ \&\ \ c <= d$
by(*simp add*: *psubset-eq expand-set-eq less-le-not-le*)(*blast intro*: *order-trans*)

lemma *atLeastAtMost-singleton-iff*[*simp*]:
$\{a\ ..\ b\} = \{c\} \longleftrightarrow a = b \wedge b = c$
proof
assume $\{a..b\} = \{c\}$
hence $\neg\ (\neg\ a \leq b)$ **unfolding** *atLeastatMost-empty-iff*[*symmetric*] **by** *simp*
moreover with ‹$\{a..b\} = \{c\}$› **have** $c \leq a \wedge b \leq c$ **by** *auto*
ultimately show $a = b \wedge b = c$ **by** *auto*
qed *simp*

end

lemma (**in** *linorder*) *atLeastLessThan-subset-iff*:
$\{a..<b\} <= \{c..<d\} \Longrightarrow b <= a \mid c<=a\ \&\ b<=d$
apply (*auto simp*:*subset-eq Ball-def*)
apply(*frule-tac x*=*a* **in** *spec*)
apply(*erule-tac x*=*d* **in** *allE*)
apply (*simp add*: *less-imp-le*)
done

38.3.2 Intersection

context *linorder*
begin

lemma *Int-atLeastAtMost*[*simp*]: $\{a..b\}\ Int\ \{c..d\} = \{max\ a\ c\ ..\ min\ b\ d\}$
by *auto*

lemma *Int-atLeastAtMostR1*[*simp*]: $\{..b\}\ Int\ \{c..d\} = \{c\ ..\ min\ b\ d\}$
by *auto*

lemma *Int-atLeastAtMostR2*[*simp*]: $\{a..\}\ Int\ \{c..d\} = \{max\ a\ c\ ..\ d\}$
by *auto*

lemma *Int-atLeastAtMostL1*[*simp*]: $\{a..b\}\ Int\ \{..d\} = \{a\ ..\ min\ b\ d\}$
by *auto*

lemma *Int-atLeastAtMostL2*[*simp*]: $\{a..b\}$ *Int* $\{c..\} = \{max\ a\ c\ ..\ b\}$
by *auto*

lemma *Int-atLeastLessThan*[*simp*]: $\{a..<b\}$ *Int* $\{c..<d\} = \{max\ a\ c\ ..<\ min\ b\ d\}$
by *auto*

lemma *Int-greaterThanAtMost*[*simp*]: $\{a<..b\}$ *Int* $\{c<..d\} = \{max\ a\ c\ <..\ min\ b\ d\}$
by *auto*

lemma *Int-greaterThanLessThan*[*simp*]: $\{a<..<b\}$ *Int* $\{c<..<d\} = \{max\ a\ c\ <..<\ min\ b\ d\}$
by *auto*

end

38.4 Intervals of natural numbers

38.4.1 The Constant *lessThan*

lemma *lessThan-0* [*simp*]: *lessThan* $(0::nat) = \{\}$
by (*simp add*: *lessThan-def*)

lemma *lessThan-Suc*: *lessThan* $(Suc\ k) =$ *insert* k (*lessThan* k)
by (*simp add*: *lessThan-def less-Suc-eq*, *blast*)

lemma *lessThan-Suc-atMost*: *lessThan* $(Suc\ k) =$ *atMost* k
by (*simp add*: *lessThan-def atMost-def less-Suc-eq-le*)

lemma *UN-lessThan-UNIV*: $(UN\ m::nat.\ lessThan\ m) = UNIV$
by *blast*

38.4.2 The Constant *greaterThan*

lemma *greaterThan-0* [*simp*]: *greaterThan* $0 =$ *range Suc*
apply (*simp add*: *greaterThan-def*)
apply (*blast dest*: *gr0-conv-Suc* [*THEN iffD1*])
done

lemma *greaterThan-Suc*: *greaterThan* $(Suc\ k) =$ *greaterThan* $k - \{Suc\ k\}$
apply (*simp add*: *greaterThan-def*)
apply (*auto elim*: *linorder-neqE*)
done

lemma *INT-greaterThan-UNIV*: $(INT\ m::nat.\ greaterThan\ m) = \{\}$
by *blast*

38.4.3 The Constant *atLeast*

lemma *atLeast-0* [*simp*]: *atLeast* $(0::nat) = UNIV$

by (*unfold atLeast-def UNIV-def*, *simp*)

lemma *atLeast-Suc*: *atLeast* (*Suc k*) = *atLeast k* − {*k*}
apply (*simp add*: *atLeast-def*)
apply (*simp add*: *Suc-le-eq*)
apply (*simp add*: *order-le-less*, *blast*)
done

lemma *atLeast-Suc-greaterThan*: *atLeast* (*Suc k*) = *greaterThan k*
by (*auto simp add*: *greaterThan-def atLeast-def less-Suc-eq-le*)

lemma *UN-atLeast-UNIV*: (*UN m*::*nat*. *atLeast m*) = *UNIV*
by *blast*

38.4.4 The Constant *atMost*

lemma *atMost-0* [*simp*]: *atMost* (*0*::*nat*) = {*0*}
by (*simp add*: *atMost-def*)

lemma *atMost-Suc*: *atMost* (*Suc k*) = *insert* (*Suc k*) (*atMost k*)
apply (*simp add*: *atMost-def*)
apply (*simp add*: *less-Suc-eq order-le-less*, *blast*)
done

lemma *UN-atMost-UNIV*: (*UN m*::*nat*. *atMost m*) = *UNIV*
by *blast*

38.4.5 The Constant *atLeastLessThan*

The orientation of the following 2 rules is tricky. The lhs is defined in terms of the rhs. Hence the chosen orientation makes sense in this theory — the reverse orientation complicates proofs (eg nontermination). But outside, when the definition of the lhs is rarely used, the opposite orientation seems preferable because it reduces a specific concept to a more general one.

lemma *atLeast0LessThan*: {*0*::*nat*..<*n*} = {..<*n*}
by(*simp add*:*lessThan-def atLeastLessThan-def*)

lemma *atLeast0AtMost*: {*0*..*n*::*nat*} = {..*n*}
by(*simp add*:*atMost-def atLeastAtMost-def*)

declare *atLeast0LessThan*[*symmetric*, *code-unfold*]
atLeast0AtMost[*symmetric*, *code-unfold*]

lemma *atLeastLessThan0*: {*m*..<*0*::*nat*} = {}
by (*simp add*: *atLeastLessThan-def*)

38.4.6 Intervals of nats with *Suc*

Not a simprule because the RHS is too messy.

lemma *atLeastLessThanSuc*:
$\{m..<Suc\ n\} = (if\ m \leq n\ then\ insert\ n\ \{m..<n\}\ else\ \{\})$
by (*auto simp add*: *atLeastLessThan-def*)

lemma *atLeastLessThan-singleton* [*simp*]: $\{m..<Suc\ m\} = \{m\}$
by (*auto simp add*: *atLeastLessThan-def*)

lemma *atLeastLessThanSuc-atLeastAtMost*: $\{l..<Suc\ u\} = \{l..u\}$
by (*simp add*: *lessThan-Suc-atMost atLeastAtMost-def atLeastLessThan-def*)

lemma *atLeastSucAtMost-greaterThanAtMost*: $\{Suc\ l..u\} = \{l<..u\}$
by (*simp add*: *atLeast-Suc-greaterThan atLeastAtMost-def*
greaterThanAtMost-def)

lemma *atLeastSucLessThan-greaterThanLessThan*: $\{Suc\ l..<u\} = \{l<..<u\}$
by (*simp add*: *atLeast-Suc-greaterThan atLeastLessThan-def*
greaterThanLessThan-def)

lemma *atLeastAtMostSuc-conv*: $m \leq Suc\ n \Longrightarrow \{m..Suc\ n\} = insert\ (Suc\ n)\ \{m..n\}$
by (*auto simp add*: *atLeastAtMost-def*)

lemma *atLeastLessThan-add-Un*: $i \leq j \Longrightarrow \{i..<j+k\} = \{i..<j\} \cup \{j..<j+k::nat\}$
apply (*induct k*)
apply (*simp-all add*: *atLeastLessThanSuc*)
done

38.4.7 Image

lemma *image-add-atLeastAtMost*:
$(\%n::nat.\ n+k)\ `\ \{i..j\} = \{i+k..j+k\}$ (**is** *?A* = *?B*)
proof
show *?A* $\subseteq$ *?B* **by** *auto*
next
show *?B* $\subseteq$ *?A*
proof
fix *n* **assume** *a*: *n* : *?B*
hence $n - k : \{i..j\}$ **by** *auto*
moreover have $n = (n - k) + k$ **using** *a* **by** *auto*
ultimately show *n* : *?A* **by** *blast*
qed
qed

lemma *image-add-atLeastLessThan*:
$(\%n::nat.\ n+k)\ `\ \{i..<j\} = \{i+k..<j+k\}$ (**is** *?A* = *?B*)
proof

```
  show ?A ⊆ ?B by auto
next
  show ?B ⊆ ?A
  proof
    fix n assume a: n : ?B
    hence n − k : {i..<j} by auto
    moreover have n = (n − k) + k using a by auto
    ultimately show n : ?A by blast
  qed
qed

corollary image-Suc-atLeastAtMost[simp]:
  Suc ‘ {i..j} = {Suc i..Suc j}
using image-add-atLeastAtMost[where k=Suc 0] by simp

corollary image-Suc-atLeastLessThan[simp]:
  Suc ‘ {i..<j} = {Suc i..<Suc j}
using image-add-atLeastLessThan[where k=Suc 0] by simp

lemma image-add-int-atLeastLessThan:
    (%x. x + (l::int)) ‘ {0..<u−l} = {l..<u}
  apply (auto simp add: image-def)
  apply (rule-tac x = x − l in bexI)
  apply auto
  done

context ordered-ab-group-add
begin

lemma
  fixes x :: 'a
  shows image-uminus-greaterThan[simp]: uminus ‘ {x<..} = {..<−x}
  and image-uminus-atLeast[simp]: uminus ‘ {x..} = {..−x}
proof safe
  fix y assume y < −x
  hence ∗: x < −y using neg-less-iff-less[of −y x] by simp
  have − (−y) ∈ uminus ‘ {x<..}
    by (rule imageI) (simp add: ∗)
  thus y ∈ uminus ‘ {x<..} by simp
next
  fix y assume y ≤ −x
  have − (−y) ∈ uminus ‘ {x..}
    by (rule imageI) (insert ‹y ≤ −x› [THEN le-imp-neg-le], simp)
  thus y ∈ uminus ‘ {x..} by simp
qed simp-all

lemma
  fixes x :: 'a
  shows image-uminus-lessThan[simp]: uminus ‘ {..<x} = {−x<..}
```

and *image-uminus-atMost*[*simp*]: *uminus* ‘ $\{..x\} = \{-x..\}$
proof −
have *uminus* ‘ $\{..<x\} = uminus$ ‘ *uminus* ‘ $\{-x<..\}$
and *uminus* ‘ $\{..x\} = uminus$ ‘ *uminus* ‘ $\{-x..\}$ **by** *simp-all*
thus *uminus* ‘ $\{..<x\} = \{-x<..\}$ **and** *uminus* ‘ $\{..x\} = \{-x..\}$
by (*simp-all add*: *image-image*
del: *image-uminus-greaterThan image-uminus-atLeast*)
qed

lemma
fixes $x :: 'a$
shows *image-uminus-atLeastAtMost*[*simp*]: *uminus* ‘ $\{x..y\} = \{-y..-x\}$
and *image-uminus-greaterThanAtMost*[*simp*]: *uminus* ‘ $\{x<..y\} = \{-y..<-x\}$
and *image-uminus-atLeastLessThan*[*simp*]: *uminus* ‘ $\{x..<y\} = \{-y<..-x\}$
and *image-uminus-greaterThanLessThan*[*simp*]: *uminus* ‘ $\{x<..<y\} = \{-y<..<-x\}$
by (*simp-all add*: *atLeastAtMost-def greaterThanAtMost-def atLeastLessThan-def*
greaterThanLessThan-def image-Int[*OF inj-uminus*] *Int-commute*)
end

38.4.8 Finiteness

lemma *finite-lessThan* [*iff*]: **fixes** $k :: nat$ **shows** *finite* $\{..<k\}$
by (*induct k*) (*simp-all add*: *lessThan-Suc*)

lemma *finite-atMost* [*iff*]: **fixes** $k :: nat$ **shows** *finite* $\{..k\}$
by (*induct k*) (*simp-all add*: *atMost-Suc*)

lemma *finite-greaterThanLessThan* [*iff*]:
fixes $l :: nat$ **shows** *finite* $\{l<..<u\}$
by (*simp add*: *greaterThanLessThan-def*)

lemma *finite-atLeastLessThan* [*iff*]:
fixes $l :: nat$ **shows** *finite* $\{l..<u\}$
by (*simp add*: *atLeastLessThan-def*)

lemma *finite-greaterThanAtMost* [*iff*]:
fixes $l :: nat$ **shows** *finite* $\{l<..u\}$
by (*simp add*: *greaterThanAtMost-def*)

lemma *finite-atLeastAtMost* [*iff*]:
fixes $l :: nat$ **shows** *finite* $\{l..u\}$
by (*simp add*: *atLeastAtMost-def*)

A bounded set of natural numbers is finite.

lemma *bounded-nat-set-is-finite*:
$(ALL\ i{:}N.\ i < (n::nat)) ==>$ *finite N*
apply (*rule finite-subset*)
apply (*rule-tac* [*2*] *finite-lessThan*, *auto*)
done

A set of natural numbers is finite iff it is bounded.

lemma *finite-nat-set-iff-bounded*:
finite(*N*::*nat set*) = (*EX m*. *ALL n*:*N*. *n*<*m*) (**is** *?F* = *?B*)
proof
assume *f*:*?F* **show** *?B*
using *Max-ge*[*OF* ‹*?F*›, *simplified less-Suc-eq-le*[*symmetric*]] **by** *blast*
next
assume *?B* **show** *?F* **using** ‹*?B*› **by**(*blast intro*:*bounded-nat-set-is-finite*)
qed

lemma *finite-nat-set-iff-bounded-le*:
finite(*N*::*nat set*) = (*EX m*. *ALL n*:*N*. *n*<=*m*)
apply(*simp add*:*finite-nat-set-iff-bounded*)
apply(*blast dest*:*less-imp-le-nat le-imp-less-Suc*)
done

lemma *finite-less-ub*:
!!*f*::*nat*=>*nat*. (!!*n*. *n* ≤ *f n*) ==> *finite* {*n*. *f n* ≤ *u*}
by (*rule-tac B*={..*u*} **in** *finite-subset*, *auto intro*: *order-trans*)

Any subset of an interval of natural numbers the size of the subset is exactly that interval.

lemma *subset-card-intvl-is-intvl*:
A <= {*k*..<*k*+*card A*} ⟹ *A* = {*k*..<*k*+*card A*} (**is** *PROP ?P*)
proof *cases*
assume *finite A*
thus *PROP ?P*
proof(*induct A rule*:*finite-linorder-max-induct*)
case *empty* **thus** *?case* **by** *auto*
next
case (*insert b A*)
moreover hence *b* ~: *A* **by** *auto*
moreover have *A* <= {*k*..<*k*+*card A*} **and** *b* = *k*+*card A*
using ‹*b* ~: *A*› *insert* **by** *fastsimp*+
ultimately show *?case* **by** *auto*
qed
next
assume ~*finite A* **thus** *PROP ?P* **by** *simp*
qed

38.4.9 Proving Inclusions and Equalities between Unions

lemma *UN-le-eq-Un0*:
(⋃*i*≤*n*::*nat*. *M i*) = (⋃*i*∈{*1*..*n*}. *M i*) ∪ *M 0* (**is** *?A* = *?B*)
proof
show *?A* <= *?B*
proof
fix *x* **assume** *x* : *?A*
then obtain *i* **where** *i*: *i*≤*n x* : *M i* **by** *auto*

```
    show x : ?B
    proof(cases i)
      case 0 with i show ?thesis by simp
    next
      case (Suc j) with i show ?thesis by auto
    qed
  qed
next
  show ?B <= ?A by auto
qed

lemma UN-le-add-shift:
  (⋃ i≤n::nat. M(i+k)) = (⋃ i∈{k..n+k}. M i) (is ?A = ?B)
proof
  show ?A <= ?B by fastsimp
next
  show ?B <= ?A
  proof
    fix x assume x : ?B
    then obtain i where i: i : {k..n+k} x : M(i) by auto
    hence i−k≤n & x : M((i−k)+k) by auto
    thus x : ?A by blast
  qed
qed

lemma UN-UN-finite-eq: (⋃ n::nat. ⋃ i∈{0..<n}. A i) = (⋃ n. A n)
  by (auto simp add: atLeast0LessThan)

lemma UN-finite-subset: (!!n::nat. (⋃ i∈{0..<n}. A i) ⊆ C) ⟹ (⋃ n. A n) ⊆ C
  by (subst UN-UN-finite-eq [symmetric]) blast

lemma UN-finite2-subset:
      (!!n::nat. (⋃ i∈{0..<n}. A i) ⊆ (⋃ i∈{0..<n+k}. B i)) ⟹ (⋃ n. A n) ⊆
(⋃ n. B n)
  apply (rule UN-finite-subset)
  apply (subst UN-UN-finite-eq [symmetric, of B])
  apply blast
  done

lemma UN-finite2-eq:
  (!!n::nat. (⋃ i∈{0..<n}. A i) = (⋃ i∈{0..<n+k}. B i)) ⟹ (⋃ n. A n) = (⋃ n.
B n)
  apply (rule subset-antisym)
   apply (rule UN-finite2-subset, blast)
 apply (rule UN-finite2-subset [where k=k])
 apply (force simp add: atLeastLessThan-add-Un [of 0])
 done
```

38.4.10 Cardinality

lemma *card-lessThan* [*simp*]: *card* {..<*u*} = *u*
by (*induct u*, *simp-all add*: *lessThan-Suc*)

lemma *card-atMost* [*simp*]: *card* {..*u*} = *Suc u*
by (*simp add*: *lessThan-Suc-atMost* [*THEN sym*])

lemma *card-atLeastLessThan* [*simp*]: *card* {*l*..<*u*} = *u* − *l*
apply (*subgoal-tac card* {*l*..<*u*} = *card* {..<*u*−*l*})
apply (*erule ssubst*, *rule card-lessThan*)
apply (*subgoal-tac* (%*x*. *x* + *l*) ' {..<*u*−*l*} = {*l*..<*u*})
apply (*erule subst*)
apply (*rule card-image*)
apply (*simp add*: *inj-on-def*)
apply (*auto simp add*: *image-def atLeastLessThan-def lessThan-def*)
apply (*rule-tac x* = *x* − *l* **in** *exI*)
apply *arith*
done

lemma *card-atLeastAtMost* [*simp*]: *card* {*l*..*u*} = *Suc u* − *l*
by (*subst atLeastLessThanSuc-atLeastAtMost* [*THEN sym*], *simp*)

lemma *card-greaterThanAtMost* [*simp*]: *card* {*l*<..*u*} = *u* − *l*
by (*subst atLeastSucAtMost-greaterThanAtMost* [*THEN sym*], *simp*)

lemma *card-greaterThanLessThan* [*simp*]: *card* {*l*<..<*u*} = *u* − *Suc l*
by (*subst atLeastSucLessThan-greaterThanLessThan* [*THEN sym*], *simp*)

lemma *ex-bij-betw-nat-finite*:
finite M ⟹ ∃ *h*. *bij-betw h* {*0*..<*card M*} *M*
apply(*drule finite-imp-nat-seg-image-inj-on*)
apply(*auto simp*:*atLeast0LessThan*[*symmetric*] *lessThan-def*[*symmetric*] *card-image bij-betw-def*)
done

lemma *ex-bij-betw-finite-nat*:
finite M ⟹ ∃ *h*. *bij-betw h M* {*0*..<*card M*}
by (*blast dest*: *ex-bij-betw-nat-finite bij-betw-inv*)

lemma *finite-same-card-bij*:
finite A ⟹ *finite B* ⟹ *card A* = *card B* ⟹ *EX h*. *bij-betw h A B*
apply(*drule ex-bij-betw-finite-nat*)
apply(*drule ex-bij-betw-nat-finite*)
apply(*auto intro*!:*bij-betw-trans*)
done

lemma *ex-bij-betw-nat-finite-1*:
finite M ⟹ ∃ *h*. *bij-betw h* {*1* .. *card M*} *M*
by (*rule finite-same-card-bij*) *auto*

38.5 Intervals of integers

lemma *atLeastLessThanPlusOne-atLeastAtMost-int*: $\{l..<u+1\} = \{l..(u::int)\}$
by (*auto simp add*: *atLeastAtMost-def atLeastLessThan-def*)

lemma *atLeastPlusOneAtMost-greaterThanAtMost-int*: $\{l+1..u\} = \{l<..(u::int)\}$
by (*auto simp add*: *atLeastAtMost-def greaterThanAtMost-def*)

lemma *atLeastPlusOneLessThan-greaterThanLessThan-int*:
$\{l+1..<u\} = \{l<..<u::int\}$
by (*auto simp add*: *atLeastLessThan-def greaterThanLessThan-def*)

38.5.1 Finiteness

lemma *image-atLeastZeroLessThan-int*: $0 \leq u \Longrightarrow$
$\{(0::int)..<u\} = int\ `\ \{..<nat\ u\}$
apply (*unfold image-def lessThan-def*)
apply *auto*
apply (*rule-tac x = nat x* **in** *exI*)
apply (*auto simp add*: *zless-nat-eq-int-zless* [*THEN sym*])
done

lemma *finite-atLeastZeroLessThan-int*: *finite* $\{(0::int)..<u\}$
apply (*case-tac* $0 \leq u$)
apply (*subst image-atLeastZeroLessThan-int*, *assumption*)
apply (*rule finite-imageI*)
apply *auto*
done

lemma *finite-atLeastLessThan-int* [*iff*]: *finite* $\{l..<u::int\}$
apply (*subgoal-tac* $(\%x.\ x + l)\ `\ \{0..<u-l\} = \{l..<u\}$)
apply (*erule subst*)
apply (*rule finite-imageI*)
apply (*rule finite-atLeastZeroLessThan-int*)
apply (*rule image-add-int-atLeastLessThan*)
done

lemma *finite-atLeastAtMost-int* [*iff*]: *finite* $\{l..(u::int)\}$
by (*subst atLeastLessThanPlusOne-atLeastAtMost-int* [*THEN sym*], *simp*)

lemma *finite-greaterThanAtMost-int* [*iff*]: *finite* $\{l<..(u::int)\}$
by (*subst atLeastPlusOneAtMost-greaterThanAtMost-int* [*THEN sym*], *simp*)

lemma *finite-greaterThanLessThan-int* [*iff*]: *finite* $\{l<..<u::int\}$
by (*subst atLeastPlusOneLessThan-greaterThanLessThan-int* [*THEN sym*], *simp*)

38.5.2 Cardinality

lemma *card-atLeastZeroLessThan-int*: *card* $\{(0::int)..<u\} = nat\ u$
apply (*case-tac* $0 \leq u$)

apply (*subst image-atLeastZeroLessThan-int*, *assumption*)
apply (*subst card-image*)
apply (*auto simp add*: *inj-on-def*)
done

lemma *card-atLeastLessThan-int* [*simp*]: *card* {*l*..<*u*} = *nat* (*u* − *l*)
apply (*subgoal-tac card* {*l*..<*u*} = *card* {*0*..<*u*−*l*})
apply (*erule ssubst*, *rule card-atLeastZeroLessThan-int*)
apply (*subgoal-tac* (%*x*. *x* + *l*) ' {*0*..<*u*−*l*} = {*l*..<*u*})
apply (*erule subst*)
apply (*rule card-image*)
apply (*simp add*: *inj-on-def*)
apply (*rule image-add-int-atLeastLessThan*)
done

lemma *card-atLeastAtMost-int* [*simp*]: *card* {*l*..*u*} = *nat* (*u* − *l* + *1*)
apply (*subst atLeastLessThanPlusOne-atLeastAtMost-int* [*THEN sym*])
apply (*auto simp add*: *algebra-simps*)
done

lemma *card-greaterThanAtMost-int* [*simp*]: *card* {*l*<..*u*} = *nat* (*u* − *l*)
by (*subst atLeastPlusOneAtMost-greaterThanAtMost-int* [*THEN sym*], *simp*)

lemma *card-greaterThanLessThan-int* [*simp*]: *card* {*l*<..<*u*} = *nat* (*u* − (*l* + *1*))
by (*subst atLeastPlusOneLessThan-greaterThanLessThan-int* [*THEN sym*], *simp*)

lemma *finite-M-bounded-by-nat*: *finite* {*k*. *P k* ∧ *k* < (*i*::*nat*)}
proof −
have {*k*. *P k* ∧ *k* < *i*} ⊆ {..<*i*} **by** *auto*
with *finite-lessThan*[*of i*] **show** *?thesis* **by** (*simp add*: *finite-subset*)
qed

lemma *card-less*:
assumes *zero-in-M*: *0* ∈ *M*
shows *card* {*k* ∈ *M*. *k* < *Suc i*} ≠ *0*
proof −
from *zero-in-M* **have** {*k* ∈ *M*. *k* < *Suc i*} ≠ {} **by** *auto*
with *finite-M-bounded-by-nat* **show** *?thesis* **by** (*auto simp add*: *card-eq-0-iff*)
qed

lemma *card-less-Suc2*: *0* ∉ *M* ⟹ *card* {*k*. *Suc k* ∈ *M* ∧ *k* < *i*} = *card* {*k* ∈ *M*. *k* < *Suc i*}
apply (*rule card-bij-eq* [*of Suc - - λx*. *x* − *Suc 0*])
apply *simp*
apply *fastsimp*
apply *auto*
apply (*rule inj-on-diff-nat*)
apply *auto*
apply (*case-tac x*)

apply *auto*
apply (*case-tac xa*)
apply *auto*
apply (*case-tac xa*)
apply *auto*
done

lemma *card-less-Suc*:
  assumes *zero-in-M*: $0 \in M$
    shows $Suc\ (card\ \{k.\ Suc\ k \in M \wedge k < i\}) = card\ \{k \in M.\ k < Suc\ i\}$
proof −
  from *assms* **have** *a*: $0 \in \{k \in M.\ k < Suc\ i\}$ **by** *simp*
  hence *c*: $\{k \in M.\ k < Suc\ i\} = insert\ 0\ (\{k \in M.\ k < Suc\ i\} - \{0\})$
    by (*auto simp only*: *insert-Diff*)
  have *b*: $\{k \in M.\ k < Suc\ i\} - \{0\} = \{k \in M - \{0\}.\ k < Suc\ i\}$ **by** *auto*
  from *finite-M-bounded-by-nat*[*of* $\lambda x.\ x \in M\ Suc\ i$] **have** $Suc\ (card\ \{k.\ Suc\ k \in M \wedge k < i\}) = card\ (insert\ 0\ (\{k \in M.\ k < Suc\ i\} - \{0\}))$
    apply (*subst card-insert*)
    apply *simp-all*
    apply (*subst b*)
    apply (*subst card-less-Suc2*[*symmetric*])
    apply *simp-all*
    done
  with *c* **show** *?thesis* **by** *simp*
qed

38.6 Lemmas useful with the summation operator setsum

For examples, see Algebra/poly/UnivPoly2.thy

38.6.1 Disjoint Unions

Singletons and open intervals

lemma *ivl-disj-un-singleton*:
  $\{l::'a::linorder\}\ Un\ \{l<..\} = \{l..\}$
  $\{..<u\}\ Un\ \{u::'a::linorder\} = \{..u\}$
  $(l::'a::linorder) < u \Longrightarrow \{l\}\ Un\ \{l<..<u\} = \{l..<u\}$
  $(l::'a::linorder) < u \Longrightarrow \{l<..<u\}\ Un\ \{u\} = \{l<..u\}$
  $(l::'a::linorder) <= u \Longrightarrow \{l\}\ Un\ \{l<..u\} = \{l..u\}$
  $(l::'a::linorder) <= u \Longrightarrow \{l..<u\}\ Un\ \{u\} = \{l..u\}$
by *auto*

One- and two-sided intervals

lemma *ivl-disj-un-one*:
  $(l::'a::linorder) < u \Longrightarrow \{..l\}\ Un\ \{l<..<u\} = \{..<u\}$
  $(l::'a::linorder) <= u \Longrightarrow \{..<l\}\ Un\ \{l..<u\} = \{..<u\}$
  $(l::'a::linorder) <= u \Longrightarrow \{..l\}\ Un\ \{l<..u\} = \{..u\}$
  $(l::'a::linorder) <= u \Longrightarrow \{..<l\}\ Un\ \{l..u\} = \{..u\}$

$(l::'a::linorder) <= u ==> \{l<..u\}\ Un\ \{u<..\} = \{l<..\}$
$(l::'a::linorder) < u ==> \{l<..<u\}\ Un\ \{u..\} = \{l<..\}$
$(l::'a::linorder) <= u ==> \{l..u\}\ Un\ \{u<..\} = \{l..\}$
$(l::'a::linorder) <= u ==> \{l..<u\}\ Un\ \{u..\} = \{l..\}$
by *auto*

Two- and two-sided intervals

lemma *ivl-disj-un-two*:
$[|\ (l::'a::linorder) < m;\ m <= u\ |] ==> \{l<..<m\}\ Un\ \{m..<u\} = \{l<..<u\}$
$[|\ (l::'a::linorder) <= m;\ m < u\ |] ==> \{l<..m\}\ Un\ \{m<..<u\} = \{l<..<u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l..<m\}\ Un\ \{m..<u\} = \{l..<u\}$
$[|\ (l::'a::linorder) <= m;\ m < u\ |] ==> \{l..m\}\ Un\ \{m<..<u\} = \{l..<u\}$
$[|\ (l::'a::linorder) < m;\ m <= u\ |] ==> \{l<..<m\}\ Un\ \{m..u\} = \{l<..u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l<..m\}\ Un\ \{m<..u\} = \{l<..u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l..<m\}\ Un\ \{m..u\} = \{l..u\}$
$[|\ (l::'a::linorder) <= m;\ m <= u\ |] ==> \{l..m\}\ Un\ \{m<..u\} = \{l..u\}$
by *auto*

lemmas *ivl-disj-un = ivl-disj-un-singleton ivl-disj-un-one ivl-disj-un-two*

38.6.2 Disjoint Intersections

One- and two-sided intervals

lemma *ivl-disj-int-one*:
$\{..l::'a::order\}\ Int\ \{l<..<u\} = \{\}$
$\{..<l\}\ Int\ \{l..<u\} = \{\}$
$\{..l\}\ Int\ \{l<..u\} = \{\}$
$\{..<l\}\ Int\ \{l..u\} = \{\}$
$\{l<..u\}\ Int\ \{u<..\} = \{\}$
$\{l<..<u\}\ Int\ \{u..\} = \{\}$
$\{l..u\}\ Int\ \{u<..\} = \{\}$
$\{l..<u\}\ Int\ \{u..\} = \{\}$
by *auto*

Two- and two-sided intervals

lemma *ivl-disj-int-two*:
$\{l::'a::order<..<m\}\ Int\ \{m..<u\} = \{\}$
$\{l<..m\}\ Int\ \{m<..<u\} = \{\}$
$\{l..<m\}\ Int\ \{m..<u\} = \{\}$
$\{l..m\}\ Int\ \{m<..<u\} = \{\}$
$\{l<..<m\}\ Int\ \{m..u\} = \{\}$
$\{l<..m\}\ Int\ \{m<..u\} = \{\}$
$\{l..<m\}\ Int\ \{m..u\} = \{\}$
$\{l..m\}\ Int\ \{m<..u\} = \{\}$
by *auto*

lemmas *ivl-disj-int = ivl-disj-int-one ivl-disj-int-two*

38.6.3 Some Differences

lemma *ivl-diff*[*simp*]:
$i \leq n \Longrightarrow \{i..<m\} - \{i..<n\} = \{n..<(m::'a::linorder)\}$
by(*auto*)

38.6.4 Some Subset Conditions

lemma *ivl-subset* [*simp*,*no-atp*]:
$(\{i..<j\} \subseteq \{m..<n\}) = (j \leq i \mid m \leq i \ \& \ j \leq (n::'a::linorder))$
apply(*auto simp*:*linorder-not-le*)
apply(*rule ccontr*)
apply(*insert linorder-le-less-linear*[*of i n*])
apply(*clarsimp simp*:*linorder-not-le*)
apply(*fastsimp*)
done

38.7 Summation indexed over intervals

syntax
-from-to-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*SUM - = -..-./ -*) [*0*,*0*,*0*,*10*] *10*)
-from-upto-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*SUM - = -..<-./ -*) [*0*,*0*,*0*,*10*] *10*)
-upt-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*SUM -<-./ -*) [*0*,*0*,*10*] *10*)
-upto-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*SUM -<=-./ -*) [*0*,*0*,*10*] *10*)
syntax (*xsymbols*)
-from-to-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑ *- = -..-./ -*) [*0*,*0*,*0*,*10*] *10*)
-from-upto-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑ *- = -..<-./ -*) [*0*,*0*,*0*,*10*] *10*)
-upt-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑ *-<-./ -*) [*0*,*0*,*10*] *10*)
-upto-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑ *-≤-./ -*) [*0*,*0*,*10*] *10*)
syntax (*HTML* **output**)
-from-to-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑ *- = -..-./ -*) [*0*,*0*,*0*,*10*] *10*)
-from-upto-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑ *- = -..<-./ -*) [*0*,*0*,*0*,*10*] *10*)
-upt-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑ *-<-./ -*) [*0*,*0*,*10*] *10*)
-upto-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*3*∑ *-≤-./ -*) [*0*,*0*,*10*] *10*)
syntax (*latex-sum* **output**)
-from-to-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*3*∑*-* *- = _ -*) [*0*,*0*,*0*,*10*] *10*)
-from-upto-setsum :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*3*∑*-*$^{<-}$ *- = _ -*) [*0*,*0*,*0*,*10*] *10*)
-upt-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*3*∑*-* *- < _ -*) [*0*,*0*,*10*] *10*)
-upto-setsum :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*3*∑*-* *- ≤ _ -*) [*0*,*0*,*10*] *10*)

translations
∑ *x=a..b. t* == *CONST setsum* (%*x. t*) {*a..b*}
∑ *x=a..<b. t* == *CONST setsum* (%*x. t*) {*a..<b*}

$\sum i{\leq}n.\ t == CONST\ setsum\ (\lambda i.\ t)\ \{..n\}$
$\sum i{<}n.\ t == CONST\ setsum\ (\lambda i.\ t)\ \{..{<}n\}$

The above introduces some pretty alternative syntaxes for summation over intervals:

Old	New	LaTeX
$\sum x{\in}\{a..b\}.\ e$	$\sum x = a..b.\ e$	$\sum_{x=a}^{b} e$
$\sum x{\in}\{a..{<}b\}.\ e$	$\sum x = a..{<}b.\ e$	$\sum_{x=a}^{<b} e$
$\sum x{\in}\{..b\}.\ e$	$\sum x{\leq}b.\ e$	$\sum_{x \leq b} e$
$\sum x{\in}\{..{<}b\}.\ e$	$\sum x{<}b.\ e$	$\sum_{x < b} e$

The left column shows the term before introduction of the new syntax, the middle column shows the new (default) syntax, and the right column shows a special syntax. The latter is only meaningful for latex output and has to be activated explicitly by setting the print mode to *latex-sum* (e.g. via *mode = latex-sum* in antiquotations). It is not the default LaTeX output because it only works well with italic-style formulae, not tt-style.

Note that for uniformity on *nat* it is better to use $\sum x = 0..{<}n.\ e$ rather than $\sum x{<}n.\ e$: *setsum* may not provide all lemmas available for $\{m..{<}n\}$ also in the special form for $\{..{<}n\}$.

This congruence rule should be used for sums over intervals as the standard theorem *setsum-cong* does not work well with the simplifier who adds the unsimplified premise $x \in B$ to the context.

lemma *setsum-ivl-cong*:
$[\![a = c;\ b = d;\ !!x.\ [\![c \leq x;\ x < d]\!] \Longrightarrow f\ x = g\ x]\!] \Longrightarrow$
$setsum\ f\ \{a..{<}b\} = setsum\ g\ \{c..{<}d\}$
by(*rule setsum-cong*, *simp-all*)

lemma *setsum-atMost-Suc*[*simp*]: $(\sum i \leq Suc\ n.\ f\ i) = (\sum i \leq n.\ f\ i) + f(Suc\ n)$
by (*simp add*:*atMost-Suc add-ac*)

lemma *setsum-lessThan-Suc*[*simp*]: $(\sum i < Suc\ n.\ f\ i) = (\sum i < n.\ f\ i) + f\ n$
by (*simp add*:*lessThan-Suc add-ac*)

lemma *setsum-cl-ivl-Suc*[*simp*]:
$setsum\ f\ \{m..Suc\ n\} = (if\ Suc\ n < m\ then\ 0\ else\ setsum\ f\ \{m..n\} + f(Suc\ n))$
by (*auto simp*:*add-ac atLeastAtMostSuc-conv*)

lemma *setsum-op-ivl-Suc*[*simp*]:
$setsum\ f\ \{m..{<}Suc\ n\} = (if\ n < m\ then\ 0\ else\ setsum\ f\ \{m..{<}n\} + f(n))$
by (*auto simp*:*add-ac atLeastLessThanSuc*)

lemma *setsum-head*:
fixes n :: *nat*
assumes *mn*: $m <= n$
shows $(\sum x{\in}\{m..n\}.\ P\ x) = P\ m + (\sum x{\in}\{m<..n\}.\ P\ x)$ (**is** *?lhs* = *?rhs*)
proof −
from *mn*
have $\{m..n\} = \{m\} \cup \{m<..n\}$
by (*auto intro*: *ivl-disj-un-singleton*)
hence *?lhs* = $(\sum x{\in}\{m\} \cup \{m<..n\}.\ P\ x)$
by (*simp add*: *atLeast0LessThan*)
also have ... = *?rhs* **by** *simp*
finally show *?thesis* **.**
qed

lemma *setsum-head-Suc*:
$m \leq n \Longrightarrow$ *setsum f* $\{m..n\} = f\ m +$ *setsum f* $\{Suc\ m..n\}$
by (*simp add*: *setsum-head atLeastSucAtMost-greaterThanAtMost*)

lemma *setsum-head-upt-Suc*:
$m < n \Longrightarrow$ *setsum f* $\{m..<n\} = f\ m +$ *setsum f* $\{Suc\ m..<n\}$
apply(*insert setsum-head-Suc*[*of m n* − *Suc 0 f*])
apply (*simp add*: *atLeastLessThanSuc-atLeastAtMost*[*symmetric*] *algebra-simps*)
done

lemma *setsum-ub-add-nat*: **assumes** $(m{::}nat) \leq n + 1$
shows *setsum f* $\{m..n + p\} =$ *setsum f* $\{m..n\} +$ *setsum f* $\{n + 1..n + p\}$
proof−
have $\{m\ ..\ n{+}p\} = \{m..n\} \cup \{n{+}1..n{+}p\}$ **using** ‹$m \leq n{+}1$› **by** *auto*
thus *?thesis* **by** (*auto simp*: *ivl-disj-int setsum-Un-disjoint*
atLeastSucAtMost-greaterThanAtMost)
qed

lemma *setsum-add-nat-ivl*: ⟦ $m \leq n$; $n \leq p$ ⟧ $\Longrightarrow$
setsum f $\{m..<n\}$ + *setsum f* $\{n..<p\} =$ *setsum f* $\{m..<p{::}nat\}$
by (*simp add*:*setsum-Un-disjoint*[*symmetric*] *ivl-disj-int ivl-disj-un*)

lemma *setsum-diff-nat-ivl*:
fixes f :: $nat \Rightarrow {'a}{::}ab\text{-}group\text{-}add$
shows ⟦ $m \leq n$; $n \leq p$ ⟧ $\Longrightarrow$
setsum f $\{m..<p\} -$ *setsum f* $\{m..<n\} =$ *setsum f* $\{n..<p\}$
using *setsum-add-nat-ivl* [*of m n p f*,*symmetric*]
apply (*simp add*: *add-ac*)
done

lemma *setsum-natinterval-difff*:
fixes $f{::}\ nat \Rightarrow ({'a}{::}ab\text{-}group\text{-}add)$
shows *setsum* $(\lambda k.\ f\ k - f(k + 1))$ $\{(m{::}nat)\ ..\ n\} =$
(*if* $m <= n$ *then* $f\ m - f(n + 1)$ *else* 0)
by (*induct n*, *auto simp add*: *algebra-simps not-le le-Suc-eq*)

lemmas *setsum-restrict-set'* = *setsum-restrict-set*[*unfolded Int-def*]

lemma *setsum-setsum-restrict*:
finite S ⟹ *finite T* ⟹ *setsum* (λ*x*. *setsum* (λ*y*. *f x y*) {*y*. *y*∈ *T* ∧ *R x y*}) *S*
= *setsum* (λ*y*. *setsum* (λ*x*. *f x y*) {*x*. *x* ∈ *S* ∧ *R x y*}) *T*
by (*simp add*: *setsum-restrict-set'*[*unfolded mem-def*] *mem-def*)
(*rule setsum-commute*)

lemma *setsum-image-gen*: **assumes** *fS*: *finite S*
shows *setsum g S* = *setsum* (λ*y*. *setsum g* {*x*. *x* ∈ *S* ∧ *f x* = *y*}) (*f* ' *S*)
proof−
{ fix *x* **assume** *x* ∈ *S* **then have** {*y*. *y*∈ *f*'*S* ∧ *f x* = *y*} = {*f x*} **by** *auto* **}**
hence *setsum g S* = *setsum* (λ*x*. *setsum* (λ*y*. *g x*) {*y*. *y*∈ *f*'*S* ∧ *f x* = *y*}) *S*
by *simp*
also have ... = *setsum* (λ*y*. *setsum g* {*x*. *x* ∈ *S* ∧ *f x* = *y*}) (*f* ' *S*)
by (*rule setsum-setsum-restrict*[*OF fS finite-imageI*[*OF fS*]])
finally show *?thesis* **.**
qed

lemma *setsum-le-included*:
fixes *f* :: *'a* ⇒ *'b*::*ordered-comm-monoid-add*
assumes *finite s finite t*
and ∀ *y*∈*t*. *0* ≤ *g y* (∀ *x*∈*s*. ∃ *y*∈*t*. *i y* = *x* ∧ *f x* ≤ *g y*)
shows *setsum f s* ≤ *setsum g t*
proof −
have *setsum f s* ≤ *setsum* (λ*y*. *setsum g* {*x*. *x*∈*t* ∧ *i x* = *y*}) *s*
proof (*rule setsum-mono*)
fix *y* **assume** *y* ∈ *s*
with *assms* **obtain** *z* **where** *z*: *z* ∈ *t y* = *i z f y* ≤ *g z* **by** *auto*
with *assms* **show** *f y* ≤ *setsum g* {*x* ∈ *t*. *i x* = *y*} (**is** *?A y* ≤ *?B y*)
using *order-trans*[*of ?A* (*i z*) *setsum g* {*z*} *?B* (*i z*), *intro*]
by (*auto intro*!: *setsum-mono2*)
qed
also have ... ≤ *setsum* (λ*y*. *setsum g* {*x*. *x*∈*t* ∧ *i x* = *y*}) (*i* ' *t*)
using *assms*(*2*−*4*) **by** (*auto intro*!: *setsum-mono2 setsum-nonneg*)
also have ... ≤ *setsum g t*
using *assms* **by** (*auto simp*: *setsum-image-gen*[*symmetric*])
finally show *?thesis* **.**
qed

lemma *setsum-multicount-gen*:
assumes *finite s finite t* ∀ *j*∈*t*. (*card* {*i*∈*s*. *R i j*} = *k j*)
shows *setsum* (λ*i*. (*card* {*j*∈*t*. *R i j*})) *s* = *setsum k t* (**is** *?l* = *?r*)
proof−
have *?l* = *setsum* (λ*i*. *setsum* (λ*x.1*) {*j*∈*t*. *R i j*}) *s* **by** *auto*
also have ... = *?r* **unfolding** *setsum-setsum-restrict*[*OF assms*(*1*−*2*)]
using *assms*(*3*) **by** *auto*
finally show *?thesis* **.**

qed

lemma *setsum-multicount*:
assumes *finite S finite T* $\forall j{\in}T.$ (*card* $\{i{\in}S.\ R\ i\ j\} = k$)
shows *setsum* $(\lambda i.\ card\ \{j{\in}T.\ R\ i\ j\})\ S = k * card\ T$ (**is** *?l* = *?r*)
proof−
have *?l* = *setsum* $(\lambda i.\ k)\ T$ **by**(*rule setsum-multicount-gen*)(*auto simp*:*assms*)
also have ... = *?r* **by**(*simp add*: *mult-commute*)
finally show *?thesis* **by** *auto*
qed

38.8 Shifting bounds

lemma *setsum-shift-bounds-nat-ivl*:
setsum f $\{m{+}k..{<}n{+}k\}$ = *setsum* $(\%i.\ f(i + k))\{m..{<}n{::}nat\}$
by (*induct n*, *auto simp*:*atLeastLessThanSuc*)

lemma *setsum-shift-bounds-cl-nat-ivl*:
setsum f $\{m{+}k..n{+}k\}$ = *setsum* $(\%i.\ f(i + k))\{m..n{::}nat\}$
apply (*insert setsum-reindex*[*OF inj-on-add-nat*, **where** *h*=*f* **and** $B = \{m..n\}$])
apply (*simp add*:*image-add-atLeastAtMost o-def*)
done

corollary *setsum-shift-bounds-cl-Suc-ivl*:
setsum f $\{Suc\ m..Suc\ n\}$ = *setsum* $(\%i.\ f(Suc\ i))\{m..n\}$
by (*simp add*:*setsum-shift-bounds-cl-nat-ivl*[**where** *k*=*Suc 0*, *simplified*])

corollary *setsum-shift-bounds-Suc-ivl*:
setsum f $\{Suc\ m..{<}Suc\ n\}$ = *setsum* $(\%i.\ f(Suc\ i))\{m..{<}n\}$
by (*simp add*:*setsum-shift-bounds-nat-ivl*[**where** *k*=*Suc 0*, *simplified*])

lemma *setsum-shift-lb-Suc0-0*:
$f(0{::}nat) = (0{::}nat) \Longrightarrow$ *setsum f* $\{Suc\ 0..k\}$ = *setsum f* $\{0..k\}$
by(*simp add*:*setsum-head-Suc*)

lemma *setsum-shift-lb-Suc0-0-upt*:
$f(0{::}nat) = 0 \Longrightarrow$ *setsum f* $\{Suc\ 0..{<}k\}$ = *setsum f* $\{0..{<}k\}$
apply(*cases k*)**apply** *simp*
apply(*simp add*:*setsum-head-upt-Suc*)
done

38.9 The formula for geometric sums

lemma *geometric-sum*:
assumes $x \neq 1$
shows $(\sum i{=}0..{<}n.\ x\ \hat{}\ i) = (x\ \hat{}\ n - 1)\ /\ (x - 1{::}'a{::}field)$
proof −
from *assms* **obtain** *y* **where** $y = x - 1$ **and** $y \neq 0$ **by** *simp-all*
moreover have $(\sum i{=}0..{<}n.\ (y + 1)\ \hat{}\ i) = ((y + 1)\ \hat{}\ n - 1)\ /\ y$
proof (*induct n*)

case *0* **then show** *?case* **by** *simp*
next
case (*Suc n*)
moreover with ‹*y* ≠ *0*› **have** (*1* + *y*) ^ *n* = (*y* * *inverse y*) * (*1* + *y*) ^ *n*
by *simp*
ultimately show *?case* **by** (*simp add*: *field-simps divide-inverse*)
qed
ultimately show *?thesis* **by** *simp*
qed

38.10 The formula for arithmetic sums

lemma *gauss-sum*:
((*1*::'*a*::*comm-semiring-1*) + *1*)*(∑ *i*∈{*1*..*n*}. *of-nat i*) =
of-nat n∗((*of-nat n*)+*1*)
proof (*induct n*)
case *0*
show *?case* **by** *simp*
next
case (*Suc n*)
then show *?case* **by** (*simp add*: *algebra-simps*)
qed

theorem *arith-series-general*:
((*1*::'*a*::*comm-semiring-1*) + *1*) * (∑ *i*∈{..<*n*}. *a* + *of-nat i* * *d*) =
of-nat n * (*a* + (*a* + *of-nat*(*n* − *1*)∗*d*))
proof *cases*
assume *ngt1*: *n* > *1*
let *?I* = λ*i*. *of-nat i* **and** *?n* = *of-nat n*
have
(∑ *i*∈{..<*n*}. *a*+*?I i*∗*d*) =
((∑ *i*∈{..<*n*}. *a*) + (∑ *i*∈{..<*n*}. *?I i*∗*d*))
by (*rule setsum-addf*)
also from *ngt1* **have** ... = *?n*∗*a* + (∑ *i*∈{..<*n*}. *?I i*∗*d*) **by** *simp*
also from *ngt1* **have** ... = (*?n*∗*a* + *d*∗(∑ *i*∈{*1*..<*n*}. *?I i*))
unfolding *One-nat-def*
by (*simp add*: *setsum-right-distrib atLeast0LessThan*[*symmetric*] *setsum-shift-lb-Suc0-0-upt*
mult-ac)
also have (*1*+*1*)∗... = (*1*+*1*)∗*?n*∗*a* + *d*∗(*1*+*1*)∗(∑ *i*∈{*1*..<*n*}. *?I i*)
by (*simp add*: *left-distrib right-distrib*)
also from *ngt1* **have** {*1*..<*n*} = {*1*..*n* − *1*}
by (*cases n*) (*auto simp*: *atLeastLessThanSuc-atLeastAtMost*)
also from *ngt1*
have (*1*+*1*)∗*?n*∗*a* + *d*∗(*1*+*1*)∗(∑ *i*∈{*1*..*n* − *1*}. *?I i*) = ((*1*+*1*)∗*?n*∗*a* + *d*∗*?I*
(*n* − *1*)∗*?I n*)
by (*simp only*: *mult-ac gauss-sum* [*of n* − *1*], *unfold One-nat-def*)
(*simp add*: *mult-ac trans* [*OF add-commute of-nat-Suc* [*symmetric*]])
finally show *?thesis* **by** (*simp add*: *algebra-simps*)
next

assume $\neg(n > 1)$
hence $n = 1 \vee n = 0$ **by** *auto*
thus *?thesis* **by** (*auto simp*: *algebra-simps*)
qed

lemma *arith-series-nat*:
$Suc\ (Suc\ 0) * (\sum i{\in}\{..{<}n\}.\ a{+}i{*}d) = n * (a + (a{+}(n - 1){*}d))$
proof −
have
$((1{::}nat) + 1) * (\sum i{\in}\{..{<}n{::}nat\}.\ a + of\text{-}nat(i){*}d) =$
$of\text{-}nat(n) * (a + (a + of\text{-}nat(n - 1){*}d))$
by (*rule arith-series-general*)
thus *?thesis*
unfolding *One-nat-def* **by** *auto*
qed

lemma *arith-series-int*:
$(2{::}int) * (\sum i{\in}\{..{<}n\}.\ a + of\text{-}nat\ i * d) =$
$of\text{-}nat\ n * (a + (a + of\text{-}nat(n - 1){*}d))$
proof −
have
$((1{::}int) + 1) * (\sum i{\in}\{..{<}n\}.\ a + of\text{-}nat\ i * d) =$
$of\text{-}nat(n) * (a + (a + of\text{-}nat(n - 1){*}d))$
by (*rule arith-series-general*)
thus *?thesis* **by** *simp*
qed

lemma *sum-diff-distrib*:
fixes $P{::}nat{\Rightarrow}nat$
shows
$\forall x.\ Q\ x \leq P\ x \implies$
$(\sum x{<}n.\ P\ x) - (\sum x{<}n.\ Q\ x) = (\sum x{<}n.\ P\ x - Q\ x)$
proof (*induct n*)
case *0* **show** *?case* **by** *simp*
next
case (*Suc n*)

let *?lhs* $= (\sum x{<}n.\ P\ x) - (\sum x{<}n.\ Q\ x)$
let *?rhs* $= \sum x{<}n.\ P\ x - Q\ x$

from *Suc* **have** *?lhs* $=$ *?rhs* **by** *simp*
moreover
from *Suc* **have** *?lhs* $+ P\ n - Q\ n =$ *?rhs* $+ (P\ n - Q\ n)$ **by** *simp*
moreover
from *Suc* **have**
$(\sum x{<}n.\ P\ x) + P\ n - ((\sum x{<}n.\ Q\ x) + Q\ n) =$ *?rhs* $+ (P\ n - Q\ n)$
by (*subst diff-diff-left*[*symmetric*],
subst diff-add-assoc2)
(*auto simp*: *diff-add-assoc2 intro*: *setsum-mono*)

ultimately
show *?case* **by** *simp*
qed

38.11 Products indexed over intervals

syntax
-from-to-setprod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*PROD - = -..-./ -*) [*0*,*0*,*0*,*10*] *10*)
-from-upto-setprod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*PROD - = -..<-./ -*) [*0*,*0*,*0*,*10*] *10*)
-upt-setprod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*PROD -<-./ -*) [*0*,*0*,*10*] *10*)
-upto-setprod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*PROD -<=-./ -*) [*0*,*0*,*10*] *10*)
syntax (*xsymbols*)
-from-to-setprod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*∏ - = -..-./ -*) [*0*,*0*,*0*,*10*] *10*)
-from-upto-setprod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*∏ - = -..<-./ -*) [*0*,*0*,*0*,*10*] *10*)
-upt-setprod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*∏ -<-./ -*) [*0*,*0*,*10*] *10*)
-upto-setprod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*∏ -≤-./ -*) [*0*,*0*,*10*] *10*)
syntax (*HTML* **output**)
-from-to-setprod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*∏ - = -..-./ -*) [*0*,*0*,*0*,*10*] *10*)
-from-upto-setprod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*∏ - = -..<-./ -*) [*0*,*0*,*0*,*10*] *10*)
-upt-setprod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*∏ -<-./ -*) [*0*,*0*,*10*] *10*)
-upto-setprod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b* ((*∏ -≤-./ -*) [*0*,*0*,*10*] *10*)
syntax (*latex-prod* **output**)
-from-to-setprod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*∏*_- = _ -) [*0*,*0*,*0*,*10*] *10*)
-from-upto-setprod :: *idt* ⇒ *'a* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*∏*^<-_- = _ -) [*0*,*0*,*0*,*10*] *10*)
-upt-setprod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*∏*_- < _ -) [*0*,*0*,*10*] *10*)
-upto-setprod :: *idt* ⇒ *'a* ⇒ *'b* ⇒ *'b*
((*∏*_- ≤ _ -) [*0*,*0*,*10*] *10*)

translations
∏ x=a..b. t == *CONST setprod* (*%x. t*) {*a..b*}
∏ x=a..<b. t == *CONST setprod* (*%x. t*) {*a..<b*}
∏ i≤n. t == *CONST setprod* (*λi. t*) {*..n*}
∏ i<n. t == *CONST setprod* (*λi. t*) {*..<n*}

38.12 Transfer setup

lemma *transfer-nat-int-set-functions*:
{*..n*} = *nat* ' {*0..int n*}
{*m..n*} = *nat* ' {*int m..int n*}
apply (*auto simp add*: *image-def*)
apply (*rule-tac x* = *int x* **in** *bexI*)
apply *auto*
apply (*rule-tac x* = *int x* **in** *bexI*)
apply *auto*

done

lemma *transfer-nat-int-set-function-closures*:
$x >= 0 \Longrightarrow$ *nat-set* $\{x..y\}$
by (*simp add*: *nat-set-def*)

declare *transfer-morphism-nat-int*[*transfer add*
return: *transfer-nat-int-set-functions*
transfer-nat-int-set-function-closures
]

lemma *transfer-int-nat-set-functions*:
is-nat m $\Longrightarrow$ *is-nat n* $\Longrightarrow$ $\{m..n\} =$ *int* ' $\{$*nat m*..*nat n*$\}$
by (*simp only*: *is-nat-def transfer-nat-int-set-functions*
transfer-nat-int-set-function-closures
transfer-nat-int-set-return-embed nat-0-le
cong: *transfer-nat-int-set-cong*)

lemma *transfer-int-nat-set-function-closures*:
is-nat x $\Longrightarrow$ *nat-set* $\{x..y\}$
by (*simp only*: *transfer-nat-int-set-function-closures is-nat-def*)

declare *transfer-morphism-int-nat*[*transfer add*
return: *transfer-int-nat-set-functions*
transfer-int-nat-set-function-closures
]

end

39 Presburger: Decision Procedure for Presburger Arithmetic

theory *Presburger*
imports *Groebner-Basis SetInterval*
uses
Tools/Qelim/qelim.ML
Tools/Qelim/cooper-procedure.ML
(*Tools/Qelim/cooper.ML*)
begin

39.1 The $-\infty$ and $+\infty$ Properties

lemma *minf*:
$[\![\exists (z ::'a::linorder).\forall x<z.\ P\ x = P'\ x; \exists z.\forall x<z.\ Q\ x = Q'\ x]\!]$
$\Longrightarrow \exists z.\forall x<z.\ (P\ x \wedge Q\ x) = (P'\ x \wedge Q'\ x)$
$[\![\exists (z ::'a::linorder).\forall x<z.\ P\ x = P'\ x; \exists z.\forall x<z.\ Q\ x = Q'\ x]\!]$
$\Longrightarrow \exists z.\forall x<z.\ (P\ x \vee Q\ x) = (P'\ x \vee Q'\ x)$

$\exists(z::'a::\{linorder\}).\forall x<z.(x = t) = False$
$\exists(z::'a::\{linorder\}).\forall x<z.(x \neq t) = True$
$\exists(z::'a::\{linorder\}).\forall x<z.(x < t) = True$
$\exists(z::'a::\{linorder\}).\forall x<z.(x \leq t) = True$
$\exists(z::'a::\{linorder\}).\forall x<z.(x > t) = False$
$\exists(z::'a::\{linorder\}).\forall x<z.(x \geq t) = False$
$\exists z.\forall(x::'a::\{linorder,plus,Rings.dvd\})<z.\ (d\ dvd\ x + s) = (d\ dvd\ x + s)$
$\exists z.\forall(x::'a::\{linorder,plus,Rings.dvd\})<z.\ (\neg\ d\ dvd\ x + s) = (\neg\ d\ dvd\ x + s)$
$\exists z.\forall x<z.\ F = F$
by ((*erule exE*, *erule exE*,*rule-tac x*=*min z za* **in** *exI*,*simp*)+, (*rule-tac x*=*t* **in** *exI*,*fastsimp*)+) *simp-all*

lemma *pinf*:
$[\![\exists(z::'a::linorder).\forall x>z.\ P\ x = P'\ x;\ \exists z.\forall x>z.\ Q\ x = Q'\ x]\!]$
$\Longrightarrow \exists z.\forall x>z.\ (P\ x \wedge Q\ x) = (P'\ x \wedge Q'\ x)$
$[\![\exists(z::'a::linorder).\forall x>z.\ P\ x = P'\ x;\ \exists z.\forall x>z.\ Q\ x = Q'\ x]\!]$
$\Longrightarrow \exists z.\forall x>z.\ (P\ x \vee Q\ x) = (P'\ x \vee Q'\ x)$
$\exists(z::'a::\{linorder\}).\forall x>z.(x = t) = False$
$\exists(z::'a::\{linorder\}).\forall x>z.(x \neq t) = True$
$\exists(z::'a::\{linorder\}).\forall x>z.(x < t) = False$
$\exists(z::'a::\{linorder\}).\forall x>z.(x \leq t) = False$
$\exists(z::'a::\{linorder\}).\forall x>z.(x > t) = True$
$\exists(z::'a::\{linorder\}).\forall x>z.(x \geq t) = True$
$\exists z.\forall(x::'a::\{linorder,plus,Rings.dvd\})>z.\ (d\ dvd\ x + s) = (d\ dvd\ x + s)$
$\exists z.\forall(x::'a::\{linorder,plus,Rings.dvd\})>z.\ (\neg\ d\ dvd\ x + s) = (\neg\ d\ dvd\ x + s)$
$\exists z.\forall x>z.\ F = F$
by ((*erule exE*, *erule exE*,*rule-tac x*=*max z za* **in** *exI*,*simp*)+,(*rule-tac x*=*t* **in** *exI*,*fastsimp*)+) *simp-all*

lemma *inf-period*:
$[\![\forall x\ k.\ P\ x = P\ (x - k*D);\ \forall x\ k.\ Q\ x = Q\ (x - k*D)]\!]$
$\Longrightarrow \forall x\ k.\ (P\ x \wedge Q\ x) = (P\ (x - k*D) \wedge Q\ (x - k*D))$
$[\![\forall x\ k.\ P\ x = P\ (x - k*D);\ \forall x\ k.\ Q\ x = Q\ (x - k*D)]\!]$
$\Longrightarrow \forall x\ k.\ (P\ x \vee Q\ x) = (P\ (x - k*D) \vee Q\ (x - k*D))$
$(d::'a::\{comm\text{-}ring,Rings.dvd\})\ dvd\ D \Longrightarrow \forall x\ k.\ (d\ dvd\ x + t) = (d\ dvd\ (x - k*D) + t)$
$(d::'a::\{comm\text{-}ring,Rings.dvd\})\ dvd\ D \Longrightarrow \forall x\ k.\ (\neg d\ dvd\ x + t) = (\neg d\ dvd\ (x - k*D) + t)$
$\forall x\ k.\ F = F$
apply (*auto elim*!: *dvdE simp add*: *algebra-simps*)
unfolding *mult-assoc* [*symmetric*] *left-distrib* [*symmetric*] *left-diff-distrib* [*symmetric*]
unfolding *dvd-def mult-commute* [*of d*]
by *auto*

39.2 The A and B sets

lemma *bset*:
$[\![\forall x.(\forall j \in \{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow P\ x \longrightarrow P(x - D)\ ;$
$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow Q\ x \longrightarrow Q(x - D)]\!] \Longrightarrow$

$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (P\ x \wedge Q\ x) \longrightarrow (P(x - D) \wedge Q\ (x - D))$
$[\![\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow P\ x \longrightarrow P(x - D)\ ;$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow Q\ x \longrightarrow Q(x - D)]\!] \Longrightarrow$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (P\ x \vee Q\ x) \longrightarrow (P(x - D) \vee Q\ (x - D))$
$[\![D{>}0; t - 1{\in} B]\!] \Longrightarrow (\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x = t) \longrightarrow (x - D = t))$
$[\![D{>}0\ ;\ t \in B]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x \neq t) \longrightarrow (x - D \neq t))$
$D{>}0 \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x < t) \longrightarrow (x - D < t))$
$D{>}0 \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x \leq t) \longrightarrow (x - D \leq t))$
$[\![D{>}0\ ;\ t \in B]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x > t) \longrightarrow (x - D > t))$
$[\![D{>}0\ ;\ t - 1 \in B]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x \geq t) \longrightarrow (x - D \geq t))$
$d\ dvd\ D \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (d\ dvd\ x{+}t) \longrightarrow (d\ dvd\ (x - D) + t))$
$d\ dvd\ D \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (\neg d\ dvd\ x{+}t) \longrightarrow (\neg\ d\ dvd\ (x - D) + t))$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow F \longrightarrow F$

proof (*blast*, *blast*)
assume *dp*: $D > 0$ **and** *tB*: $t - 1{\in} B$
show $(\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x = t) \longrightarrow (x - D = t))$
apply (*rule allI*, *rule impI*,*erule ballE*[**where** $x{=}1$],*erule ballE*[**where** $x{=}t - 1$])
apply *algebra* **using** *dp tB* **by** *simp-all*
next
assume *dp*: $D > 0$ **and** *tB*: $t \in B$
show $(\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x \neq t) \longrightarrow (x - D \neq t))$
apply (*rule allI*, *rule impI*,*erule ballE*[**where** $x{=}D$],*erule ballE*[**where** $x{=}t$])
apply *algebra*
using *dp tB* **by** *simp-all*
next
assume *dp*: $D > 0$ **thus** $(\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x < t) \longrightarrow (x - D < t))$ **by** *arith*
next
assume *dp*: $D > 0$ **thus** $\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x \leq t) \longrightarrow (x - D \leq t)$ **by** *arith*
next
assume *dp*: $D > 0$ **and** *tB*:$t \in B$
{fix x **assume** *nob*: $\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j$ **and** g: $x > t$ **and** ng: $\neg\ (x - D) > t$
hence $x - t \leq D$ **and** $1 \leq x - t$ **by** *simp+*
hence $\exists j \in \{1 \,..\, D\}.\ x - t = j$ **by** *auto*
hence $\exists j \in \{1 \,..\, D\}.\ x = t + j$ **by** (*simp add*: *algebra-simps*)
with *nob tB* **have** *False* **by** *simp***}**

thus $\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x > t) \longrightarrow (x - D > t)$ **by** *blast*
next
assume *dp*: $D > 0$ **and** *tB*:$t - 1 \in B$
{fix x **assume** *nob*: $\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j$ **and** *g*: $x \geq t$ **and** *ng*: $\neg\ (x - D) \geq t$
hence $x - (t - 1) \leq D$ **and** $1 \leq x - (t - 1)$ **by** *simp+*
hence $\exists j \in \{1 \,..\, D\}.\ x - (t - 1) = j$ **by** *auto*
hence $\exists j \in \{1 \,..\, D\}.\ x = (t - 1) + j$ **by** (*simp add*: *algebra-simps*)
with *nob tB* **have** *False* **by** *simp***}**
thus $\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (x \geq t) \longrightarrow (x - D \geq t)$ **by** *blast*
next
assume *d*: *d dvd D*
{fix x **assume** *H*: *d dvd* $x + t$ **with** *d* **have** *d dvd* $(x - D) + t$ **by** *algebra***}**
thus $\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (d\ dvd\ x{+}t) \longrightarrow (d\ dvd\ (x - D) + t)$ **by** *simp*
next
assume *d*: *d dvd D*
{fix x **assume** *H*: $\neg(d\ dvd\ x + t)$ **with** *d* **have** $\neg\ d\ dvd\ (x - D) + t$
by (*clarsimp simp add*: *dvd-def*,*erule-tac x*= *ka* + *k* **in** *allE*,*simp add*: *algebra-simps*)**}**
thus $\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}B.\ x \neq b + j) \longrightarrow (\neg d\ dvd\ x{+}t) \longrightarrow (\neg d\ dvd\ (x - D) + t)$ **by** *auto*
qed *blast*

lemma *aset*:
$[\![\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow P\ x \longrightarrow P(x + D)\ ;$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow Q\ x \longrightarrow Q(x + D)]\!] \Longrightarrow$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (P\ x \wedge Q\ x) \longrightarrow (P(x + D) \wedge Q\ (x + D))$
$[\![\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow P\ x \longrightarrow P(x + D)\ ;$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow Q\ x \longrightarrow Q(x + D)]\!] \Longrightarrow$
$\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (P\ x \vee Q\ x) \longrightarrow (P(x + D) \vee Q\ (x + D))$
$[\![D{>}0;\ t + 1 \in A]\!] \Longrightarrow (\forall x.(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x = t) \longrightarrow (x + D = t))$
$[\![D{>}0\ ;\ t \in A]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \neq t) \longrightarrow (x + D \neq t))$
$[\![D{>}0;\ t{\in}\ A]\!] \Longrightarrow (\forall (x{::}int).\ (\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x < t) \longrightarrow (x + D < t))$
$[\![D{>}0;\ t + 1 \in A]\!] \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \leq t) \longrightarrow (x + D \leq t))$
$D{>}0 \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x > t) \longrightarrow (x + D > t))$
$D{>}0 \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \geq t) \longrightarrow (x + D \geq t))$
$d\ dvd\ D \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (d\ dvd\ x{+}t) \longrightarrow (d\ dvd\ (x + D) + t))$
$d\ dvd\ D \Longrightarrow (\forall (x{::}int).(\forall j{\in}\{1 \,..\, D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (\neg d\ dvd\ x{+}t) \longrightarrow (\neg\ d\ dvd\ (x + D) + t))$

$\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow F \longrightarrow F$
proof (*blast*, *blast*)
assume *dp*: $D > 0$ **and** *tA*: $t + 1 \in A$
show $(\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x = t) \longrightarrow (x + D = t))$
apply (*rule allI*, *rule impI*,*erule ballE*[**where** $x{=}1$],*erule ballE*[**where** $x{=}t + 1$])
using *dp tA* **by** *simp-all*
next
assume *dp*: $D > 0$ **and** *tA*: $t \in A$
show $(\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \neq t) \longrightarrow (x + D \neq t))$
apply (*rule allI*, *rule impI*,*erule ballE*[**where** $x{=}D$],*erule ballE*[**where** $x{=}t$])
using *dp tA* **by** *simp-all*
next
assume *dp*: $D > 0$ **thus** $(\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x > t) \longrightarrow (x + D > t))$ **by** *arith*
next
assume *dp*: $D > 0$ **thus** $\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \geq t) \longrightarrow (x + D \geq t)$ **by** *arith*
next
assume *dp*: $D > 0$ **and** *tA*:$t \in A$
{fix *x* **assume** *nob*: $\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j$ **and** *g*: $x < t$ **and** *ng*: $\neg\ (x + D) < t$
hence $t - x \leq D$ **and** $1 \leq t - x$ **by** *simp+*
hence $\exists j \in \{1\ ..\ D\}.\ t - x = j$ **by** *auto*
hence $\exists j \in \{1\ ..\ D\}.\ x = t - j$ **by** (*auto simp add*: *algebra-simps*)
with *nob tA* **have** *False* **by** *simp***}**
thus $\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x < t) \longrightarrow (x + D < t)$ **by** *blast*
next
assume *dp*: $D > 0$ **and** *tA*:$t + 1 \in A$
{fix *x* **assume** *nob*: $\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j$ **and** *g*: $x \leq t$ **and** *ng*: $\neg\ (x + D) \leq t$
hence $(t + 1) - x \leq D$ **and** $1 \leq (t + 1) - x$ **by** (*simp-all add*: *algebra-simps*)
hence $\exists j \in \{1\ ..\ D\}.\ (t + 1) - x = j$ **by** *auto*
hence $\exists j \in \{1\ ..\ D\}.\ x = (t + 1) - j$ **by** (*auto simp add*: *algebra-simps*)
with *nob tA* **have** *False* **by** *simp***}**
thus $\forall x.(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (x \leq t) \longrightarrow (x + D \leq t)$ **by** *blast*
next
assume *d*: *d dvd D*
{fix *x* **assume** *H*: *d dvd x* + *t* **with** *d* **have** *d dvd* $(x + D) + t$
by (*clarsimp simp add*: *dvd-def*,*rule-tac x*= *ka* + *k* **in** *exI*,*simp add*: *algebra-simps*)**}**
thus $\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (d\ dvd\ x{+}t) \longrightarrow (d\ dvd\ (x + D) + t)$ **by** *simp*
next
assume *d*: *d dvd D*
{fix *x* **assume** *H*: $\neg(d\ dvd\ x + t)$ **with** *d* **have** $\neg d\ dvd\ (x + D) + t$
by (*clarsimp simp add*: *dvd-def*,*erule-tac x*= *ka* − *k* **in** *allE*,*simp add*: *algebra-simps*)**}**
thus $\forall (x{::}int).(\forall j{\in}\{1\ ..\ D\}.\ \forall b{\in}A.\ x \neq b - j) \longrightarrow (\neg d\ dvd\ x{+}t) \longrightarrow (\neg d\ dvd$

$(x + D) + t)$ **by** *auto*
qed *blast*

39.3 Cooper's Theorem $-\infty$ and $+\infty$ Version

39.3.1 First some trivial facts about periodic sets or predicates

lemma *periodic-finite-ex*:
assumes *dpos*: $(0::int) < d$ **and** *modd*: *ALL x k*. $P\ x = P(x - k*d)$
shows $(EX\ x.\ P\ x) = (EX\ j : \{1..d\}.\ P\ j)$
(**is** *?LHS* = *?RHS*)
proof
assume *?LHS*
then obtain x **where** P: $P\ x$ **..**
have $x\ mod\ d = x - (x\ div\ d)*d$ **by**(*simp add*:*zmod-zdiv-equality mult-ac eq-diff-eq*)
hence *Pmod*: $P\ x = P(x\ mod\ d)$ **using** *modd* **by** *simp*
show *?RHS*
proof (*cases*)
assume $x\ mod\ d = 0$
hence $P\ 0$ **using** P *Pmod* **by** *simp*
moreover have $P\ 0 = P(0 - (-1)*d)$ **using** *modd* **by** *blast*
ultimately have $P\ d$ **by** *simp*
moreover have $d : \{1..d\}$ **using** *dpos* **by** *simp*
ultimately show *?RHS* **..**
next
assume *not0*: $x\ mod\ d \neq 0$
have $P(x\ mod\ d)$ **using** *dpos P Pmod* **by** *simp*
moreover have $x\ mod\ d : \{1..d\}$
proof −
from *dpos* **have** $0 \leq x\ mod\ d$ **by**(*rule pos-mod-sign*)
moreover from *dpos* **have** $x\ mod\ d < d$ **by**(*rule pos-mod-bound*)
ultimately show *?thesis* **using** *not0* **by** *simp*
qed
ultimately show *?RHS* **..**
qed
qed *auto*

39.3.2 The $-\infty$ Version

lemma *decr-lemma*: $0 < (d::int) \Longrightarrow x - (abs(x-z)+1) * d < z$
by(*induct rule*: *int-gr-induct*,*simp-all add*:*int-distrib*)

lemma *incr-lemma*: $0 < (d::int) \Longrightarrow z < x + (abs(x-z)+1) * d$
by(*induct rule*: *int-gr-induct*, *simp-all add*:*int-distrib*)

lemma *decr-mult-lemma*:
assumes *dpos*: $(0::int) < d$ **and** *minus*: $\forall x.\ P\ x \longrightarrow P(x - d)$ **and** *knneg*: $0 <= k$
shows *ALL x*. $P\ x \longrightarrow P(x - k*d)$
using *knneg*

proof (*induct rule*:*int-ge-induct*)
case *base* **thus** *?case* **by** *simp*
next
case (*step i*)
{fix *x*
have *P x* ⟶ *P* (*x* − *i* ∗ *d*) **using** *step.hyps* **by** *blast*
also have ... ⟶ *P*(*x* − (*i* + *1*) ∗ *d*) **using** *minus*[*THEN spec*, *of x* − *i* ∗ *d*]
by (*simp add*: *algebra-simps*)
ultimately have *P x* ⟶ *P*(*x* − (*i* + *1*) ∗ *d*) **by** *blast***}**
thus *?case* **..**
qed

lemma *minusinfinity*:
assumes *dpos*: *0* < *d* **and**
P1eqP1: *ALL x k*. *P1 x* = *P1*(*x* − *k*∗*d*) **and** *ePeqP1*: *EX z*::*int*. *ALL x*. *x* < *z* ⟶ (*P x* = *P1 x*)
shows (*EX x*. *P1 x*) ⟶ (*EX x*. *P x*)
proof
assume *eP1*: *EX x*. *P1 x*
then obtain *x* **where** *P1*: *P1 x* **..**
from *ePeqP1* **obtain** *z* **where** *P1eqP*: *ALL x*. *x* < *z* ⟶ (*P x* = *P1 x*) **..**
let *?w* = *x* − (*abs*(*x*−*z*)+*1*) ∗ *d*
from *dpos* **have** *w*: *?w* < *z* **by**(*rule decr-lemma*)
have *P1 x* = *P1 ?w* **using** *P1eqP1* **by** *blast*
also have ... = *P*(*?w*) **using** *w P1eqP* **by** *blast*
finally have *P ?w* **using** *P1* **by** *blast*
thus *EX x*. *P x* **..**
qed

lemma *cpmi*:
assumes *dp*: *0* < *D* **and** *p1*:∃ *z*. ∀ *x*< *z*. *P x* = *P*′ *x*
and *nb*:∀ *x*.(∀ *j*∈ {*1*..*D*}. ∀(*b*::*int*) ∈ *B*. *x* ≠ *b*+*j*) −−> *P* (*x*) −−> *P* (*x* − *D*)
and *pd*: ∀ *x k*. *P*′ *x* = *P*′ (*x*−*k*∗*D*)
shows (∃ *x*. *P x*) = ((∃ *j*∈ {*1*..*D*} . *P*′ *j*) | (∃ *j* ∈ {*1*..*D*}.∃ *b*∈ *B*. *P* (*b*+*j*)))
(**is** *?L* = (*?R1* ∨ *?R2*))
proof−
{assume *?R2* **hence** *?L* **by** *blast***}**
moreover
{assume *H*:*?R1* **hence** *?L* **using** *minusinfinity*[*OF dp pd p1*] *periodic-finite-ex*[*OF dp pd*] **by** *simp***}**
moreover
{ fix *x*
assume *P*: *P x* **and** *H*: ¬ *?R2*
{fix *y* **assume** ¬ (∃*j*∈{*1*..*D*}. ∃ *b*∈*B*. *P* (*b* + *j*)) **and** *P*: *P y*
hence ~(*EX* (*j*::*int*) : {*1*..*D*}. *EX* (*b*::*int*) : *B*. *y* = *b*+*j*) **by** *auto*
with *nb P* **have** *P* (*y* − *D*) **by** *auto* **}**
hence *ALL x*.~(*EX* (*j*::*int*) : {*1*..*D*}. *EX* (*b*::*int*) : *B*. *P*(*b*+*j*)) −−> *P* (*x*) −−> *P* (*x* − *D*) **by** *blast*

with *H P* **have** *th*: $\forall x.\ P\ x \longrightarrow P\ (x - D)$ **by** *auto*
from *p1* **obtain** *z* **where** *z*: *ALL x. x < z −−> (P x = P' x)* **by** *blast*
let *?y* = $x - (|x - z| + 1)*D$
have *zp*: $0 <= (|x - z| + 1)$ **by** *arith*
from *dp* **have** *yz*: *?y < z* **using** *decr-lemma*[*OF dp*] **by** *simp*
from *z*[*rule-format*, *OF yz*] *decr-mult-lemma*[*OF dp th zp*, *rule-format*, *OF P*]
have *th2*: *P' ?y* **by** *auto*
with *periodic-finite-ex*[*OF dp pd*]
have *?R1* **by** *blast*}
ultimately show *?thesis* **by** *blast*
qed

39.3.3 The $+\infty$ Version

lemma *plusinfinity*:
assumes *dpos*: $(0::int) < d$ **and**
P1eqP1: $\forall x\ k.\ P'\ x = P'(x - k*d)$ **and** *ePeqP1*: $\exists\ z.\ \forall\ x>z.\ P\ x = P'\ x$
shows $(\exists\ x.\ P'\ x) \longrightarrow (\exists\ x.\ P\ x)$
proof
assume *eP1*: *EX x. P' x*
then obtain *x* **where** *P1*: *P' x* **..**
from *ePeqP1* **obtain** *z* **where** *P1eqP*: $\forall x>z.\ P\ x = P'\ x$ **..**
let *?w'* = $x + (abs(x-z)+1) * d$
let *?w* = $x - (-(abs(x-z) + 1))*d$
have *ww'*[*simp*]: *?w = ?w'* **by** (*simp add*: *algebra-simps*)
from *dpos* **have** *w*: *?w > z* **by**(*simp only*: *ww' incr-lemma*)
hence *P' x = P' ?w* **using** *P1eqP1* **by** *blast*
also have *... = P*(*?w*) **using** *w P1eqP* **by** *blast*
finally have *P ?w* **using** *P1* **by** *blast*
thus *EX x. P x* **..**
qed

lemma *incr-mult-lemma*:
assumes *dpos*: $(0::int) < d$ **and** *plus*: *ALL x::int. P x* $\longrightarrow$ $P(x + d)$ **and** *knneg*:
$0 <= k$
shows *ALL x. P x* $\longrightarrow$ $P(x + k*d)$
using *knneg*
proof (*induct rule*:*int-ge-induct*)
case *base* **thus** *?case* **by** *simp*
next
case (*step i*)
{fix *x*
have *P x* $\longrightarrow$ *P* $(x + i * d)$ **using** *step.hyps* **by** *blast*
also have *...* $\longrightarrow$ $P(x + (i + 1) * d)$ **using** *plus*[*THEN spec*, *of x + i * d*]
by (*simp add*:*int-distrib zadd-ac*)
ultimately have *P x* $\longrightarrow$ $P(x + (i + 1) * d)$ **by** *blast*}
thus *?case* **..**
qed

lemma *cppi*:
assumes *dp*: $0 < D$ **and** *p1*:$\exists z.\ \forall\ x> z.\ P\ x = P'\ x$
and *nb*:$\forall x.(\forall\ j\in \{1..D\}.\ \forall (b::int) \in A.\ x \neq b - j) --> P\ (x) --> P\ (x + D)$
and *pd*: $\forall\ x\ k.\ P'\ x = P'\ (x-k*D)$
shows $(\exists x.\ P\ x) = ((\exists\ j\in \{1..D\}\ .\ P'\ j) \mid (\exists\ j \in \{1..D\}.\exists\ b\in A.\ P\ (b - j)))$
(**is** *?L* = (*?R1* $\vee$ *?R2*))
proof−
{assume *?R2* **hence** *?L* **by** *blast***}**
moreover
{assume *H*:*?R1* **hence** *?L* **using** *plusinfinity*[*OF dp pd p1*] *periodic-finite-ex*[*OF dp pd*] **by** *simp***}**
moreover
{ fix *x*
assume *P*: *P x* **and** *H*: ¬ *?R2*
{fix *y* **assume** ¬ $(\exists j\in\{1..D\}.\ \exists\ b\in A.\ P\ (b - j))$ **and** *P*: *P y*
hence ~(*EX* (*j*::*int*) : $\{1..D\}$. *EX* (*b*::*int*) : *A*. $y = b - j$) **by** *auto*
with *nb P* **have** *P* $(y + D)$ **by** *auto* **}**
hence *ALL x*.~(*EX* (*j*::*int*) : $\{1..D\}$. *EX* (*b*::*int*) : *A*. $P(b-j)) --> P\ (x) --> P\ (x + D)$ **by** *blast*
with *H P* **have** *th*: $\forall x.\ P\ x \longrightarrow P\ (x + D)$ **by** *auto*
from *p1* **obtain** *z* **where** *z*: *ALL x*. $x > z --> (P\ x = P'\ x)$ **by** *blast*
let *?y* = $x + (|x - z| + 1)*D$
have *zp*: $0 <= (|x - z| + 1)$ **by** *arith*
from *dp* **have** *yz*: *?y* > *z* **using** *incr-lemma*[*OF dp*] **by** *simp*
from *z*[*rule-format*, *OF yz*] *incr-mult-lemma*[*OF dp th zp*, *rule-format*, *OF P*]
have *th2*: *P' ?y* **by** *auto*
with *periodic-finite-ex*[*OF dp pd*]
have *?R1* **by** *blast***}**
ultimately show *?thesis* **by** *blast*
qed

lemma *simp-from-to*: $\{i..j::int\} = (if\ j < i\ then\ \{\}\ else\ insert\ i\ \{i+1..j\})$
apply(*simp add*:*atLeastAtMost-def atLeast-def atMost-def*)
apply(*fastsimp*)
done

theorem *unity-coeff-ex*: $(\exists (x::'a::\{semiring\text{-}0,Rings.dvd\}).\ P\ (l * x)) \equiv (\exists x.\ l\ dvd\ (x + 0) \wedge P\ x)$
apply (*rule eq-reflection* [*symmetric*])
apply (*rule iffI*)
defer
apply (*erule exE*)
apply (*rule-tac* $x = l * x$ **in** *exI*)
apply (*simp add*: *dvd-def*)
apply (*rule-tac* $x = x$ **in** *exI*, *simp*)
apply (*erule exE*)
apply (*erule conjE*)
apply *simp*

apply (*erule dvdE*)
apply (*rule-tac* $x = k$ **in** *exI*)
apply *simp*
done

lemma *zdvd-mono*: **assumes** *not0*: $(k::int) \neq 0$
shows $((m::int)\ dvd\ t) \equiv (k*m\ dvd\ k*t)$
using *not0* **by** (*simp add*: *dvd-def*)

lemma *uminus-dvd-conv*: $(d\ dvd\ (t::int)) \equiv (-d\ dvd\ t)\ (d\ dvd\ (t::int)) \equiv (d\ dvd\ -t)$
by *simp-all*

Theorems for transforming predicates on nat to predicates on *int*

lemma *zdiff-int-split*: $P\ (int\ (x - y)) =$
$((y \leq x \longrightarrow P\ (int\ x - int\ y)) \wedge (x < y \longrightarrow P\ 0))$
by (*cases* $y \leq x$) (*simp-all add*: *zdiff-int*)

lemma *number-of1*: $(0::int) <= number\text{-}of\ n \Longrightarrow (0::int) <= number\text{-}of\ (Int.Bit0\ n) \wedge (0::int) <= number\text{-}of\ (Int.Bit1\ n)$
by *simp*

lemma *number-of2*: $(0::int) <= Numeral0$ **by** *simp*

Specific instances of congruence rules, to prevent simplifier from looping.

theorem *imp-le-cong*: $(0 <= x \Longrightarrow P = P') \Longrightarrow (0 <= (x::int) \longrightarrow P) = (0 <= x \longrightarrow P')$ **by** *simp*

theorem *conj-le-cong*: $(0 <= x \Longrightarrow P = P') \Longrightarrow (0 <= (x::int) \wedge P) = (0 <= x \wedge P')$
by (*simp cong*: *conj-cong*)

use *Tools/Qelim/cooper.ML*

setup *Cooper.setup*

method-setup *presburger* = *Cooper.method Cooper's algorithm for Presburger arithmetic*

declare *dvd-eq-mod-eq-0*[*symmetric*, *presburger*]
declare *mod-1*[*presburger*]
declare *mod-0*[*presburger*]
declare *mod-by-1*[*presburger*]
declare *zmod-zero*[*presburger*]
declare *zmod-self*[*presburger*]
declare *mod-self*[*presburger*]
declare *mod-by-0*[*presburger*]
declare *mod-div-trivial*[*presburger*]

declare *div-mod-equality2*[*presburger*]
declare *div-mod-equality*[*presburger*]
declare *mod-div-equality2*[*presburger*]
declare *mod-div-equality*[*presburger*]
declare *mod-mult-self1*[*presburger*]
declare *mod-mult-self2*[*presburger*]
declare *zdiv-zmod-equality2*[*presburger*]
declare *zdiv-zmod-equality*[*presburger*]
declare *mod2-Suc-Suc*[*presburger*]
lemma [*presburger*]: (*a*::*int*) *div 0 = 0* **and** [*presburger*]: *a mod 0 = a*
by *simp-all*

lemma [*presburger*, *algebra*]: *m mod 2 = (1*::*nat*) ⟷ ¬ *2 dvd m* **by** *presburger*
lemma [*presburger*, *algebra*]: *m mod 2 = Suc 0* ⟷ ¬ *2 dvd m* **by** *presburger*
lemma [*presburger*, *algebra*]: *m mod* (*Suc* (*Suc 0*)) = (*1*::*nat*) ⟷ ¬ *2 dvd m* **by** *presburger*
lemma [*presburger*, *algebra*]: *m mod* (*Suc* (*Suc 0*)) = *Suc 0* ⟷ ¬ *2 dvd m* **by** *presburger*
lemma [*presburger*, *algebra*]: *m mod 2 = (1*::*int*) ⟷ ¬ *2 dvd m* **by** *presburger*

end

40 Hilbert-Choice: Hilbert's Epsilon-Operator and the Axiom of Choice

theory *Hilbert-Choice*
imports *Nat Wellfounded Plain*
uses (*Tools/meson.ML*) (*Tools/choice-specification.ML*)
begin

40.1 Hilbert's epsilon

axiomatization *Eps* :: (*'a* => *bool*) => *'a* **where**
someI: *P x* ==> *P* (*Eps P*)

syntax (*epsilon*)
-Eps :: [*pttrn*, *bool*] => *'a* ((*3*ϵ *-./ -*) [*0*, *10*] *10*)
syntax (*HOL*)
-Eps :: [*pttrn*, *bool*] => *'a* ((*3*@ *-./ -*) [*0*, *10*] *10*)
syntax
-Eps :: [*pttrn*, *bool*] => *'a* ((*3SOME -./ -*) [*0*, *10*] *10*)
translations
SOME x. P == *CONST Eps* (%*x. P*)

print-translation ⟪
[(@{*const-syntax Eps*}, *fn* [*Abs abs*] =>
let val (*x*, *t*) = *atomic-abs-tr' abs*

in Syntax.const @{syntax-const -Eps} $ x $ t end)]
⟩⟩ — to avoid eta-contraction of body

definition *inv-into* :: *'a set => ('a => 'b) => ('b => 'a)* **where**
inv-into A f == %x. SOME y. y : A & f y = x

abbreviation *inv* :: *('a => 'b) => ('b => 'a)* **where**
inv == inv-into UNIV

40.2 Hilbert's Epsilon-operator

Easier to apply than *someI* if the witness comes from an existential formula

lemma *someI-ex* [*elim?*]: *∃ x. P x ==> P (SOME x. P x)*
apply (*erule exE*)
apply (*erule someI*)
done

Easier to apply than *someI* because the conclusion has only one occurrence of *P*.

lemma *someI2*: *[| P a; !!x. P x ==> Q x |] ==> Q (SOME x. P x)*
by (*blast intro*: *someI*)

Easier to apply than *someI2* if the witness comes from an existential formula

lemma *someI2-ex*: *[| ∃ a. P a; !!x. P x ==> Q x |] ==> Q (SOME x. P x)*
by (*blast intro*: *someI2*)

lemma *some-equality* [*intro*]:
[| P a; !!x. P x ==> x=a |] ==> (SOME x. P x) = a
by (*blast intro*: *someI2*)

lemma *some1-equality*: *[| EX!x. P x; P a |] ==> (SOME x. P x) = a*
by *blast*

lemma *some-eq-ex*: *P (SOME x. P x) = (∃ x. P x)*
by (*blast intro*: *someI*)

lemma *some-eq-trivial* [*simp*]: *(SOME y. y=x) = x*
apply (*rule some-equality*)
apply (*rule refl*, *assumption*)
done

lemma *some-sym-eq-trivial* [*simp*]: *(SOME y. x=y) = x*
apply (*rule some-equality*)
apply (*rule refl*)
apply (*erule sym*)
done

40.3 Axiom of Choice, Proved Using the Description Operator

Used in *Tools/meson.ML*

lemma *choice*: $\forall x.\ \exists y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall x.\ Q\ x\ (f\ x)$
by (*fast elim*: *someI*)

lemma *bchoice*: $\forall x{\in}S.\ \exists y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall x{\in}S.\ Q\ x\ (f\ x)$
by (*fast elim*: *someI*)

40.4 Function Inverse

lemma *inv-def*: *inv f* = (%*y*. *SOME x*. *f x* = *y*)
by(*simp add*: *inv-into-def*)

lemma *inv-into-into*: *x* : *f* ' *A* ==> *inv-into A f x* : *A*
apply (*simp add*: *inv-into-def*)
apply (*fast intro*: *someI2*)
done

lemma *inv-id* [*simp*]: *inv id* = *id*
by (*simp add*: *inv-into-def id-def*)

lemma *inv-into-f-f* [*simp*]:
[| *inj-on f A*; *x* : *A* |] ==> *inv-into A f* (*f x*) = *x*
apply (*simp add*: *inv-into-def inj-on-def*)
apply (*blast intro*: *someI2*)
done

lemma *inv-f-f*: *inj f* ==> *inv f* (*f x*) = *x*
by *simp*

lemma *f-inv-into-f*: *y* : *f*'*A* ==> *f* (*inv-into A f y*) = *y*
apply (*simp add*: *inv-into-def*)
apply (*fast intro*: *someI2*)
done

lemma *inv-into-f-eq*: [| *inj-on f A*; *x* : *A*; *f x* = *y* |] ==> *inv-into A f y* = *x*
apply (*erule subst*)
apply (*fast intro*: *inv-into-f-f*)
done

lemma *inv-f-eq*: [| *inj f*; *f x* = *y* |] ==> *inv f y* = *x*
by (*simp add*:*inv-into-f-eq*)

lemma *inj-imp-inv-eq*: [| *inj f*; *ALL x*. *f*(*g x*) = *x* |] ==> *inv f* = *g*
by (*blast intro*: *ext inv-into-f-eq*)

But is it useful?

lemma *inj-transfer*:
  assumes *injf*: *inj f* **and** *minor*: !!*y*. *y* ∈ *range*(*f*) ==> *P*(*inv f y*)
  shows *P x*
proof −
  have *f x* ∈ *range f* **by** *auto*
  hence *P*(*inv f* (*f x*)) **by** (*rule minor*)
  thus *P x* **by** (*simp add*: *inv-into-f-f* [*OF injf*])
qed

lemma *inj-iff*: (*inj f*) = (*inv f o f* = *id*)
apply (*simp add*: *o-def expand-fun-eq*)
apply (*blast intro*: *inj-on-inverseI inv-into-f-f*)
done

lemma *inv-o-cancel*[*simp*]: *inj f* ==> *inv f o f* = *id*
by (*simp add*: *inj-iff*)

lemma *o-inv-o-cancel*[*simp*]: *inj f* ==> *g o inv f o f* = *g*
by (*simp add*: *o-assoc*[*symmetric*])

lemma *inv-into-image-cancel*[*simp*]:
  inj-on f A ==> *S* <= *A* ==> *inv-into A f* ‘ *f* ‘ *S* = *S*
by(*fastsimp simp*: *image-def*)

lemma *inj-imp-surj-inv*: *inj f* ==> *surj* (*inv f*)
by (*blast intro*: *surjI inv-into-f-f*)

lemma *surj-f-inv-f*: *surj f* ==> *f*(*inv f y*) = *y*
by (*simp add*: *f-inv-into-f surj-range*)

lemma *inv-into-injective*:
  assumes *eq*: *inv-into A f x* = *inv-into A f y*
    and *x*: *x*: *f*‘*A*
    and *y*: *y*: *f*‘*A*
  shows *x*=*y*
proof −
  have *f* (*inv-into A f x*) = *f* (*inv-into A f y*) **using** *eq* **by** *simp*
  thus *?thesis* **by** (*simp add*: *f-inv-into-f x y*)
qed

lemma *inj-on-inv-into*: *B* <= *f*‘*A* ==> *inj-on* (*inv-into A f*) *B*
by (*blast intro*: *inj-onI dest*: *inv-into-injective injD*)

lemma *bij-betw-inv-into*: *bij-betw f A B* ==> *bij-betw* (*inv-into A f*) *B A*
by (*auto simp add*: *bij-betw-def inj-on-inv-into*)

lemma *surj-imp-inj-inv*: *surj f* ==> *inj* (*inv f*)
by (*simp add*: *inj-on-inv-into surj-range*)

lemma *surj-iff*: $(surj\ f) = (f\ o\ inv\ f = id)$
apply (*simp add*: *o-def expand-fun-eq*)
apply (*blast intro*: *surjI surj-f-inv-f*)
done

lemma *surj-imp-inv-eq*: $[|\ surj\ f;\ \forall x.\ g(f\ x) = x\ |] ==> inv\ f = g$
apply (*rule ext*)
apply (*drule-tac* $x = inv\ f\ x$ **in** *spec*)
apply (*simp add*: *surj-f-inv-f*)
done

lemma *bij-imp-bij-inv*: $bij\ f ==> bij\ (inv\ f)$
by (*simp add*: *bij-def inj-imp-surj-inv surj-imp-inj-inv*)

lemma *inv-equality*: $[|\ !!x.\ g\ (f\ x) = x;\ \ !!y.\ f\ (g\ y) = y\ |] ==> inv\ f = g$
apply (*rule ext*)
apply (*auto simp add*: *inv-into-def*)
done

lemma *inv-inv-eq*: $bij\ f ==> inv\ (inv\ f) = f$
apply (*rule inv-equality*)
apply (*auto simp add*: *bij-def surj-f-inv-f*)
done

lemma *inv-into-comp*:
$[|\ inj\text{-}on\ f\ (g\ `\ A);\ inj\text{-}on\ g\ A;\ x : f\ `\ g\ `\ A\ |] ==>$
$inv\text{-}into\ A\ (f\ o\ g)\ x = (inv\text{-}into\ A\ g\ o\ inv\text{-}into\ (g\ `\ A)\ f)\ x$
apply (*rule inv-into-f-eq*)
apply (*fast intro*: *comp-inj-on*)
apply (*simp add*: *inv-into-into*)
apply (*simp add*: *f-inv-into-f inv-into-into*)
done

lemma *o-inv-distrib*: $[|\ bij\ f;\ bij\ g\ |] ==> inv\ (f\ o\ g) = inv\ g\ o\ inv\ f$
apply (*rule inv-equality*)
apply (*auto simp add*: *bij-def surj-f-inv-f*)
done

lemma *image-surj-f-inv-f*: $surj\ f ==> f\ `\ (inv\ f\ `\ A) = A$
by (*simp add*: *image-eq-UN surj-f-inv-f*)

lemma *image-inv-f-f*: $inj\ f ==> (inv\ f)\ `\ (f\ `\ A) = A$
by (*simp add*: *image-eq-UN*)

lemma *inv-image-comp*: $inj\ f ==> inv\ f\ `\ (f`X) = X$
by (*auto simp add*: *image-def*)

lemma *bij-image-Collect-eq*: *bij f ==> f ' Collect P = {y. P (inv f y)}*
apply *auto*
apply (*force simp add*: *bij-is-inj*)
apply (*blast intro*: *bij-is-surj* [*THEN surj-f-inv-f*, *symmetric*])
done

lemma *bij-vimage-eq-inv-image*: *bij f ==> f -' A = inv f ' A*
apply (*auto simp add*: *bij-is-surj* [*THEN surj-f-inv-f*])
apply (*blast intro*: *bij-is-inj* [*THEN inv-into-f-f*, *symmetric*])
done

lemma *finite-fun-UNIVD1*:
  assumes *fin*: *finite* (*UNIV* :: (*'a* ⇒ *'b*) *set*)
  and *card*: *card* (*UNIV* :: *'b set*) ≠ *Suc 0*
  shows *finite* (*UNIV* :: *'a set*)
proof −
  from *fin* **have** *finb*: *finite* (*UNIV* :: *'b set*) **by** (*rule finite-fun-UNIVD2*)
  with *card* **have** *card* (*UNIV* :: *'b set*) ≥ *Suc* (*Suc 0*)
    by (*cases card* (*UNIV* :: *'b set*)) (*auto simp add*: *card-eq-0-iff*)
  then obtain *n* **where** *card* (*UNIV* :: *'b set*) = *Suc* (*Suc n*) *n* = *card* (*UNIV* :: *'b set*) − *Suc* (*Suc 0*) **by** *auto*
   then obtain *b1 b2* **where** *b1b2*: (*b1* :: *'b*) ≠ (*b2* :: *'b*) **by** (*auto simp add*: *card-Suc-eq*)
  from *fin* **have** *finite* (*range* (λ*f* :: *'a* ⇒ *'b*. *inv f b1*)) **by** (*rule finite-imageI*)
  moreover have *UNIV* = *range* (λ*f* :: *'a* ⇒ *'b*. *inv f b1*)
  proof (*rule UNIV-eq-I*)
    fix *x* :: *'a*
    from *b1b2* **have** *x* = *inv* (λ*y*. *if y* = *x then b1 else b2*) *b1* **by** (*simp add*: *inv-into-def*)
    thus *x* ∈ *range* (λ*f*::*'a* ⇒ *'b*. *inv f b1*) **by** *blast*
  qed
  ultimately show *finite* (*UNIV* :: *'a set*) **by** *simp*
qed

40.5 Other Consequences of Hilbert's Epsilon

Hilbert's Epsilon and the *split* Operator

Looping simprule

lemma *split-paired-Eps*: (*SOME x*. *P x*) = (*SOME* (*a*,*b*). *P*(*a*,*b*))
  by *simp*

lemma *Eps-split*: *Eps* (*split P*) = (*SOME xy*. *P* (*fst xy*) (*snd xy*))
  by (*simp add*: *split-def*)

lemma *Eps-split-eq* [*simp*]: (@(*x'*,*y'*). *x* = *x'* & *y* = *y'*) = (*x*,*y*)
  by *blast*

A relation is wellfounded iff it has no infinite descending chain

lemma *wf-iff-no-infinite-down-chain*:
wf r = (~(∃f. ∀i. (f(Suc i),f i) ∈ r))
apply (*simp only*: *wf-eq-minimal*)
apply (*rule iffI*)
apply (*rule notI*)
apply (*erule exE*)
apply (*erule-tac x = {w. ∃i. w=f i} in allE*, *blast*)
apply (*erule contrapos-np*, *simp*, *clarify*)
apply (*subgoal-tac ∀n. nat-rec x (%i y. @z. z:Q & (z,y) :r) n ∈ Q*)
apply (*rule-tac x = nat-rec x (%i y. @z. z:Q & (z,y) :r) in exI*)
apply (*rule allI*, *simp*)
apply (*rule someI2-ex*, *blast*, *blast*)
apply (*rule allI*)
apply (*induct-tac n*, *simp-all*)
apply (*rule someI2-ex*, *blast+*)
done

lemma *wf-no-infinite-down-chainE*:
assumes *wf r* **obtains** *k* **where** *(f (Suc k), f k) ∉ r*
using ‹*wf r*› *wf-iff-no-infinite-down-chain*[*of r*] **by** *blast*

A dynamically-scoped fact for TFL

lemma *tfl-some*: *∀P x. P x --> P (Eps P)*
by (*blast intro*: *someI*)

40.6 Least value operator

definition
LeastM :: ['*a* => '*b*::*ord*, '*a* => *bool*] => '*a* **where**
LeastM m P == SOME x. P x & (∀y. P y --> m x <= m y)

syntax
-LeastM :: [*pttrn*, '*a* => '*b*::*ord*, *bool*] => '*a* (*LEAST - WRT -. - [0, 4, 10] 10*)
translations
LEAST x WRT m. P == CONST LeastM m (%x. P)

lemma *LeastMI2*:
P x ==> (!!y. P y ==> m x <= m y)
==> (!!x. P x ==> ∀y. P y --> m x ≤ m y ==> Q x)
==> Q (LeastM m P)
apply (*simp add*: *LeastM-def*)
apply (*rule someI2-ex*, *blast*, *blast*)
done

lemma *LeastM-equality*:
P k ==> (!!x. P x ==> m k <= m x)
==> m (LEAST x WRT m. P x) = (m k::'a::order)
apply (*rule LeastMI2*, *assumption*, *blast*)

apply (*blast intro*!: *order-antisym*)
done

lemma *wf-linord-ex-has-least*:
wf r ==> ∀ *x y*. ((*x*,*y*):*r*^+) = ((*y*,*x*)~:*r*^*) ==> *P k*
==> ∃ *x*. *P x* & (!*y*. *P y* --> (*m x*,*m y*):*r*^*)
apply (*drule wf-trancl* [*THEN wf-eq-minimal* [*THEN iffD1*]])
apply (*drule-tac x* = *m*'*Collect P* **in** *spec*, *force*)
done

lemma *ex-has-least-nat*:
P k ==> ∃ *x*. *P x* & (∀ *y*. *P y* --> *m x* <= (*m y*::*nat*))
apply (*simp only*: *pred-nat-trancl-eq-le* [*symmetric*])
apply (*rule wf-pred-nat* [*THEN wf-linord-ex-has-least*])
apply (*simp add*: *less-eq linorder-not-le pred-nat-trancl-eq-le*, *assumption*)
done

lemma *LeastM-nat-lemma*:
P k ==> *P* (*LeastM m P*) & (∀ *y*. *P y* --> *m* (*LeastM m P*) <= (*m y*::*nat*))
apply (*simp add*: *LeastM-def*)
apply (*rule someI-ex*)
apply (*erule ex-has-least-nat*)
done

lemmas *LeastM-natI* = *LeastM-nat-lemma* [*THEN conjunct1*, *standard*]

lemma *LeastM-nat-le*: *P x* ==> *m* (*LeastM m P*) <= (*m x*::*nat*)
by (*rule LeastM-nat-lemma* [*THEN conjunct2*, *THEN spec*, *THEN mp*], *assumption*, *assumption*)

40.7 Greatest value operator

definition
GreatestM :: ['*a* => '*b*::*ord*, '*a* => *bool*] => '*a* **where**
GreatestM m P == *SOME x*. *P x* & (∀ *y*. *P y* --> *m y* <= *m x*)

definition
Greatest :: ('*a*::*ord* => *bool*) => '*a* (**binder** *GREATEST* 10) **where**
Greatest == *GreatestM* (%*x*. *x*)

syntax
-*GreatestM* :: [*pttrn*, '*a* => '*b*::*ord*, *bool*] => '*a*
(*GREATEST* - *WRT* -. - [0, 4, 10] 10)
translations
GREATEST x WRT m. *P* == *CONST GreatestM m* (%*x*. *P*)

lemma *GreatestMI2*:
P x ==> (!!*y*. *P y* ==> *m y* <= *m x*)
==> (!!*x*. *P x* ==> ∀ *y*. *P y* --> *m y* ≤ *m x* ==> *Q x*)

```
    ==> Q (GreatestM m P)
  apply (simp add: GreatestM-def)
  apply (rule someI2-ex, blast, blast)
  done

lemma GreatestM-equality:
 P k ==> (!!x. P x ==> m x <= m k)
    ==> m (GREATEST x WRT m. P x) = (m k::'a::order)
  apply (rule-tac m = m in GreatestMI2, assumption, blast)
  apply (blast intro!: order-antisym)
  done

lemma Greatest-equality:
  P (k::'a::order) ==> (!!x. P x ==> x <= k) ==> (GREATEST x. P x) = k
  apply (simp add: Greatest-def)
  apply (erule GreatestM-equality, blast)
  done

lemma ex-has-greatest-nat-lemma:
  P k ==> ∀x. P x --> (∃y. P y & ~ ((m y::nat) <= m x))
    ==> ∃y. P y & ~ (m y < m k + n)
  apply (induct n, force)
  apply (force simp add: le-Suc-eq)
  done

lemma ex-has-greatest-nat:
  P k ==> ∀y. P y --> m y < b
    ==> ∃x. P x & (∀y. P y --> (m y::nat) <= m x)
  apply (rule ccontr)
  apply (cut-tac P = P and n = b - m k in ex-has-greatest-nat-lemma)
    apply (subgoal-tac [3] m k <= b, auto)
  done

lemma GreatestM-nat-lemma:
  P k ==> ∀y. P y --> m y < b
    ==> P (GreatestM m P) & (∀y. P y --> (m y::nat) <= m (GreatestM m
P))
  apply (simp add: GreatestM-def)
  apply (rule someI-ex)
  apply (erule ex-has-greatest-nat, assumption)
  done

lemmas GreatestM-natI = GreatestM-nat-lemma [THEN conjunct1, standard]

lemma GreatestM-nat-le:
  P x ==> ∀y. P y --> m y < b
    ==> (m x::nat) <= m (GreatestM m P)
  apply (blast dest: GreatestM-nat-lemma [THEN conjunct2, THEN spec, of P])
  done
```

Specialization to *GREATEST*.

lemma *GreatestI*: $P\ (k::nat) ==> \forall y.\ P\ y --> y < b ==> P\ (GREATEST\ x.\ P\ x)$
apply (*simp add*: *Greatest-def*)
apply (*rule GreatestM-natI*, *auto*)
done

lemma *Greatest-le*:
$P\ x ==> \forall y.\ P\ y --> y < b ==> (x::nat) <= (GREATEST\ x.\ P\ x)$
apply (*simp add*: *Greatest-def*)
apply (*rule GreatestM-nat-le*, *auto*)
done

40.8 The Meson proof procedure

40.8.1 Negation Normal Form

de Morgan laws

lemma *meson-not-conjD*: $\sim(P\&Q) ==> \sim P \mid \sim Q$
and *meson-not-disjD*: $\sim(P|Q) ==> \sim P\ \&\ \sim Q$
and *meson-not-notD*: $\sim\sim P ==> P$
and *meson-not-allD*: $!!P.\ \sim(\forall x.\ P(x)) ==> \exists x.\ \sim P(x)$
and *meson-not-exD*: $!!P.\ \sim(\exists x.\ P(x)) ==> \forall x.\ \sim P(x)$
by *fast+*

Removal of $-->$ and $<->$ (positive and negative occurrences)

lemma *meson-imp-to-disjD*: $P-->Q ==> \sim P \mid Q$
and *meson-not-impD*: $\sim(P-->Q) ==> P\ \&\ \sim Q$
and *meson-iff-to-disjD*: $P=Q ==> (\sim P \mid Q)\ \&\ (\sim Q \mid P)$
and *meson-not-iffD*: $\sim(P=Q) ==> (P \mid Q)\ \&\ (\sim P \mid \sim Q)$
— Much more efficient than $P \wedge \neg Q \vee Q \wedge \neg P$ for computing CNF
and *meson-not-refl-disj-D*: $x \sim= x \mid P ==> P$
by *fast+*

40.8.2 Pulling out the existential quantifiers

Conjunction

lemma *meson-conj-exD1*: $!!P\ Q.\ (\exists x.\ P(x))\ \&\ Q ==> \exists x.\ P(x)\ \&\ Q$
and *meson-conj-exD2*: $!!P\ Q.\ P\ \&\ (\exists x.\ Q(x)) ==> \exists x.\ P\ \&\ Q(x)$
by *fast+*

Disjunction

lemma *meson-disj-exD*: $!!P\ Q.\ (\exists x.\ P(x)) \mid (\exists x.\ Q(x)) ==> \exists x.\ P(x) \mid Q(x)$
— DO NOT USE with forall-Skolemization: makes fewer schematic variables!!
— With ex-Skolemization, makes fewer Skolem constants
and *meson-disj-exD1*: $!!P\ Q.\ (\exists x.\ P(x)) \mid Q ==> \exists x.\ P(x) \mid Q$
and *meson-disj-exD2*: $!!P\ Q.\ P \mid (\exists x.\ Q(x)) ==> \exists x.\ P \mid Q(x)$
by *fast+*

40.8.3 Generating clauses for the Meson Proof Procedure

Disjunctions

lemma *meson-disj-assoc*: $(P|Q)|R ==> P|(Q|R)$
and *meson-disj-comm*: $P|Q ==> Q|P$
and *meson-disj-FalseD1*: $False|P ==> P$
and *meson-disj-FalseD2*: $P|False ==> P$
by *fast+*

40.9 Lemmas for Meson, the Model Elimination Procedure

Generation of contrapositives

Inserts negated disjunct after removing the negation; P is a literal. Model elimination requires assuming the negation of every attempted subgoal, hence the negated disjuncts.

lemma *make-neg-rule*: $~P|Q ==> ((~P==>P) ==> Q)$
by *blast*

Version for Plaisted's "Postive refinement" of the Meson procedure

lemma *make-refined-neg-rule*: $~P|Q ==> (P ==> Q)$
by *blast*

P should be a literal

lemma *make-pos-rule*: $P|Q ==> ((P==>~P) ==> Q)$
by *blast*

Versions of *make-neg-rule* and *make-pos-rule* that don't insert new assumptions, for ordinary resolution.

lemmas *make-neg-rule'* = *make-refined-neg-rule*

lemma *make-pos-rule'*: $[|P|Q; ~P|] ==> Q$
by *blast*

Generation of a goal clause – put away the final literal

lemma *make-neg-goal*: $~P ==> ((~P==>P) ==> False)$
by *blast*

lemma *make-pos-goal*: $P ==> ((P==>~P) ==> False)$
by *blast*

40.9.1 Lemmas for Forward Proof

There is a similarity to congruence rules

lemma *conj-forward*: $[|\ P'\&Q';\ \ P' ==> P;\ \ Q' ==> Q\ |] ==> P\&Q$
by *blast*

lemma *disj-forward*: [| $P'|Q'$; $P' ==> P$; $Q' ==> Q$ |] ==> $P|Q$
by *blast*

lemma *disj-forward2*:
[| $P'|Q'$; $P' ==> P$; [| Q'; $P==>False$ |] ==> Q |] ==> $P|Q$
apply *blast*
done

lemma *all-forward*: [| $\forall x.\ P'(x)$; !!$x.\ P'(x) ==> P(x)$ |] ==> $\forall x.\ P(x)$
by *blast*

lemma *ex-forward*: [| $\exists x.\ P'(x)$; !!$x.\ P'(x) ==> P(x)$ |] ==> $\exists x.\ P(x)$
by *blast*

40.10 Meson package

use *Tools/meson.ML*

setup *Meson.setup*

40.11 Specification package – Hilbertized version

lemma *exE-some*: [| $Ex\ P$; $c == Eps\ P$ |] ==> $P\ c$
by (*simp only*: *someI-ex*)

use *Tools/choice-specification.ML*

end

41 Sledgehammer: Sledgehammer: Isabelle–ATP Linkup

theory *Sledgehammer*
imports *Plain Hilbert-Choice*
uses
"~~/*src/Tools/Metis/metis.ML*
Tools/Sledgehammer/sledgehammer-util.ML
(*Tools/Sledgehammer/sledgehammer-fol-clause.ML*)
(*Tools/Sledgehammer/sledgehammer-fact-preprocessor.ML*)
(*Tools/Sledgehammer/sledgehammer-hol-clause.ML*)
(*Tools/Sledgehammer/sledgehammer-proof-reconstruct.ML*)
(*Tools/Sledgehammer/sledgehammer-fact-filter.ML*)
(*Tools/ATP-Manager/atp-manager.ML*)
(*Tools/ATP-Manager/atp-systems.ML*)
(*Tools/Sledgehammer/sledgehammer-fact-minimizer.ML*)

(*Tools/Sledgehammer/sledgehammer-isar.ML*)
(*Tools/Sledgehammer/meson-tactic.ML*)
(*Tools/Sledgehammer/metis-tactics.ML*)
begin

definition *COMBI* :: $'a \Rightarrow 'a$ **where**
[*no-atp*]: *COMBI P* ≡ *P*

definition *COMBK* :: $'a \Rightarrow 'b \Rightarrow 'a$ **where**
[*no-atp*]: *COMBK P Q* ≡ *P*

definition *COMBB* :: $('b => 'c) \Rightarrow ('a => 'b) \Rightarrow 'a \Rightarrow 'c$ **where** [*no-atp*]:
COMBB P Q R ≡ *P* (*Q R*)

definition *COMBC* :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow 'b \Rightarrow 'a \Rightarrow 'c$ **where**
[*no-atp*]: *COMBC P Q R* ≡ *P R Q*

definition *COMBS* :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'c$ **where**
[*no-atp*]: *COMBS P Q R* ≡ *P R* (*Q R*)

definition *fequal* :: $'a \Rightarrow 'a \Rightarrow$ *bool* **where** [*no-atp*]:
fequal X Y ≡ (*X* = *Y*)

lemma *fequal-imp-equal* [*no-atp*]: *fequal X Y* ⟹ *X* = *Y*
by (*simp add*: *fequal-def*)

lemma *equal-imp-fequal* [*no-atp*]: *X* = *Y* ⟹ *fequal X Y*
by (*simp add*: *fequal-def*)

These two represent the equivalence between Boolean equality and iff. They can't be converted to clauses automatically, as the iff would be expanded...

lemma *iff-positive*: *P* ∨ *Q* ∨ *P* = *Q*
by *blast*

lemma *iff-negative*: ¬ *P* ∨ ¬ *Q* ∨ *P* = *Q*
by *blast*

Theorems for translation to combinators

lemma *abs-S* [*no-atp*]: λ*x*. (*f x*) (*g x*) ≡ *COMBS f g*
apply (*rule eq-reflection*)
apply (*rule ext*)
apply (*simp add*: *COMBS-def*)
done

lemma *abs-I* [*no-atp*]: λ*x*. *x* ≡ *COMBI*
apply (*rule eq-reflection*)
apply (*rule ext*)
apply (*simp add*: *COMBI-def*)
done

lemma *abs-K* [*no-atp*]: $\lambda x.\ y \equiv COMBK\ y$
apply (*rule eq-reflection*)
apply (*rule ext*)
apply (*simp add*: *COMBK-def*)
done

lemma *abs-B* [*no-atp*]: $\lambda x.\ a\ (g\ x) \equiv COMBB\ a\ g$
apply (*rule eq-reflection*)
apply (*rule ext*)
apply (*simp add*: *COMBB-def*)
done

lemma *abs-C* [*no-atp*]: $\lambda x.\ (f\ x)\ b \equiv COMBC\ f\ b$
apply (*rule eq-reflection*)
apply (*rule ext*)
apply (*simp add*: *COMBC-def*)
done

41.1 Setup of external ATPs

use *Tools/Sledgehammer/sledgehammer-fol-clause.ML*
use *Tools/Sledgehammer/sledgehammer-fact-preprocessor.ML*
setup *Sledgehammer-Fact-Preprocessor.setup*
use *Tools/Sledgehammer/sledgehammer-hol-clause.ML*
use *Tools/Sledgehammer/sledgehammer-proof-reconstruct.ML*
use *Tools/Sledgehammer/sledgehammer-fact-filter.ML*
use *Tools/ATP-Manager/atp-manager.ML*
use *Tools/ATP-Manager/atp-systems.ML*
setup *ATP-Systems.setup*
use *Tools/Sledgehammer/sledgehammer-fact-minimizer.ML*
use *Tools/Sledgehammer/sledgehammer-isar.ML*
setup *Sledgehammer-Isar.setup*

41.2 The MESON prover

use *Tools/Sledgehammer/meson-tactic.ML*
setup *Meson-Tactic.setup*

41.3 The Metis prover

use *Tools/Sledgehammer/metis-tactics.ML*
setup *Metis-Tactics.setup*

end

42 Recdef: TFL: recursive function definitions

theory *Recdef*
imports *FunDef Plain*
uses
(*Tools/TFL/casesplit.ML*)
(*Tools/TFL/utils.ML*)
(*Tools/TFL/usyntax.ML*)
(*Tools/TFL/dcterm.ML*)
(*Tools/TFL/thms.ML*)
(*Tools/TFL/rules.ML*)
(*Tools/TFL/thry.ML*)
(*Tools/TFL/tfl.ML*)
(*Tools/TFL/post.ML*)
(*Tools/recdef.ML*)
begin

inductive
wfrec-rel :: $('a * 'a)$ *set* => $(('a$ => $'b)$ => $'a$ => $'b)$ => $'a$ => $'b$ => *bool*
for R :: $('a * 'a)$ *set*
and F :: $('a$ => $'b)$ => $'a$ => $'b$
where
wfrecI: *ALL z. (z, x) : R* --> *wfrec-rel R F z (g z)* ==>
wfrec-rel R F x (F g x)

definition
cut :: $('a$ => $'b)$ => $('a * 'a)$*set* => $'a$ => $'a$ => $'b$ **where**
cut f r x == *(%y. if (y,x):r then f y else undefined)*

definition
adm-wf :: $('a * 'a)$ *set* => $(('a$ => $'b)$ => $'a$ => $'b)$ => *bool* **where**
adm-wf R F == *ALL f g x.*
(ALL z. (z, x) : R --> *f z = g z)* --> *F f x = F g x*

definition
wfrec :: $('a * 'a)$ *set* => $(('a$ => $'b)$ => $'a$ => $'b)$ => $'a$ => $'b$ **where**
[*code del*]: *wfrec R F* == *%x. THE y. wfrec-rel R (%f x. F (cut f R x) x) x y*

42.1 Well-Founded Recursion

cut

lemma *cuts-eq*: *(cut f r x = cut g r x)* = *(ALL y. (y,x):r* --> *f(y)=g(y))*
by (*simp add*: *expand-fun-eq cut-def*)

lemma *cut-apply*: *(x,a):r* ==> *(cut f r a)(x)* = *f(x)*
by (*simp add*: *cut-def*)

Inductive characterization of wfrec combinator; for details see: John Harrison, "Inductive definitions: automation and application"

lemma *wfrec-unique*: [| *adm-wf R F*; *wf R* |] ==> *EX*! *y*. *wfrec-rel R F x y*
apply (*simp add*: *adm-wf-def*)
apply (*erule-tac a*=*x* **in** *wf-induct*)
apply (*rule ex1I*)
apply (*rule-tac g* = %*x*. *THE y*. *wfrec-rel R F x y* **in** *wfrec-rel.wfrecI*)
apply (*fast dest*!: *theI'*)
apply (*erule wfrec-rel.cases*, *simp*)
apply (*erule allE*, *erule allE*, *erule allE*, *erule mp*)
apply (*fast intro*: *the-equality* [*symmetric*])
done

lemma *adm-lemma*: *adm-wf R* (%*f x*. *F* (*cut f R x*) *x*)
apply (*simp add*: *adm-wf-def*)
apply (*intro strip*)
apply (*rule cuts-eq* [*THEN iffD2*, *THEN subst*], *assumption*)
apply (*rule refl*)
done

lemma *wfrec*: *wf*(*r*) ==> *wfrec r H a* = *H* (*cut* (*wfrec r H*) *r a*) *a*
apply (*simp add*: *wfrec-def*)
apply (*rule adm-lemma* [*THEN wfrec-unique*, *THEN the1-equality*], *assumption*)
apply (*rule wfrec-rel.wfrecI*)
apply (*intro strip*)
apply (*erule adm-lemma* [*THEN wfrec-unique*, *THEN theI'*])
done

* This form avoids giant explosions in proofs. NOTE USE OF ==

lemma *def-wfrec*: [| *f*==*wfrec r H*; *wf*(*r*) |] ==> *f*(*a*) = *H* (*cut f r a*) *a*
apply *auto*
apply (*blast intro*: *wfrec*)
done

lemma *tfl-wf-induct*: *ALL R*. *wf R* −−>
(*ALL P*. (*ALL x*. (*ALL y*. (*y*,*x*):*R* −−> *P y*) −−> *P x*) −−> (*ALL x*. *P x*))
apply *clarify*
apply (*rule-tac r* = *R* **and** *P* = *P* **and** *a* = *x* **in** *wf-induct*, *assumption*, *blast*)
done

lemma *tfl-cut-apply*: *ALL f R*. (*x*,*a*):*R* −−> (*cut f R a*)(*x*) = *f*(*x*)
apply *clarify*
apply (*rule cut-apply*, *assumption*)
done

lemma *tfl-wfrec*:
ALL M R f. (*f*=*wfrec R M*) −−> *wf R* −−> (*ALL x*. *f x* = *M* (*cut f R x*) *x*)
apply *clarify*
apply (*erule wfrec*)

done

lemma *tfl-eq-True*: $(x = True) \longrightarrow x$
by *blast*

lemma *tfl-rev-eq-mp*: $(x = y) \longrightarrow y \longrightarrow x$
by *blast*

lemma *tfl-simp-thm*: $(x \longrightarrow y) \longrightarrow (x = x') \longrightarrow (x' \longrightarrow y)$
by *blast*

lemma *tfl-P-imp-P-iff-True*: $P \Longrightarrow P = True$
by *blast*

lemma *tfl-imp-trans*: $(A \longrightarrow B) \Longrightarrow (B \longrightarrow C) \Longrightarrow (A \longrightarrow C)$
by *blast*

lemma *tfl-disj-assoc*: $(a \vee b) \vee c == a \vee (b \vee c)$
by *simp*

lemma *tfl-disjE*: $P \vee Q \Longrightarrow P \longrightarrow R \Longrightarrow Q \longrightarrow R \Longrightarrow R$
by *blast*

lemma *tfl-exE*: $\exists x.\ P\ x \Longrightarrow \forall x.\ P\ x \longrightarrow Q \Longrightarrow Q$
by *blast*

use *Tools/TFL/casesplit.ML*
use *Tools/TFL/utils.ML*
use *Tools/TFL/usyntax.ML*
use *Tools/TFL/dcterm.ML*
use *Tools/TFL/thms.ML*
use *Tools/TFL/rules.ML*
use *Tools/TFL/thry.ML*
use *Tools/TFL/tfl.ML*
use *Tools/TFL/post.ML*
use *Tools/recdef.ML*
setup *Recdef.setup*

Wellfoundedness of *same-fst*

definition
same-fst :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow ('b * 'b)set) \Rightarrow (('a*'b)*('a*'b))set$
where
same-fst P R == $\{((x',y'),(x,y)) .\ x'=x\ \&\ P\ x\ \&\ (y',y) : R\ x\}$
— For *rec-def* declarations where the first n parameters stay unchanged in the recursive call.

lemma *same-fstI* [*intro*!]:
$[|\ P\ x;\ (y',y) : R\ x\ |] \Longrightarrow ((x,y'),(x,y)) : same\text{-}fst\ P\ R$
by (*simp add*: *same-fst-def*)

lemma *wf-same-fst*:
  assumes *prem*: (!!x. $P\ x$ ==> *wf*($R\ x$))
  shows *wf*(*same-fst* $P\ R$)
apply (*simp cong del*: *imp-cong add*: *wf-def same-fst-def*)
apply (*intro strip*)
apply (*rename-tac a b*)
apply (*case-tac wf* ($R\ a$))
 apply (*erule-tac* $a = b$ **in** *wf-induct*, *blast*)
apply (*blast intro*: *prem*)
done

Rule setup

lemmas [*recdef-simp*] =
  inv-image-def
  measure-def
  lex-prod-def
  same-fst-def
  less-Suc-eq [*THEN iffD2*]

lemmas [*recdef-cong*] =
  if-cong let-cong image-cong INT-cong UN-cong bex-cong ball-cong imp-cong

lemmas [*recdef-wf*] =
  wf-trancl
  wf-less-than
  wf-lex-prod
  wf-inv-image
  wf-measure
  wf-pred-nat
  wf-same-fst
  wf-empty

end

43 List: The datatype of finite lists

theory *List*
imports *Plain Presburger Sledgehammer Recdef*
uses (*Tools/list-code.ML*)
begin

datatype *'a list* =
    Nil ([])
  | *Cons 'a 'a list* (**infixr** # *65*)

syntax
  — list Enumeration

-list :: *args* => *'a list* ([(-)])

translations
$[x, xs] == x\#[xs]$
$[x] == x\#[]$

43.1 Basic list processing functions

primrec
hd :: *'a list* ⇒ *'a* **where**
hd (*x* # *xs*) = *x*

primrec
tl :: *'a list* ⇒ *'a list* **where**
tl [] = []
| *tl* (*x* # *xs*) = *xs*

primrec
last :: *'a list* ⇒ *'a* **where**
last (*x* # *xs*) = (*if xs* = [] *then x else last xs*)

primrec
butlast :: *'a list* ⇒ *'a list* **where**
butlast []= []
| *butlast* (*x* # *xs*) = (*if xs* = [] *then* [] *else x* # *butlast xs*)

primrec
set :: *'a list* ⇒ *'a set* **where**
set [] = {}
| *set* (*x* # *xs*) = *insert x* (*set xs*)

primrec
map :: (*'a* ⇒ *'b*) ⇒ *'a list* ⇒ *'b list* **where**
map f [] = []
| *map f* (*x* # *xs*) = *f x* # *map f xs*

primrec
append :: *'a list* ⇒ *'a list* ⇒ *'a list* (**infixr** @ *65*) **where**
append-Nil:[] @ *ys* = *ys*
| *append-Cons*: (*x*#*xs*) @ *ys* = *x* # *xs* @ *ys*

primrec
rev :: *'a list* ⇒ *'a list* **where**
rev [] = []
| *rev* (*x* # *xs*) = *rev xs* @ [*x*]

primrec
filter:: (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *'a list* **where**
filter P [] = []

| *filter P (x # xs) = (if P x then x # filter P xs else filter P xs)*

syntax
— Special syntax for filter
-filter :: [*pttrn*, *'a list*, *bool*] => *'a list* ((*1*[-<−-./ -]))

translations
[*x<−xs . P*]== *CONST filter (%x. P) xs*

syntax (*xsymbols*)
-filter :: [*pttrn*, *'a list*, *bool*] => *'a list*((*1*[-←- ./ -]))
syntax (*HTML* **output**)
-filter :: [*pttrn*, *'a list*, *bool*] => *'a list*((*1*[-←- ./ -]))

primrec
foldl :: (*'b* ⇒ *'a* ⇒ *'b*) ⇒ *'b* ⇒ *'a list* ⇒ *'b* **where**
foldl-Nil: *foldl f a [] = a*
| *foldl-Cons*: *foldl f a (x # xs) = foldl f (f a x) xs*

primrec
foldr :: (*'a* ⇒ *'b* ⇒ *'b*) ⇒ *'a list* ⇒ *'b* ⇒ *'b* **where**
foldr f [] a = a
| *foldr f (x # xs) a = f x (foldr f xs a)*

primrec
concat:: *'a list list* ⇒ *'a list* **where**
concat [] = []
| *concat (x # xs) = x @ concat xs*

primrec (**in** *monoid-add*)
listsum :: *'a list* ⇒ *'a* **where**
listsum [] = 0
| *listsum (x # xs) = x + listsum xs*

primrec
drop:: *nat* ⇒ *'a list* ⇒ *'a list* **where**
drop-Nil: *drop n [] = []*
| *drop-Cons*: *drop n (x # xs) = (case n of 0 ⇒ x # xs | Suc m ⇒ drop m xs)*
— Warning: simpset does not contain this definition, but separate theorems for *n = 0* and *n = Suc k*

primrec
take:: *nat* ⇒ *'a list* ⇒ *'a list* **where**
take-Nil:*take n [] = []*
| *take-Cons*: *take n (x # xs) = (case n of 0 ⇒ [] | Suc m ⇒ x # take m xs)*
— Warning: simpset does not contain this definition, but separate theorems for *n = 0* and *n = Suc k*

primrec

nth :: *'a list => nat => 'a* (**infixl** ! *100*) **where**
nth-Cons: (*x # xs*) ! *n* = (*case n of 0 ⇒ x | Suc k ⇒ xs ! k*)
— Warning: simpset does not contain this definition, but separate theorems for *n = 0* and *n = Suc k*

primrec
list-update :: *'a list ⇒ nat ⇒ 'a ⇒ 'a list* **where**
list-update [] *i v* = []
| *list-update* (*x # xs*) *i v* = (*case i of 0 ⇒ v # xs | Suc j ⇒ x # list-update xs j v*)

nonterminals *lupdbinds lupdbind*

syntax
-lupdbind:: ['*a*, '*a*] => *lupdbind* ((*2*- :=/ -))
:: *lupdbind* => *lupdbinds* (-)
-lupdbinds :: [*lupdbind*, *lupdbinds*] => *lupdbinds* (-,/ -)
-LUpdate :: ['*a*, *lupdbinds*] => '*a* (-/[(-)] [*900*,*0*] *900*)

translations
-LUpdate xs (*-lupdbinds b bs*) == *-LUpdate* (*-LUpdate xs b*) *bs*
xs[*i*:=*x*] == *CONST list-update xs i x*

primrec
takeWhile :: (*'a ⇒ bool*) ⇒ *'a list ⇒ 'a list* **where**
takeWhile P [] = []
| *takeWhile P* (*x # xs*) = (*if P x then x # takeWhile P xs else* [])

primrec
dropWhile :: (*'a ⇒ bool*) ⇒ *'a list ⇒ 'a list* **where**
dropWhile P [] = []
| *dropWhile P* (*x # xs*) = (*if P x then dropWhile P xs else x # xs*)

primrec
zip :: *'a list ⇒ 'b list ⇒* (*'a × 'b*) *list* **where**
zip xs [] = []
| *zip-Cons*: *zip xs* (*y # ys*) = (*case xs of* [] => [] | *z # zs* => (*z*, *y*) *# zip zs ys*)
— Warning: simpset does not contain this definition, but separate theorems for *xs* = [] and *xs* = *z # zs*

primrec
upt :: *nat ⇒ nat ⇒ nat list* ((*1*[-..</-'])) **where**
upt-0: [*i*..<*0*] = []
| *upt-Suc*: [*i*..<(*Suc j*)] = (*if i <= j then* [*i*..<*j*] @ [*j*] *else* [])

primrec
distinct :: *'a list ⇒ bool* **where**
distinct [] ⟷ *True*

| *distinct* (*x* # *xs*) ⟷ *x* ∉ *set xs* ∧ *distinct xs*

primrec
remdups :: *'a list* ⇒ *'a list* **where**
remdups [] = []
| *remdups* (*x* # *xs*) = (*if x* ∈ *set xs then remdups xs else x* # *remdups xs*)

definition
insert :: *'a* ⇒ *'a list* ⇒ *'a list* **where**
insert x xs = (*if x* ∈ *set xs then xs else x* # *xs*)

hide-const (**open**) *insert*
hide-fact (**open**) *insert-def*

primrec
remove1 :: *'a* ⇒ *'a list* ⇒ *'a list* **where**
remove1 x [] = []
| *remove1 x* (*y* # *xs*) = (*if x* = *y then xs else y* # *remove1 x xs*)

primrec
removeAll :: *'a* ⇒ *'a list* ⇒ *'a list* **where**
removeAll x [] = []
| *removeAll x* (*y* # *xs*) = (*if x* = *y then removeAll x xs else y* # *removeAll x xs*)

primrec
replicate :: *nat* ⇒ *'a* ⇒ *'a list* **where**
replicate-0: *replicate 0 x* = []
| *replicate-Suc*: *replicate* (*Suc n*) *x* = *x* # *replicate n x*

Function *size* is overloaded for all datatypes. Users may refer to the list version as *length*.

abbreviation
length :: *'a list* ⇒ *nat* **where**
length ≡ *size*

definition
rotate1 :: *'a list* ⇒ *'a list* **where**
rotate1 xs = (*case xs of* [] ⇒ [] | *x*#*xs* ⇒ *xs* @ [*x*])

definition
rotate :: *nat* ⇒ *'a list* ⇒ *'a list* **where**
rotate n = *rotate1* ^^ *n*

definition
list-all2 :: (*'a* => *'b* => *bool*) => *'a list* => *'b list* => *bool* **where**
[*code del*]: *list-all2 P xs ys* =
(*length xs* = *length ys* ∧ (∀(*x*, *y*) ∈ *set* (*zip xs ys*). *P x y*))

definition

sublist :: *'a list* => *nat set* => *'a list* **where**
sublist xs A = *map fst* (*filter* (λ*p*. *snd p* ∈ *A*) (*zip xs* [*0*..<*size xs*]))

primrec
splice :: *'a list* ⇒ *'a list* ⇒ *'a list* **where**
splice [] *ys* = *ys*
| *splice* (*x* # *xs*) *ys* = (*if ys* = [] *then x* # *xs else x* # *hd ys* # *splice xs* (*tl ys*))
— Warning: simpset does not contain the second eqn but a derived one.

Figure 1 shows characteristic examples that should give an intuitive understanding of the above functions.

The following simple sort functions are intended for proofs, not for efficient implementations.

context *linorder*
begin

fun *sorted* :: *'a list* ⇒ *bool* **where**
sorted [] ⟷ *True* |
sorted [*x*] ⟷ *True* |
sorted (*x*#*y*#*zs*) ⟷ *x* <= *y* ∧ *sorted* (*y*#*zs*)

primrec *insort-key* :: (*'b* ⇒ *'a*) ⇒ *'b* ⇒ *'b list* ⇒ *'b list* **where**
insort-key f x [] = [*x*] |
insort-key f x (*y*#*ys*) = (*if f x* ≤ *f y then* (*x*#*y*#*ys*) *else y*#(*insort-key f x ys*))

definition *sort-key* :: (*'b* ⇒ *'a*) ⇒ *'b list* ⇒ *'b list* **where**
sort-key f xs = *foldr* (*insort-key f*) *xs* []

abbreviation *sort* ≡ *sort-key* (λ*x*. *x*)
abbreviation *insort* ≡ *insort-key* (λ*x*. *x*)

definition *insort-insert* :: *'a* ⇒ *'a list* ⇒ *'a list* **where**
insort-insert x xs = (*if x* ∈ *set xs then xs else insort x xs*)

end

43.1.1 List comprehension

Input syntax for Haskell-like list comprehension notation. Typical example: [(*x*,*y*). *x* ← *xs*, *y* ← *ys*, *x* ≠ *y*], the list of all pairs of distinct elements from *xs* and *ys*. The syntax is as in Haskell, except that | becomes a dot (like in Isabelle's set comprehension): [*e*. *x* ← *xs*, ...] rather than `[e| x <- xs, ...]`.
The qualifiers after the dot are

generators *p* ← *xs*, where *p* is a pattern and *xs* an expression of list type,
or

$[a, b] @ [c, d] = [a, b, c, d]$
length $[a, b, c] = 3$
set $[a, b, c] = \{a, b, c\}$
map f $[a, b, c] = [f\ a, f\ b, f\ c]$
rev $[a, b, c] = [c, b, a]$
hd $[a, b, c, d] = a$
tl $[a, b, c, d] = [b, c, d]$
last $[a, b, c, d] = d$
butlast $[a, b, c, d] = [a, b, c]$
filter $(\lambda n{::}nat.\ n{<}2)\ [0,2,1] = [0,1]$
concat $[[a, b], [c, d, e], [], [f]] = [a, b, c, d, e, f]$
foldl f x $[a, b, c] = f\ (f\ (f\ x\ a)\ b)\ c$
foldr f $[a, b, c]\ x = f\ a\ (f\ b\ (f\ c\ x))$
zip $[a, b, c]\ [x, y, z] = [(a, x), (b, y), (c, z)]$
zip $[a, b]\ [x, y, z] = [(a, x), (b, y)]$
splice $[a, b, c]\ [x, y, z] = [a, x, b, y, c, z]$
splice $[a, b, c, d]\ [x, y] = [a, x, b, y, c, d]$
take 2 $[a, b, c, d] = [a, b]$
take 6 $[a, b, c, d] = [a, b, c, d]$
drop 2 $[a, b, c, d] = [c, d]$
drop 6 $[a, b, c, d] = []$
takeWhile $(\lambda n.\ n < 3)\ [1, 2, 3, 0] = [1, 2]$
dropWhile $(\lambda n.\ n < 3)\ [1, 2, 3, 0] = [3, 0]$
distinct $[2, 0, 1]$
remdups $[2, 0, 2, 1, 2] = [0, 1, 2]$
List.insert 2 $[0, 1, 2] = [0, 1, 2]$
List.insert 3 $[0, 1, 2] = [3, 0, 1, 2]$
remove1 2 $[2, 0, 2, 1, 2] = [0, 2, 1, 2]$
removeAll 2 $[2, 0, 2, 1, 2] = [0, 1]$
$[a, b, c, d]\ !\ 2 = c$
$[a, b, c, d][2 := x] = [a, b, x, d]$
sublist $[a, b, c, d, e]\ \{0, 2, 3\} = [a, c, d]$
rotate1 $[a, b, c, d] = [b, c, d, a]$
rotate 3 $[a, b, c, d] = [d, a, b, c]$
replicate 4 a $= [a, a, a, a]$
$[2..{<}5] = [2, 3, 4]$
listsum $[1, 2, 3] = 6$

Figure 1: Characteristic examples

guards b, where b is a boolean expression.

Just like in Haskell, list comprehension is just a shorthand. To avoid misunderstandings, the translation into desugared form is not reversed upon output. Note that the translation of $[e.\ x \leftarrow xs]$ is optmized to $map\ (\lambda x.\ e)\ xs$.

It is easy to write short list comprehensions which stand for complex expressions. During proofs, they may become unreadable (and mangled). In such cases it can be advisable to introduce separate definitions for the list comprehensions in question.

nonterminals *lc-qual lc-quals*

syntax
-listcompr :: *'a* ⇒ *lc-qual* ⇒ *lc-quals* ⇒ *'a list* ([- . --)
-lc-gen :: *'a* ⇒ *'a list* ⇒ *lc-qual* (- <− -)
-lc-test :: *bool* ⇒ *lc-qual* (-)

-lc-end :: *lc-quals* (])
-lc-quals :: *lc-qual* ⇒ *lc-quals* ⇒ *lc-quals* (, --)
-lc-abs :: *'a* => *'b list* => *'b list*

syntax (*xsymbols*)
-lc-gen :: *'a* ⇒ *'a list* ⇒ *lc-qual* (- ← -)
syntax (*HTML* **output**)
-lc-gen :: *'a* ⇒ *'a list* ⇒ *lc-qual* (- ← -)

parse-translation (**advanced**) ⟪
```
let
  val NilC = Syntax.const @{const-syntax Nil};
  val ConsC = Syntax.const @{const-syntax Cons};
  val mapC = Syntax.const @{const-syntax map};
  val concatC = Syntax.const @{const-syntax concat};
  val IfC = Syntax.const @{const-syntax If};

  fun singl x = ConsC $ x $ NilC;

  fun pat-tr ctxt p e opti = (* %x. case x of p => e | - => [] *)
    let
      val x = Free (Name.variant (fold Term.add-free-names [p, e] []) x, dummyT);
      val e = if opti then singl e else e;
      val case1 = Syntax.const @{syntax-const -case1} $ p $ e;
      val case2 =
        Syntax.const @{syntax-const -case1} $
          Syntax.const @{const-syntax dummy-pattern} $ NilC;
      val cs = Syntax.const @{syntax-const -case2} $ case1 $ case2;
      val ft = Datatype-Case.case-tr false Datatype.info-of-constr ctxt [x, cs];
```

```
  in lambda x ft end;

 fun abs-tr ctxt (p as Free (s, T)) e opti =
       let
         val thy = ProofContext.theory-of ctxt;
         val s' = Sign.intern-const thy s;
       in
         if Sign.declared-const thy s'
         then (pat-tr ctxt p e opti, false)
         else (lambda p e, true)
       end
   | abs-tr ctxt p e opti = (pat-tr ctxt p e opti, false);

 fun lc-tr ctxt [e, Const (@{syntax-const -lc-test}, -) $ b, qs] =
       let
         val res =
           (case qs of
              Const (@{syntax-const -lc-end}, -) => singl e
            | Const (@{syntax-const -lc-quals}, -) $ q $ qs => lc-tr ctxt [e, q, qs]);
       in IfC $ b $ res $ NilC end
   | lc-tr ctxt
         [e, Const (@{syntax-const -lc-gen}, -) $ p $ es,
           Const(@{syntax-const -lc-end}, -)] =
       (case abs-tr ctxt p e true of
         (f, true) => mapC $ f $ es
       | (f, false) => concatC $ (mapC $ f $ es))
   | lc-tr ctxt
         [e, Const (@{syntax-const -lc-gen}, -) $ p $ es,
           Const (@{syntax-const -lc-quals}, -) $ q $ qs] =
       let val e' = lc-tr ctxt [e, q, qs];
       in concatC $ (mapC $ (fst (abs-tr ctxt p e' false)) $ es) end;

in [(@{syntax-const -listcompr}, lc-tr)] end
⟩

term [(x,y,z). b]
term [(x,y,z). x←xs]
term [e x y. x←xs, y←ys]
term [(x,y,z). x<a, x>b]
term [(x,y,z). x←xs, x>b]
term [(x,y,z). x<a, x←xs]
term [(x,y). Cons True x ← xs]
term [(x,y,z). Cons x [] ← xs]
term [(x,y,z). x<a, x>b, x=d]
term [(x,y,z). x<a, x>b, y←ys]
term [(x,y,z). x<a, x←xs,y>b]
term [(x,y,z). x<a, x←xs, y←ys]
term [(x,y,z). x←xs, x>b, y<a]
term [(x,y,z). x←xs, x>b, y←ys]
```

term [(*x*,*y*,*z*). *x*←*xs*, *y*←*ys*,*y*>*x*]
term [(*x*,*y*,*z*). *x*←*xs*, *y*←*ys*,*z*←*zs*]

43.1.2 [] and *op #*

lemma *not-Cons-self* [*simp*]:
xs ≠ *x* # *xs*
by (*induct xs*) *auto*

lemmas *not-Cons-self2* [*simp*] = *not-Cons-self* [*symmetric*]

lemma *neq-Nil-conv*: (*xs* ≠ []) = (∃ *y ys*. *xs* = *y* # *ys*)
by (*induct xs*) *auto*

lemma *length-induct*:
(⋀*xs*. ∀ *ys*. *length ys* < *length xs* ⟶ *P ys* ⟹ *P xs*) ⟹ *P xs*
by (*rule measure-induct* [*of length*]) *iprover*

lemma *list-nonempty-induct* [*consumes 1*, *case-names single cons*]:
assumes *xs* ≠ []
assumes *single*: ⋀*x*. *P* [*x*]
assumes *cons*: ⋀*x xs*. *xs* ≠ [] ⟹ *P xs* ⟹ *P* (*x* # *xs*)
shows *P xs*
using ‹*xs* ≠ []› **proof** (*induct xs*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*) **show** *?case* **proof** (*cases xs*)
case *Nil* **with** *single* **show** *?thesis* **by** *simp*
next
case *Cons* **then have** *xs* ≠ [] **by** *simp*
moreover with *Cons.hyps* **have** *P xs* **.**
ultimately show *?thesis* **by** (*rule cons*)
qed
qed

43.1.3 *length*

Needs to come before @ because of theorem *append-eq-append-conv*.

lemma *length-append* [*simp*]: *length* (*xs* @ *ys*) = *length xs* + *length ys*
by (*induct xs*) *auto*

lemma *length-map* [*simp*]: *length* (*map f xs*) = *length xs*
by (*induct xs*) *auto*

lemma *length-rev* [*simp*]: *length* (*rev xs*) = *length xs*
by (*induct xs*) *auto*

lemma *length-tl* [*simp*]: *length* (*tl xs*) = *length xs* − *1*
by (*cases xs*) *auto*

lemma *length-0-conv* [*iff*]: (*length xs = 0*) = (*xs = []*)
by (*induct xs*) *auto*

lemma *length-greater-0-conv* [*iff*]: (*0 < length xs*) = (*xs ≠ []*)
by (*induct xs*) *auto*

lemma *length-pos-if-in-set*: *x : set xs ⟹ length xs > 0*
by *auto*

lemma *length-Suc-conv*:
(*length xs = Suc n*) = (∃ *y ys*. *xs = y # ys ∧ length ys = n*)
by (*induct xs*) *auto*

lemma *Suc-length-conv*:
(*Suc n = length xs*) = (∃ *y ys*. *xs = y # ys ∧ length ys = n*)
apply (*induct xs*, *simp*, *simp*)
apply *blast*
done

lemma *impossible-Cons*: *length xs <= length ys ==> xs = x # ys = False*
by (*induct xs*) *auto*

lemma *list-induct2* [*consumes 1*, *case-names Nil Cons*]:
length xs = length ys ⟹ P [] [] ⟹
(⋀*x xs y ys*. *length xs = length ys ⟹ P xs ys ⟹ P* (*x#xs*) (*y#ys*))
⟹ P xs ys
proof (*induct xs arbitrary*: *ys*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs ys*) **then show** *?case* **by** (*cases ys*) *simp-all*
qed

lemma *list-induct3* [*consumes 2*, *case-names Nil Cons*]:
length xs = length ys ⟹ length ys = length zs ⟹ P [] [] [] ⟹
(⋀*x xs y ys z zs*. *length xs = length ys ⟹ length ys = length zs ⟹ P xs ys zs*
⟹ P (*x#xs*) (*y#ys*) (*z#zs*))
⟹ P xs ys zs
proof (*induct xs arbitrary*: *ys zs*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs ys zs*) **then show** *?case* **by** (*cases ys*, *simp-all*)
(*cases zs*, *simp-all*)
qed

lemma *list-induct4* [*consumes 3*, *case-names Nil Cons*]:
length xs = length ys ⟹ length ys = length zs ⟹ length zs = length ws ⟹
P [] [] [] [] ⟹ (⋀*x xs y ys z zs w ws*. *length xs = length ys ⟹*
length ys = length zs ⟹ length zs = length ws ⟹ P xs ys zs ws ⟹

```
   P (x#xs) (y#ys) (z#zs) (w#ws)) ⟹ P xs ys zs ws
proof (induct xs arbitrary: ys zs ws)
  case Nil then show ?case by simp
next
  case (Cons x xs ys zs ws) then show ?case by ((cases ys, simp-all), (cases
zs,simp-all)) (cases ws, simp-all)
qed

lemma list-induct2':
  ⟦ P [] [];
  ⋀x xs. P (x#xs) [];
  ⋀y ys. P [] (y#ys);
   ⋀x xs y ys. P xs ys  ⟹ P (x#xs) (y#ys) ⟧
 ⟹ P xs ys
by (induct xs arbitrary: ys) (case-tac x, auto)+

lemma neq-if-length-neq: length xs ≠ length ys ⟹ (xs = ys) == False
by (rule Eq-FalseI) auto

simproc-setup list-neq ((xs::'a list) = ys) = ⟪
(*
Reduces xs=ys to False if xs and ys cannot be of the same length.
This is the case if the atomic sublists of one are a submultiset
of those of the other list and there are fewer Cons's in one than the other.
*)

let

fun len (Const(@{const-name Nil},-)) acc = acc
  | len (Const(@{const-name Cons},-) $ - $ xs) (ts,n) = len xs (ts,n+1)
  | len (Const(@{const-name append},-) $ xs $ ys) acc = len xs (len ys acc)
  | len (Const(@{const-name rev},-) $ xs) acc = len xs acc
  | len (Const(@{const-name map},-) $ - $ xs) acc = len xs acc
  | len t (ts,n) = (t::ts,n);

fun list-neq - ss ct =
  let
    val (Const(-,eqT) $ lhs $ rhs) = Thm.term-of ct;
    val (ls,m) = len lhs ([],0) and (rs,n) = len rhs ([],0);
    fun prove-neq() =
      let
        val Type(-,listT::-) = eqT;
        val size = HOLogic.size-const listT;
        val eq-len = HOLogic.mk-eq (size $ lhs, size $ rhs);
        val neq-len = HOLogic.mk-Trueprop (HOLogic.Not $ eq-len);
        val thm = Goal.prove (Simplifier.the-context ss) [] [] neq-len
          (K (simp-tac (Simplifier.inherit-context ss @{simpset}) 1));
      in SOME (thm RS @{thm neq-if-length-neq}) end
  in
```

```
    if m < n andalso submultiset (op aconv) (ls,rs) orelse
       n < m andalso submultiset (op aconv) (rs,ls)
    then prove-neq() else NONE
  end;
in list-neq end;
⟩⟩
```

43.1.4 @ − append

lemma *append-assoc* [*simp*]: $(xs \mathbin{@} ys) \mathbin{@} zs = xs \mathbin{@} (ys \mathbin{@} zs)$
by (*induct xs*) *auto*

lemma *append-Nil2* [*simp*]: $xs \mathbin{@} [] = xs$
by (*induct xs*) *auto*

lemma *append-is-Nil-conv* [*iff*]: $(xs \mathbin{@} ys = []) = (xs = [] \wedge ys = [])$
by (*induct xs*) *auto*

lemma *Nil-is-append-conv* [*iff*]: $([] = xs \mathbin{@} ys) = (xs = [] \wedge ys = [])$
by (*induct xs*) *auto*

lemma *append-self-conv* [*iff*]: $(xs \mathbin{@} ys = xs) = (ys = [])$
by (*induct xs*) *auto*

lemma *self-append-conv* [*iff*]: $(xs = xs \mathbin{@} ys) = (ys = [])$
by (*induct xs*) *auto*

lemma *append-eq-append-conv* [*simp*, *no-atp*]:
length xs = *length ys* ∨ *length us* = *length vs*
==> (*xs*@*us* = *ys*@*vs*) = (*xs*=*ys* ∧ *us*=*vs*)
apply (*induct xs arbitrary*: *ys*)
apply (*case-tac ys*, *simp*, *force*)
apply (*case-tac ys*, *force*, *simp*)
done

lemma *append-eq-append-conv2*: $(xs \mathbin{@} ys = zs \mathbin{@} ts) =$
$(\exists us.\ xs = zs \mathbin{@} us \wedge us \mathbin{@} ys = ts \mid xs \mathbin{@} us = zs \wedge ys = us \mathbin{@} ts)$
apply (*induct xs arbitrary*: *ys zs ts*)
apply *fastsimp*
apply(*case-tac zs*)
apply *simp*
apply *fastsimp*
done

lemma *same-append-eq* [*iff*, *induct-simp*]: $(xs \mathbin{@} ys = xs \mathbin{@} zs) = (ys = zs)$
by *simp*

lemma *append1-eq-conv* [*iff*]: $(xs \mathbin{@} [x] = ys \mathbin{@} [y]) = (xs = ys \wedge x = y)$
by *simp*

lemma *append-same-eq* [*iff*, *induct-simp*]: $(ys \mathbin{@} xs = zs \mathbin{@} xs) = (ys = zs)$
by *simp*

lemma *append-self-conv2* [*iff*]: $(xs \mathbin{@} ys = ys) = (xs = [])$
using *append-same-eq* [*of* - - []] **by** *auto*

lemma *self-append-conv2* [*iff*]: $(ys = xs \mathbin{@} ys) = (xs = [])$
using *append-same-eq* [*of* []] **by** *auto*

lemma *hd-Cons-tl* [*simp*,*no-atp*]: $xs \neq [] \Longrightarrow hd\ xs \# tl\ xs = xs$
by (*induct xs*) *auto*

lemma *hd-append*: $hd\ (xs \mathbin{@} ys) = (if\ xs = []\ then\ hd\ ys\ else\ hd\ xs)$
by (*induct xs*) *auto*

lemma *hd-append2* [*simp*]: $xs \neq [] \Longrightarrow hd\ (xs \mathbin{@} ys) = hd\ xs$
by (*simp add*: *hd-append split*: *list.split*)

lemma *tl-append*: $tl\ (xs \mathbin{@} ys) = (case\ xs\ of\ [] \Rightarrow tl\ ys \mid z\#zs \Rightarrow zs \mathbin{@} ys)$
by (*simp split*: *list.split*)

lemma *tl-append2* [*simp*]: $xs \neq [] \Longrightarrow tl\ (xs \mathbin{@} ys) = tl\ xs \mathbin{@} ys$
by (*simp add*: *tl-append split*: *list.split*)

lemma *Cons-eq-append-conv*: $x\#xs = ys@zs =$
$(ys = [] \ \&\ x\#xs = zs \mid (EX\ ys'.\ x\#ys' = ys \ \&\ xs = ys'@zs))$
by(*cases ys*) *auto*

lemma *append-eq-Cons-conv*: $(ys@zs = x\#xs) =$
$(ys = [] \ \&\ zs = x\#xs \mid (EX\ ys'.\ ys = x\#ys' \ \&\ ys'@zs = xs))$
by(*cases ys*) *auto*

Trivial rules for solving @-equations automatically.

lemma *eq-Nil-appendI*: $xs = ys \Longrightarrow xs = [] \mathbin{@} ys$
by *simp*

lemma *Cons-eq-appendI*:
$[|\ x \# xs1 = ys;\ xs = xs1 \mathbin{@} zs\ |] \Longrightarrow x \# xs = ys \mathbin{@} zs$
by (*drule sym*) *simp*

lemma *append-eq-appendI*:
$[|\ xs \mathbin{@} xs1 = zs;\ ys = xs1 \mathbin{@} us\ |] \Longrightarrow xs \mathbin{@} ys = zs \mathbin{@} us$
by (*drule sym*) *simp*

Simplification procedure for all list equalities. Currently only tries to rearrange @ to see if - both lists end in a singleton list, - or both lists end in the same list.

ML ⟪
```
local

fun last (cons as Const(@{const-name Cons},-) $ - $ xs) =
  (case xs of Const(@{const-name Nil},-) => cons | - => last xs)
  | last (Const(@{const-name append},-) $ - $ ys) = last ys
  | last t = t;

fun list1 (Const(@{const-name Cons},-) $ - $ Const(@{const-name Nil},-)) = true
  | list1 - = false;

fun butlast ((cons as Const(@{const-name Cons},-) $ x) $ xs) =
  (case xs of Const(@{const-name Nil},-) => xs | - => cons $ butlast xs)
  | butlast ((app as Const(@{const-name append},-) $ xs) $ ys) = app $ butlast ys
  | butlast xs = Const(@{const-name Nil},fastype-of xs);

val rearr-ss = HOL-basic-ss addsimps [@{thm append-assoc},
  @{thm append-Nil}, @{thm append-Cons}];

fun list-eq ss (F as (eq as Const(-,eqT)) $ lhs $ rhs) =
  let
    val lastl = last lhs and lastr = last rhs;
    fun rearr conv =
      let
        val lhs1 = butlast lhs and rhs1 = butlast rhs;
        val Type(-,listT::-) = eqT
        val appT = [listT,listT] ---> listT
        val app = Const(@{const-name append},appT)
        val F2 = eq $ (app$lhs1$lastl) $ (app$rhs1$lastr)
        val eq = HOLogic.mk-Trueprop (HOLogic.mk-eq (F,F2));
        val thm = Goal.prove (Simplifier.the-context ss) [] [] eq
          (K (simp-tac (Simplifier.inherit-context ss rearr-ss) 1));
      in SOME ((conv RS (thm RS trans)) RS eq-reflection) end;

  in
    if list1 lastl andalso list1 lastr then rearr @{thm append1-eq-conv}
    else if lastl aconv lastr then rearr @{thm append-same-eq}
    else NONE
  end;

in

val list-eq-simproc =
  Simplifier.simproc @{theory} list-eq [(xs::'a list) = ys] (K list-eq);

end;

Addsimprocs [list-eq-simproc];
```
⟫

43.1.5 *map*

lemma *map-ext*: $(!!x.\ x : set\ xs \longrightarrow f\ x = g\ x) \Longrightarrow map\ f\ xs = map\ g\ xs$
by (*induct xs*) *simp-all*

lemma *map-ident* [*simp*]: $map\ (\lambda x.\ x) = (\lambda xs.\ xs)$
by (*rule ext*, *induct-tac xs*) *auto*

lemma *map-append* [*simp*]: $map\ f\ (xs\ @\ ys) = map\ f\ xs\ @\ map\ f\ ys$
by (*induct xs*) *auto*

lemma *map-map* [*simp*]: $map\ f\ (map\ g\ xs) = map\ (f \circ g)\ xs$
by (*induct xs*) *auto*

lemma *map-comp-map*[*simp*]: $((map\ f)\ o\ (map\ g)) = map(f\ o\ g)$
apply(*rule ext*)
apply(*simp*)
done

lemma *rev-map*: $rev\ (map\ f\ xs) = map\ f\ (rev\ xs)$
by (*induct xs*) *auto*

lemma *map-eq-conv*[*simp*]: $(map\ f\ xs = map\ g\ xs) = (!x : set\ xs.\ f\ x = g\ x)$
by (*induct xs*) *auto*

lemma *map-cong* [*fundef-cong*, *recdef-cong*]:
$xs = ys \Longrightarrow (!!x.\ x : set\ ys \Longrightarrow f\ x = g\ x) \Longrightarrow map\ f\ xs = map\ g\ ys$
— a congruence rule for *map*
by *simp*

lemma *map-is-Nil-conv* [*iff*]: $(map\ f\ xs = []) = (xs = [])$
by (*cases xs*) *auto*

lemma *Nil-is-map-conv* [*iff*]: $([] = map\ f\ xs) = (xs = [])$
by (*cases xs*) *auto*

lemma *map-eq-Cons-conv*:
$(map\ f\ xs = y\#ys) = (\exists z\ zs.\ xs = z\#zs \wedge f\ z = y \wedge map\ f\ zs = ys)$
by (*cases xs*) *auto*

lemma *Cons-eq-map-conv*:
$(x\#xs = map\ f\ ys) = (\exists z\ zs.\ ys = z\#zs \wedge x = f\ z \wedge xs = map\ f\ zs)$
by (*cases ys*) *auto*

lemmas *map-eq-Cons-D* = *map-eq-Cons-conv* [*THEN iffD1*]
lemmas *Cons-eq-map-D* = *Cons-eq-map-conv* [*THEN iffD1*]
declare *map-eq-Cons-D* [*dest*!] *Cons-eq-map-D* [*dest*!]

lemma *ex-map-conv*:
$(EX\ xs.\ ys = map\ f\ xs) = (ALL\ y : set\ ys.\ EX\ x.\ y = f\ x)$

by(*induct ys*, *auto simp add*: *Cons-eq-map-conv*)

lemma *map-eq-imp-length-eq*:
assumes *map f xs* = *map g ys*
shows *length xs* = *length ys*
using *assms* **proof** (*induct ys arbitrary*: *xs*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons y ys*) **then obtain** *z zs* **where** *xs*: *xs* = *z* # *zs* **by** *auto*
from *Cons xs* **have** *map f zs* = *map g ys* **by** *simp*
moreover with *Cons* **have** *length zs* = *length ys* **by** *blast*
with *xs* **show** *?case* **by** *simp*
qed

lemma *map-inj-on*:
[| *map f xs* = *map f ys*; *inj-on f* (*set xs Un set ys*) |]
==> *xs* = *ys*
apply(*frule map-eq-imp-length-eq*)
apply(*rotate-tac* −*1*)
apply(*induct rule*:*list-induct2*)
apply *simp*
apply(*simp*)
apply (*blast intro*:*sym*)
done

lemma *inj-on-map-eq-map*:
inj-on f (*set xs Un set ys*) ⟹ (*map f xs* = *map f ys*) = (*xs* = *ys*)
by(*blast dest*:*map-inj-on*)

lemma *map-injective*:
map f xs = *map f ys* ==> *inj f* ==> *xs* = *ys*
by (*induct ys arbitrary*: *xs*) (*auto dest*!:*injD*)

lemma *inj-map-eq-map*[*simp*]: *inj f* ⟹ (*map f xs* = *map f ys*) = (*xs* = *ys*)
by(*blast dest*:*map-injective*)

lemma *inj-mapI*: *inj f* ==> *inj* (*map f*)
by (*iprover dest*: *map-injective injD intro*: *inj-onI*)

lemma *inj-mapD*: *inj* (*map f*) ==> *inj f*
apply (*unfold inj-on-def*, *clarify*)
apply (*erule-tac x* = [*x*] **in** *ballE*)
apply (*erule-tac x* = [*y*] **in** *ballE*, *simp*, *blast*)
apply *blast*
done

lemma *inj-map*[*iff*]: *inj* (*map f*) = *inj f*
by (*blast dest*: *inj-mapD intro*: *inj-mapI*)

lemma *inj-on-mapI*: *inj-on f* ($\bigcup$(*set* ' *A*)) $\Longrightarrow$ *inj-on* (*map f*) *A*
apply(*rule inj-onI*)
apply(*erule map-inj-on*)
apply(*blast intro*:*inj-onI dest*:*inj-onD*)
done

lemma *map-idI*: ($\bigwedge$*x*. *x* ∈ *set xs* $\Longrightarrow$ *f x* = *x*) $\Longrightarrow$ *map f xs* = *xs*
by (*induct xs*, *auto*)

lemma *map-fun-upd* [*simp*]: *y* ∉ *set xs* $\Longrightarrow$ *map* (*f*(*y*:=*v*)) *xs* = *map f xs*
by (*induct xs*) *auto*

lemma *map-fst-zip*[*simp*]:
length xs = *length ys* $\Longrightarrow$ *map fst* (*zip xs ys*) = *xs*
by (*induct rule*:*list-induct2*, *simp-all*)

lemma *map-snd-zip*[*simp*]:
length xs = *length ys* $\Longrightarrow$ *map snd* (*zip xs ys*) = *ys*
by (*induct rule*:*list-induct2*, *simp-all*)

43.1.6 *rev*

lemma *rev-append* [*simp*]: *rev* (*xs* @ *ys*) = *rev ys* @ *rev xs*
by (*induct xs*) *auto*

lemma *rev-rev-ident* [*simp*]: *rev* (*rev xs*) = *xs*
by (*induct xs*) *auto*

lemma *rev-swap*: (*rev xs* = *ys*) = (*xs* = *rev ys*)
by *auto*

lemma *rev-is-Nil-conv* [*iff*]: (*rev xs* = []) = (*xs* = [])
by (*induct xs*) *auto*

lemma *Nil-is-rev-conv* [*iff*]: ([] = *rev xs*) = (*xs* = [])
by (*induct xs*) *auto*

lemma *rev-singleton-conv* [*simp*]: (*rev xs* = [*x*]) = (*xs* = [*x*])
by (*cases xs*) *auto*

lemma *singleton-rev-conv* [*simp*]: ([*x*] = *rev xs*) = (*xs* = [*x*])
by (*cases xs*) *auto*

lemma *rev-is-rev-conv* [*iff*]: (*rev xs* = *rev ys*) = (*xs* = *ys*)
apply (*induct xs arbitrary*: *ys*, *force*)
apply (*case-tac ys*, *simp*, *force*)
done

lemma *inj-on-rev*[*iff*]: *inj-on rev A*

by(*simp add*:*inj-on-def*)

lemma *rev-induct* [*case-names Nil snoc*]:
[| *P* []; !!*x xs*. *P xs* ==> *P* (*xs* @ [*x*]) |] ==> *P xs*
apply(*simplesubst rev-rev-ident*[*symmetric*])
apply(*rule-tac list* = *rev xs* **in** *list.induct*, *simp-all*)
done

lemma *rev-exhaust* [*case-names Nil snoc*]:
(*xs* = [] ==> *P*) ==>(!!*ys y*. *xs* = *ys* @ [*y*] ==> *P*) ==> *P*
by (*induct xs rule*: *rev-induct*) *auto*

lemmas *rev-cases* = *rev-exhaust*

lemma *rev-eq-Cons-iff*[*iff*]: (*rev xs* = *y*#*ys*) = (*xs* = *rev ys* @ [*y*])
by(*rule rev-cases*[*of xs*]) *auto*

43.1.7 *set*

lemma *finite-set* [*iff*]: *finite* (*set xs*)
by (*induct xs*) *auto*

lemma *set-append* [*simp*]: *set* (*xs* @ *ys*) = (*set xs* ∪ *set ys*)
by (*induct xs*) *auto*

lemma *hd-in-set*[*simp*]: *xs* ≠ [] ⟹ *hd xs* : *set xs*
by(*cases xs*) *auto*

lemma *set-subset-Cons*: *set xs* ⊆ *set* (*x* # *xs*)
by *auto*

lemma *set-ConsD*: *y* ∈ *set* (*x* # *xs*) ⟹ *y*=*x* ∨ *y* ∈ *set xs*
by *auto*

lemma *set-empty* [*iff*]: (*set xs* = {}) = (*xs* = [])
by (*induct xs*) *auto*

lemma *set-empty2*[*iff*]: ({} = *set xs*) = (*xs* = [])
by(*induct xs*) *auto*

lemma *set-rev* [*simp*]: *set* (*rev xs*) = *set xs*
by (*induct xs*) *auto*

lemma *set-map* [*simp*]: *set* (*map f xs*) = *f*‘(*set xs*)
by (*induct xs*) *auto*

lemma *set-filter* [*simp*]: *set* (*filter P xs*) = {*x*. *x* : *set xs* ∧ *P x*}
by (*induct xs*) *auto*

lemma *set-upt* [*simp*]: $set[i..<j] = \{i..<j\}$
by (*induct j*) (*simp-all add*: *atLeastLessThanSuc*)

lemma *split-list*: $x : set\ xs \Longrightarrow \exists\, ys\ zs.\ xs = ys\ @\ x\ \#\ zs$
proof (*induct xs*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case *Cons* **thus** *?case* **by** (*auto intro*: *Cons-eq-appendI*)
qed

lemma *in-set-conv-decomp*: $x \in set\ xs \longleftrightarrow (\exists\, ys\ zs.\ xs = ys\ @\ x\ \#\ zs)$
  by (*auto elim*: *split-list*)

lemma *split-list-first*: $x : set\ xs \Longrightarrow \exists\, ys\ zs.\ xs = ys\ @\ x\ \#\ zs \wedge x \notin set\ ys$
proof (*induct xs*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*Cons a xs*)
  show *?case*
  proof *cases*
    assume $x = a$ **thus** *?case* **using** *Cons* **by** *fastsimp*
  next
    assume $x \neq a$ **thus** *?case* **using** *Cons* **by**(*fastsimp intro*!: *Cons-eq-appendI*)
  qed
qed

lemma *in-set-conv-decomp-first*:
  $(x : set\ xs) = (\exists\, ys\ zs.\ xs = ys\ @\ x\ \#\ zs \wedge x \notin set\ ys)$
  by (*auto dest*!: *split-list-first*)

lemma *split-list-last*: $x : set\ xs \Longrightarrow \exists\, ys\ zs.\ xs = ys\ @\ x\ \#\ zs \wedge x \notin set\ zs$
proof (*induct xs rule*:*rev-induct*)
  case *Nil* **thus** *?case* **by** *simp*
next
  case (*snoc a xs*)
  show *?case*
  proof *cases*
    assume $x = a$ **thus** *?case* **using** *snoc* **by** *simp* (*metis ex-in-conv set-empty2*)
  next
    assume $x \neq a$ **thus** *?case* **using** *snoc* **by** *fastsimp*
  qed
qed

lemma *in-set-conv-decomp-last*:
  $(x : set\ xs) = (\exists\, ys\ zs.\ xs = ys\ @\ x\ \#\ zs \wedge x \notin set\ zs)$
  by (*auto dest*!: *split-list-last*)

lemma *split-list-prop*: $\exists\, x \in set\ xs.\ P\ x \Longrightarrow \exists\, ys\ x\ zs.\ xs = ys\ @\ x\ \#\ zs\ \&\ P\ x$

proof (*induct xs*)
case *Nil* **thus** *?case* **by** *simp*
next
case *Cons* **thus** *?case*
by(*simp add*:*Bex-def*)(*metis append-Cons append.simps*(*1*))
qed

lemma *split-list-propE*:
assumes ∃ *x* ∈ *set xs*. *P x*
obtains *ys x zs* **where** *xs* = *ys* @ *x* # *zs* **and** *P x*
using *split-list-prop* [*OF assms*] **by** *blast*

lemma *split-list-first-prop*:
∃ *x* ∈ *set xs*. *P x* ⟹
∃ *ys x zs*. *xs* = *ys*@*x*#*zs* ∧ *P x* ∧ (∀ *y* ∈ *set ys*. ¬ *P y*)
proof (*induct xs*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons x xs*)
show *?case*
proof *cases*
assume *P x*
thus *?thesis* **by** *simp*
(*metis Un-upper1 contra-subsetD in-set-conv-decomp-first self-append-conv2 set-append*)
next
assume ¬ *P x*
hence ∃ *x*∈*set xs*. *P x* **using** *Cons*(*2*) **by** *simp*
thus *?thesis* **using** ‹¬ *P x*› *Cons*(*1*) **by** (*metis append-Cons set-ConsD*)
qed
qed

lemma *split-list-first-propE*:
assumes ∃ *x* ∈ *set xs*. *P x*
obtains *ys x zs* **where** *xs* = *ys* @ *x* # *zs* **and** *P x* **and** ∀ *y* ∈ *set ys*. ¬ *P y*
using *split-list-first-prop* [*OF assms*] **by** *blast*

lemma *split-list-first-prop-iff*:
(∃ *x* ∈ *set xs*. *P x*) ⟷
(∃ *ys x zs*. *xs* = *ys*@*x*#*zs* ∧ *P x* ∧ (∀ *y* ∈ *set ys*. ¬ *P y*))
by (*rule*, *erule split-list-first-prop*) *auto*

lemma *split-list-last-prop*:
∃ *x* ∈ *set xs*. *P x* ⟹
∃ *ys x zs*. *xs* = *ys*@*x*#*zs* ∧ *P x* ∧ (∀ *z* ∈ *set zs*. ¬ *P z*)
proof(*induct xs rule*:*rev-induct*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*snoc x xs*)

show *?case*
proof *cases*
assume $P\ x$ **thus** *?thesis* **by** (*metis emptyE set-empty*)
next
assume $\neg\ P\ x$
hence $\exists\, x{\in}set\ xs.\ P\ x$ **using** *snoc*(*2*) **by** *simp*
thus *?thesis* **using** ‹$\neg\ P\ x$› *snoc*(*1*) **by** *fastsimp*
qed
qed

lemma *split-list-last-propE*:
assumes $\exists\, x \in set\ xs.\ P\ x$
obtains *ys x zs* **where** $xs = ys\ @\ x\ \#\ zs$ **and** $P\ x$ **and** $\forall\, z \in set\ zs.\ \neg\ P\ z$
using *split-list-last-prop* [*OF assms*] **by** *blast*

lemma *split-list-last-prop-iff*:
$(\exists\, x \in set\ xs.\ P\ x) \longleftrightarrow$
$(\exists\, ys\ x\ zs.\ xs = ys@x\#zs \wedge P\ x \wedge (\forall\, z \in set\ zs.\ \neg\ P\ z))$
by (*metis split-list-last-prop* [**where** *P*=*P*] *in-set-conv-decomp*)

lemma *finite-list*: *finite A* ==> *EX xs. set xs* = *A*
by (*erule finite-induct*)
(*auto simp add*: *set.simps*(*2*) [*symmetric*] *simp del*: *set.simps*(*2*))

lemma *card-length*: *card* (*set xs*) ≤ *length xs*
by (*induct xs*) (*auto simp add*: *card-insert-if*)

lemma *set-minus-filter-out*:
set xs − {*y*} = *set* (*filter* (λ*x*. ¬ (*x* = *y*)) *xs*)
by (*induct xs*) *auto*

43.1.8 *filter*

lemma *filter-append* [*simp*]: *filter P* (*xs* @ *ys*) = *filter P xs* @ *filter P ys*
by (*induct xs*) *auto*

lemma *rev-filter*: *rev* (*filter P xs*) = *filter P* (*rev xs*)
by (*induct xs*) *simp-all*

lemma *filter-filter* [*simp*]: *filter P* (*filter Q xs*) = *filter* (λ*x*. *Q x* ∧ *P x*) *xs*
by (*induct xs*) *auto*

lemma *length-filter-le* [*simp*]: *length* (*filter P xs*) ≤ *length xs*
by (*induct xs*) (*auto simp add*: *le-SucI*)

lemma *sum-length-filter-compl*:
length(*filter P xs*) + *length*(*filter* (%*x*. ~*P x*) *xs*) = *length xs*
by(*induct xs*) *simp-all*

lemma *filter-True* [*simp*]: $\forall x \in set\ xs.\ P\ x \Longrightarrow filter\ P\ xs = xs$
by (*induct xs*) *auto*

lemma *filter-False* [*simp*]: $\forall x \in set\ xs.\ \neg\ P\ x \Longrightarrow filter\ P\ xs = []$
by (*induct xs*) *auto*

lemma *filter-empty-conv*: $(filter\ P\ xs = []) = (\forall x{\in}set\ xs.\ \neg\ P\ x)$
by (*induct xs*) *simp-all*

lemma *filter-id-conv*: $(filter\ P\ xs = xs) = (\forall x{\in}set\ xs.\ P\ x)$
apply (*induct xs*)
apply *auto*
apply(*cut-tac P=P* **and** *xs=xs* **in** *length-filter-le*)
apply *simp*
done

lemma *filter-map*:
$filter\ P\ (map\ f\ xs) = map\ f\ (filter\ (P\ o\ f)\ xs)$
by (*induct xs*) *simp-all*

lemma *length-filter-map*[*simp*]:
$length\ (filter\ P\ (map\ f\ xs)) = length(filter\ (P\ o\ f)\ xs)$
by (*simp add*:*filter-map*)

lemma *filter-is-subset* [*simp*]: $set\ (filter\ P\ xs) \leq set\ xs$
by *auto*

lemma *length-filter-less*:
$[\![\ x : set\ xs;\ \sim\ P\ x\ ]\!] \Longrightarrow length(filter\ P\ xs) < length\ xs$
proof (*induct xs*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons x xs*) **thus** *?case*
apply (*auto split*:*split-if-asm*)
using *length-filter-le*[*of P xs*] **apply** *arith*
done
qed

lemma *length-filter-conv-card*:
$length(filter\ p\ xs) = card\{i.\ i < length\ xs\ \&\ p(xs!i)\}$
proof (*induct xs*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons x xs*)
let *?S* $= \{i.\ i < length\ xs\ \&\ p(xs!i)\}$
have *fin*: *finite ?S* **by**(*fast intro*: *bounded-nat-set-is-finite*)
show *?case* (**is** *?l* $=$ *card ?S'*)
proof (*cases*)
assume *p x*

```
    hence eq: ?S' = insert 0 (Suc ` ?S)
      by(auto simp: image-def split:nat.split dest:gr0-implies-Suc)
    have length (filter p (x # xs)) = Suc(card ?S)
      using Cons ‹p x› by simp
    also have ... = Suc(card(Suc ` ?S)) using fin
      by (simp add: card-image inj-Suc)
    also have ... = card ?S' using eq fin
      by (simp add:card-insert-if) (simp add:image-def)
    finally show ?thesis .
  next
    assume ¬ p x
    hence eq: ?S' = Suc ` ?S
      by(auto simp add: image-def split:nat.split elim:lessE)
    have length (filter p (x # xs)) = card ?S
      using Cons ‹¬ p x› by simp
    also have ... = card(Suc ` ?S) using fin
      by (simp add: card-image inj-Suc)
    also have ... = card ?S' using eq fin
      by (simp add:card-insert-if)
    finally show ?thesis .
  qed
qed

lemma Cons-eq-filterD:
 x#xs = filter P ys ⟹
  ∃us vs. ys = us @ x # vs ∧ (∀u∈set us. ¬ P u) ∧ P x ∧ xs = filter P vs
  (is - ⟹ ∃us vs. ?P ys us vs)
proof(induct ys)
  case Nil thus ?case by simp
next
  case (Cons y ys)
  show ?case (is ∃x. ?Q x)
  proof cases
    assume Py: P y
    show ?thesis
    proof cases
      assume x = y
      with Py Cons.prems have ?Q [] by simp
      then show ?thesis ..
    next
      assume x ≠ y
      with Py Cons.prems show ?thesis by simp
    qed
  next
    assume ¬ P y
    with Cons obtain us vs where ?P (y#ys) (y#us) vs by fastsimp
    then have ?Q (y#us) by simp
    then show ?thesis ..
  qed
```

qed

lemma *filter-eq-ConsD*:
filter P ys = x#xs ⟹
∃ us vs. ys = us @ x # vs ∧ (∀ u∈set us. ¬ P u) ∧ P x ∧ xs = filter P vs
by(*rule Cons-eq-filterD*) *simp*

lemma *filter-eq-Cons-iff*:
(filter P ys = x#xs) =
(∃ us vs. ys = us @ x # vs ∧ (∀ u∈set us. ¬ P u) ∧ P x ∧ xs = filter P vs)
by(*auto dest*:*filter-eq-ConsD*)

lemma *Cons-eq-filter-iff*:
(x#xs = filter P ys) =
(∃ us vs. ys = us @ x # vs ∧ (∀ u∈set us. ¬ P u) ∧ P x ∧ xs = filter P vs)
by(*auto dest*:*Cons-eq-filterD*)

lemma *filter-cong*[*fundef-cong*, *recdef-cong*]:
xs = ys ⟹ (⋀x. x ∈ set ys ⟹ P x = Q x) ⟹ filter P xs = filter Q ys
apply *simp*
apply(*erule thin-rl*)
by (*induct ys*) *simp-all*

43.1.9 List partitioning

primrec *partition* :: *('a ⇒ bool) ⇒'a list ⇒ 'a list × 'a list* **where**
partition P [] = ([], [])
| partition P (x # xs) =
(let (yes, no) = partition P xs
in if P x then (x # yes, no) else (yes, x # no))

lemma *partition-filter1*:
fst (partition P xs) = filter P xs
by (*induct xs*) (*auto simp add*: *Let-def split-def*)

lemma *partition-filter2*:
snd (partition P xs) = filter (Not o P) xs
by (*induct xs*) (*auto simp add*: *Let-def split-def*)

lemma *partition-P*:
assumes *partition P xs = (yes, no)*
shows *(∀ p ∈ set yes. P p) ∧ (∀ p ∈ set no. ¬ P p)*
proof −
from *assms* **have** *yes = fst (partition P xs)* **and** *no = snd (partition P xs)*
by *simp-all*
then show *?thesis* **by** (*simp-all add*: *partition-filter1 partition-filter2*)
qed

lemma *partition-set*:

assumes *partition P xs* = (*yes*, *no*)
shows *set yes* ∪ *set no* = *set xs*
proof −
from *assms* **have** *yes* = *fst* (*partition P xs*) **and** *no* = *snd* (*partition P xs*)
by *simp-all*
then show *?thesis* **by** (*auto simp add*: *partition-filter1 partition-filter2*)
qed

lemma *partition-filter-conv*[*simp*]:
partition f xs = (*filter f xs*,*filter* (*Not o f*) *xs*)
unfolding *partition-filter2*[*symmetric*]
unfolding *partition-filter1*[*symmetric*] **by** *simp*

declare *partition.simps*[*simp del*]

43.1.10 *concat*

lemma *concat-append* [*simp*]: *concat* (*xs* @ *ys*) = *concat xs* @ *concat ys*
by (*induct xs*) *auto*

lemma *concat-eq-Nil-conv* [*simp*]: (*concat xss* = []) = (∀ *xs* ∈ *set xss*. *xs* = [])
by (*induct xss*) *auto*

lemma *Nil-eq-concat-conv* [*simp*]: ([] = *concat xss*) = (∀ *xs* ∈ *set xss*. *xs* = [])
by (*induct xss*) *auto*

lemma *set-concat* [*simp*]: *set* (*concat xs*) = (*UN x*:*set xs*. *set x*)
by (*induct xs*) *auto*

lemma *concat-map-singleton*[*simp*]: *concat*(*map* (%*x*. [*f x*]) *xs*) = *map f xs*
by (*induct xs*) *auto*

lemma *map-concat*: *map f* (*concat xs*) = *concat* (*map* (*map f*) *xs*)
by (*induct xs*) *auto*

lemma *filter-concat*: *filter p* (*concat xs*) = *concat* (*map* (*filter p*) *xs*)
by (*induct xs*) *auto*

lemma *rev-concat*: *rev* (*concat xs*) = *concat* (*map rev* (*rev xs*))
by (*induct xs*) *auto*

43.1.11 *nth*

lemma *nth-Cons-0* [*simp*, *code*]: (*x* # *xs*)!*0* = *x*
by *auto*

lemma *nth-Cons-Suc* [*simp*, *code*]: (*x* # *xs*)!(*Suc n*) = *xs*!*n*
by *auto*

declare *nth.simps* [*simp del*]

lemma *nth-append*:
(*xs* @ *ys*)!*n* = (*if n* < *length xs then xs*!*n else ys*!(*n* − *length xs*))
apply (*induct xs arbitrary*: *n*, *simp*)
apply (*case-tac n*, *auto*)
done

lemma *nth-append-length* [*simp*]: (*xs* @ *x* # *ys*) ! *length xs* = *x*
by (*induct xs*) *auto*

lemma *nth-append-length-plus*[*simp*]: (*xs* @ *ys*) ! (*length xs* + *n*) = *ys* ! *n*
by (*induct xs*) *auto*

lemma *nth-map* [*simp*]: *n* < *length xs* ==> (*map f xs*)!*n* = *f*(*xs*!*n*)
apply (*induct xs arbitrary*: *n*, *simp*)
apply (*case-tac n*, *auto*)
done

lemma *hd-conv-nth*: *xs* ≠ [] ⟹ *hd xs* = *xs*!*0*
by(*cases xs*) *simp-all*

lemma *list-eq-iff-nth-eq*:
(*xs* = *ys*) = (*length xs* = *length ys* ∧ (*ALL i*<*length xs*. *xs*!*i* = *ys*!*i*))
apply(*induct xs arbitrary*: *ys*)
apply *force*
apply(*case-tac ys*)
apply *simp*
apply(*simp add*:*nth-Cons split*:*nat.split*)**apply** *blast*
done

lemma *set-conv-nth*: *set xs* = {*xs*!*i* | *i*. *i* < *length xs*}
apply (*induct xs*, *simp*, *simp*)
apply *safe*
apply (*metis nat-case-0 nth.simps zero-less-Suc*)
apply (*metis less-Suc-eq-0-disj nth-Cons-Suc*)
apply (*case-tac i*, *simp*)
apply (*metis diff-Suc-Suc nat-case-Suc nth.simps zero-less-diff*)
done

lemma *in-set-conv-nth*: (*x* ∈ *set xs*) = (∃ *i* < *length xs*. *xs*!*i* = *x*)
by(*auto simp*:*set-conv-nth*)

lemma *list-ball-nth*: [| *n* < *length xs*; !*x* : *set xs*. *P x*|] ==> *P*(*xs*!*n*)
by (*auto simp add*: *set-conv-nth*)

lemma *nth-mem* [*simp*]: *n* < *length xs* ==> *xs*!*n* : *set xs*
by (*auto simp add*: *set-conv-nth*)

lemma *all-nth-imp-all-set*:
[| !*i* < *length xs*. *P*(*xs*!*i*); *x* : *set xs*|] ==> *P x*
by (*auto simp add*: *set-conv-nth*)

lemma *all-set-conv-all-nth*:
(∀ *x* ∈ *set xs*. *P x*) = (∀ *i*. *i* < *length xs* −−> *P* (*xs* ! *i*))
by (*auto simp add*: *set-conv-nth*)

lemma *rev-nth*:
n < *size xs* ⟹ *rev xs* ! *n* = *xs* ! (*length xs* − *Suc n*)
proof (*induct xs arbitrary*: *n*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons x xs*)
hence *n*: *n* < *Suc* (*length xs*) **by** *simp*
moreover
{ assume *n* < *length xs*
with *n* **obtain** *n′* **where** *length xs* − *n* = *Suc n′*
by (*cases length xs* − *n*, *auto*)
moreover
then have *length xs* − *Suc n* = *n′* **by** *simp*
ultimately
have *xs* ! (*length xs* − *Suc n*) = (*x* # *xs*) ! (*length xs* − *n*) **by** *simp*
}
ultimately
show *?case* **by** (*clarsimp simp add*: *Cons nth-append*)
qed

lemma *Skolem-list-nth*:
(*ALL i*<*k*. *EX x*. *P i x*) = (*EX xs*. *size xs* = *k* & (*ALL i*<*k*. *P i* (*xs*!*i*)))
(**is** - = (*EX xs*. *?P k xs*))
proof(*induct k*)
case *0* **show** *?case* **by** *simp*
next
case (*Suc k*)
show *?case* (**is** *?L* = *?R* **is** - = (*EX xs*. *?P′ xs*))
proof
assume *?R* **thus** *?L* **using** *Suc* **by** *auto*
next
assume *?L*
with *Suc* **obtain** *x xs* **where** *?P k xs* & *P k x* **by** (*metis less-Suc-eq*)
hence *?P′*(*xs*@[*x*]) **by**(*simp add*:*nth-append less-Suc-eq*)
thus *?R* **..**
qed
qed

43.1.12 *list-update*

lemma *length-list-update* [*simp*]: *length*(*xs*[*i*:=*x*]) = *length xs*

by (*induct xs arbitrary*: *i*) (*auto split*: *nat.split*)

lemma *nth-list-update*:
i < *length xs*==> (*xs*[*i*:=*x*])!*j* = (*if i* = *j then x else xs*!*j*)
by (*induct xs arbitrary*: *i j*) (*auto simp add*: *nth-Cons split*: *nat.split*)

lemma *nth-list-update-eq* [*simp*]: *i* < *length xs* ==> (*xs*[*i*:=*x*])!*i* = *x*
by (*simp add*: *nth-list-update*)

lemma *nth-list-update-neq* [*simp*]: *i* ≠ *j* ==> *xs*[*i*:=*x*]!*j* = *xs*!*j*
by (*induct xs arbitrary*: *i j*) (*auto simp add*: *nth-Cons split*: *nat.split*)

lemma *list-update-id*[*simp*]: *xs*[*i* := *xs*!*i*] = *xs*
by (*induct xs arbitrary*: *i*) (*simp-all split*:*nat.splits*)

lemma *list-update-beyond*[*simp*]: *length xs* ≤ *i* ⟹ *xs*[*i*:=*x*] = *xs*
apply (*induct xs arbitrary*: *i*)
apply *simp*
apply (*case-tac i*)
apply *simp-all*
done

lemma *list-update-nonempty*[*simp*]: *xs*[*k*:=*x*] = [] ⟷ *xs*=[]
by(*metis length-0-conv length-list-update*)

lemma *list-update-same-conv*:
i < *length xs* ==> (*xs*[*i* := *x*] = *xs*) = (*xs*!*i* = *x*)
by (*induct xs arbitrary*: *i*) (*auto split*: *nat.split*)

lemma *list-update-append1*:
i < *size xs* ⟹ (*xs* @ *ys*)[*i*:=*x*] = *xs*[*i*:=*x*] @ *ys*
apply (*induct xs arbitrary*: *i*, *simp*)
apply(*simp split*:*nat.split*)
done

lemma *list-update-append*:
(*xs* @ *ys*) [*n*:= *x*] =
(*if n* < *length xs then xs*[*n*:= *x*] @ *ys else xs* @ (*ys* [*n*−*length xs*:= *x*]))
by (*induct xs arbitrary*: *n*) (*auto split*:*nat.splits*)

lemma *list-update-length* [*simp*]:
(*xs* @ *x* # *ys*)[*length xs* := *y*] = (*xs* @ *y* # *ys*)
by (*induct xs*, *auto*)

lemma *map-update*: *map f* (*xs*[*k*:= *y*]) = (*map f xs*)[*k* := *f y*]
by(*induct xs arbitrary*: *k*)(*auto split*:*nat.splits*)

lemma *rev-update*:
k < *length xs* ⟹ *rev* (*xs*[*k*:= *y*]) = (*rev xs*)[*length xs* − *k* − *1* := *y*]

by (*induct xs arbitrary*: *k*) (*auto simp*: *list-update-append split*:*nat.splits*)

lemma *update-zip*:
(*zip xs ys*)[*i*:=*xy*] = *zip* (*xs*[*i*:=*fst xy*]) (*ys*[*i*:=*snd xy*])
by (*induct ys arbitrary*: *i xy xs*) (*auto*, *case-tac xs*, *auto split*: *nat.split*)

lemma *set-update-subset-insert*: *set*(*xs*[*i*:=*x*]) <= *insert x* (*set xs*)
by (*induct xs arbitrary*: *i*) (*auto split*: *nat.split*)

lemma *set-update-subsetI*: [| *set xs* <= *A*; *x*:*A* |] ==> *set*(*xs*[*i* := *x*]) <= *A*
by (*blast dest*!: *set-update-subset-insert* [*THEN subsetD*])

lemma *set-update-memI*: *n* < *length xs* ⟹ *x* ∈ *set* (*xs*[*n* := *x*])
by (*induct xs arbitrary*: *n*) (*auto split*:*nat.splits*)

lemma *list-update-overwrite*[*simp*]:
xs [*i* := *x*, *i* := *y*] = *xs* [*i* := *y*]
apply (*induct xs arbitrary*: *i*) **apply** *simp*
apply (*case-tac i*, *simp-all*)
done

lemma *list-update-swap*:
i ≠ *i'* ⟹ *xs* [*i* := *x*, *i'* := *x'*] = *xs* [*i'* := *x'*, *i* := *x*]
apply (*induct xs arbitrary*: *i i'*)
apply *simp*
apply (*case-tac i*, *case-tac i'*)
apply *auto*
apply (*case-tac i'*)
apply *auto*
done

lemma *list-update-code* [*code*]:
[][*i* := *y*] = []
(*x* # *xs*)[*0* := *y*] = *y* # *xs*
(*x* # *xs*)[*Suc i* := *y*] = *x* # *xs*[*i* := *y*]
by *simp-all*

43.1.13 *last* and *butlast*

lemma *last-snoc* [*simp*]: *last* (*xs* @ [*x*]) = *x*
by (*induct xs*) *auto*

lemma *butlast-snoc* [*simp*]: *butlast* (*xs* @ [*x*]) = *xs*
by (*induct xs*) *auto*

lemma *last-ConsL*: *xs* = [] ⟹ *last*(*x*#*xs*) = *x*
by(*simp add*:*last.simps*)

lemma *last-ConsR*: *xs* ≠ [] ⟹ *last*(*x*#*xs*) = *last xs*

by(*simp add*:*last.simps*)

lemma *last-append*: *last*(*xs* @ *ys*) = (*if ys* = [] *then last xs else last ys*)
by (*induct xs*) (*auto*)

lemma *last-appendL*[*simp*]: *ys* = [] ⟹ *last*(*xs* @ *ys*) = *last xs*
by(*simp add*:*last-append*)

lemma *last-appendR*[*simp*]: *ys* ≠ [] ⟹ *last*(*xs* @ *ys*) = *last ys*
by(*simp add*:*last-append*)

lemma *hd-rev*: *xs* ≠ [] ⟹ *hd*(*rev xs*) = *last xs*
by(*rule rev-exhaust*[*of xs*]) *simp-all*

lemma *last-rev*: *xs* ≠ [] ⟹ *last*(*rev xs*) = *hd xs*
by(*cases xs*) *simp-all*

lemma *last-in-set*[*simp*]: *as* ≠ [] ⟹ *last as* ∈ *set as*
by (*induct as*) *auto*

lemma *length-butlast* [*simp*]: *length* (*butlast xs*) = *length xs* − *1*
by (*induct xs rule*: *rev-induct*) *auto*

lemma *butlast-append*:
butlast (*xs* @ *ys*) = (*if ys* = [] *then butlast xs else xs* @ *butlast ys*)
by (*induct xs arbitrary*: *ys*) *auto*

lemma *append-butlast-last-id* [*simp*]:
xs ≠ [] ==> *butlast xs* @ [*last xs*] = *xs*
by (*induct xs*) *auto*

lemma *in-set-butlastD*: *x* : *set* (*butlast xs*) ==> *x* : *set xs*
by (*induct xs*) (*auto split*: *split-if-asm*)

lemma *in-set-butlast-appendI*:
x : *set* (*butlast xs*) | *x* : *set* (*butlast ys*) ==> *x* : *set* (*butlast* (*xs* @ *ys*))
by (*auto dest*: *in-set-butlastD simp add*: *butlast-append*)

lemma *last-drop*[*simp*]: *n* < *length xs* ⟹ *last* (*drop n xs*) = *last xs*
apply (*induct xs arbitrary*: *n*)
apply *simp*
apply (*auto split*:*nat.split*)
done

lemma *last-conv-nth*: *xs*≠[] ⟹ *last xs* = *xs*!(*length xs* − *1*)
by(*induct xs*)(*auto simp*:*neq-Nil-conv*)

lemma *butlast-conv-take*: *butlast xs* = *take* (*length xs* − *1*) *xs*
by (*induct xs*, *simp*, *case-tac xs*, *simp-all*)

lemma *last-list-update*:
xs ≠ [] ⟹ *last*(*xs*[*k*:=*x*]) = (*if k* = *size xs* − *1 then x else last xs*)
by (*auto simp*: *last-conv-nth*)

lemma *butlast-list-update*:
butlast(*xs*[*k*:=*x*]) =
(*if k* = *size xs* − *1 then butlast xs else* (*butlast xs*)[*k*:=*x*])
apply(*cases xs rule*:*rev-cases*)
apply *simp*
apply(*simp add*:*list-update-append split*:*nat.splits*)
done

lemma *last-map*:
xs ≠ [] ⟹ *last* (*map f xs*) = *f* (*last xs*)
by (*cases xs rule*: *rev-cases*) *simp-all*

lemma *map-butlast*:
map f (*butlast xs*) = *butlast* (*map f xs*)
by (*induct xs*) *simp-all*

43.1.14 *take* and *drop*

lemma *take-0* [*simp*]: *take 0 xs* = []
by (*induct xs*) *auto*

lemma *drop-0* [*simp*]: *drop 0 xs* = *xs*
by (*induct xs*) *auto*

lemma *take-Suc-Cons* [*simp*]: *take* (*Suc n*) (*x* # *xs*) = *x* # *take n xs*
by *simp*

lemma *drop-Suc-Cons* [*simp*]: *drop* (*Suc n*) (*x* # *xs*) = *drop n xs*
by *simp*

declare *take-Cons* [*simp del*] **and** *drop-Cons* [*simp del*]

lemma *take-1-Cons* [*simp*]: *take 1* (*x* # *xs*) = [*x*]
unfolding *One-nat-def* **by** *simp*

lemma *drop-1-Cons* [*simp*]: *drop 1* (*x* # *xs*) = *xs*
unfolding *One-nat-def* **by** *simp*

lemma *take-Suc*: *xs* ~= [] ==> *take* (*Suc n*) *xs* = *hd xs* # *take n* (*tl xs*)
by(*clarsimp simp add*:*neq-Nil-conv*)

lemma *drop-Suc*: *drop* (*Suc n*) *xs* = *drop n* (*tl xs*)
by(*cases xs*, *simp-all*)

lemma *take-tl*: *take n (tl xs) = tl (take (Suc n) xs)*
by (*induct xs arbitrary*: *n*) *simp-all*

lemma *drop-tl*: *drop n (tl xs) = tl(drop n xs)*
by(*induct xs arbitrary*: *n*, *simp-all add*:*drop-Cons drop-Suc split*:*nat.split*)

lemma *tl-take*: *tl (take n xs) = take (n − 1) (tl xs)*
by (*cases n*, *simp*, *cases xs*, *auto*)

lemma *tl-drop*: *tl (drop n xs) = drop n (tl xs)*
by (*simp only*: *drop-tl*)

lemma *nth-via-drop*: *drop n xs = y#ys ⟹ xs!n = y*
apply (*induct xs arbitrary*: *n*, *simp*)
apply(*simp add*:*drop-Cons nth-Cons split*:*nat.splits*)
done

lemma *take-Suc-conv-app-nth*:
i < length xs ⟹ take (Suc i) xs = take i xs @ [xs!i]
apply (*induct xs arbitrary*: *i*, *simp*)
apply (*case-tac i*, *auto*)
done

lemma *drop-Suc-conv-tl*:
i < length xs ⟹ (xs!i) # (drop (Suc i) xs) = drop i xs
apply (*induct xs arbitrary*: *i*, *simp*)
apply (*case-tac i*, *auto*)
done

lemma *length-take* [*simp*]: *length (take n xs) = min (length xs) n*
by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *length-drop* [*simp*]: *length (drop n xs) = (length xs − n)*
by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *take-all* [*simp*]: *length xs <= n ==> take n xs = xs*
by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *drop-all* [*simp*]: *length xs <= n ==> drop n xs = []*
by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *take-append* [*simp*]:
take n (xs @ ys) = (take n xs @ take (n − length xs) ys)
by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *drop-append* [*simp*]:
drop n (xs @ ys) = drop n xs @ drop (n − length xs) ys
by (*induct n arbitrary*: *xs*) (*auto*, *case-tac xs*, *auto*)

lemma *take-take* [*simp*]: *take n* (*take m xs*) = *take* (*min n m*) *xs*
apply (*induct m arbitrary*: *xs n*, *auto*)
apply (*case-tac xs*, *auto*)
apply (*case-tac n*, *auto*)
done

lemma *drop-drop* [*simp*]: *drop n* (*drop m xs*) = *drop* (*n* + *m*) *xs*
apply (*induct m arbitrary*: *xs*, *auto*)
apply (*case-tac xs*, *auto*)
done

lemma *take-drop*: *take n* (*drop m xs*) = *drop m* (*take* (*n* + *m*) *xs*)
apply (*induct m arbitrary*: *xs n*, *auto*)
apply (*case-tac xs*, *auto*)
done

lemma *drop-take*: *drop n* (*take m xs*) = *take* (*m*−*n*) (*drop n xs*)
apply(*induct xs arbitrary*: *m n*)
apply *simp*
apply(*simp add*: *take-Cons drop-Cons split*:*nat.split*)
done

lemma *append-take-drop-id* [*simp*]: *take n xs* @ *drop n xs* = *xs*
apply (*induct n arbitrary*: *xs*, *auto*)
apply (*case-tac xs*, *auto*)
done

lemma *take-eq-Nil*[*simp*]: (*take n xs* = []) = (*n* = *0* ∨ *xs* = [])
apply(*induct xs arbitrary*: *n*)
apply *simp*
apply(*simp add*:*take-Cons split*:*nat.split*)
done

lemma *drop-eq-Nil*[*simp*]: (*drop n xs* = []) = (*length xs* <= *n*)
apply(*induct xs arbitrary*: *n*)
apply *simp*
apply(*simp add*:*drop-Cons split*:*nat.split*)
done

lemma *take-map*: *take n* (*map f xs*) = *map f* (*take n xs*)
apply (*induct n arbitrary*: *xs*, *auto*)
apply (*case-tac xs*, *auto*)
done

lemma *drop-map*: *drop n* (*map f xs*) = *map f* (*drop n xs*)
apply (*induct n arbitrary*: *xs*, *auto*)
apply (*case-tac xs*, *auto*)
done

lemma *rev-take*: *rev* (*take i xs*) = *drop* (*length xs* − *i*) (*rev xs*)
apply (*induct xs arbitrary*: *i*, *auto*)
apply (*case-tac i*, *auto*)
done

lemma *rev-drop*: *rev* (*drop i xs*) = *take* (*length xs* − *i*) (*rev xs*)
apply (*induct xs arbitrary*: *i*, *auto*)
apply (*case-tac i*, *auto*)
done

lemma *nth-take* [*simp*]: *i* < *n* ==> (*take n xs*)!*i* = *xs*!*i*
apply (*induct xs arbitrary*: *i n*, *auto*)
apply (*case-tac n*, *blast*)
apply (*case-tac i*, *auto*)
done

lemma *nth-drop* [*simp*]:
n + *i* <= *length xs* ==> (*drop n xs*)!*i* = *xs*!(*n* + *i*)
apply (*induct n arbitrary*: *xs i*, *auto*)
apply (*case-tac xs*, *auto*)
done

lemma *butlast-take*:
n <= *length xs* ==> *butlast* (*take n xs*) = *take* (*n* − *1*) *xs*
by (*simp add*: *butlast-conv-take min-max.inf-absorb1 min-max.inf-absorb2*)

lemma *butlast-drop*: *butlast* (*drop n xs*) = *drop n* (*butlast xs*)
by (*simp add*: *butlast-conv-take drop-take add-ac*)

lemma *take-butlast*: *n* < *length xs* ==> *take n* (*butlast xs*) = *take n xs*
by (*simp add*: *butlast-conv-take min-max.inf-absorb1*)

lemma *drop-butlast*: *drop n* (*butlast xs*) = *butlast* (*drop n xs*)
by (*simp add*: *butlast-conv-take drop-take add-ac*)

lemma *hd-drop-conv-nth*: ⟦ *xs* ≠ []; *n* < *length xs* ⟧ ⟹ *hd*(*drop n xs*) = *xs*!*n*
by(*simp add*: *hd-conv-nth*)

lemma *set-take-subset-set-take*:
m <= *n* ⟹ *set*(*take m xs*) <= *set*(*take n xs*)
by(*induct xs arbitrary*: *m n*)(*auto simp*:*take-Cons split*:*nat.split*)

lemma *set-take-subset*: *set*(*take n xs*) ⊆ *set xs*
by(*induct xs arbitrary*: *n*)(*auto simp*:*take-Cons split*:*nat.split*)

lemma *set-drop-subset*: *set*(*drop n xs*) ⊆ *set xs*
by(*induct xs arbitrary*: *n*)(*auto simp*:*drop-Cons split*:*nat.split*)

lemma *set-drop-subset-set-drop*:

m >= *n* ⟹ *set*(*drop m xs*) <= *set*(*drop n xs*)
apply(*induct xs arbitrary*: *m n*)
apply(*auto simp*:*drop-Cons split*:*nat.split*)
apply (*metis set-drop-subset subset-iff*)
done

lemma *in-set-takeD*: *x* : *set*(*take n xs*) ⟹ *x* : *set xs*
using *set-take-subset* **by** *fast*

lemma *in-set-dropD*: *x* : *set*(*drop n xs*) ⟹ *x* : *set xs*
using *set-drop-subset* **by** *fast*

lemma *append-eq-conv-conj*:
(*xs* @ *ys* = *zs*) = (*xs* = *take* (*length xs*) *zs* ∧ *ys* = *drop* (*length xs*) *zs*)
apply (*induct xs arbitrary*: *zs*, *simp*, *clarsimp*)
apply (*case-tac zs*, *auto*)
done

lemma *take-add*:
i+*j* ≤ *length*(*xs*) ⟹ *take* (*i*+*j*) *xs* = *take i xs* @ *take j* (*drop i xs*)
apply (*induct xs arbitrary*: *i*, *auto*)
apply (*case-tac i*, *simp-all*)
done

lemma *append-eq-append-conv-if*:
(*xs*$_1$ @ *xs*$_2$ = *ys*$_1$ @ *ys*$_2$) =
(*if size xs*$_1$ ≤ *size ys*$_1$
then xs$_1$ = *take* (*size xs*$_1$) *ys*$_1$ ∧ *xs*$_2$ = *drop* (*size xs*$_1$) *ys*$_1$ @ *ys*$_2$
else take (*size ys*$_1$) *xs*$_1$ = *ys*$_1$ ∧ *drop* (*size ys*$_1$) *xs*$_1$ @ *xs*$_2$ = *ys*$_2$)
apply(*induct xs*$_1$ *arbitrary*: *ys*$_1$)
apply *simp*
apply(*case-tac ys*$_1$)
apply *simp-all*
done

lemma *take-hd-drop*:
n < *length xs* ⟹ *take n xs* @ [*hd* (*drop n xs*)] = *take* (*Suc n*) *xs*
apply(*induct xs arbitrary*: *n*)
apply *simp*
apply(*simp add*:*drop-Cons split*:*nat.split*)
done

lemma *id-take-nth-drop*:
i < *length xs* ⟹ *xs* = *take i xs* @ *xs*!*i* # *drop* (*Suc i*) *xs*
proof −
assume *si*: *i* < *length xs*
hence *xs* = *take* (*Suc i*) *xs* @ *drop* (*Suc i*) *xs* **by** *auto*
moreover
from *si* **have** *take* (*Suc i*) *xs* = *take i xs* @ [*xs*!*i*]

apply (*rule-tac take-Suc-conv-app-nth*) **by** *arith*
ultimately show *?thesis* **by** *auto*
qed

lemma *upd-conv-take-nth-drop*:
i < *length xs* ⟹ *xs*[*i*:=*a*] = *take i xs* @ *a* # *drop* (*Suc i*) *xs*
proof −
assume *i*: *i* < *length xs*
have *xs*[*i*:=*a*] = (*take i xs* @ *xs*!*i* # *drop* (*Suc i*) *xs*)[*i*:=*a*]
by(*rule arg-cong*[*OF id-take-nth-drop*[*OF i*]])
also have ... = *take i xs* @ *a* # *drop* (*Suc i*) *xs*
using *i* **by** (*simp add*: *list-update-append*)
finally show *?thesis* **.**
qed

lemma *nth-drop'*:
i < *length xs* ⟹ *xs* ! *i* # *drop* (*Suc i*) *xs* = *drop i xs*
apply (*induct i arbitrary*: *xs*)
apply (*simp add*: *neq-Nil-conv*)
apply (*erule exE*)+
apply *simp*
apply (*case-tac xs*)
apply *simp-all*
done

43.1.15 *takeWhile* and *dropWhile*

lemma *length-takeWhile-le*: *length* (*takeWhile P xs*) ≤ *length xs*
by (*induct xs*) *auto*

lemma *takeWhile-dropWhile-id* [*simp*]: *takeWhile P xs* @ *dropWhile P xs* = *xs*
by (*induct xs*) *auto*

lemma *takeWhile-append1* [*simp*]:
[| *x*:*set xs*; ~*P*(*x*)|] ==> *takeWhile P* (*xs* @ *ys*) = *takeWhile P xs*
by (*induct xs*) *auto*

lemma *takeWhile-append2* [*simp*]:
(!!*x*. *x* : *set xs* ==> *P x*) ==> *takeWhile P* (*xs* @ *ys*) = *xs* @ *takeWhile P ys*
by (*induct xs*) *auto*

lemma *takeWhile-tail*: ¬ *P x* ==> *takeWhile P* (*xs* @ (*x*#*l*)) = *takeWhile P xs*
by (*induct xs*) *auto*

lemma *takeWhile-nth*: *j* < *length* (*takeWhile P xs*) ⟹ *takeWhile P xs* ! *j* = *xs*
! *j*
apply (*subst* (*3*) *takeWhile-dropWhile-id*[*symmetric*]) **unfolding** *nth-append* **by**
auto

lemma *dropWhile-nth*: $j < length\ (dropWhile\ P\ xs) \Longrightarrow dropWhile\ P\ xs\ !\ j = xs\ !\ (j + length\ (takeWhile\ P\ xs))$
apply $(subst\ (3)\ takeWhile\text{-}dropWhile\text{-}id[symmetric])$ **unfolding** *nth-append* **by** *auto*

lemma *length-dropWhile-le*: $length\ (dropWhile\ P\ xs) \leq length\ xs$
by $(induct\ xs)\ auto$

lemma *dropWhile-append1* [*simp*]:
$[|\ x : set\ xs; \sim P(x)|] ==> dropWhile\ P\ (xs\ @\ ys) = (dropWhile\ P\ xs)@ys$
by $(induct\ xs)\ auto$

lemma *dropWhile-append2* [*simp*]:
$(!!x.\ x{:}set\ xs ==> P(x)) ==> dropWhile\ P\ (xs\ @\ ys) = dropWhile\ P\ ys$
by $(induct\ xs)\ auto$

lemma *set-takeWhileD*: $x : set\ (takeWhile\ P\ xs) ==> x : set\ xs \wedge P\ x$
by $(induct\ xs)\ (auto\ split{:}\ split\text{-}if\text{-}asm)$

lemma *takeWhile-eq-all-conv*[*simp*]:
$(takeWhile\ P\ xs = xs) = (\forall x \in set\ xs.\ P\ x)$
by$(induct\ xs,\ auto)$

lemma *dropWhile-eq-Nil-conv*[*simp*]:
$(dropWhile\ P\ xs = []) = (\forall x \in set\ xs.\ P\ x)$
by$(induct\ xs,\ auto)$

lemma *dropWhile-eq-Cons-conv*:
$(dropWhile\ P\ xs = y\#ys) = (xs = takeWhile\ P\ xs\ @\ y\ \#\ ys\ \&\ \neg\ P\ y)$
by$(induct\ xs,\ auto)$

lemma *distinct-takeWhile*[*simp*]: $distinct\ xs ==> distinct\ (takeWhile\ P\ xs)$
by $(induct\ xs)\ (auto\ dest{:}\ set\text{-}takeWhileD)$

lemma *distinct-dropWhile*[*simp*]: $distinct\ xs ==> distinct\ (dropWhile\ P\ xs)$
by $(induct\ xs)\ auto$

lemma *takeWhile-map*: $takeWhile\ P\ (map\ f\ xs) = map\ f\ (takeWhile\ (P \circ f)\ xs)$
by $(induct\ xs)\ auto$

lemma *dropWhile-map*: $dropWhile\ P\ (map\ f\ xs) = map\ f\ (dropWhile\ (P \circ f)\ xs)$
by $(induct\ xs)\ auto$

lemma *takeWhile-eq-take*: $takeWhile\ P\ xs = take\ (length\ (takeWhile\ P\ xs))\ xs$
by $(induct\ xs)\ auto$

lemma *dropWhile-eq-drop*: $dropWhile\ P\ xs = drop\ (length\ (takeWhile\ P\ xs))\ xs$
by $(induct\ xs)\ auto$

lemma *hd-dropWhile*:
dropWhile P xs ≠ [] ⟹ ¬ *P* (*hd* (*dropWhile P xs*))
using *assms* **by** (*induct xs*) *auto*

lemma *takeWhile-eq-filter*:
assumes ⋀ *x*. *x* ∈ *set* (*dropWhile P xs*) ⟹ ¬ *P x*
shows *takeWhile P xs* = *filter P xs*
proof −
have *A*: *filter P xs* = *filter P* (*takeWhile P xs* @ *dropWhile P xs*)
by *simp*
have *B*: *filter P* (*dropWhile P xs*) = []
unfolding *filter-empty-conv* **using** *assms* **by** *blast*
have *filter P xs* = *takeWhile P xs*
unfolding *A filter-append B*
by (*auto simp add*: *filter-id-conv dest*: *set-takeWhileD*)
thus *?thesis* **..**
qed

lemma *takeWhile-eq-take-P-nth*:
⟦ ⋀ *i*. ⟦ *i* < *n* ; *i* < *length xs* ⟧ ⟹ *P* (*xs* ! *i*) ; *n* < *length xs* ⟹ ¬ *P* (*xs* ! *n*)
⟧ ⟹
takeWhile P xs = *take n xs*
proof (*induct xs arbitrary*: *n*)
case (*Cons x xs*)
thus *?case*
proof (*cases n*)
case (*Suc n′*) **note** *this*[*simp*]
have *P x* **using** *Cons.prems*(*1*)[*of 0*] **by** *simp*
moreover have *takeWhile P xs* = *take n′ xs*
proof (*rule Cons.hyps*)
case *goal1* **thus** *P* (*xs* ! *i*) **using** *Cons.prems*(*1*)[*of Suc i*] **by** *simp*
next case *goal2* **thus** *?case* **using** *Cons* **by** *auto*
qed
ultimately show *?thesis* **by** *simp*
qed *simp*
qed *simp*

lemma *nth-length-takeWhile*:
length (*takeWhile P xs*) < *length xs* ⟹ ¬ *P* (*xs* ! *length* (*takeWhile P xs*))
by (*induct xs*) *auto*

lemma *length-takeWhile-less-P-nth*:
assumes *all*: ⋀ *i*. *i* < *j* ⟹ *P* (*xs* ! *i*) **and** *j* ≤ *length xs*
shows *j* ≤ *length* (*takeWhile P xs*)
proof (*rule classical*)
assume ¬ *?thesis*
hence *length* (*takeWhile P xs*) < *length xs* **using** *assms* **by** *simp*
thus *?thesis* **using** *all* ‹¬ *?thesis*› *nth-length-takeWhile*[*of P xs*] **by** *auto*
qed

The following two lemmmas could be generalized to an arbitrary property.

lemma *takeWhile-neq-rev*: ⟦*distinct xs*; *x* ∈ *set xs*⟧ ⟹
takeWhile (λ*y*. *y* ≠ *x*) (*rev xs*) = *rev* (*tl* (*dropWhile* (λ*y*. *y* ≠ *x*) *xs*))
by(*induct xs*) (*auto simp*: *takeWhile-tail*[**where** *l*=[]])

lemma *dropWhile-neq-rev*: ⟦*distinct xs*; *x* ∈ *set xs*⟧ ⟹
dropWhile (λ*y*. *y* ≠ *x*) (*rev xs*) = *x* # *rev* (*takeWhile* (λ*y*. *y* ≠ *x*) *xs*)
apply(*induct xs*)
apply *simp*
apply *auto*
apply(*subst dropWhile-append2*)
apply *auto*
done

lemma *takeWhile-not-last*:
⟦ *xs* ≠ []; *distinct xs*⟧ ⟹ *takeWhile* (λ*y*. *y* ≠ *last xs*) *xs* = *butlast xs*
apply(*induct xs*)
apply *simp*
apply(*case-tac xs*)
apply(*auto*)
done

lemma *takeWhile-cong* [*fundef-cong*, *recdef-cong*]:
[| *l* = *k*; !!*x*. *x* : *set l* ==> *P x* = *Q x* |]
==> *takeWhile P l* = *takeWhile Q k*
by (*induct k arbitrary*: *l*) (*simp-all*)

lemma *dropWhile-cong* [*fundef-cong*, *recdef-cong*]:
[| *l* = *k*; !!*x*. *x* : *set l* ==> *P x* = *Q x* |]
==> *dropWhile P l* = *dropWhile Q k*
by (*induct k arbitrary*: *l*, *simp-all*)

43.1.16 *zip*

lemma *zip-Nil* [*simp*]: *zip* [] *ys* = []
by (*induct ys*) *auto*

lemma *zip-Cons-Cons* [*simp*]: *zip* (*x* # *xs*) (*y* # *ys*) = (*x*, *y*) # *zip xs ys*
by *simp*

declare *zip-Cons* [*simp del*]

lemma [*code*]:
zip [] *ys* = []
zip xs [] = []
zip (*x* # *xs*) (*y* # *ys*) = (*x*, *y*) # *zip xs ys*
by (*fact zip-Nil zip.simps*(*1*) *zip-Cons-Cons*)+

lemma *zip-Cons1*:

zip (x#xs) ys = (case ys of [] ⇒ [] | y#ys ⇒ (x,y)#zip xs ys)
by(*auto split*:*list.split*)

lemma *length-zip* [*simp*]:
length (zip xs ys) = min (length xs) (length ys)
by (*induct xs ys rule*:*list-induct2'*) *auto*

lemma *zip-obtain-same-length*:
assumes ⋀*zs ws n. length zs = length ws ⟹ n = min (length xs) (length ys)*
⟹ zs = take n xs ⟹ ws = take n ys ⟹ P (zip zs ws)
shows *P (zip xs ys)*
proof −
let *?n = min (length xs) (length ys)*
have *P (zip (take ?n xs) (take ?n ys))*
by (*rule assms*) *simp-all*
moreover have *zip xs ys = zip (take ?n xs) (take ?n ys)*
proof (*induct xs arbitrary*: *ys*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*) **then show** *?case* **by** (*cases ys*) *simp-all*
qed
ultimately show *?thesis* **by** *simp*
qed

lemma *zip-append1*:
zip (xs @ ys) zs =
zip xs (take (length xs) zs) @ zip ys (drop (length xs) zs)
by (*induct xs zs rule*:*list-induct2'*) *auto*

lemma *zip-append2*:
zip xs (ys @ zs) =
zip (take (length ys) xs) ys @ zip (drop (length ys) xs) zs
by (*induct xs ys rule*:*list-induct2'*) *auto*

lemma *zip-append* [*simp*]:
[| length xs = length us; length ys = length vs |] ==>
zip (xs@ys) (us@vs) = zip xs us @ zip ys vs
by (*simp add*: *zip-append1*)

lemma *zip-rev*:
length xs = length ys ==> zip (rev xs) (rev ys) = rev (zip xs ys)
by (*induct rule*:*list-induct2*, *simp-all*)

lemma *zip-map-map*:
zip (map f xs) (map g ys) = map (λ (x, y). (f x, g y)) (zip xs ys)
proof (*induct xs arbitrary*: *ys*)
case (*Cons x xs*) **note** *Cons-x-xs = Cons.hyps*
show *?case*
proof (*cases ys*)

```
    case (Cons y ys')
    show ?thesis unfolding Cons using Cons-x-xs by simp
  qed simp
qed simp

lemma zip-map1:
  zip (map f xs) ys = map (λ(x, y). (f x, y)) (zip xs ys)
using zip-map-map[of f xs λx. x ys] by simp

lemma zip-map2:
  zip xs (map f ys) = map (λ(x, y). (x, f y)) (zip xs ys)
using zip-map-map[of λx. x xs f ys] by simp

lemma map-zip-map:
  map f (zip (map g xs) ys) = map (%(x,y). f(g x, y)) (zip xs ys)
unfolding zip-map1 by auto

lemma map-zip-map2:
  map f (zip xs (map g ys)) = map (%(x,y). f(x, g y)) (zip xs ys)
unfolding zip-map2 by auto
```

Courtesy of Andreas Lochbihler:

```
lemma zip-same-conv-map: zip xs xs = map (λx. (x, x)) xs
by(induct xs) auto

lemma nth-zip [simp]:
[| i < length xs; i < length ys|] ==> (zip xs ys)!i = (xs!i, ys!i)
apply (induct ys arbitrary: i xs, simp)
apply (case-tac xs)
 apply (simp-all add: nth.simps split: nat.split)
done

lemma set-zip:
set (zip xs ys) = {(xs!i, ys!i) | i. i < min (length xs) (length ys)}
by(simp add: set-conv-nth cong: rev-conj-cong)

lemma zip-same: ((a,b) ∈ set (zip xs xs)) = (a ∈ set xs ∧ a = b)
by(induct xs) auto

lemma zip-update:
  zip (xs[i:=x]) (ys[i:=y]) = (zip xs ys)[i:=(x,y)]
by(rule sym, simp add: update-zip)

lemma zip-replicate [simp]:
  zip (replicate i x) (replicate j y) = replicate (min i j) (x,y)
apply (induct i arbitrary: j, auto)
apply (case-tac j, auto)
done
```

lemma *take-zip*:
take n (zip xs ys) = zip (take n xs) (take n ys)
apply (*induct n arbitrary*: *xs ys*)
apply *simp*
apply (*case-tac xs*, *simp*)
apply (*case-tac ys*, *simp-all*)
done

lemma *drop-zip*:
drop n (zip xs ys) = zip (drop n xs) (drop n ys)
apply (*induct n arbitrary*: *xs ys*)
apply *simp*
apply (*case-tac xs*, *simp*)
apply (*case-tac ys*, *simp-all*)
done

lemma *zip-takeWhile-fst*: *zip (takeWhile P xs) ys = takeWhile (P ∘ fst) (zip xs ys)*
proof (*induct xs arbitrary*: *ys*)
case (*Cons x xs*) **thus** *?case* **by** (*cases ys*) *auto*
qed *simp*

lemma *zip-takeWhile-snd*: *zip xs (takeWhile P ys) = takeWhile (P ∘ snd) (zip xs ys)*
proof (*induct xs arbitrary*: *ys*)
case (*Cons x xs*) **thus** *?case* **by** (*cases ys*) *auto*
qed *simp*

lemma *set-zip-leftD*:
(x,y)∈ set (zip xs ys) ⟹ x ∈ set xs
by (*induct xs ys rule*:*list-induct2'*) *auto*

lemma *set-zip-rightD*:
(x,y)∈ set (zip xs ys) ⟹ y ∈ set ys
by (*induct xs ys rule*:*list-induct2'*) *auto*

lemma *in-set-zipE*:
(x,y) : set(zip xs ys) ⟹ (⟦ x : set xs; y : set ys ⟧ ⟹ R) ⟹ R
by(*blast dest*: *set-zip-leftD set-zip-rightD*)

lemma *zip-map-fst-snd*:
zip (map fst zs) (map snd zs) = zs
by (*induct zs*) *simp-all*

lemma *zip-eq-conv*:
length xs = length ys ⟹ zip xs ys = zs ⟷ map fst zs = xs ∧ map snd zs = ys
by (*auto simp add*: *zip-map-fst-snd*)

lemma *distinct-zipI1*:

```
  distinct xs ⟹ distinct (zip xs ys)
proof (induct xs arbitrary: ys)
  case (Cons x xs)
  show ?case
  proof (cases ys)
    case (Cons y ys')
    have (x, y) ∉ set (zip xs ys')
      using Cons.prems by (auto simp: set-zip)
    thus ?thesis
      unfolding Cons zip-Cons-Cons distinct.simps
      using Cons.hyps Cons.prems by simp
  qed simp
qed simp

lemma distinct-zipI2:
  distinct xs ⟹ distinct (zip xs ys)
proof (induct xs arbitrary: ys)
  case (Cons x xs)
  show ?case
  proof (cases ys)
    case (Cons y ys')
     have (x, y) ∉ set (zip xs ys')
      using Cons.prems by (auto simp: set-zip)
    thus ?thesis
      unfolding Cons zip-Cons-Cons distinct.simps
      using Cons.hyps Cons.prems by simp
  qed simp
qed simp
```

43.1.17 *list-all2*

```
lemma list-all2-lengthD [intro?]:
  list-all2 P xs ys ==> length xs = length ys
by (simp add: list-all2-def)

lemma list-all2-Nil [iff, code]: list-all2 P [] ys = (ys = [])
by (simp add: list-all2-def)

lemma list-all2-Nil2 [iff, code]: list-all2 P xs [] = (xs = [])
by (simp add: list-all2-def)

lemma list-all2-Cons [iff, code]:
  list-all2 P (x # xs) (y # ys) = (P x y ∧ list-all2 P xs ys)
by (auto simp add: list-all2-def)

lemma list-all2-Cons1:
list-all2 P (x # xs) ys = (∃z zs. ys = z # zs ∧ P x z ∧ list-all2 P xs zs)
by (cases ys) auto
```

lemma *list-all2-Cons2*:
list-all2 P xs (*y # ys*) = (∃ *z zs*. *xs* = *z # zs* ∧ *P z y* ∧ *list-all2 P zs ys*)
by (*cases xs*) *auto*

lemma *list-all2-rev* [*iff*]:
list-all2 P (*rev xs*) (*rev ys*) = *list-all2 P xs ys*
by (*simp add*: *list-all2-def zip-rev cong*: *conj-cong*)

lemma *list-all2-rev1*:
list-all2 P (*rev xs*) *ys* = *list-all2 P xs* (*rev ys*)
by (*subst list-all2-rev* [*symmetric*]) *simp*

lemma *list-all2-append1*:
list-all2 P (*xs* @ *ys*) *zs* =
(*EX us vs*. *zs* = *us* @ *vs* ∧ *length us* = *length xs* ∧ *length vs* = *length ys* ∧
list-all2 P xs us ∧ *list-all2 P ys vs*)
apply (*simp add*: *list-all2-def zip-append1*)
apply (*rule iffI*)
apply (*rule-tac x* = *take* (*length xs*) *zs* **in** *exI*)
apply (*rule-tac x* = *drop* (*length xs*) *zs* **in** *exI*)
apply (*force split*: *nat-diff-split simp add*: *min-def*, *clarify*)
apply (*simp add*: *ball-Un*)
done

lemma *list-all2-append2*:
list-all2 P xs (*ys* @ *zs*) =
(*EX us vs*. *xs* = *us* @ *vs* ∧ *length us* = *length ys* ∧ *length vs* = *length zs* ∧
list-all2 P us ys ∧ *list-all2 P vs zs*)
apply (*simp add*: *list-all2-def zip-append2*)
apply (*rule iffI*)
apply (*rule-tac x* = *take* (*length ys*) *xs* **in** *exI*)
apply (*rule-tac x* = *drop* (*length ys*) *xs* **in** *exI*)
apply (*force split*: *nat-diff-split simp add*: *min-def*, *clarify*)
apply (*simp add*: *ball-Un*)
done

lemma *list-all2-append*:
length xs = *length ys* ⟹
list-all2 P (*xs*@*us*) (*ys*@*vs*) = (*list-all2 P xs ys* ∧ *list-all2 P us vs*)
by (*induct rule*:*list-induct2*, *simp-all*)

lemma *list-all2-appendI* [*intro?*, *trans*]:
⟦ *list-all2 P a b*; *list-all2 P c d* ⟧ ⟹ *list-all2 P* (*a*@*c*) (*b*@*d*)
by (*simp add*: *list-all2-append list-all2-lengthD*)

lemma *list-all2-conv-all-nth*:
list-all2 P xs ys =
(*length xs* = *length ys* ∧ (∀ *i* < *length xs*. *P* (*xs*!*i*) (*ys*!*i*)))
by (*force simp add*: *list-all2-def set-zip*)

lemma *list-all2-trans*:
assumes *tr*: !!*a b c*. *P1 a b ==> P2 b c ==> P3 a c*
shows !!*bs cs*. *list-all2 P1 as bs ==> list-all2 P2 bs cs ==> list-all2 P3 as cs*
(**is** !!*bs cs*. *PROP ?Q as bs cs*)
proof (*induct as*)
fix *x xs bs* **assume** *I1*: !!*bs cs*. *PROP ?Q xs bs cs*
show !!*cs*. *PROP ?Q* (*x # xs*) *bs cs*
proof (*induct bs*)
fix *y ys cs* **assume** *I2*: !!*cs*. *PROP ?Q* (*x # xs*) *ys cs*
show *PROP ?Q* (*x # xs*) (*y # ys*) *cs*
by (*induct cs*) (*auto intro*: *tr I1 I2*)
qed *simp*
qed *simp*

lemma *list-all2-all-nthI* [*intro?*]:
length a = length b ⟹ (⋀*n*. *n < length a* ⟹ *P* (*a*!*n*) (*b*!*n*)) ⟹ *list-all2 P a b*
by (*simp add*: *list-all2-conv-all-nth*)

lemma *list-all2I*:
∀ *x* ∈ *set* (*zip a b*). *split P x* ⟹ *length a = length b* ⟹ *list-all2 P a b*
by (*simp add*: *list-all2-def*)

lemma *list-all2-nthD*:
⟦ *list-all2 P xs ys*; *p < size xs* ⟧ ⟹ *P* (*xs*!*p*) (*ys*!*p*)
by (*simp add*: *list-all2-conv-all-nth*)

lemma *list-all2-nthD2*:
⟦*list-all2 P xs ys*; *p < size ys*⟧ ⟹ *P* (*xs*!*p*) (*ys*!*p*)
by (*frule list-all2-lengthD*) (*auto intro*: *list-all2-nthD*)

lemma *list-all2-map1*:
list-all2 P (*map f as*) *bs* = *list-all2* (λ*x y*. *P* (*f x*) *y*) *as bs*
by (*simp add*: *list-all2-conv-all-nth*)

lemma *list-all2-map2*:
list-all2 P as (*map f bs*) = *list-all2* (λ*x y*. *P x* (*f y*)) *as bs*
by (*auto simp add*: *list-all2-conv-all-nth*)

lemma *list-all2-refl* [*intro?*]:
(⋀*x*. *P x x*) ⟹ *list-all2 P xs xs*
by (*simp add*: *list-all2-conv-all-nth*)

lemma *list-all2-update-cong*:
⟦ *i*<*size xs*; *list-all2 P xs ys*; *P x y* ⟧ ⟹ *list-all2 P* (*xs*[*i*:=*x*]) (*ys*[*i*:=*y*])
by (*simp add*: *list-all2-conv-all-nth nth-list-update*)

lemma *list-all2-update-cong2*:
⟦*list-all2 P xs ys*; *P x y*; *i < length ys*⟧ ⟹ *list-all2 P* (*xs*[*i*:=*x*]) (*ys*[*i*:=*y*])

by (*simp add*: *list-all2-lengthD list-all2-update-cong*)

lemma *list-all2-takeI* [*simp*,*intro?*]:
list-all2 P xs ys ⟹ *list-all2 P* (*take n xs*) (*take n ys*)
apply (*induct xs arbitrary*: *n ys*)
apply *simp*
apply (*clarsimp simp add*: *list-all2-Cons1*)
apply (*case-tac n*)
apply *auto*
done

lemma *list-all2-dropI* [*simp*,*intro?*]:
list-all2 P as bs ⟹ *list-all2 P* (*drop n as*) (*drop n bs*)
apply (*induct as arbitrary*: *n bs*, *simp*)
apply (*clarsimp simp add*: *list-all2-Cons1*)
apply (*case-tac n*, *simp*, *simp*)
done

lemma *list-all2-mono* [*intro?*]:
list-all2 P xs ys ⟹ (⋀*xs ys*. *P xs ys* ⟹ *Q xs ys*) ⟹ *list-all2 Q xs ys*
apply (*induct xs arbitrary*: *ys*, *simp*)
apply (*case-tac ys*, *auto*)
done

lemma *list-all2-eq*:
xs = *ys* ⟷ *list-all2* (*op* =) *xs ys*
by (*induct xs ys rule*: *list-induct2'*) *auto*

43.1.18 *foldl* and *foldr*

lemma *foldl-append* [*simp*]:
foldl f a (*xs* @ *ys*) = *foldl f* (*foldl f a xs*) *ys*
by (*induct xs arbitrary*: *a*) *auto*

lemma *foldr-append*[*simp*]: *foldr f* (*xs* @ *ys*) *a* = *foldr f xs* (*foldr f ys a*)
by (*induct xs*) *auto*

lemma *foldr-map*: *foldr g* (*map f xs*) *a* = *foldr* (*g o f*) *xs a*
by(*induct xs*) *simp-all*

For efficient code generation: avoid intermediate list.

lemma *foldl-map*[*code-unfold*]:
foldl g a (*map f xs*) = *foldl* (%*a x*. *g a* (*f x*)) *a xs*
by(*induct xs arbitrary*:*a*) *simp-all*

lemma *foldl-apply*:
assumes ⋀*x*. *x* ∈ *set xs* ⟹ *f x* ∘ *h* = *h* ∘ *g x*
shows *foldl* (λ*s x*. *f x s*) (*h s*) *xs* = *h* (*foldl* (λ*s x*. *g x s*) *s xs*)
by (*rule sym*, *insert assms*, *induct xs arbitrary*: *s*) (*simp-all add*: *expand-fun-eq*)

lemma *foldl-cong* [*fundef-cong*, *recdef-cong*]:
[| *a* = *b*; *l* = *k*; !!*a x*. *x* : *set l* ==> *f a x* = *g a x* |]
==> *foldl f a l* = *foldl g b k*
by (*induct k arbitrary*: *a b l*) *simp-all*

lemma *foldr-cong* [*fundef-cong*, *recdef-cong*]:
[| *a* = *b*; *l* = *k*; !!*a x*. *x* : *set l* ==> *f x a* = *g x a* |]
==> *foldr f l a* = *foldr g k b*
by (*induct k arbitrary*: *a b l*) *simp-all*

lemma *foldl-fun-comm*:
assumes ⋀*x y s*. *f* (*f s x*) *y* = *f* (*f s y*) *x*
shows *f* (*foldl f s xs*) *x* = *foldl f* (*f s x*) *xs*
by (*induct xs arbitrary*: *s*)
(*simp-all add*: *assms*)

lemma (**in** *semigroup-add*) *foldl-assoc*:
shows *foldl op*+ (*x*+*y*) *zs* = *x* + (*foldl op*+ *y zs*)
by (*induct zs arbitrary*: *y*) (*simp-all add*:*add-assoc*)

lemma (**in** *monoid-add*) *foldl-absorb0*:
shows *x* + (*foldl op*+ *0 zs*) = *foldl op*+ *x zs*
by (*induct zs*) (*simp-all add*:*foldl-assoc*)

lemma *foldl-rev*:
assumes ⋀*x y s*. *f* (*f s x*) *y* = *f* (*f s y*) *x*
shows *foldl f s* (*rev xs*) = *foldl f s xs*
proof (*induct xs arbitrary*: *s*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*) **with** *assms* **show** *?case* **by** (*simp add*: *foldl-fun-comm*)
qed

The "First Duality Theorem" in Bird & Wadler:

lemma *foldl-foldr1-lemma*:
foldl op + *a xs* = *a* + *foldr op* + *xs* (*0*::*'a*::*monoid-add*)
by (*induct xs arbitrary*: *a*) (*auto simp*:*add-assoc*)

corollary *foldl-foldr1*:
foldl op + *0 xs* = *foldr op* + *xs* (*0*::*'a*::*monoid-add*)
by (*simp add*:*foldl-foldr1-lemma*)

The "Third Duality Theorem" in Bird & Wadler:

lemma *foldr-foldl*: *foldr f xs a* = *foldl* (%*x y*. *f y x*) *a* (*rev xs*)
by (*induct xs*) *auto*

lemma *foldl-foldr*: *foldl f a xs* = *foldr* (%*x y*. *f y x*) (*rev xs*) *a*
by (*simp add*: *foldr-foldl* [*of* %*x y*. *f y x rev xs*])

lemma (**in** *ab-semigroup-add*) *foldr-conv-foldl*: *foldr op + xs a = foldl op + a xs*
by (*induct xs*, *auto simp add*: *foldl-assoc add-commute*)

Note: $n \leq$ *foldl* (*op +*) *n ns* looks simpler, but is more difficult to use because it requires an additional transitivity step.

lemma *start-le-sum*: (*m*::*nat*) <= *n* ==> *m* <= *foldl* (*op +*) *n ns*
by (*induct ns arbitrary*: *n*) *auto*

lemma *elem-le-sum*: (*n*::*nat*) : *set ns* ==> *n* <= *foldl* (*op +*) *0 ns*
by (*force intro*: *start-le-sum simp add*: *in-set-conv-decomp*)

lemma *sum-eq-0-conv* [*iff*]:
(*foldl* (*op +*) (*m*::*nat*) *ns* = *0*) = (*m* = *0* ∧ (∀ *n* ∈ *set ns*. *n* = *0*))
by (*induct ns arbitrary*: *m*) *auto*

lemma *foldr-invariant*:
⟦*Q x* ; ∀ *x*∈ *set xs*. *P x*; ∀ *x y*. *P x* ∧ *Q y* ⟶ *Q* (*f x y*) ⟧ ⟹ *Q* (*foldr f xs x*)
by (*induct xs*, *simp-all*)

lemma *foldl-invariant*:
⟦*Q x* ; ∀ *x*∈ *set xs*. *P x*; ∀ *x y*. *P x* ∧ *Q y* ⟶ *Q* (*f y x*) ⟧ ⟹ *Q* (*foldl f x xs*)
by (*induct xs arbitrary*: *x*, *simp-all*)

lemma *foldl-weak-invariant*:
assumes *P s*
and ⋀*s x*. *x* ∈ *set xs* ⟹ *P s* ⟹ *P* (*f s x*)
shows *P* (*foldl f s xs*)
using *assms* **by** (*induct xs arbitrary*: *s*) *simp-all*

foldl and *concat*

lemma *foldl-conv-concat*:
foldl (*op* @) *xs xss* = *xs* @ *concat xss*
proof (*induct xss arbitrary*: *xs*)
case *Nil* **show** *?case* **by** *simp*
next
interpret *monoid-add op* @ [] **proof qed** *simp-all*
case *Cons* **then show** *?case* **by** (*simp add*: *foldl-absorb0*)
qed

lemma *concat-conv-foldl*: *concat xss* = *foldl* (*op* @) [] *xss*
by (*simp add*: *foldl-conv-concat*)

fold and *foldl*

lemma (**in** *fun-left-comm*) *fold-set-remdups*:
fold f y (*set xs*) = *foldl* (λ*y x*. *f x y*) *y* (*remdups xs*)
by (*rule sym*, *induct xs arbitrary*: *y*) (*simp-all add*: *fold-fun-comm insert-absorb*)

lemma (**in** *fun-left-comm-idem*) *fold-set*:

fold f y (set xs) = foldl (λy x. f x y) y xs
by (*rule sym*, *induct xs arbitrary*: *y*) (*simp-all add*: *fold-fun-comm*)

lemma (**in** *ab-semigroup-idem-mult*) *fold1-set*:
assumes *xs ≠ []*
shows *fold1 times (set xs) = foldl times (hd xs) (tl xs)*
proof −
interpret *fun-left-comm-idem times* **by** (*fact fun-left-comm-idem*)
from *assms* **obtain** *y ys* **where** *xs*: *xs = y # ys*
by (*cases xs*) *auto*
show *?thesis*
proof (*cases set ys = {}*)
case *True* **with** *xs* **show** *?thesis* **by** *simp*
next
case *False*
then have *fold1 times (insert y (set ys)) = fold times y (set ys)*
by (*simp only*: *finite-set fold1-eq-fold-idem*)
with *xs* **show** *?thesis* **by** (*simp add*: *fold-set mult-commute*)
qed
qed

lemma (**in** *lattice*) *Inf-fin-set-fold* [*code-unfold*]:
Inf-fin (set (x # xs)) = foldl inf x xs
proof −
interpret *ab-semigroup-idem-mult inf* :: *'a ⇒ 'a ⇒ 'a*
by (*fact ab-semigroup-idem-mult-inf*)
show *?thesis*
by (*simp add*: *Inf-fin-def fold1-set del*: *set.simps*)
qed

lemma (**in** *lattice*) *Sup-fin-set-fold* [*code-unfold*]:
Sup-fin (set (x # xs)) = foldl sup x xs
proof −
interpret *ab-semigroup-idem-mult sup* :: *'a ⇒ 'a ⇒ 'a*
by (*fact ab-semigroup-idem-mult-sup*)
show *?thesis*
by (*simp add*: *Sup-fin-def fold1-set del*: *set.simps*)
qed

lemma (**in** *linorder*) *Min-fin-set-fold* [*code-unfold*]:
Min (set (x # xs)) = foldl min x xs
proof −
interpret *ab-semigroup-idem-mult min* :: *'a ⇒ 'a ⇒ 'a*
by (*fact ab-semigroup-idem-mult-min*)
show *?thesis*
by (*simp add*: *Min-def fold1-set del*: *set.simps*)
qed

lemma (**in** *linorder*) *Max-fin-set-fold* [*code-unfold*]:

Max (*set* (*x* # *xs*)) = *foldl max x xs*
proof −
interpret *ab-semigroup-idem-mult max* :: *'a* ⇒ *'a* ⇒ *'a*
by (*fact ab-semigroup-idem-mult-max*)
show *?thesis*
by (*simp add*: *Max-def fold1-set del*: *set.simps*)
qed

lemma (**in** *complete-lattice*) *Inf-set-fold* [*code-unfold*]:
Inf (*set xs*) = *foldl inf top xs*
proof −
interpret *fun-left-comm-idem inf* :: *'a* ⇒ *'a* ⇒ *'a*
by (*fact fun-left-comm-idem-inf*)
show *?thesis* **by** (*simp add*: *Inf-fold-inf fold-set inf-commute*)
qed

lemma (**in** *complete-lattice*) *Sup-set-fold* [*code-unfold*]:
Sup (*set xs*) = *foldl sup bot xs*
proof −
interpret *fun-left-comm-idem sup* :: *'a* ⇒ *'a* ⇒ *'a*
by (*fact fun-left-comm-idem-sup*)
show *?thesis* **by** (*simp add*: *Sup-fold-sup fold-set sup-commute*)
qed

lemma (**in** *complete-lattice*) *INFI-set-fold*:
INFI (*set xs*) *f* = *foldl* (λ*y x*. *inf* (*f x*) *y*) *top xs*
unfolding *INFI-def set-map* [*symmetric*] *Inf-set-fold foldl-map*
by (*simp add*: *inf-commute*)

lemma (**in** *complete-lattice*) *SUPR-set-fold*:
SUPR (*set xs*) *f* = *foldl* (λ*y x*. *sup* (*f x*) *y*) *bot xs*
unfolding *SUPR-def set-map* [*symmetric*] *Sup-set-fold foldl-map*
by (*simp add*: *sup-commute*)

43.1.19 List summation: *listsum* and $\sum$

lemma *listsum-append* [*simp*]: *listsum* (*xs* @ *ys*) = *listsum xs* + *listsum ys*
by (*induct xs*) (*simp-all add*:*add-assoc*)

lemma *listsum-rev* [*simp*]:
fixes *xs* :: *'a*::*comm-monoid-add list*
shows *listsum* (*rev xs*) = *listsum xs*
by (*induct xs*) (*simp-all add*:*add-ac*)

lemma *listsum-map-remove1*:
fixes *f* :: *'a* ⇒ (*'b*::*comm-monoid-add*)
shows *x* : *set xs* ⟹ *listsum*(*map f xs*) = *f x* + *listsum*(*map f* (*remove1 x xs*))
by (*induct xs*)(*auto simp add*:*add-ac*)

lemma *list-size-conv-listsum*:
list-size f xs = listsum (map f xs) + size xs
by(*induct xs*) *auto*

lemma *listsum-foldr*: *listsum xs = foldr (op +) xs 0*
by (*induct xs*) *auto*

lemma *length-concat*: *length (concat xss) = listsum (map length xss)*
by (*induct xss*) *simp-all*

lemma *listsum-map-filter*:
fixes *f* :: *'a ⇒ 'b* :: *comm-monoid-add*
assumes ⋀ *x*. ⟦ *x ∈ set xs* ; ¬ *P x* ⟧ ⟹ *f x = 0*
shows *listsum (map f (filter P xs)) = listsum (map f xs)*
using *assms* **by** (*induct xs*) *auto*

For efficient code generation — *listsum* is not tail recursive but *foldl* is.

lemma *listsum*[*code-unfold*]: *listsum xs = foldl (op +) 0 xs*
by(*simp add*:*listsum-foldr foldl-foldr1*)

lemma *distinct-listsum-conv-Setsum*:
distinct xs ⟹ *listsum xs = Setsum*(*set xs*)
by (*induct xs*) *simp-all*

lemma *listsum-eq-0-nat-iff-nat*[*simp*]:
listsum ns = (0::*nat*) ⟷ (∀ *n ∈ set ns. n = 0*)
by(*simp add*: *listsum*)

lemma *elem-le-listsum-nat*: *k*<*size ns* ⟹ *ns*!*k* <= *listsum*(*ns*::*nat list*)
apply(*induct ns arbitrary*: *k*)
apply *simp*
apply(*fastsimp simp add*:*nth-Cons split*: *nat.split*)
done

lemma *listsum-update-nat*: *k*<*size ns* ⟹
listsum (*ns*[*k* := (*n*::*nat*)]) = *listsum ns + n − ns*!*k*
apply(*induct ns arbitrary*:*k*)
apply (*auto split*:*nat.split*)
apply(*drule elem-le-listsum-nat*)
apply *arith*
done

Some syntactic sugar for summing a function over a list:

syntax
-listsum :: *pttrn => 'a list => 'b => 'b* (*3SUM -<−-. -*) [*0*, *51*, *10*] *10*)
syntax (*xsymbols*)
-listsum :: *pttrn => 'a list => 'b => 'b* ((*3*∑ *-←-. -*) [*0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
-listsum :: *pttrn => 'a list => 'b => 'b* ((*3*∑ *-←-. -*) [*0*, *51*, *10*] *10*)

translations — Beware of argument permutation!
SUM x<−xs. b == CONST listsum (CONST map (%x. b) xs)
$\sum x{\leftarrow}xs.\ b$ == *CONST listsum (CONST map (%x. b) xs)*

lemma *listsum-triv*: $(\sum x{\leftarrow}xs.\ r) = \mathit{of\text{-}nat}\ (\mathit{length}\ xs) * r$
by (*induct xs*) (*simp-all add*: *left-distrib*)

lemma *listsum-0* [*simp*]: $(\sum x{\leftarrow}xs.\ 0) = 0$
by (*induct xs*) (*simp-all add*: *left-distrib*)

For non-Abelian groups *xs* needs to be reversed on one side:

lemma *uminus-listsum-map*:
fixes *f* :: *'a* ⇒ *'b*::*ab-group-add*
shows − *listsum* (*map f xs*) = (*listsum* (*map* (*uminus o f*) *xs*))
by (*induct xs*) *simp-all*

lemma *listsum-addf*:
fixes *f g* :: *'a* ⇒ *'b*::*comm-monoid-add*
shows $(\sum x{\leftarrow}xs.\ f\ x + g\ x) = \mathit{listsum}\ (\mathit{map}\ f\ xs) + \mathit{listsum}\ (\mathit{map}\ g\ xs)$
by (*induct xs*) (*simp-all add*: *algebra-simps*)

lemma *listsum-subtractf*:
fixes *f g* :: *'a* ⇒ *'b*::*ab-group-add*
shows $(\sum x{\leftarrow}xs.\ f\ x - g\ x) = \mathit{listsum}\ (\mathit{map}\ f\ xs) - \mathit{listsum}\ (\mathit{map}\ g\ xs)$
by (*induct xs*) *simp-all*

lemma *listsum-const-mult*:
fixes *f* :: *'a* ⇒ *'b*::*semiring-0*
shows $(\sum x{\leftarrow}xs.\ c * f\ x) = c * (\sum x{\leftarrow}xs.\ f\ x)$
by (*induct xs*, *simp-all add*: *algebra-simps*)

lemma *listsum-mult-const*:
fixes *f* :: *'a* ⇒ *'b*::*semiring-0*
shows $(\sum x{\leftarrow}xs.\ f\ x * c) = (\sum x{\leftarrow}xs.\ f\ x) * c$
by (*induct xs*, *simp-all add*: *algebra-simps*)

lemma *listsum-abs*:
fixes *xs* :: *'a*::*ordered-ab-group-add-abs list*
shows $|\mathit{listsum}\ xs| \leq \mathit{listsum}\ (\mathit{map}\ \mathit{abs}\ xs)$
by (*induct xs*, *simp*, *simp add*: *order-trans* [*OF abs-triangle-ineq*])

lemma *listsum-mono*:
fixes *f g* :: *'a* ⇒ *'b*::{*comm-monoid-add*, *ordered-ab-semigroup-add*}
shows $(\bigwedge x.\ x \in \mathit{set}\ xs \Longrightarrow f\ x \leq g\ x) \Longrightarrow (\sum x{\leftarrow}xs.\ f\ x) \leq (\sum x{\leftarrow}xs.\ g\ x)$
by (*induct xs*, *simp*, *simp add*: *add-mono*)

43.1.20 *upt*

lemma *upt-rec*[*code*]: [*i*..<*j*] = (*if* *i*<*j* *then* *i*#[*Suc* *i*..<*j*] *else* [])
— simp does not terminate!
by (*induct* *j*) *auto*

lemmas *upt-rec-number-of*[*simp*] = *upt-rec*[*of number-of m number-of n*, *standard*]

lemma *upt-conv-Nil* [*simp*]: *j* <= *i* ==> [*i*..<*j*] = []
by (*subst upt-rec*) *simp*

lemma *upt-eq-Nil-conv*[*simp*]: ([*i*..<*j*] = []) = (*j* = *0* ∨ *j* <= *i*)
by(*induct* *j*)*simp-all*

lemma *upt-eq-Cons-conv*:
([*i*..<*j*] = *x*#*xs*) = (*i* < *j* & *i* = *x* & [*i*+*1*..<*j*] = *xs*)
apply(*induct* *j* *arbitrary*: *x* *xs*)
apply *simp*
apply(*clarsimp* *simp* *add*: *append-eq-Cons-conv*)
apply *arith*
done

lemma *upt-Suc-append*: *i* <= *j* ==> [*i*..<(*Suc* *j*)] = [*i*..<*j*]@[*j*]
— Only needed if *upt-Suc* is deleted from the simpset.
by *simp*

lemma *upt-conv-Cons*: *i* < *j* ==> [*i*..<*j*] = *i* # [*Suc* *i*..<*j*]
by (*simp* *add*: *upt-rec*)

lemma *upt-add-eq-append*: *i*<=*j* ==> [*i*..<*j*+*k*] = [*i*..<*j*]@[*j*..<*j*+*k*]
— LOOPS as a simprule, since *j* <= *j*.
by (*induct* *k*) *auto*

lemma *length-upt* [*simp*]: *length* [*i*..<*j*] = *j* − *i*
by (*induct* *j*) (*auto* *simp* *add*: *Suc-diff-le*)

lemma *nth-upt* [*simp*]: *i* + *k* < *j* ==> [*i*..<*j*] ! *k* = *i* + *k*
apply (*induct* *j*)
apply (*auto* *simp* *add*: *less-Suc-eq* *nth-append* *split*: *nat-diff-split*)
done

lemma *hd-upt*[*simp*]: *i* < *j* ⟹ *hd*[*i*..<*j*] = *i*
by(*simp* *add*:*upt-conv-Cons*)

lemma *last-upt*[*simp*]: *i* < *j* ⟹ *last*[*i*..<*j*] = *j* − *1*
apply(*cases* *j*)
apply *simp*
by(*simp* *add*:*upt-Suc-append*)

lemma *take-upt* [*simp*]: *i+m <= n ==> take m [i..<n] = [i..<i+m]*
apply (*induct m arbitrary*: *i*, *simp*)
apply (*subst upt-rec*)
apply (*rule sym*)
apply (*subst upt-rec*)
apply (*simp del*: *upt.simps*)
done

lemma *drop-upt*[*simp*]: *drop m [i..<j] = [i+m..<j]*
apply(*induct j*)
apply *auto*
done

lemma *map-Suc-upt*: *map Suc [m..<n] = [Suc m..<Suc n]*
by (*induct n*) *auto*

lemma *nth-map-upt*: *i < n−m ==> (map f [m..<n]) ! i = f(m+i)*
apply (*induct n m arbitrary*: *i rule*: *diff-induct*)
prefer *3* **apply** (*subst map-Suc-upt*[*symmetric*])
apply (*auto simp add*: *less-diff-conv nth-upt*)
done

lemma *nth-take-lemma*:
k <= length xs ==> k <= length ys ==>
(!!i. i < k −−> xs!i = ys!i) ==> take k xs = take k ys
apply (*atomize*, *induct k arbitrary*: *xs ys*)
apply (*simp-all add*: *less-Suc-eq-0-disj all-conj-distrib*, *clarify*)

Both lists must be non-empty

apply (*case-tac xs*, *simp*)
apply (*case-tac ys*, *clarify*)
apply (*simp* (*no-asm-use*))
apply *clarify*

prenexing's needed, not miniscoping

apply (*simp* (*no-asm-use*) *add*: *all-simps* [*symmetric*] *del*: *all-simps*)
apply *blast*
done

lemma *nth-equalityI*:
[| *length xs = length ys*; *ALL i < length xs. xs!i = ys!i* |] *==> xs = ys*
apply (*frule nth-take-lemma* [*OF le-refl eq-imp-le*])
apply (*simp-all add*: *take-all*)
done

lemma *map-nth*:
map (*λi. xs ! i*) [*0..<length xs*] = *xs*
by (*rule nth-equalityI*, *auto*)

lemma *list-all2-antisym*:
 $[\![(\bigwedge x\ y.\ [\![P\ x\ y;\ Q\ y\ x]\!] \Longrightarrow x = y);$ *list-all2 P xs ys*; *list-all2 Q ys xs* $]\!]$
 $\Longrightarrow$ *xs* = *ys*
 apply (*simp add*: *list-all2-conv-all-nth*)
 apply (*rule nth-equalityI*, *blast*, *simp*)
 done

lemma *take-equalityI*: $(\forall i.$ *take i xs* = *take i ys*$)$ ==> *xs* = *ys*
— The famous take-lemma.
apply (*drule-tac x* = *max* (*length xs*) (*length ys*) **in** *spec*)
apply (*simp add*: *le-max-iff-disj take-all*)
done

lemma *take-Cons'*:
 take n (*x* # *xs*) = (*if n* = *0 then* [] *else x* # *take* (*n* − *1*) *xs*)
by (*cases n*) *simp-all*

lemma *drop-Cons'*:
 drop n (*x* # *xs*) = (*if n* = *0 then x* # *xs else drop* (*n* − *1*) *xs*)
by (*cases n*) *simp-all*

lemma *nth-Cons'*: (*x* # *xs*)!*n* = (*if n* = *0 then x else xs*!(*n* − *1*))
by (*cases n*) *simp-all*

lemmas *take-Cons-number-of* = *take-Cons'*[*of number-of v*,*standard*]
lemmas *drop-Cons-number-of* = *drop-Cons'*[*of number-of v*,*standard*]
lemmas *nth-Cons-number-of* = *nth-Cons'*[*of - - number-of v*,*standard*]

declare *take-Cons-number-of* [*simp*]
 drop-Cons-number-of [*simp*]
 nth-Cons-number-of [*simp*]

43.1.21 *upto*: interval-list on *int*

function *upto* :: *int* ⇒ *int* ⇒ *int list* ((*1*[-../-])) **where**
upto i j = (*if i* ≤ *j then i* # [*i*+*1*..*j*] *else* [])
by *auto*
termination
by(*relation measure*(%(*i*::*int*,*j*). *nat*(*j* − *i* + *1*))) *auto*

declare *upto.simps*[*code*, *simp del*]

lemmas *upto-rec-number-of*[*simp*] =
 upto.simps[*of number-of m number-of n*, *standard*]

lemma *upto-empty*[*simp*]: *j* < *i* ⟹ [*i*..*j*] = []
by(*simp add*: *upto.simps*)

lemma *set-upto*[*simp*]: *set*[*i..j*] = {*i..j*}
apply(*induct i j rule*:*upto.induct*)
apply(*simp add*: *upto.simps simp-from-to*)
done

43.1.22 *distinct* and *remdups*

lemma *distinct-append* [*simp*]:
distinct (*xs* @ *ys*) = (*distinct xs* ∧ *distinct ys* ∧ *set xs* ∩ *set ys* = {})
by (*induct xs*) *auto*

lemma *distinct-rev*[*simp*]: *distinct*(*rev xs*) = *distinct xs*
by(*induct xs*) *auto*

lemma *set-remdups* [*simp*]: *set* (*remdups xs*) = *set xs*
by (*induct xs*) (*auto simp add*: *insert-absorb*)

lemma *distinct-remdups* [*iff*]: *distinct* (*remdups xs*)
by (*induct xs*) *auto*

lemma *distinct-remdups-id*: *distinct xs* ==> *remdups xs* = *xs*
by (*induct xs*, *auto*)

lemma *remdups-id-iff-distinct* [*simp*]: *remdups xs* = *xs* ⟷ *distinct xs*
by (*metis distinct-remdups distinct-remdups-id*)

lemma *finite-distinct-list*: *finite A* ⟹ *EX xs*. *set xs* = *A* & *distinct xs*
by (*metis distinct-remdups finite-list set-remdups*)

lemma *remdups-eq-nil-iff* [*simp*]: (*remdups x* = []) = (*x* = [])
by (*induct x*, *auto*)

lemma *remdups-eq-nil-right-iff* [*simp*]: ([] = *remdups x*) = (*x* = [])
by (*induct x*, *auto*)

lemma *length-remdups-leq*[*iff*]: *length*(*remdups xs*) <= *length xs*
by (*induct xs*) *auto*

lemma *length-remdups-eq*[*iff*]:
(*length* (*remdups xs*) = *length xs*) = (*remdups xs* = *xs*)
apply(*induct xs*)
apply *auto*
apply(*subgoal-tac length* (*remdups xs*) <= *length xs*)
apply *arith*
apply(*rule length-remdups-leq*)
done

lemma *remdups-filter*: *remdups*(*filter P xs*) = *filter P* (*remdups xs*)

apply(*induct xs*)
apply *auto*
done

lemma *distinct-map*:
  distinct(*map f xs*) = (*distinct xs* & *inj-on f* (*set xs*))
by (*induct xs*) *auto*

lemma *distinct-filter* [*simp*]: *distinct xs* ==> *distinct* (*filter P xs*)
by (*induct xs*) *auto*

lemma *distinct-upt*[*simp*]: *distinct*[*i*..<*j*]
by (*induct j*) *auto*

lemma *distinct-upto*[*simp*]: *distinct*[*i*..*j*]
apply(*induct i j rule*:*upto.induct*)
apply(*subst upto.simps*)
apply(*simp*)
done

lemma *distinct-take*[*simp*]: *distinct xs* ⟹ *distinct* (*take i xs*)
apply(*induct xs arbitrary*: *i*)
 apply *simp*
apply (*case-tac i*)
 apply *simp-all*
apply(*blast dest*:*in-set-takeD*)
done

lemma *distinct-drop*[*simp*]: *distinct xs* ⟹ *distinct* (*drop i xs*)
apply(*induct xs arbitrary*: *i*)
 apply *simp*
apply (*case-tac i*)
 apply *simp-all*
done

lemma *distinct-list-update*:
assumes *d*: *distinct xs* **and** *a*: *a* ∉ *set xs* − {*xs*!*i*}
shows *distinct* (*xs*[*i*:=*a*])
proof (*cases i* < *length xs*)
  case *True*
  with *a* **have** *a* ∉ *set* (*take i xs* @ *xs* ! *i* # *drop* (*Suc i*) *xs*) − {*xs*!*i*}
    apply (*drule-tac id-take-nth-drop*) **by** *simp*
  with *d True* **show** *?thesis*
    apply (*simp add*: *upd-conv-take-nth-drop*)
    apply (*drule subst* [*OF id-take-nth-drop*]) **apply** *assumption*
    apply *simp* **apply** (*cases a* = *xs*!*i*) **apply** *simp* **by** *blast*
next
  case *False* **with** *d* **show** *?thesis* **by** *auto*

qed

lemma *distinct-concat*:
assumes *distinct xs*
and $\bigwedge$ *ys*. *ys* $\in$ *set xs* $\Longrightarrow$ *distinct ys*
and $\bigwedge$ *ys zs*. $[\![$ *ys* $\in$ *set xs* ; *zs* $\in$ *set xs* ; *ys* $\neq$ *zs* $]\!] \Longrightarrow$ *set ys* $\cap$ *set zs* = {}
shows *distinct* (*concat xs*)
using *assms* **by** (*induct xs*) *auto*

It is best to avoid this indexed version of distinct, but sometimes it is useful.

lemma *distinct-conv-nth*:
distinct xs = (∀ *i* < *size xs*. ∀ *j* < *size xs*. *i* ≠ *j* −−> *xs*!*i* ≠ *xs*!*j*)
apply (*induct xs*, *simp*, *simp*)
apply (*rule iffI*, *clarsimp*)
apply (*case-tac i*)
apply (*case-tac j*, *simp*)
apply (*simp add*: *set-conv-nth*)
apply (*case-tac j*)
apply (*clarsimp simp add*: *set-conv-nth*, *simp*)
apply (*rule conjI*)

apply (*clarsimp simp add*: *set-conv-nth*)
apply (*erule-tac x* = *0* **in** *allE*, *simp*)
apply (*erule-tac x* = *Suc i* **in** *allE*, *simp*, *clarsimp*)

apply (*erule-tac x* = *Suc i* **in** *allE*, *simp*)
apply (*erule-tac x* = *Suc j* **in** *allE*, *simp*)
done

lemma *nth-eq-iff-index-eq*:
$[\![$ *distinct xs*; *i* < *length xs*; *j* < *length xs* $]\!] \Longrightarrow$ (*xs*!*i* = *xs*!*j*) = (*i* = *j*)
by(*auto simp*: *distinct-conv-nth*)

lemma *distinct-card*: *distinct xs* ==> *card* (*set xs*) = *size xs*
by (*induct xs*) *auto*

lemma *card-distinct*: *card* (*set xs*) = *size xs* ==> *distinct xs*
proof (*induct xs*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons x xs*)
show *?case*
proof (*cases x* ∈ *set xs*)
case *False* **with** *Cons* **show** *?thesis* **by** *simp*
next
case *True* **with** *Cons.prems*
have *card* (*set xs*) = *Suc* (*length xs*)
by (*simp add*: *card-insert-if split*: *split-if-asm*)
moreover have *card* (*set xs*) ≤ *length xs* **by** (*rule card-length*)

```
    ultimately have False by simp
    thus ?thesis ..
  qed
qed

lemma not-distinct-decomp: ~ distinct ws ==> EX xs ys zs y. ws = xs@[y]@ys@[y]@zs
apply (induct n == length ws arbitrary:ws) apply simp
apply(case-tac ws) apply simp
apply (simp split:split-if-asm)
apply (metis Cons-eq-appendI eq-Nil-appendI split-list)
done

lemma length-remdups-concat:
 length(remdups(concat xss)) = card(⋃ xs ∈ set xss. set xs)
by(simp add: set-concat distinct-card[symmetric])

lemma length-remdups-card-conv: length(remdups xs) = card(set xs)
proof −
  have xs: concat[xs] = xs by simp
  from length-remdups-concat[of [xs]] show ?thesis unfolding xs by simp
qed

lemma remdups-remdups:
  remdups (remdups xs) = remdups xs
  by (induct xs) simp-all

lemma distinct-butlast:
  assumes xs ≠ [] and distinct xs
  shows distinct (butlast xs)
proof −
  from ‹xs ≠ []› obtain ys y where xs = ys @ [y] by (cases xs rule: rev-cases)
auto
  with ‹distinct xs› show ?thesis by simp
qed
```

43.1.23 *insert*

```
lemma in-set-insert [simp]:
  x ∈ set xs ⟹ List.insert x xs = xs
  by (simp add: List.insert-def)

lemma not-in-set-insert [simp]:
  x ∉ set xs ⟹ List.insert x xs = x # xs
  by (simp add: List.insert-def)

lemma insert-Nil [simp]:
  List.insert x [] = [x]
  by simp
```

lemma *set-insert* [*simp*]:
set (*List.insert x xs*) = *insert x* (*set xs*)
by (*auto simp add*: *List.insert-def*)

lemma *distinct-insert* [*simp*]:
distinct xs ⟹ *distinct* (*List.insert x xs*)
by (*simp add*: *List.insert-def*)

lemma *insert-remdups*:
List.insert x (*remdups xs*) = *remdups* (*List.insert x xs*)
by (*simp add*: *List.insert-def*)

lemma *distinct-induct* [*consumes 1*, *case-names Nil insert*]:
assumes *distinct xs*
assumes *P* []
assumes *insert*: ⋀*x xs*. *distinct xs* ⟹ *x* ∉ *set xs*
⟹ *P xs* ⟹ *P* (*List.insert x xs*)
shows *P xs*
using ‹*distinct xs*› **proof** (*induct xs*)
case *Nil* **from** ‹*P* []› **show** *?case* **.**
next
case (*Cons x xs*)
then have *distinct xs* **and** *x* ∉ *set xs* **and** *P xs* **by** *simp-all*
with *insert* **have** *P* (*List.insert x xs*) **.**
with ‹*x* ∉ *set xs*› **show** *?case* **by** *simp*
qed

43.1.24 *remove1*

lemma *remove1-append*:
remove1 x (*xs* @ *ys*) =
(*if x* ∈ *set xs then remove1 x xs* @ *ys else xs* @ *remove1 x ys*)
by (*induct xs*) *auto*

lemma *remove1-commute*: *remove1 x* (*remove1 y zs*) = *remove1 y* (*remove1 x zs*)
by (*induct zs*) *auto*

lemma *in-set-remove1*[*simp*]:
a ≠ *b* ⟹ *a* : *set*(*remove1 b xs*) = (*a* : *set xs*)
apply (*induct xs*)
apply *auto*
done

lemma *set-remove1-subset*: *set*(*remove1 x xs*) <= *set xs*
apply(*induct xs*)
apply *simp*
apply *simp*
apply *blast*
done

lemma *set-remove1-eq* [*simp*]: *distinct xs ==> set*(*remove1 x xs*) = *set xs* − {*x*}
apply(*induct xs*)
apply *simp*
apply *simp*
apply *blast*
done

lemma *length-remove1*:
length(*remove1 x xs*) = (*if x* : *set xs then length xs* − *1 else length xs*)
apply (*induct xs*)
apply (*auto dest*!:*length-pos-if-in-set*)
done

lemma *remove1-filter-not*[*simp*]:
¬ *P x* ⟹ *remove1 x* (*filter P xs*) = *filter P xs*
by(*induct xs*) *auto*

lemma *notin-set-remove1*[*simp*]: *x* ~: *set xs ==> x* ~: *set*(*remove1 y xs*)
apply(*insert set-remove1-subset*)
apply *fast*
done

lemma *distinct-remove1*[*simp*]: *distinct xs ==> distinct*(*remove1 x xs*)
by (*induct xs*) *simp-all*

lemma *remove1-remdups*:
distinct xs ⟹ *remove1 x* (*remdups xs*) = *remdups* (*remove1 x xs*)
by (*induct xs*) *simp-all*

lemma *remove1-idem*:
assumes *x* ∉ *set xs*
shows *remove1 x xs* = *xs*
using *assms* **by** (*induct xs*) *simp-all*

43.1.25 *removeAll*

lemma *removeAll-filter-not-eq*:
removeAll x = *filter* (λ*y*. *x* ≠ *y*)
proof
fix *xs*
show *removeAll x xs* = *filter* (λ*y*. *x* ≠ *y*) *xs*
by (*induct xs*) *auto*
qed

lemma *removeAll-append*[*simp*]:
removeAll x (*xs* @ *ys*) = *removeAll x xs* @ *removeAll x ys*
by (*induct xs*) *auto*

lemma *set-removeAll*[*simp*]: *set*(*removeAll x xs*) = *set xs* − {*x*}
by (*induct xs*) *auto*

lemma *removeAll-id*[*simp*]: *x* ∉ *set xs* ⟹ *removeAll x xs* = *xs*
by (*induct xs*) *auto*

lemma *removeAll-filter-not*[*simp*]:
¬ *P x* ⟹ *removeAll x* (*filter P xs*) = *filter P xs*
by(*induct xs*) *auto*

lemma *distinct-removeAll*:
distinct xs ⟹ *distinct* (*removeAll x xs*)
by (*simp add*: *removeAll-filter-not-eq*)

lemma *distinct-remove1-removeAll*:
distinct xs ==> *remove1 x xs* = *removeAll x xs*
by (*induct xs*) *simp-all*

lemma *map-removeAll-inj-on*: *inj-on f* (*insert x* (*set xs*)) ⟹
map f (*removeAll x xs*) = *removeAll* (*f x*) (*map f xs*)
by (*induct xs*) (*simp-all add*:*inj-on-def*)

lemma *map-removeAll-inj*: *inj f* ⟹
map f (*removeAll x xs*) = *removeAll* (*f x*) (*map f xs*)
by(*metis map-removeAll-inj-on subset-inj-on subset-UNIV*)

43.1.26 *replicate*

lemma *length-replicate* [*simp*]: *length* (*replicate n x*) = *n*
by (*induct n*) *auto*

lemma *Ex-list-of-length*: ∃ *xs*. *length xs* = *n*
by (*rule exI*[*of - replicate n undefined*]) *simp*

lemma *map-replicate* [*simp*]: *map f* (*replicate n x*) = *replicate n* (*f x*)
by (*induct n*) *auto*

lemma *map-replicate-const*:
map (λ *x*. *k*) *lst* = *replicate* (*length lst*) *k*
by (*induct lst*) *auto*

lemma *replicate-app-Cons-same*:
(*replicate n x*) @ (*x* # *xs*) = *x* # *replicate n x* @ *xs*
by (*induct n*) *auto*

lemma *rev-replicate* [*simp*]: *rev* (*replicate n x*) = *replicate n x*
apply (*induct n*, *simp*)

apply (*simp add*: *replicate-app-Cons-same*)
done

lemma *replicate-add*: *replicate* (*n* + *m*) *x* = *replicate n x* @ *replicate m x*
by (*induct n*) *auto*

Courtesy of Matthias Daum:

lemma *append-replicate-commute*:
replicate n x @ *replicate k x* = *replicate k x* @ *replicate n x*
apply (*simp add*: *replicate-add* [*THEN sym*])
apply (*simp add*: *add-commute*)
done

Courtesy of Andreas Lochbihler:

lemma *filter-replicate*:
filter P (*replicate n x*) = (*if P x then replicate n x else* [])
by(*induct n*) *auto*

lemma *hd-replicate* [*simp*]: *n* ≠ *0* ==> *hd* (*replicate n x*) = *x*
by (*induct n*) *auto*

lemma *tl-replicate* [*simp*]: *n* ≠ *0* ==> *tl* (*replicate n x*) = *replicate* (*n* − *1*) *x*
by (*induct n*) *auto*

lemma *last-replicate* [*simp*]: *n* ≠ *0* ==> *last* (*replicate n x*) = *x*
by (*atomize* (*full*), *induct n*) *auto*

lemma *nth-replicate*[*simp*]: *i* < *n* ==> (*replicate n x*)!*i* = *x*
apply (*induct n arbitrary*: *i*, *simp*)
apply (*simp add*: *nth-Cons split*: *nat.split*)
done

Courtesy of Matthias Daum (2 lemmas):

lemma *take-replicate*[*simp*]: *take i* (*replicate k x*) = *replicate* (*min i k*) *x*
apply (*case-tac k* ≤ *i*)
apply (*simp add*: *min-def*)
apply (*drule not-leE*)
apply (*simp add*: *min-def*)
apply (*subgoal-tac replicate k x* = *replicate i x* @ *replicate* (*k* − *i*) *x*)
apply *simp*
apply (*simp add*: *replicate-add* [*symmetric*])
done

lemma *drop-replicate*[*simp*]: *drop i* (*replicate k x*) = *replicate* (*k*−*i*) *x*
apply (*induct k arbitrary*: *i*)
apply *simp*
apply *clarsimp*
apply (*case-tac i*)
apply *simp*

apply *clarsimp*
done

lemma *set-replicate-Suc*: *set* (*replicate* (*Suc n*) *x*) = {*x*}
by (*induct n*) *auto*

lemma *set-replicate* [*simp*]: *n* ≠ *0* ==> *set* (*replicate n x*) = {*x*}
by (*fast dest*!: *not0-implies-Suc intro*!: *set-replicate-Suc*)

lemma *set-replicate-conv-if*: *set* (*replicate n x*) = (*if n = 0 then* {} *else* {*x*})
by *auto*

lemma *in-set-replicateD*: *x* : *set* (*replicate n y*) ==> *x* = *y*
by (*simp add*: *set-replicate-conv-if split*: *split-if-asm*)

lemma *replicate-append-same*:
replicate i x @ [*x*] = *x* # *replicate i x*
by (*induct i*) *simp-all*

lemma *map-replicate-trivial*:
map (λ*i*. *x*) [*0*..<*i*] = *replicate i x*
by (*induct i*) (*simp-all add*: *replicate-append-same*)

lemma *concat-replicate-trivial*[*simp*]:
concat (*replicate i* []) = []
by (*induct i*) (*auto simp add*: *map-replicate-const*)

lemma *replicate-empty*[*simp*]: (*replicate n x* = []) ⟷ *n=0*
by (*induct n*) *auto*

lemma *empty-replicate*[*simp*]: ([] = *replicate n x*) ⟷ *n=0*
by (*induct n*) *auto*

lemma *replicate-eq-replicate*[*simp*]:
(*replicate m x* = *replicate n y*) ⟷ (*m=n* & (*m≠0* ⟶ *x=y*))
apply(*induct m arbitrary*: *n*)
apply *simp*
apply(*induct-tac n*)
apply *auto*
done

43.1.27 *rotate1* and *rotate*

lemma *rotate-simps*[*simp*]: *rotate1* [] = [] ∧ *rotate1* (*x*#*xs*) = *xs* @ [*x*]
by(*simp add*:*rotate1-def*)

lemma *rotate0*[*simp*]: *rotate 0* = *id*
by(*simp add*:*rotate-def*)

lemma *rotate-Suc*[*simp*]: *rotate* (*Suc n*) *xs* = *rotate1*(*rotate n xs*)
by(*simp add*:*rotate-def*)

lemma *rotate-add*:
  rotate (*m*+*n*) = *rotate m o rotate n*
by(*simp add*:*rotate-def funpow-add*)

lemma *rotate-rotate*: *rotate m* (*rotate n xs*) = *rotate* (*m*+*n*) *xs*
by(*simp add*:*rotate-add*)

lemma *rotate1-rotate-swap*: *rotate1* (*rotate n xs*) = *rotate n* (*rotate1 xs*)
by(*simp add*:*rotate-def funpow-swap1*)

lemma *rotate1-length01*[*simp*]: *length xs* <= *1* ⟹ *rotate1 xs* = *xs*
by(*cases xs*) *simp-all*

lemma *rotate-length01*[*simp*]: *length xs* <= *1* ⟹ *rotate n xs* = *xs*
apply(*induct n*)
 apply *simp*
apply (*simp add*:*rotate-def*)
done

lemma *rotate1-hd-tl*: *xs* ≠ [] ⟹ *rotate1 xs* = *tl xs* @ [*hd xs*]
by(*simp add*:*rotate1-def split*:*list.split*)

lemma *rotate-drop-take*:
  rotate n xs = *drop* (*n mod length xs*) *xs* @ *take* (*n mod length xs*) *xs*
apply(*induct n*)
 apply *simp*
apply(*simp add*:*rotate-def*)
apply(*cases xs* = [])
 apply (*simp*)
apply(*case-tac n mod length xs* = *0*)
 apply(*simp add*:*mod-Suc*)
 apply(*simp add*: *rotate1-hd-tl drop-Suc take-Suc*)
apply(*simp add*:*mod-Suc rotate1-hd-tl drop-Suc*[*symmetric*] *drop-tl*[*symmetric*]
          take-hd-drop linorder-not-le)
done

lemma *rotate-conv-mod*: *rotate n xs* = *rotate* (*n mod length xs*) *xs*
by(*simp add*:*rotate-drop-take*)

lemma *rotate-id*[*simp*]: *n mod length xs* = *0* ⟹ *rotate n xs* = *xs*
by(*simp add*:*rotate-drop-take*)

lemma *length-rotate1*[*simp*]: *length*(*rotate1 xs*) = *length xs*
by(*simp add*:*rotate1-def split*:*list.split*)

lemma *length-rotate*[*simp*]: *length*(*rotate n xs*) = *length xs*
by (*induct n arbitrary*: *xs*) (*simp-all add*:*rotate-def*)

lemma *distinct1-rotate*[*simp*]: *distinct*(*rotate1 xs*) = *distinct xs*
by(*simp add*:*rotate1-def split*:*list.split*) *blast*

lemma *distinct-rotate*[*simp*]: *distinct*(*rotate n xs*) = *distinct xs*
by (*induct n*) (*simp-all add*:*rotate-def*)

lemma *rotate-map*: *rotate n* (*map f xs*) = *map f* (*rotate n xs*)
by(*simp add*:*rotate-drop-take take-map drop-map*)

lemma *set-rotate1*[*simp*]: *set*(*rotate1 xs*) = *set xs*
by(*simp add*:*rotate1-def split*:*list.split*)

lemma *set-rotate*[*simp*]: *set*(*rotate n xs*) = *set xs*
by (*induct n*) (*simp-all add*:*rotate-def*)

lemma *rotate1-is-Nil-conv*[*simp*]: (*rotate1 xs* = []) = (*xs* = [])
by(*simp add*:*rotate1-def split*:*list.split*)

lemma *rotate-is-Nil-conv*[*simp*]: (*rotate n xs* = []) = (*xs* = [])
by (*induct n*) (*simp-all add*:*rotate-def*)

lemma *rotate-rev*:
  rotate n (*rev xs*) = *rev*(*rotate* (*length xs* − (*n mod length xs*)) *xs*)
apply(*simp add*:*rotate-drop-take rev-drop rev-take*)
apply(*cases length xs* = *0*)
 apply *simp*
apply(*cases n mod length xs* = *0*)
 apply *simp*
apply(*simp add*:*rotate-drop-take rev-drop rev-take*)
done

lemma *hd-rotate-conv-nth*: *xs* ≠ [] ⟹ *hd*(*rotate n xs*) = *xs*!(*n mod length xs*)
apply(*simp add*:*rotate-drop-take hd-append hd-drop-conv-nth hd-conv-nth*)
apply(*subgoal-tac length xs* ≠ *0*)
 prefer *2* **apply** *simp*
using *mod-less-divisor*[*of length xs n*] **by** *arith*

43.1.28 *sublist* — a generalization of *nth* to sets

lemma *sublist-empty* [*simp*]: *sublist xs* {} = []
by (*auto simp add*: *sublist-def*)

lemma *sublist-nil* [*simp*]: *sublist* [] *A* = []
by (*auto simp add*: *sublist-def*)

lemma *length-sublist*:

length(*sublist xs I*) = *card*{*i*. *i* < *length xs* ∧ *i* : *I*}
by(*simp add*: *sublist-def length-filter-conv-card cong*:*conj-cong*)

lemma *sublist-shift-lemma-Suc*:
map fst (*filter* (%*p*. *P*(*Suc*(*snd p*))) (*zip xs is*)) =
map fst (*filter* (%*p*. *P*(*snd p*)) (*zip xs* (*map Suc is*)))
apply(*induct xs arbitrary*: *is*)
apply *simp*
apply (*case-tac is*)
apply *simp*
apply *simp*
done

lemma *sublist-shift-lemma*:
map fst [*p*<−*zip xs* [*i*..<*i* + *length xs*] . *snd p* : *A*] =
map fst [*p*<−*zip xs* [*0*..<*length xs*] . *snd p* + *i* : *A*]
by (*induct xs rule*: *rev-induct*) (*simp-all add*: *add-commute*)

lemma *sublist-append*:
sublist (*l* @ *l'*) *A* = *sublist l A* @ *sublist l'* {*j*. *j* + *length l* : *A*}
apply (*unfold sublist-def*)
apply (*induct l' rule*: *rev-induct*, *simp*)
apply (*simp add*: *upt-add-eq-append*[*of 0*] *zip-append sublist-shift-lemma*)
apply (*simp add*: *add-commute*)
done

lemma *sublist-Cons*:
sublist (*x* # *l*) *A* = (*if 0*:*A then* [*x*] *else* []) @ *sublist l* {*j*. *Suc j* : *A*}
apply (*induct l rule*: *rev-induct*)
apply (*simp add*: *sublist-def*)
apply (*simp del*: *append-Cons add*: *append-Cons*[*symmetric*] *sublist-append*)
done

lemma *set-sublist*: *set*(*sublist xs I*) = {*xs*!*i*|*i*. *i*<*size xs* ∧ *i* ∈ *I*}
apply(*induct xs arbitrary*: *I*)
apply(*auto simp*: *sublist-Cons nth-Cons split*:*nat.split dest*!: *gr0-implies-Suc*)
done

lemma *set-sublist-subset*: *set*(*sublist xs I*) ⊆ *set xs*
by(*auto simp add*:*set-sublist*)

lemma *notin-set-sublistI*[*simp*]: *x* ∉ *set xs* ⟹ *x* ∉ *set*(*sublist xs I*)
by(*auto simp add*:*set-sublist*)

lemma *in-set-sublistD*: *x* ∈ *set*(*sublist xs I*) ⟹ *x* ∈ *set xs*
by(*auto simp add*:*set-sublist*)

lemma *sublist-singleton* [*simp*]: *sublist* [*x*] *A* = (*if 0* : *A then* [*x*] *else* [])
by (*simp add*: *sublist-Cons*)

lemma *distinct-sublistI*[*simp*]: *distinct xs* $\Longrightarrow$ *distinct*(*sublist xs I*)
apply(*induct xs arbitrary*: *I*)
apply *simp*
apply(*auto simp add*:*sublist-Cons*)
done

lemma *sublist-upt-eq-take* [*simp*]: *sublist l* {..<*n*} = *take n l*
apply (*induct l rule*: *rev-induct*, *simp*)
apply (*simp split*: *nat-diff-split add*: *sublist-append*)
done

lemma *filter-in-sublist*:
distinct xs $\Longrightarrow$ *filter* (%*x*. *x* $\in$ *set*(*sublist xs s*)) *xs* = *sublist xs s*
proof (*induct xs arbitrary*: *s*)
case *Nil* **thus** *?case* **by** *simp*
next
case (*Cons a xs*)
moreover hence !*x*. *x*: *set xs* $\longrightarrow$ *x* $\neq$ *a* **by** *auto*
ultimately show *?case* **by**(*simp add*: *sublist-Cons cong*:*filter-cong*)
qed

43.1.29 *splice*

lemma *splice-Nil2* [*simp*, *code*]:
splice xs [] = *xs*
by (*cases xs*) *simp-all*

lemma *splice-Cons-Cons* [*simp*, *code*]:
splice (*x*#*xs*) (*y*#*ys*) = *x* # *y* # *splice xs ys*
by *simp*

declare *splice.simps*(*2*) [*simp del*, *code del*]

lemma *length-splice*[*simp*]: *length*(*splice xs ys*) = *length xs* + *length ys*
apply(*induct xs arbitrary*: *ys*) **apply** *simp*
apply(*case-tac ys*)
apply *auto*
done

43.1.30 Transpose

function *transpose* **where**
transpose [] = [] |
transpose ([] # *xss*) = *transpose xss* |
transpose ((*x*#*xs*) # *xss*) =
(*x* # [*h*. (*h*#*t*) $\leftarrow$ *xss*]) # *transpose* (*xs* # [*t*. (*h*#*t*) $\leftarrow$ *xss*])
by *pat-completeness auto*

```
lemma transpose-aux-filter-head:
  concat (map (list-case [] (λh t. [h])) xss) =
  map (λxs. hd xs) [ys←xss . ys ≠ []]
  by (induct xss) (auto split: list.split)

lemma transpose-aux-filter-tail:
  concat (map (list-case [] (λh t. [t])) xss) =
  map (λxs. tl xs) [ys←xss . ys ≠ []]
  by (induct xss) (auto split: list.split)

lemma transpose-aux-max:
  max (Suc (length xs)) (foldr (λxs. max (length xs)) xss 0) =
  Suc (max (length xs) (foldr (λx. max (length x − Suc 0)) [ys←xss . ys≠[]] 0))
  (is max - ?foldB = Suc (max - ?foldA))
proof (cases [ys←xss . ys≠[]] = [])
  case True
  hence foldr (λxs. max (length xs)) xss 0 = 0
  proof (induct xss)
    case (Cons x xs)
    moreover hence x = [] by (cases x) auto
    ultimately show ?case by auto
  qed simp
  thus ?thesis using True by simp
next
  case False

  have foldA: ?foldA = foldr (λx. max (length x)) [ys←xss . ys ≠ []] 0 − 1
    by (induct xss) auto
  have foldB: ?foldB = foldr (λx. max (length x)) [ys←xss . ys ≠ []] 0
    by (induct xss) auto

  have 0 < ?foldB
  proof −
    from False
    obtain z zs where zs: [ys←xss . ys ≠ []] = z#zs by (auto simp: neq-Nil-conv)
    hence z ∈ set ([ys←xss . ys ≠ []]) by auto
    hence z ≠ [] by auto
    thus ?thesis
      unfolding foldB zs
      by (auto simp: max-def intro: less-le-trans)
  qed
  thus ?thesis
    unfolding foldA foldB max-Suc-Suc[symmetric]
    by simp
qed

termination transpose
  by (relation measure (λxs. foldr (λxs. max (length xs)) xs 0 + length xs))
```

(*auto simp*: *transpose-aux-filter-tail foldr-map comp-def transpose-aux-max less-Suc-eq-le*)

lemma *transpose-empty*: (*transpose xs* = []) ⟷ (∀ *x* ∈ *set xs*. *x* = [])
by (*induct rule*: *transpose.induct*) *simp-all*

lemma *length-transpose*:
fixes *xs* :: *'a list list*
shows *length* (*transpose xs*) = *foldr* (λ*xs*. *max* (*length xs*)) *xs 0*
by (*induct rule*: *transpose.induct*)
(*auto simp*: *transpose-aux-filter-tail foldr-map comp-def transpose-aux-max max-Suc-Suc*[*symmetric*] *simp del*: *max-Suc-Suc*)

lemma *nth-transpose*:
fixes *xs* :: *'a list list*
assumes *i* < *length* (*transpose xs*)
shows *transpose xs* ! *i* = *map* (λ*xs*. *xs* ! *i*) [*ys* ← *xs*. *i* < *length ys*]
using *assms* **proof** (*induct arbitrary*: *i rule*: *transpose.induct*)
case (*3 x xs xss*)
def *XS* == (*x* # *xs*) # *xss*
hence [*simp*]: *XS* ≠ [] **by** *auto*
thus *?case*
proof (*cases i*)
case *0*
thus *?thesis* **by** (*simp add*: *transpose-aux-filter-head hd-conv-nth*)
next
case (*Suc j*)
have ∗: ⋀*xss*. *xs* # *map tl xss* = *map tl* ((*x*#*xs*)#*xss*) **by** *simp*
have ∗∗: ⋀*xss*. (*x*#*xs*) # *filter* (λ*ys*. *ys* ≠ []) *xss* = *filter* (λ*ys*. *ys* ≠ []) ((*x*#*xs*)#*xss*) **by** *simp*
{ fix *x* **have** *Suc j* < *length x* ⟷ *x* ≠ [] ∧ *j* < *length x* − *Suc 0*
by (*cases x*) *simp-all*
} note ∗∗∗ = *this*

have *j-less*: *j* < *length* (*transpose* (*xs* # *concat* (*map* (*list-case* [] (λ*h t*. [*t*])) *xss*)))
using *3.prems* **by** (*simp add*: *transpose-aux-filter-tail length-transpose Suc*)

show *?thesis*
unfolding *transpose.simps* ‹*i* = *Suc j*› *nth-Cons-Suc 3.hyps*[*OF j-less*]
apply (*auto simp*: *transpose-aux-filter-tail filter-map comp-def length-transpose* ∗ ∗∗ ∗∗∗ *XS-def*[*symmetric*])
apply (*rule-tac y*=*x* **in** *list.exhaust*)
by *auto*
qed
qed *simp-all*

lemma *transpose-map-map*:
transpose (*map* (*map f*) *xs*) = *map* (*map f*) (*transpose xs*)

proof (*rule nth-equalityI*, *safe*)
have [*simp*]: *length* (*transpose* (*map* (*map f*) *xs*)) = *length* (*transpose xs*)
by (*simp add*: *length-transpose foldr-map comp-def*)
show *length* (*transpose* (*map* (*map f*) *xs*)) = *length* (*map* (*map f*) (*transpose xs*)) **by** *simp*

fix *i* **assume** *i* < *length* (*transpose* (*map* (*map f*) *xs*))
thus *transpose* (*map* (*map f*) *xs*) ! *i* = *map* (*map f*) (*transpose xs*) ! *i*
by (*simp add*: *nth-transpose filter-map comp-def*)
qed

43.1.31 (In)finiteness

lemma *finite-maxlen*:
finite (*M*::'*a list set*) ==> *EX n*. *ALL s*:*M*. *size s* < *n*
proof (*induct rule*: *finite.induct*)
case *emptyI* **show** *?case* **by** *simp*
next
case (*insertI M xs*)
then obtain *n* **where** ∀ *s*∈*M*. *length s* < *n* **by** *blast*
hence *ALL s*:*insert xs M*. *size s* < *max n* (*size xs*) + *1* **by** *auto*
thus *?case* **..**
qed

lemma *finite-lists-length-eq*:
assumes *finite A*
shows *finite* {*xs*. *set xs* ⊆ *A* ∧ *length xs* = *n*} (**is** *finite* (*?S n*))
proof(*induct n*)
case *0* **show** *?case* **by** *simp*
next
case (*Suc n*)
have *?S* (*Suc n*) = (⋃ *x*∈*A*. (λ*xs*. *x*#*xs*) ‘ *?S n*)
by (*auto simp*:*length-Suc-conv*)
then show *?case* **using** ‹*finite A*›
by (*auto intro*: *finite-imageI Suc*)
qed

lemma *finite-lists-length-le*:
assumes *finite A* **shows** *finite* {*xs*. *set xs* ⊆ *A* ∧ *length xs* ≤ *n*}
(**is** *finite ?S*)
proof−
have *?S* = (⋃ *n*∈{*0*..*n*}. {*xs*. *set xs* ⊆ *A* ∧ *length xs* = *n*}) **by** *auto*
thus *?thesis* **by** (*auto intro*: *finite-lists-length-eq*[*OF* ‹*finite A*›])
qed

lemma *infinite-UNIV-listI*: ~ *finite*(*UNIV*::'*a list set*)
apply(*rule notI*)
apply(*drule finite-maxlen*)
apply (*metis UNIV-I length-replicate less-not-refl*)

done

43.2 Sorting

Currently it is not shown that *sort* returns a permutation of its input because the nicest proof is via multisets, which are not yet available. Alternatively one could define a function that counts the number of occurrences of an element in a list and use that instead of multisets to state the correctness property.

context *linorder*
begin

lemma *length-insert*[*simp*] : *length* (*insort-key f x xs*) = *Suc* (*length xs*)
by (*induct xs*, *auto*)

lemma *insort-left-comm*:
insort x (*insort y xs*) = *insort y* (*insort x xs*)
by (*induct xs*) *auto*

lemma *fun-left-comm-insort*:
fun-left-comm insort
proof
qed (*fact insort-left-comm*)

lemma *sort-key-simps* [*simp*]:
sort-key f [] = []
sort-key f (*x*#*xs*) = *insort-key f x* (*sort-key f xs*)
by (*simp-all add*: *sort-key-def*)

lemma *sort-foldl-insort*:
sort xs = *foldl* (λ*ys x*. *insort x ys*) [] *xs*
by (*simp add*: *sort-key-def foldr-foldl foldl-rev insort-left-comm*)

lemma *length-sort*[*simp*]: *length* (*sort-key f xs*) = *length xs*
by (*induct xs*, *auto*)

lemma *sorted-Cons*: *sorted* (*x*#*xs*) = (*sorted xs* & (*ALL y*:*set xs*. *x* <= *y*))
apply(*induct xs arbitrary*: *x*) **apply** *simp*
by *simp* (*blast intro*: *order-trans*)

lemma *sorted-append*:
sorted (*xs*@*ys*) = (*sorted xs* & *sorted ys* & (∀ *x* ∈ *set xs*. ∀ *y* ∈ *set ys*. *x*≤*y*))
by (*induct xs*) (*auto simp add*:*sorted-Cons*)

lemma *sorted-nth-mono*:
sorted xs ⟹ *i* ≤ *j* ⟹ *j* < *length xs* ⟹ *xs*![*i* ≤ *xs*!*j*
by (*induct xs arbitrary*: *i j*) (*auto simp*:*nth-Cons*′ *sorted-Cons*)

lemma *sorted-rev-nth-mono*:
sorted (*rev xs*) $\Longrightarrow$ $i \leq j \Longrightarrow j <$ *length xs* $\Longrightarrow$ *xs*!*j* $\leq$ *xs*!*i*
using *sorted-nth-mono*[*of rev xs length xs* $-$ *j* $-$ *1 length xs* $-$ *i* $-$ *1*]
rev-nth[*of length xs* $-$ *i* $-$ *1 xs*] *rev-nth*[*of length xs* $-$ *j* $-$ *1 xs*]
by *auto*

lemma *sorted-nth-monoI*:
($\bigwedge$ *i j*. ⟦ $i \leq j$; $j <$ *length xs* ⟧ $\Longrightarrow$ *xs* ! *i* $\leq$ *xs* ! *j*) $\Longrightarrow$ *sorted xs*
proof (*induct xs*)
case (*Cons x xs*)
have *sorted xs*
proof (*rule Cons.hyps*)
fix *i j* **assume** $i \leq j$ **and** $j <$ *length xs*
with *Cons.prems*[*of Suc i Suc j*]
show *xs* ! *i* $\leq$ *xs* ! *j* **by** *auto*
qed
moreover
{
fix *y* **assume** $y \in$ *set xs*
then obtain *j* **where** $j <$ *length xs* **and** *xs* ! *j* $= y$
unfolding *in-set-conv-nth* **by** *blast*
with *Cons.prems*[*of 0 Suc j*]
have $x \leq y$
by *auto*
}
ultimately
show *?case*
unfolding *sorted-Cons* **by** *auto*
qed *simp*

lemma *sorted-equals-nth-mono*:
sorted xs = ($\forall$ *j* $<$ *length xs*. $\forall$ *i* $\leq$ *j*. *xs* ! *i* $\leq$ *xs* ! *j*)
by (*auto intro*: *sorted-nth-monoI sorted-nth-mono*)

lemma *set-insort*: *set*(*insort-key f x xs*) = *insert x* (*set xs*)
by (*induct xs*) *auto*

lemma *set-sort*[*simp*]: *set*(*sort-key f xs*) = *set xs*
by (*induct xs*) (*simp-all add*:*set-insort*)

lemma *distinct-insort*: *distinct* (*insort-key f x xs*) = ($x \notin$ *set xs* $\wedge$ *distinct xs*)
by(*induct xs*)(*auto simp*:*set-insort*)

lemma *distinct-sort*[*simp*]: *distinct* (*sort-key f xs*) = *distinct xs*
by(*induct xs*)(*simp-all add*:*distinct-insort set-sort*)

lemma *sorted-insort-key*: *sorted* (*map f* (*insort-key f x xs*)) = *sorted* (*map f xs*)
by(*induct xs*)(*auto simp*:*sorted-Cons set-insort*)

lemma *sorted-insort*: *sorted* (*insort x xs*) = *sorted xs*
using *sorted-insort-key*[**where** *f*=λ*x*. *x*] **by** *simp*

theorem *sorted-sort-key*[*simp*]: *sorted* (*map f* (*sort-key f xs*))
by(*induct xs*)(*auto simp*:*sorted-insort-key*)

theorem *sorted-sort*[*simp*]: *sorted* (*sort xs*)
by(*induct xs*)(*auto simp*:*sorted-insort*)

lemma *sorted-butlast*:
  assumes *xs* ≠ [] **and** *sorted xs*
  shows *sorted* (*butlast xs*)
proof −
  from ‹*xs* ≠ []› **obtain** *ys y* **where** *xs* = *ys* @ [*y*] **by** (*cases xs rule*: *rev-cases*)
auto
  with ‹*sorted xs*› **show** *?thesis* **by** (*simp add*: *sorted-append*)
qed

lemma *insort-not-Nil* [*simp*]:
  insort-key f a xs ≠ []
  by (*induct xs*) *simp-all*

lemma *insort-is-Cons*: ∀ *x*∈*set xs*. *f a* ≤ *f x* ⟹ *insort-key f a xs* = *a* # *xs*
by (*cases xs*) *auto*

lemma *sorted-remove1*: *sorted xs* ⟹ *sorted* (*remove1 a xs*)
by(*induct xs*)(*auto simp add*: *sorted-Cons*)

lemma *insort-key-remove1*: ⟦ *a* ∈ *set xs*; *sorted* (*map f xs*) ; *inj-on f* (*set xs*) ⟧
  ⟹ *insort-key f a* (*remove1 a xs*) = *xs*
proof (*induct xs*)
  case (*Cons x xs*)
  thus *?case*
  proof (*cases x* = *a*)
    case *False*
    hence *f x* ≠ *f a* **using** *Cons.prems* **by** *auto*
    hence *f x* < *f a* **using** *Cons.prems* **by** (*auto simp*: *sorted-Cons*)
    thus *?thesis* **using** *Cons* **by** (*auto simp*: *sorted-Cons insort-is-Cons*)
  qed (*auto simp*: *sorted-Cons insort-is-Cons*)
qed *simp*

lemma *insort-remove1*: ⟦ *a* ∈ *set xs*; *sorted xs* ⟧ ⟹ *insort a* (*remove1 a xs*) = *xs*
using *insort-key-remove1*[**where** *f*=λ*x*. *x*] **by** *simp*

lemma *sorted-remdups*[*simp*]:
  sorted l ⟹ *sorted* (*remdups l*)
by (*induct l*) (*auto simp*: *sorted-Cons*)

lemma *sorted-distinct-set-unique*:

assumes *sorted xs distinct xs sorted ys distinct ys set xs = set ys*
shows *xs = ys*
proof −
from *assms* **have** *1*: *length xs = length ys* **by** (*auto dest*!: *distinct-card*)
from *assms* **show** *?thesis*
proof(*induct rule*:*list-induct2*[*OF 1*])
case *1* **show** *?case* **by** *simp*
next
case *2* **thus** *?case* **by** (*simp add*:*sorted-Cons*)
(*metis Diff-insert-absorb antisym insertE insert-iff*)
qed
qed

lemma *map-sorted-distinct-set-unique*:
assumes *inj-on f* (*set xs* ∪ *set ys*)
assumes *sorted* (*map f xs*) *distinct* (*map f xs*)
sorted (*map f ys*) *distinct* (*map f ys*)
assumes *set xs = set ys*
shows *xs = ys*
proof −
from *assms* **have** *map f xs = map f ys*
by (*simp add*: *sorted-distinct-set-unique*)
moreover with ‹*inj-on f* (*set xs* ∪ *set ys*)› **show** *xs = ys*
by (*blast intro*: *map-inj-on*)
qed

lemma *finite-sorted-distinct-unique*:
shows *finite A* ⟹ *EX*! *xs*. *set xs = A* & *sorted xs* & *distinct xs*
apply(*drule finite-distinct-list*)
apply *clarify*
apply(*rule-tac a*=*sort xs* **in** *ex1I*)
apply (*auto simp*: *sorted-distinct-set-unique*)
done

lemma *sorted-take*:
sorted xs ⟹ *sorted* (*take n xs*)
proof (*induct xs arbitrary*: *n rule*: *sorted.induct*)
case *1* **show** *?case* **by** *simp*
next
case *2* **show** *?case* **by** (*cases n*) *simp-all*
next
case (*3 x y xs*)
then have *x* ≤ *y* **by** *simp*
show *?case* **proof** (*cases n*)
case *0* **then show** *?thesis* **by** *simp*
next
case (*Suc m*)
with *3* **have** *sorted* (*take m* (*y* # *xs*)) **by** *simp*
with *Suc* ‹*x* ≤ *y*› **show** *?thesis* **by** (*cases m*) *simp-all*

```
  qed
qed

lemma sorted-drop:
  sorted xs ⟹ sorted (drop n xs)
proof (induct xs arbitrary: n rule: sorted.induct)
  case 1 show ?case by simp
next
  case 2 show ?case by (cases n) simp-all
next
  case 3 then show ?case by (cases n) simp-all
qed

lemma sorted-dropWhile: sorted xs ⟹ sorted (dropWhile P xs)
  unfolding dropWhile-eq-drop by (rule sorted-drop)

lemma sorted-takeWhile: sorted xs ⟹ sorted (takeWhile P xs)
  apply (subst takeWhile-eq-take) by (rule sorted-take)

lemma sorted-filter:
  sorted (map f xs) ⟹ sorted (map f (filter P xs))
  by (induct xs) (simp-all add: sorted-Cons)

lemma foldr-max-sorted:
  assumes sorted (rev xs)
  shows foldr max xs y = (if xs = [] then y else max (xs ! 0) y)
using assms proof (induct xs)
  case (Cons x xs)
  moreover hence sorted (rev xs) using sorted-append by auto
  ultimately show ?case
    by (cases xs, auto simp add: sorted-append max-def)
qed simp

lemma filter-equals-takeWhile-sorted-rev:
  assumes sorted: sorted (rev (map f xs))
  shows [x ← xs. t < f x] = takeWhile (λ x. t < f x) xs
    (is filter ?P xs = ?tW)
proof (rule takeWhile-eq-filter[symmetric])
  let ?dW = dropWhile ?P xs
  fix x assume x ∈ set ?dW
  then obtain i where i: i < length ?dW and nth-i: x = ?dW ! i
    unfolding in-set-conv-nth by auto
  hence length ?tW + i < length (?tW @ ?dW)
    unfolding length-append by simp
  hence i': length (map f ?tW) + i < length (map f xs) by simp
  have (map f ?tW @ map f ?dW) ! (length (map f ?tW) + i) ≤
       (map f ?tW @ map f ?dW) ! (length (map f ?tW) + 0)
    using sorted-rev-nth-mono[OF sorted - i', of length ?tW]
    unfolding map-append[symmetric] by simp
```

hence $f\ x \leq f\ (?dW\ !\ 0)$
unfolding *nth-append-length-plus nth-i*
using *i preorder-class.le-less-trans*[*OF le0 i*] **by** *simp*
also have $\ldots \leq t$
using *hd-dropWhile*[*of ?P xs*] *le0*[*THEN preorder-class.le-less-trans*, *OF i*]
using *hd-conv-nth*[*of ?dW*] **by** *simp*
finally show $\neg\ t < f\ x$ **by** *simp*
qed

lemma *set-insort-insert*:
set (*insort-insert x xs*) = *insert x* (*set xs*)
by (*auto simp add*: *insort-insert-def set-insort*)

lemma *distinct-insort-insert*:
assumes *distinct xs*
shows *distinct* (*insort-insert x xs*)
using *assms* **by** (*induct xs*) (*auto simp add*: *insort-insert-def set-insort*)

lemma *sorted-insort-insert*:
assumes *sorted xs*
shows *sorted* (*insort-insert x xs*)
using *assms* **by** (*simp add*: *insort-insert-def sorted-insort*)

lemma *filter-insort-key-triv*:
$\neg\ P\ x \Longrightarrow$ *filter P* (*insort-key f x xs*) = *filter P xs*
by (*induct xs*) *simp-all*

lemma *filter-insort-key*:
sorted (*map f xs*) $\Longrightarrow$ *P x* $\Longrightarrow$ *filter P* (*insort-key f x xs*) = *insort-key f x* (*filter P xs*)
using *assms* **by** (*induct xs*)
(*auto simp add*: *sorted-Cons*, *subst insort-is-Cons*, *auto*)

lemma *filter-sort-key*:
filter P (*sort-key f xs*) = *sort-key f* (*filter P xs*)
by (*induct xs*) (*simp-all add*: *filter-insort-key-triv filter-insort-key*)

lemma *sorted-same* [*simp*]:
sorted [$x \leftarrow xs.\ x = f\ xs$]
proof (*induct xs arbitrary*: *f*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*)
then have *sorted* [$y \leftarrow xs\ .\ y = (\lambda xs.\ x)\ xs$] **.**
moreover from *Cons* **have** *sorted* [$y \leftarrow xs\ .\ y = (f \circ Cons\ x)\ xs$] **.**
ultimately show *?case* **by** (*simp-all add*: *sorted-Cons*)
qed

lemma *remove1-insort* [*simp*]:

remove1 x (insort x xs) = xs
by (*induct xs*) *simp-all*

end

lemma *sorted-upt*[*simp*]: *sorted*[*i..<j*]
by (*induct j*) (*simp-all add*:*sorted-append*)

lemma *sorted-upto*[*simp*]: *sorted*[*i..j*]
apply(*induct i j rule*:*upto.induct*)
apply(*subst upto.simps*)
apply(*simp add*:*sorted-Cons*)
done

43.2.1 *transpose* on sorted lists

lemma *sorted-transpose*[*simp*]:
shows *sorted* (*rev* (*map length* (*transpose xs*)))
by (*auto simp*: *sorted-equals-nth-mono rev-nth nth-transpose*
length-filter-conv-card intro: *card-mono*)

lemma *transpose-max-length*:
foldr (*λxs. max* (*length xs*)) (*transpose xs*) *0* = *length* [*x* ← *xs*. *x* ≠ []]
(**is** *?L* = *?R*)
proof (*cases transpose xs* = [])
case *False*
have *?L* = *foldr max* (*map length* (*transpose xs*)) *0*
by (*simp add*: *foldr-map comp-def*)
also have ... = *length* (*transpose xs* ! *0*)
using *False sorted-transpose* **by** (*simp add*: *foldr-max-sorted*)
finally show *?thesis*
using *False* **by** (*simp add*: *nth-transpose*)
next
case *True*
hence [*x* ← *xs*. *x* ≠ []] = []
by (*auto intro*!: *filter-False simp*: *transpose-empty*)
thus *?thesis* **by** (*simp add*: *transpose-empty True*)
qed

lemma *length-transpose-sorted*:
fixes *xs* :: *'a list list*
assumes *sorted*: *sorted* (*rev* (*map length xs*))
shows *length* (*transpose xs*) = (*if xs* = [] *then 0 else length* (*xs* ! *0*))
proof (*cases xs* = [])
case *False*
thus *?thesis*
using *foldr-max-sorted*[*OF sorted*] *False*
unfolding *length-transpose foldr-map comp-def*
by *simp*

qed *simp*

lemma *nth-nth-transpose-sorted*[*simp*]:
fixes *xs* :: *'a list list*
assumes *sorted*: *sorted* (*rev* (*map length xs*))
and *i*: *i* < *length* (*transpose xs*)
and *j*: *j* < *length* [*ys* ← *xs*. *i* < *length ys*]
shows *transpose xs* ! *i* ! *j* = *xs* ! *j* ! *i*
using *j filter-equals-takeWhile-sorted-rev*[*OF sorted*, *of i*]
nth-transpose[*OF i*] *nth-map*[*OF j*]
by (*simp add*: *takeWhile-nth*)

lemma *transpose-column-length*:
fixes *xs* :: *'a list list*
assumes *sorted*: *sorted* (*rev* (*map length xs*)) **and** *i* < *length xs*
shows *length* (*filter* (λ*ys*. *i* < *length ys*) (*transpose xs*)) = *length* (*xs* ! *i*)
proof −
have *xs* ≠ [] **using** ‹*i* < *length xs*› **by** *auto*
note *filter-equals-takeWhile-sorted-rev*[*OF sorted*, *simp*]
{ fix *j* **assume** *j* ≤ *i*
note *sorted-rev-nth-mono*[*OF sorted*, *of j i*, *simplified*, *OF this* ‹*i* < *length xs*›]
} note *sortedE* = *this*[*consumes 1*]

have {*j*. *j* < *length* (*transpose xs*) ∧ *i* < *length* (*transpose xs* ! *j*)}
= {..< *length* (*xs* ! *i*)}
proof *safe*
fix *j*
assume *j* < *length* (*transpose xs*) **and** *i* < *length* (*transpose xs* ! *j*)
with *this*(*2*) *nth-transpose*[*OF this*(*1*)]
have *i* < *length* (*takeWhile* (λ*ys*. *j* < *length ys*) *xs*) **by** *simp*
from *nth-mem*[*OF this*] *takeWhile-nth*[*OF this*]
show *j* < *length* (*xs* ! *i*) **by** (*auto dest*: *set-takeWhileD*)
next
fix *j* **assume** *j* < *length* (*xs* ! *i*)
thus *j* < *length* (*transpose xs*)
using *foldr-max-sorted*[*OF sorted*] ‹*xs* ≠ []› *sortedE*[*OF le0*]
by (*auto simp*: *length-transpose comp-def foldr-map*)

have *Suc i* ≤ *length* (*takeWhile* (λ*ys*. *j* < *length ys*) *xs*)
using ‹*i* < *length xs*› ‹*j* < *length* (*xs* ! *i*)› *less-Suc-eq-le*
by (*auto intro*!: *length-takeWhile-less-P-nth dest*!: *sortedE*)
with *nth-transpose*[*OF* ‹*j* < *length* (*transpose xs*)›]
show *i* < *length* (*transpose xs* ! *j*) **by** *simp*
qed
thus *?thesis* **by** (*simp add*: *length-filter-conv-card*)
qed

lemma *transpose-column*:
fixes *xs* :: *'a list list*

```
  assumes sorted: sorted (rev (map length xs)) and i < length xs
  shows map (λys. ys ! i) (filter (λys. i < length ys) (transpose xs))
    = xs ! i (is ?R = -)
proof (rule nth-equalityI, safe)
  show length: length ?R = length (xs ! i)
    using transpose-column-length[OF assms] by simp

  fix j assume j: j < length ?R
  note * = less-le-trans[OF this, unfolded length-map, OF length-filter-le]
  from j have j-less: j < length (xs ! i) using length by simp
  have i-less-tW: Suc i ≤ length (takeWhile (λys. Suc j ≤ length ys) xs)
  proof (rule length-takeWhile-less-P-nth)
    show Suc i ≤ length xs using ‹i < length xs› by simp
    fix k assume k < Suc i
    hence k ≤ i by auto
    with sorted-rev-nth-mono[OF sorted this] ‹i < length xs›
    have length (xs ! i) ≤ length (xs ! k) by simp
    thus Suc j ≤ length (xs ! k) using j-less by simp
  qed
  have i-less-filter: i < length [ys←xs . j < length ys]
    unfolding filter-equals-takeWhile-sorted-rev[OF sorted, of j]
    using i-less-tW by (simp-all add: Suc-le-eq)
  from j show ?R ! j = xs ! i ! j
    unfolding filter-equals-takeWhile-sorted-rev[OF sorted-transpose, of i]
   by (simp add: takeWhile-nth nth-nth-transpose-sorted[OF sorted * i-less-filter])
qed

lemma transpose-transpose:
  fixes xs :: 'a list list
  assumes sorted: sorted (rev (map length xs))
  shows transpose (transpose xs) = takeWhile (λx. x ≠ []) xs (is ?L = ?R)
proof -
  have len: length ?L = length ?R
    unfolding length-transpose transpose-max-length
    using filter-equals-takeWhile-sorted-rev[OF sorted, of 0]
    by simp

  { fix i assume i < length ?R
    with less-le-trans[OF - length-takeWhile-le[of - xs]]
    have i < length xs by simp
  } note * = this
  show ?thesis
    by (rule nth-equalityI)
     (simp-all add: len nth-transpose transpose-column[OF sorted] * takeWhile-nth)
qed

theorem transpose-rectangle:
  assumes xs = [] ⟹ n = 0
  assumes rect: ⋀ i. i < length xs ⟹ length (xs ! i) = n
```

shows *transpose xs = map* (λ *i. map* (λ *j. xs* ! *j* ! *i*) [*0..<length xs*]) [*0..<n*]
(**is** *?trans = ?map*)
proof (*rule nth-equalityI*)
have *sorted* (*rev* (*map length xs*))
by (*auto simp*: *rev-nth rect intro*!: *sorted-nth-monoI*)
from *foldr-max-sorted*[*OF this*] *assms*
show *len*: *length ?trans = length ?map*
by (*simp-all add*: *length-transpose foldr-map comp-def*)
moreover
{ fix *i* **assume** *i < n* **hence** [*ys←xs . i < length ys*] = *xs*
using *rect* **by** (*auto simp*: *in-set-conv-nth intro*!: *filter-True*) **}**
ultimately show ∀ *i < length ?trans. ?trans* ! *i = ?map* ! *i*
by (*auto simp*: *nth-transpose intro*: *nth-equalityI*)
qed

43.2.2 *sorted-list-of-set*

This function maps (finite) linearly ordered sets to sorted lists. Warning: in most cases it is not a good idea to convert from sets to lists but one should convert in the other direction (via *set*).

context *linorder*
begin

definition *sorted-list-of-set* :: *'a set* ⇒ *'a list* **where**
sorted-list-of-set = Finite-Set.fold insort []

lemma *sorted-list-of-set-empty* [*simp*]:
sorted-list-of-set {} = []
by (*simp add*: *sorted-list-of-set-def*)

lemma *sorted-list-of-set-insert* [*simp*]:
assumes *finite A*
shows *sorted-list-of-set* (*insert x A*) = *insert x* (*sorted-list-of-set* (*A* − {*x*}))
proof −
interpret *fun-left-comm insort* **by** (*fact fun-left-comm-insort*)
with *assms* **show** *?thesis* **by** (*simp add*: *sorted-list-of-set-def fold-insert-remove*)
qed

lemma *sorted-list-of-set* [*simp*]:
finite A ⟹ *set* (*sorted-list-of-set A*) = *A* ∧ *sorted* (*sorted-list-of-set A*)
∧ *distinct* (*sorted-list-of-set A*)
by (*induct A rule*: *finite-induct*) (*simp-all add*: *set-insert sorted-insort distinct-insort*)

lemma *sorted-list-of-set-sort-remdups*:
sorted-list-of-set (*set xs*) = *sort* (*remdups xs*)
proof −
interpret *fun-left-comm insort* **by** (*fact fun-left-comm-insort*)
show *?thesis* **by** (*simp add*: *sort-foldl-insort sorted-list-of-set-def fold-set-remdups*)
qed

lemma *sorted-list-of-set-remove*:
assumes *finite A*
shows *sorted-list-of-set* $(A - \{x\})$ = *remove1 x* (*sorted-list-of-set A*)
proof (*cases* $x \in A$)
case *False* **with** *assms* **have** $x \notin$ *set* (*sorted-list-of-set A*) **by** *simp*
with *False* **show** *?thesis* **by** (*simp add*: *remove1-idem*)
next
case *True* **then obtain** *B* **where** *A*: A = *insert x B* **by** (*rule Set.set-insert*)
with *assms* **show** *?thesis* **by** *simp*
qed

end

lemma *sorted-list-of-set-range* [*simp*]:
sorted-list-of-set $\{m..<n\}$ = $[m..<n]$
by (*rule sorted-distinct-set-unique*) *simp-all*

43.2.3 *lists*: the list-forming operator over sets

inductive-set
lists :: *'a set* => *'a list set*
for *A* :: *'a set*
where
Nil [*intro*!]: []: *lists A*
| *Cons* [*intro*!,*no-atp*]: [| *a*: *A*; *l*: *lists A*|] ==> *a*#*l* : *lists A*

inductive-cases *listsE* [*elim*!,*no-atp*]: *x*#*l* : *lists A*
inductive-cases *listspE* [*elim*!,*no-atp*]: *listsp A* (*x* # *l*)

lemma *listsp-mono* [*mono*]: $A \leq B$ ==> *listsp A* $\leq$ *listsp B*
by (*rule predicate1I*, *erule listsp.induct*, (*blast dest*: *predicate1D*)+)

lemmas *lists-mono* = *listsp-mono* [*to-set pred-subset-eq*]

lemma *listsp-infI*:
assumes *l*: *listsp A l* **shows** *listsp B l* ==> *listsp* (*inf A B*) *l* **using** *l*
by *induct blast*+

lemmas *lists-IntI* = *listsp-infI* [*to-set*]

lemma *listsp-inf-eq* [*simp*]: *listsp* (*inf A B*) = *inf* (*listsp A*) (*listsp B*)
proof (*rule mono-inf* [**where** *f*=*listsp*, *THEN order-antisym*])
show *mono listsp* **by** (*simp add*: *mono-def listsp-mono*)
show *inf* (*listsp A*) (*listsp B*) $\leq$ *listsp* (*inf A B*) **by** (*blast intro*!: *listsp-infI predicate1I*)
qed

lemmas *listsp-conj-eq* [*simp*] = *listsp-inf-eq* [*simplified inf-fun-eq inf-bool-eq*]

lemmas *lists-Int-eq* [*simp*] = *listsp-inf-eq* [*to-set pred-equals-eq*]

lemma *append-in-listsp-conv* [*iff*]:
(*listsp A* (*xs* @ *ys*)) = (*listsp A xs* ∧ *listsp A ys*)
by (*induct xs*) *auto*

lemmas *append-in-lists-conv* [*iff*] = *append-in-listsp-conv* [*to-set*]

lemma *in-listsp-conv-set*: (*listsp A xs*) = (∀ *x* ∈ *set xs*. *A x*)
— eliminate *listsp* in favour of *set*
by (*induct xs*) *auto*

lemmas *in-lists-conv-set* = *in-listsp-conv-set* [*to-set*]

lemma *in-listspD* [*dest*!,*no-atp*]: *listsp A xs* ==> ∀ *x*∈*set xs*. *A x*
by (*rule in-listsp-conv-set* [*THEN iffD1*])

lemmas *in-listsD* [*dest*!,*no-atp*] = *in-listspD* [*to-set*]

lemma *in-listspI* [*intro*!,*no-atp*]: ∀ *x*∈*set xs*. *A x* ==> *listsp A xs*
by (*rule in-listsp-conv-set* [*THEN iffD2*])

lemmas *in-listsI* [*intro*!,*no-atp*] = *in-listspI* [*to-set*]

lemma *lists-UNIV* [*simp*]: *lists UNIV* = *UNIV*
by *auto*

43.2.4 Inductive definition for membership

inductive *ListMem* :: *'a* ⇒ *'a list* ⇒ *bool*
where
elem: *ListMem x* (*x* # *xs*)
| *insert*: *ListMem x xs* ⟹ *ListMem x* (*y* # *xs*)

lemma *ListMem-iff*: (*ListMem x xs*) = (*x* ∈ *set xs*)
apply (*rule iffI*)
apply (*induct set*: *ListMem*)
apply *auto*
apply (*induct xs*)
apply (*auto intro*: *ListMem.intros*)
done

43.2.5 Lists as Cartesian products

set-Cons A Xs: the set of lists with head drawn from *A* and tail drawn from *Xs*.

definition
set-Cons :: *'a set* ⇒ *'a list set* ⇒ *'a list set* **where**

[*code del*]: *set-Cons A XS* = {*z*. ∃ *x xs*. *z* = *x* # *xs* ∧ *x* ∈ *A* ∧ *xs* ∈ *XS*}

lemma *set-Cons-sing-Nil* [*simp*]: *set-Cons A* {[]} = (%*x*. [*x*])‘*A*
by (*auto simp add*: *set-Cons-def*)

Yields the set of lists, all of the same length as the argument and with elements drawn from the corresponding element of the argument.

primrec
listset :: *'a set list* ⇒ *'a list set* **where**
listset [] = {[]}
| *listset* (*A* # *As*) = *set-Cons A* (*listset As*)

43.3 Relations on Lists

43.3.1 Length Lexicographic Ordering

These orderings preserve well-foundedness: shorter lists precede longer lists. These ordering are not used in dictionaries.

primrec — The lexicographic ordering for lists of the specified length
lexn :: (*'a* × *'a*) *set* ⇒ *nat* ⇒ (*'a list* × *'a list*) *set* **where**
lexn r 0 = {}
| *lexn r* (*Suc n*) = (*prod-fun* (%(*x*, *xs*). *x*#*xs*) (%(*x*, *xs*). *x*#*xs*) ‘ (*r* <∗*lex*∗> *lexn r n*)) *Int*
{(*xs*, *ys*). *length xs* = *Suc n* ∧ *length ys* = *Suc n*}

definition
lex :: (*'a* × *'a*) *set* ⇒ (*'a list* × *'a list*) *set* **where**
[*code del*]: *lex r* = (⋃ *n*. *lexn r n*) — Holds only between lists of the same length

definition
lenlex :: (*'a* × *'a*) *set* => (*'a list* × *'a list*) *set* **where**
[*code del*]: *lenlex r* = *inv-image* (*less-than* <∗*lex*∗> *lex r*) (λ*xs*. (*length xs*, *xs*))
— Compares lists by their length and then lexicographically

lemma *wf-lexn*: *wf r* ==> *wf* (*lexn r n*)
apply (*induct n*, *simp*, *simp*)
apply(*rule wf-subset*)
prefer *2* **apply** (*rule Int-lower1*)
apply(*rule wf-prod-fun-image*)
prefer *2* **apply** (*rule inj-onI*, *auto*)
done

lemma *lexn-length*:
(*xs*, *ys*) : *lexn r n* ==> *length xs* = *n* ∧ *length ys* = *n*
by (*induct n arbitrary*: *xs ys*) *auto*

lemma *wf-lex* [*intro*!]: *wf r* ==> *wf* (*lex r*)
apply (*unfold lex-def*)
apply (*rule wf-UN*)

apply (*blast intro*: *wf-lexn*, *clarify*)
apply (*rename-tac m n*)
apply (*subgoal-tac m* $\neq$ *n*)
prefer *2* **apply** *blast*
apply (*blast dest*: *lexn-length not-sym*)
done

lemma *lexn-conv*:
lexn r n =
{(*xs*,*ys*). *length xs* = *n* $\wedge$ *length ys* = *n* $\wedge$
($\exists$ *xys x y xs' ys'*. *xs*= *xys* @ *x*#*xs'* $\wedge$ *ys*= *xys* @ *y* # *ys'* $\wedge$ (*x*, *y*):*r*)}
apply (*induct n*, *simp*)
apply (*simp add*: *image-Collect lex-prod-def*, *safe*, *blast*)
apply (*rule-tac x* = *ab* # *xys* **in** *exI*, *simp*)
apply (*case-tac xys*, *simp-all*, *blast*)
done

lemma *lex-conv*:
lex r =
{(*xs*,*ys*). *length xs* = *length ys* $\wedge$
($\exists$ *xys x y xs' ys'*. *xs* = *xys* @ *x* # *xs'* $\wedge$ *ys* = *xys* @ *y* # *ys'* $\wedge$ (*x*, *y*):*r*)}
by (*force simp add*: *lex-def lexn-conv*)

lemma *wf-lenlex* [*intro*!]: *wf r* ==> *wf* (*lenlex r*)
by (*unfold lenlex-def*) *blast*

lemma *lenlex-conv*:
lenlex r = {(*xs*,*ys*). *length xs* < *length ys* |
length xs = *length ys* $\wedge$ (*xs*, *ys*) : *lex r*}
by (*simp add*: *lenlex-def Id-on-def lex-prod-def inv-image-def*)

lemma *Nil-notin-lex* [*iff*]: ([], *ys*) $\notin$ *lex r*
by (*simp add*: *lex-conv*)

lemma *Nil2-notin-lex* [*iff*]: (*xs*, []) $\notin$ *lex r*
by (*simp add*:*lex-conv*)

lemma *Cons-in-lex* [*simp*]:
((*x* # *xs*, *y* # *ys*) : *lex r*) =
((*x*, *y*) : *r* $\wedge$ *length xs* = *length ys* | *x* = *y* $\wedge$ (*xs*, *ys*) : *lex r*)
apply (*simp add*: *lex-conv*)
apply (*rule iffI*)
prefer *2* **apply** (*blast intro*: *Cons-eq-appendI*, *clarify*)
apply (*case-tac xys*, *simp*, *simp*)
apply *blast*
done

43.3.2 Lexicographic Ordering

Classical lexicographic ordering on lists, ie. "a" ¡ "ab" ¡ "b". This ordering does *not* preserve well-foundedness. Author: N. Voelker, March 2005.

definition
lexord :: $('a \times 'a)$ *set* $\Rightarrow$ $('a$ *list* $\times$ $'a$ *list*$)$ *set* **where**
[*code del*]: *lexord r* = $\{(x,y)\ .\ \exists\ a\ v.\ y = x\ @\ a\ \#\ v\ \vee$
$(\exists\ u\ a\ b\ v\ w.\ (a,b) \in r \wedge x = u\ @\ (a\ \#\ v) \wedge y = u\ @\ (b\ \#\ w))\}$

lemma *lexord-Nil-left*[*simp*]: $([],y) \in$ *lexord r* = $(\exists\ a\ x.\ y = a\ \#\ x)$
by (*unfold lexord-def*, *induct-tac y*, *auto*)

lemma *lexord-Nil-right*[*simp*]: $(x,[]) \notin$ *lexord r*
by (*unfold lexord-def*, *induct-tac x*, *auto*)

lemma *lexord-cons-cons*[*simp*]:
$((a\ \#\ x,\ b\ \#\ y) \in$ *lexord r*$) = ((a,b) \in r \mid (a = b\ \&\ (x,y) \in$ *lexord r*$))$
apply (*unfold lexord-def*, *safe*, *simp-all*)
apply (*case-tac u*, *simp*, *simp*)
apply (*case-tac u*, *simp*, *clarsimp*, *blast*, *blast*, *clarsimp*)
apply (*erule-tac x*=*b* # *u* **in** *allE*)
by *force*

lemmas *lexord-simps* = *lexord-Nil-left lexord-Nil-right lexord-cons-cons*

lemma *lexord-append-rightI*: $\exists\ b\ z.\ y = b\ \#\ z \Longrightarrow (x,\ x\ @\ y) \in$ *lexord r*
by (*induct-tac x*, *auto*)

lemma *lexord-append-left-rightI*:
$(a,b) \in r \Longrightarrow (u\ @\ a\ \#\ x,\ u\ @\ b\ \#\ y) \in$ *lexord r*
by (*induct-tac u*, *auto*)

lemma *lexord-append-leftI*: $(u,v) \in$ *lexord r* $\Longrightarrow (x\ @\ u,\ x\ @\ v) \in$ *lexord r*
by (*induct x*, *auto*)

lemma *lexord-append-leftD*:
$[\![\ (x\ @\ u,\ x\ @\ v) \in$ *lexord r*; $(!\ a.\ (a,a) \notin r)\ ]\!] \Longrightarrow (u,v) \in$ *lexord r*
by (*erule rev-mp*, *induct-tac x*, *auto*)

lemma *lexord-take-index-conv*:
$((x,y) :$ *lexord r*$) =$
$((length\ x < length\ y \wedge take\ (length\ x)\ y = x) \vee$
$(\exists\, i.\ i < min(length\ x)(length\ y)\ \&\ take\ i\ x = take\ i\ y\ \&\ (x!i,y!i) \in r))$
apply (*unfold lexord-def Let-def*, *clarsimp*)
apply (*rule-tac f* = (% *a b*. *a* $\vee$ *b*) **in** *arg-cong2*)
apply *auto*
apply (*rule-tac x*=*hd* (*drop* (*length x*) *y*) **in** *exI*)
apply (*rule-tac x*=*tl* (*drop* (*length x*) *y*) **in** *exI*)
apply (*erule subst*, *simp add*: *min-def*)

apply (*rule-tac x =length u* **in** *exI*, *simp*)
apply (*rule-tac x =take i x* **in** *exI*)
apply (*rule-tac x =x ! i* **in** *exI*)
apply (*rule-tac x =y ! i* **in** *exI*, *safe*)
apply (*rule-tac x=drop* (*Suc i*) *x* **in** *exI*)
apply (*drule sym*, *simp add*: *drop-Suc-conv-tl*)
apply (*rule-tac x=drop* (*Suc i*) *y* **in** *exI*)
by (*simp add*: *drop-Suc-conv-tl*)

— lexord is extension of partial ordering List.lex
lemma *lexord-lex*: (*x*,*y*) ∈ *lex r* = ((*x*,*y*) ∈ *lexord r* ∧ *length x* = *length y*)
apply (*rule-tac x* = *y* **in** *spec*)
apply (*induct-tac x*, *clarsimp*)
by (*clarify*, *case-tac x*, *simp*, *force*)

lemma *lexord-irreflexive*: (! *x*. (*x*,*x*) ∉ *r*) ⟹ (*y*,*y*) ∉ *lexord r*
by (*induct y*, *auto*)

lemma *lexord-trans*:
⟦ (*x*, *y*) ∈ *lexord r*; (*y*, *z*) ∈ *lexord r*; *trans r* ⟧ ⟹ (*x*, *z*) ∈ *lexord r*
apply (*erule rev-mp*)+
apply (*rule-tac x* = *x* **in** *spec*)
apply (*rule-tac x* = *z* **in** *spec*)
apply (*induct-tac y*, *simp*, *clarify*)
apply (*case-tac xa*, *erule ssubst*)
apply (*erule allE*, *erule allE*) — avoid simp recursion
apply (*case-tac x*, *simp*, *simp*)
apply (*case-tac x*, *erule allE*, *erule allE*, *simp*)
apply (*erule-tac x* = *listb* **in** *allE*)
apply (*erule-tac x* = *lista* **in** *allE*, *simp*)
apply (*unfold trans-def*)
by *blast*

lemma *lexord-transI*: *trans r* ⟹ *trans* (*lexord r*)
by (*rule transI*, *drule lexord-trans*, *blast*)

lemma *lexord-linear*: (! *a b*. (*a*,*b*)∈ *r* | *a* = *b* | (*b*,*a*) ∈ *r*) ⟹ (*x*,*y*) : *lexord r* | *x* = *y* | (*y*,*x*) : *lexord r*
apply (*rule-tac x* = *y* **in** *spec*)
apply (*induct-tac x*, *rule allI*)
apply (*case-tac x*, *simp*, *simp*)
apply (*rule allI*, *case-tac x*, *simp*, *simp*)
by *blast*

43.4 Lexicographic combination of measure functions

These are useful for termination proofs

definition
measures fs = *inv-image* (*lex less-than*) (%*a*. *map* (%*f*. *f a*) *fs*)

lemma *wf-measures*[*recdef-wf*, *simp*]: *wf* (*measures fs*)
unfolding *measures-def*
by *blast*

lemma *in-measures*[*simp*]:
(*x*, *y*) ∈ *measures* [] = *False*
(*x*, *y*) ∈ *measures* (*f* # *fs*)
= (*f x* < *f y* ∨ (*f x* = *f y* ∧ (*x*, *y*) ∈ *measures fs*))
unfolding *measures-def*
by *auto*

lemma *measures-less*: *f x* < *f y* ==> (*x*, *y*) ∈ *measures* (*f*#*fs*)
by *simp*

lemma *measures-lesseq*: *f x* <= *f y* ==> (*x*, *y*) ∈ *measures fs* ==> (*x*, *y*) ∈ *measures* (*f*#*fs*)
by *auto*

43.4.1 Lifting a Relation on List Elements to the Lists

inductive-set
listrel :: ('*a* ∗ '*a*)*set* => ('*a list* ∗ '*a list*)*set*
for *r* :: ('*a* ∗ '*a*)*set*
where
Nil: ([],[]) ∈ *listrel r*
| *Cons*: [| (*x*,*y*) ∈ *r*; (*xs*,*ys*) ∈ *listrel r* |] ==> (*x*#*xs*, *y*#*ys*) ∈ *listrel r*

inductive-cases *listrel-Nil1* [*elim*!]: ([],*xs*) ∈ *listrel r*
inductive-cases *listrel-Nil2* [*elim*!]: (*xs*,[]) ∈ *listrel r*
inductive-cases *listrel-Cons1* [*elim*!]: (*y*#*ys*,*xs*) ∈ *listrel r*
inductive-cases *listrel-Cons2* [*elim*!]: (*xs*,*y*#*ys*) ∈ *listrel r*

lemma *listrel-mono*: *r* ⊆ *s* ⟹ *listrel r* ⊆ *listrel s*
apply *clarify*
apply (*erule listrel.induct*)
apply (*blast intro*: *listrel.intros*)+
done

lemma *listrel-subset*: *r* ⊆ *A* × *A* ⟹ *listrel r* ⊆ *lists A* × *lists A*
apply *clarify*
apply (*erule listrel.induct*, *auto*)
done

lemma *listrel-refl-on*: *refl-on A r* ⟹ *refl-on* (*lists A*) (*listrel r*)
apply (*simp add*: *refl-on-def listrel-subset Ball-def*)
apply (*rule allI*)
apply (*induct-tac x*)

apply (*auto intro*: *listrel.intros*)
done

lemma *listrel-sym*: *sym r* ⟹ *sym* (*listrel r*)
apply (*auto simp add*: *sym-def*)
apply (*erule listrel.induct*)
apply (*blast intro*: *listrel.intros*)+
done

lemma *listrel-trans*: *trans r* ⟹ *trans* (*listrel r*)
apply (*simp add*: *trans-def*)
apply (*intro allI*)
apply (*rule impI*)
apply (*erule listrel.induct*)
apply (*blast intro*: *listrel.intros*)+
done

theorem *equiv-listrel*: *equiv A r* ⟹ *equiv* (*lists A*) (*listrel r*)
by (*simp add*: *equiv-def listrel-refl-on listrel-sym listrel-trans*)

lemma *listrel-Nil* [*simp*]: *listrel r `` {[]} = {[]}*
by (*blast intro*: *listrel.intros*)

lemma *listrel-Cons*:
listrel r `` {x#xs} = set-Cons* (*r``{x}) (*listrel r `` {xs}*)
by (*auto simp add*: *set-Cons-def intro*: *listrel.intros*)

43.5 Size function

lemma [*measure-function*]: *is-measure f* ⟹ *is-measure* (*list-size f*)
by (*rule is-measure-trivial*)

lemma [*measure-function*]: *is-measure f* ⟹ *is-measure* (*option-size f*)
by (*rule is-measure-trivial*)

lemma *list-size-estimation*[*termination-simp*]:
x ∈ *set xs* ⟹ *y* < *f x* ⟹ *y* < *list-size f xs*
by (*induct xs*) *auto*

lemma *list-size-estimation'*[*termination-simp*]:
x ∈ *set xs* ⟹ *y* ≤ *f x* ⟹ *y* ≤ *list-size f xs*
by (*induct xs*) *auto*

lemma *list-size-map*[*simp*]: *list-size f* (*map g xs*) = *list-size* (*f o g*) *xs*
by (*induct xs*) *auto*

lemma *list-size-pointwise*[*termination-simp*]:
(⋀*x*. *x* ∈ *set xs* ⟹ *f x* < *g x*) ⟹ *list-size f xs* ≤ *list-size g xs*
by (*induct xs*) *force*+

43.6 Transfer

definition
embed-list :: *nat list* ⇒ *int list*
where
embed-list l = *map int l*

definition
nat-list :: *int list* ⇒ *bool*
where
nat-list l = *nat-set* (*set l*)

definition
return-list :: *int list* ⇒ *nat list*
where
return-list l = *map nat l*

lemma *transfer-nat-int-list-return-embed*: *nat-list l* ⟶
embed-list (*return-list l*) = *l*
unfolding *embed-list-def return-list-def nat-list-def nat-set-def*
apply (*induct l*)
apply *auto*
done

lemma *transfer-nat-int-list-functions*:
l @ *m* = *return-list* (*embed-list l* @ *embed-list m*)
[] = *return-list* []
unfolding *return-list-def embed-list-def*
apply *auto*
apply (*induct l*, *auto*)
apply (*induct m*, *auto*)
done

43.7 Code generator

43.7.1 Setup

use *Tools/list-code.ML*

code-type *list*
(*SML* - *list*)
(*OCaml* - *list*)
(*Haskell* ![(-)])
(*Scala List*[(-)])

code-const *Nil*
(*SML* [])
(*OCaml* [])
(*Haskell* [])
(*Scala Nil*)

```
code-instance list :: eq
  (Haskell -)

code-const eq-class.eq :: 'a::eq list ⇒ 'a list ⇒ bool
  (Haskell infixl 4 ==)

code-reserved SML
  list

code-reserved OCaml
  list

types-code
  list (- list)
attach (term-of) ⟪
fun term-of-list f T = HOLogic.mk-list T o map f;
⟫
attach (test) ⟪
fun gen-list' aG aT i j = frequency
  [(i, fn () =>
      let
        val (x, t) = aG j;
        val (xs, ts) = gen-list' aG aT (i-1) j
      in (x :: xs, fn () => HOLogic.cons-const aT $ t () $ ts ()) end),
   (1, fn () => ([], fn () => HOLogic.nil-const aT))] ()
and gen-list aG aT i = gen-list' aG aT i i;
⟫

consts-code Cons ((- ::/ -))

setup ⟪
let
  fun list-codegen thy defs dep thyname b t gr =
    let
      val ts = HOLogic.dest-list t;
      val (-, gr') = Codegen.invoke-tycodegen thy defs dep thyname false
        (fastype-of t) gr;
      val (ps, gr'') = fold-map
        (Codegen.invoke-codegen thy defs dep thyname false) ts gr'
    in SOME (Pretty.list [ ] ps, gr'') end handle TERM - => NONE;
in
  fold (List-Code.add-literal-list) [SML, OCaml, Haskell, Scala]
  #> Codegen.add-codegen list-codegen list-codegen
end
⟫
```

43.7.2 Generation of efficient code

definition *member* :: *'a list* ⇒ *'a* ⇒ *bool* **where**
mem-iff [*code-post*]: *member xs x* ⟷ *x* ∈ *set xs*

primrec
null:: *'a list* ⇒ *bool*
where
null [] = *True*
| *null* (*x*#*xs*) = *False*

primrec
list-inter :: *'a list* ⇒ *'a list* ⇒ *'a list*
where
list-inter [] *bs* = []
| *list-inter* (*a*#*as*) *bs* =
(*if a* ∈ *set bs then a* # *list-inter as bs else list-inter as bs*)

primrec
list-all :: (*'a* ⇒ *bool*) ⇒ (*'a list* ⇒ *bool*)
where
list-all P [] = *True*
| *list-all P* (*x*#*xs*) = (*P x* ∧ *list-all P xs*)

primrec
list-ex :: (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *bool*
where
list-ex P [] = *False*
| *list-ex P* (*x*#*xs*) = (*P x* ∨ *list-ex P xs*)

primrec
filtermap :: (*'a* ⇒ *'b option*) ⇒ *'a list* ⇒ *'b list*
where
filtermap f [] = []
| *filtermap f* (*x*#*xs*) =
(*case f x of None* ⇒ *filtermap f xs*
| *Some y* ⇒ *y* # *filtermap f xs*)

primrec
map-filter :: (*'a* ⇒ *'b*) ⇒ (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *'b list*
where
map-filter f P [] = []
| *map-filter f P* (*x*#*xs*) =
(*if P x then f x* # *map-filter f P xs else map-filter f P xs*)

primrec
length-unique :: *'a list* ⇒ *nat*
where
length-unique [] = *0*
| *length-unique* (*x*#*xs*) =

(*if* $x \in$ *set xs then length-unique xs else Suc* (*length-unique xs*))

primrec
concat-map :: (*'a => 'b list*) => *'a list => 'b list*
where
concat-map f [] = []
| *concat-map f* (*x#xs*) = *f x* @ *concat-map f xs*

Only use *member* for generating executable code. Otherwise use $x \in$ *set xs* instead — it is much easier to reason about. The same is true for *list-all* and *list-ex*: write $\forall x{\in}$*set xs* and $\exists x{\in}$*set xs* instead because the HOL quantifiers are aleady known to the automatic provers. In fact, the declarations in the code subsection make sure that $\in$, $\forall x{\in}$*set xs* and $\exists x{\in}$*set xs* are implemented efficiently.

Efficient emptyness check is implemented by *null*.

The functions *filtermap* and *map-filter* are just there to generate efficient code. Do not use them for modelling and proving.

lemma *rev-foldl-cons* [*code*]:
rev xs = *foldl* (*λxs x. x # xs*) [] *xs*
proof (*induct xs*)
case *Nil* **then show** *?case* **by** *simp*
next
case *Cons*
{
fix *x xs ys*
have *foldl* (*λxs x. x # xs*) *ys xs* @ [*x*]
= *foldl* (*λxs x. x # xs*) (*ys* @ [*x*]) *xs*
by (*induct xs arbitrary*: *ys*) *auto*
}
note *aux* = *this*
show *?case* **by** (*induct xs*) (*auto simp add*: *Cons aux*)
qed

lemmas *in-set-code* [*code-unfold*] = *mem-iff* [*symmetric*]

lemma *member-simps* [*simp*, *code*]:
member (*x # xs*) *y* ⟷ *x* = *y* ∨ *member xs y*
member [] *y* ⟷ *False*
by (*auto simp add*: *mem-iff*)

lemma *member-set*:
member = *set*
by (*simp add*: *expand-fun-eq mem-iff mem-def*)

abbreviation (*input*) *mem* :: *'a* ⇒ *'a list* ⇒ *bool* (**infixl** *mem 55*) **where**
x mem xs ≡ *member xs x* — for backward compatibility

lemma *empty-null*:

xs = [] ⟷ *null xs*
by (*cases xs*) *simp-all*

lemma [*code-unfold*]:
eq-class.eq xs [] ⟷ *null xs*
by (*simp add*: *eq empty-null*)

lemmas *null-empty* [*code-post*] =
empty-null [*symmetric*]

lemma *list-inter-conv*:
set (*list-inter xs ys*) = *set xs* ∩ *set ys*
by (*induct xs*) *auto*

lemma *list-all-iff* [*code-post*]:
list-all P xs ⟷ (∀ *x* ∈ *set xs*. *P x*)
by (*induct xs*) *auto*

lemmas *list-ball-code* [*code-unfold*] = *list-all-iff* [*symmetric*]

lemma *list-all-append* [*simp*]:
list-all P (*xs* @ *ys*) ⟷ (*list-all P xs* ∧ *list-all P ys*)
by (*induct xs*) *auto*

lemma *list-all-rev* [*simp*]:
list-all P (*rev xs*) ⟷ *list-all P xs*
by (*simp add*: *list-all-iff*)

lemma *list-all-length*:
list-all P xs ⟷ (∀ *n* < *length xs*. *P* (*xs* ! *n*))
unfolding *list-all-iff* **by** (*auto intro*: *all-nth-imp-all-set*)

lemma *list-ex-iff* [*code-post*]:
list-ex P xs ⟷ (∃ *x* ∈ *set xs*. *P x*)
by (*induct xs*) *simp-all*

lemmas *list-bex-code* [*code-unfold*] =
list-ex-iff [*symmetric*]

lemma *list-ex-length*:
list-ex P xs ⟷ (∃ *n* < *length xs*. *P* (*xs* ! *n*))
unfolding *list-ex-iff set-conv-nth* **by** *auto*

lemma *filtermap-conv*:
filtermap f xs = *map* (λ*x*. *the* (*f x*)) (*filter* (λ*x*. *f x* ≠ *None*) *xs*)
by (*induct xs*) (*simp-all split*: *option.split*)

lemma *map-filter-conv* [*simp*]:
map-filter f P xs = *map f* (*filter P xs*)

by (*induct xs*) *auto*

lemma *length-remdups-length-unique* [*code-unfold*]:
  length (*remdups xs*) = *length-unique xs*
by (*induct xs*) *simp-all*

lemma *concat-map-code*[*code-unfold*]:
  concat(*map f xs*) = *concat-map f xs*
by (*induct xs*) *simp-all*

declare *INFI-def* [*code-unfold*]
declare *SUPR-def* [*code-unfold*]

declare *set-map* [*symmetric*, *code-unfold*]

hide-const (**open**) *length-unique*

Code for bounded quantification and summation over nats.

lemma *atMost-upto* [*code-unfold*]:
  {..*n*} = *set* [*0*..<*Suc n*]
by *auto*

lemma *atLeast-upt* [*code-unfold*]:
  {..<*n*} = *set* [*0*..<*n*]
by *auto*

lemma *greaterThanLessThan-upt* [*code-unfold*]:
  {*n*<..<*m*} = *set* [*Suc n*..<*m*]
by *auto*

lemmas *atLeastLessThan-upt* [*code-unfold*] = *set-upt* [*symmetric*]

lemma *greaterThanAtMost-upt* [*code-unfold*]:
  {*n*<..*m*} = *set* [*Suc n*..<*Suc m*]
by *auto*

lemma *atLeastAtMost-upt* [*code-unfold*]:
  {*n*..*m*} = *set* [*n*..<*Suc m*]
by *auto*

lemma *all-nat-less-eq* [*code-unfold*]:
  (∀ *m*<*n*::*nat*. *P m*) ⟷ (∀ *m* ∈ {*0*..<*n*}. *P m*)
by *auto*

lemma *ex-nat-less-eq* [*code-unfold*]:
  (∃ *m*<*n*::*nat*. *P m*) ⟷ (∃ *m* ∈ {*0*..<*n*}. *P m*)
by *auto*

lemma *all-nat-less* [*code-unfold*]:

$(\forall\, m{\leq}n{::}nat.\ P\ m) \longleftrightarrow (\forall\, m \in \{0..n\}.\ P\ m)$
by *auto*

lemma *ex-nat-less* [*code-unfold*]:
$(\exists\, m{\leq}n{::}nat.\ P\ m) \longleftrightarrow (\exists\, m \in \{0..n\}.\ P\ m)$
by *auto*

lemma *setsum-set-distinct-conv-listsum*:
distinct xs $\Longrightarrow$ *setsum f* (*set xs*) = *listsum* (*map f xs*)
by (*induct xs*) *simp-all*

lemma *setsum-set-upt-conv-listsum* [*code-unfold*]:
setsum f (*set* [*m*..<*n*]) = *listsum* (*map f* [*m*..<*n*])
by (*rule setsum-set-distinct-conv-listsum*) *simp*

General equivalence between *listsum* and *setsum*

lemma *listsum-setsum-nth*:
listsum xs = ($\sum$ *i* = *0* ..< *length xs*. *xs* ! *i*)
using *setsum-set-upt-conv-listsum*[*of op* ! *xs 0 length xs*]
by (*simp add*: *map-nth*)

Code for summation over ints.

lemma *greaterThanLessThan-upto* [*code-unfold*]:
{*i*<..<*j*::*int*} = *set* [*i*+*1*..*j* − *1*]
by *auto*

lemma *atLeastLessThan-upto* [*code-unfold*]:
{*i*..<*j*::*int*} = *set* [*i*..*j* − *1*]
by *auto*

lemma *greaterThanAtMost-upto* [*code-unfold*]:
{*i*<..*j*::*int*} = *set* [*i*+*1*..*j*]
by *auto*

lemmas *atLeastAtMost-upto* [*code-unfold*] = *set-upto*[*symmetric*]

lemma *setsum-set-upto-conv-listsum* [*code-unfold*]:
setsum f (*set* [*i*..*j*::*int*]) = *listsum* (*map f* [*i*..*j*])
by (*rule setsum-set-distinct-conv-listsum*) *simp*

Optimized code for $\forall\, i{\in}\{a..b{::}int\}$ and $\forall\, n{:}\{a..{<}b{::}nat\}$ and similiarly for $\exists$.

function *all-from-to-nat* :: (*nat* ⇒ *bool*) ⇒ *nat* ⇒ *nat* ⇒ *bool* **where**
all-from-to-nat P i j =
(*if i* < *j then if P i then all-from-to-nat P* (*i*+*1*) *j else False*
else True)
by *auto*
termination
by (*relation measure*(%(*P*,*i*,*j*). *j* − *i*)) *auto*

declare *all-from-to-nat.simps*[*simp del*]

lemma *all-from-to-nat-iff-ball*:
all-from-to-nat P i j = (*ALL n* : {*i* ..< *j*}. *P n*)
proof(*induct P i j rule*:*all-from-to-nat.induct*)
case (*1 P i j*)
let *?yes* = *i* < *j* & *P i*
show *?case*
proof (*cases*)
assume *?yes*
hence *all-from-to-nat P i j* = (*P i* & *all-from-to-nat P* (*i*+*1*) *j*)
by(*simp add*: *all-from-to-nat.simps*)
also have ... = (*P i* & (*ALL n* : {*i*+*1* ..< *j*}. *P n*)) **using** ‹*?yes*› *1* **by** *simp*
also have ... = (*ALL n* : {*i* ..< *j*}. *P n*) (**is** *?L* = *?R*)
proof
assume *L*: *?L*
show *?R*
proof *clarify*
fix *n* **assume** *n*: *n* : {*i*..<*j*}
show *P n*
proof *cases*
assume *n* = *i* **thus** *P n* **using** *L* **by** *simp*
next
assume *n* ~= *i*
hence *i*+*1* <= *n* **using** *n* **by** *auto*
thus *P n* **using** *L n* **by** *simp*
qed
qed
next
assume *R*: *?R* **thus** *?L* **using** ‹*?yes*› *1* **by** *auto*
qed
finally show *?thesis* **.**
next
assume ~*?yes* **thus** *?thesis* **by**(*auto simp add*: *all-from-to-nat.simps*)
qed
qed

lemma *list-all-iff-all-from-to-nat*[*code-unfold*]:
list-all P [*i*..<*j*] = *all-from-to-nat P i j*
by(*simp add*: *all-from-to-nat-iff-ball list-all-iff*)

lemma *list-ex-iff-not-all-from-to-not-nat*[*code-unfold*]:
list-ex P [*i*..<*j*] = (~*all-from-to-nat* (%*x*. ~*P x*) *i j*)
by(*simp add*: *all-from-to-nat-iff-ball list-ex-iff*)

function *all-from-to-int* :: (*int* ⇒ *bool*) ⇒ *int* ⇒ *int* ⇒ *bool* **where**

```
all-from-to-int P i j =
 (if i <= j then if P i then all-from-to-int P (i+1) j else False
  else True)
by auto
termination
by (relation measure(%(P,i,j). nat(j - i + 1))) auto

declare all-from-to-int.simps[simp del]

lemma all-from-to-int-iff-ball:
  all-from-to-int P i j = (ALL n : {i .. j}. P n)
proof(induct P i j rule:all-from-to-int.induct)
  case (1 P i j)
  let ?yes = i <= j & P i
  show ?case
  proof (cases)
    assume ?yes
    hence all-from-to-int P i j = (P i & all-from-to-int P (i+1) j)
      by(simp add: all-from-to-int.simps)
    also have ... = (P i & (ALL n : {i+1 .. j}. P n)) using ‹?yes› 1 by simp
    also have ... = (ALL n : {i .. j}. P n) (is ?L = ?R)
    proof
      assume L: ?L
      show ?R
      proof clarify
        fix n assume n: n : {i..j}
        show P n
        proof cases
          assume n = i thus P n using L by simp
        next
          assume n ~= i
          hence i+1 <= n using n by auto
          thus P n using L n by simp
        qed
      qed
    next
      assume R: ?R thus ?L using ‹?yes› 1 by auto
    qed
    finally show ?thesis .
  next
    assume ~?yes thus ?thesis by(auto simp add: all-from-to-int.simps)
  qed
qed

lemma list-all-iff-all-from-to-int[code-unfold]:
  list-all P [i..j] = all-from-to-int P i j
by(simp add: all-from-to-int-iff-ball list-all-iff)

lemma list-ex-iff-not-all-from-to-not-int[code-unfold]:
```

list-ex P [i..j] = (~ *all-from-to-int* (%*x*. ~*P x*) *i j*)
by(*simp add*: *all-from-to-int-iff-ball list-ex-iff*)

end

44 Random: A HOL random engine

theory *Random*
imports *Code-Numeral List*
begin

notation *fcomp* (**infixl** *o*> *60*)
notation *scomp* (**infixl** *o*→ *60*)

44.1 Auxiliary functions

fun *log* :: *code-numeral* ⇒ *code-numeral* ⇒ *code-numeral* **where**
log b i = (*if b* ≤ *1* ∨ *i* < *b then 1 else 1* + *log b* (*i div b*))

definition *inc-shift* :: *code-numeral* ⇒ *code-numeral* ⇒ *code-numeral* **where**
inc-shift v k = (*if v* = *k then 1 else k* + *1*)

definition *minus-shift* :: *code-numeral* ⇒ *code-numeral* ⇒ *code-numeral* ⇒ *code-numeral* **where**
minus-shift r k l = (*if k* < *l then r* + *k* − *l else k* − *l*)

44.2 Random seeds

types *seed* = *code-numeral* × *code-numeral*

primrec *next* :: *seed* ⇒ *code-numeral* × *seed* **where**
next (*v*, *w*) = (*let*
k = *v div 53668*;
v' = *minus-shift 2147483563* ((*v mod 53668*) * *40014*) (*k* * *12211*);
l = *w div 52774*;
w' = *minus-shift 2147483399* ((*w mod 52774*) * *40692*) (*l* * *3791*);
z = *minus-shift 2147483562 v'* (*w'* + *1*) + *1*
in (*z*, (*v'*, *w'*)))

definition *split-seed* :: *seed* ⇒ *seed* × *seed* **where**
split-seed s = (*let*
(*v*, *w*) = *s*;
(*v'*, *w'*) = *snd* (*next s*);
v'' = *inc-shift 2147483562 v*;
w'' = *inc-shift 2147483398 w*
in ((*v''*, *w'*), (*v'*, *w''*)))

44.3 Base selectors

fun *iterate* :: *code-numeral* ⇒ (*'b* ⇒ *'a* ⇒ *'b* × *'a*) ⇒ *'b* ⇒ *'a* ⇒ *'b* × *'a* **where**
iterate k f x = (*if k* = *0 then Pair x else f x o*→ *iterate* (*k* − *1*) *f*)

definition *range* :: *code-numeral* ⇒ *seed* ⇒ *code-numeral* × *seed* **where**
range k = *iterate* (*log 2147483561 k*)
(λ*l*. *next o*→ (λ*v*. *Pair* (*v* + *l* ∗ *2147483561*))) *1*
o→ (λ*v*. *Pair* (*v mod k*))

lemma *range*:
k > *0* ⟹ *fst* (*range k s*) < *k*
by (*simp add*: *range-def scomp-apply split-def del*: *log.simps iterate.simps*)

definition *select* :: *'a list* ⇒ *seed* ⇒ *'a* × *seed* **where**
select xs = *range* (*Code-Numeral.of-nat* (*length xs*))
o→ (λ*k*. *Pair* (*nth xs* (*Code-Numeral.nat-of k*)))

lemma *select*:
assumes *xs* ≠ []
shows *fst* (*select xs s*) ∈ *set xs*
proof −
from *assms* **have** *Code-Numeral.of-nat* (*length xs*) > *0* **by** *simp*
with *range* **have**
fst (*range* (*Code-Numeral.of-nat* (*length xs*)) *s*) < *Code-Numeral.of-nat* (*length xs*) **by** *best*
then have
Code-Numeral.nat-of (*fst* (*range* (*Code-Numeral.of-nat* (*length xs*)) *s*)) < *length xs* **by** *simp*
then show *?thesis*
by (*simp add*: *scomp-apply split-beta select-def*)
qed

primrec *pick* :: (*code-numeral* × *'a*) *list* ⇒ *code-numeral* ⇒ *'a* **where**
pick (*x* # *xs*) *i* = (*if i* < *fst x then snd x else pick xs* (*i* − *fst x*))

lemma *pick-member*:
i < *listsum* (*map fst xs*) ⟹ *pick xs i* ∈ *set* (*map snd xs*)
by (*induct xs arbitrary*: *i*) *simp-all*

lemma *pick-drop-zero*:
pick (*filter* (λ(*k*, -). *k* > *0*) *xs*) = *pick xs*
by (*induct xs*) (*auto simp add*: *expand-fun-eq*)

lemma *pick-same*:
l < *length xs* ⟹ *Random.pick* (*map* (*Pair 1*) *xs*) (*Code-Numeral.of-nat l*) = *nth xs l*
proof (*induct xs arbitrary*: *l*)
case *Nil* **then show** *?case* **by** *simp*
next

case (*Cons x xs*) **then show** *?case* **by** (*cases l*) *simp-all*
qed

definition *select-weight* :: (*code-numeral* × *'a*) *list* ⇒ *seed* ⇒ *'a* × *seed* **where**
select-weight xs = *range* (*listsum* (*map fst xs*))
o→ (λ*k*. *Pair* (*pick xs k*))

lemma *select-weight-member*:
assumes *0* < *listsum* (*map fst xs*)
shows *fst* (*select-weight xs s*) ∈ *set* (*map snd xs*)
proof −
from *range assms*
have *fst* (*range* (*listsum* (*map fst xs*)) *s*) < *listsum* (*map fst xs*) **.**
with *pick-member*
have *pick xs* (*fst* (*range* (*listsum* (*map fst xs*)) *s*)) ∈ *set* (*map snd xs*) **.**
then show *?thesis* **by** (*simp add*: *select-weight-def scomp-def split-def*)
qed

lemma *select-weight-cons-zero*:
select-weight ((*0*, *x*) # *xs*) = *select-weight xs*
by (*simp add*: *select-weight-def*)

lemma *select-weigth-drop-zero*:
select-weight (*filter* (λ(*k*, -). *k* > *0*) *xs*) = *select-weight xs*
proof −
have *listsum* (*map fst* [(*k*, -)←*xs* . *0* < *k*]) = *listsum* (*map fst xs*)
by (*induct xs*) *auto*
then show *?thesis* **by** (*simp only*: *select-weight-def pick-drop-zero*)
qed

lemma *select-weigth-select*:
assumes *xs* ≠ []
shows *select-weight* (*map* (*Pair 1*) *xs*) = *select xs*
proof −
have *less*: ⋀*s*. *fst* (*range* (*Code-Numeral.of-nat* (*length xs*)) *s*) < *Code-Numeral.of-nat* (*length xs*)
using *assms* **by** (*intro range*) *simp*
moreover have *listsum* (*map fst* (*map* (*Pair 1*) *xs*)) = *Code-Numeral.of-nat* (*length xs*)
by (*induct xs*) *simp-all*
ultimately show *?thesis*
by (*auto simp add*: *select-weight-def select-def scomp-def split-def*
expand-fun-eq pick-same [*symmetric*])
qed

44.4 *ML* interface

code-reflect *Random-Engine*
functions *range select select-weight*

ML ⟪
structure Random-Engine =
struct

open Random-Engine;

*type seed = int * int*;

local

val seed = Unsynchronized.ref
(let
val now = Time.toMilliseconds (Time.now ());
val (q, s1) = IntInf.divMod (now, 2147483562);
val s2 = q mod 2147483398;
in (s1 + 1, s2 + 1) end);

in

fun next-seed () =
let
val (seed1, seed′) = @{code split-seed} (! seed)
val - = seed := seed′
in
seed1
end

fun run f =
let
val (x, seed′) = f (! seed);
val - = seed := seed′
in x end;

end;

end;
⟫

hide-type (**open**) *seed*
hide-const (**open**) *inc-shift minus-shift log next split-seed*
iterate range select pick select-weight
hide-fact (**open**) *range-def*

no-notation *fcomp* (**infixl** *o>* *60*)
no-notation *scomp* (**infixl** *o→* *60*)

end

45 String: Character and string types

theory *String*
imports *List*
uses
(*Tools/string-syntax.ML*)
(*Tools/string-code.ML*)
begin

45.1 Characters

datatype *nibble* =
Nibble0 | *Nibble1* | *Nibble2* | *Nibble3* | *Nibble4* | *Nibble5* | *Nibble6* | *Nibble7*
| *Nibble8* | *Nibble9* | *NibbleA* | *NibbleB* | *NibbleC* | *NibbleD* | *NibbleE* | *NibbleF*

lemma *UNIV-nibble*:
UNIV = {*Nibble0*, *Nibble1*, *Nibble2*, *Nibble3*, *Nibble4*, *Nibble5*, *Nibble6*, *Nibble7*, *Nibble8*, *Nibble9*, *NibbleA*, *NibbleB*, *NibbleC*, *NibbleD*, *NibbleE*, *NibbleF*} (**is** - = *?A*)
proof (*rule UNIV-eq-I*)
fix *x* **show** *x* ∈ *?A* **by** (*cases x*) *simp-all*
qed

instance *nibble* :: *finite*
by *default* (*simp add*: *UNIV-nibble*)

datatype *char* = *Char nibble nibble*
— Note: canonical order of character encoding coincides with standard term ordering

lemma *UNIV-char*:
UNIV = *image* (*split Char*) (*UNIV* × *UNIV*)
proof (*rule UNIV-eq-I*)
fix *x* **show** *x* ∈ *image* (*split Char*) (*UNIV* × *UNIV*) **by** (*cases x*) *auto*
qed

instance *char* :: *finite*
by *default* (*simp add*: *UNIV-char*)

lemma *size-char* [*code*, *simp*]:
size (*c*::*char*) = *0* **by** (*cases c*) *simp*

lemma *char-size* [*code*, *simp*]:
char-size (*c*::*char*) = *0* **by** (*cases c*) *simp*

primrec *nibble-pair-of-char* :: *char* ⇒ *nibble* × *nibble* **where**
nibble-pair-of-char (*Char n m*) = (*n*, *m*)

setup ⟪
let
  val nibbles = map-range (Thm.cterm-of @{theory} o HOLogic.mk-nibble) 16;
  val thms = map-product
  (fn n => fn m => Drule.instantiate' [] [SOME n, SOME m] @{thm nibble-pair-of-char.simps})
    nibbles nibbles;
in
  PureThy.note-thmss Thm.definitionK [((Binding.name nibble-pair-of-char-simps,
[]), [(thms, [])])]
  #-> (fn [(-, thms)] => fold-rev Code.add-eqn thms)
end
⟫

lemma *char-case-nibble-pair* [*code*, *code-unfold*]:
  char-case f = split f o nibble-pair-of-char
  by (*simp add*: *expand-fun-eq split*: *char.split*)

lemma *char-rec-nibble-pair* [*code*, *code-unfold*]:
  char-rec f = split f o nibble-pair-of-char
  unfolding *char-case-nibble-pair* [*symmetric*]
  by (*simp add*: *expand-fun-eq split*: *char.split*)

syntax
  -Char :: *xstr => char* (*CHR* -)

45.2 Strings

types *string = char list*

syntax
  -String :: *xstr => string* (-)

use *Tools/string-syntax.ML*
setup *String-Syntax.setup*

definition *chars* :: *string* **where**
  chars = [Char Nibble0 Nibble0, Char Nibble0 Nibble1, Char Nibble0 Nibble2,
  Char Nibble0 Nibble3, Char Nibble0 Nibble4, Char Nibble0 Nibble5,
  Char Nibble0 Nibble6, Char Nibble0 Nibble7, Char Nibble0 Nibble8,
  Char Nibble0 Nibble9, Char Nibble0 NibbleA, Char Nibble0 NibbleB,
  Char Nibble0 NibbleC, Char Nibble0 NibbleD, Char Nibble0 NibbleE,
  Char Nibble0 NibbleF, Char Nibble1 Nibble0, Char Nibble1 Nibble1,
  Char Nibble1 Nibble2, Char Nibble1 Nibble3, Char Nibble1 Nibble4,
  Char Nibble1 Nibble5, Char Nibble1 Nibble6, Char Nibble1 Nibble7,
  Char Nibble1 Nibble8, Char Nibble1 Nibble9, Char Nibble1 NibbleA,
  Char Nibble1 NibbleB, Char Nibble1 NibbleC, Char Nibble1 NibbleD,
  Char Nibble1 NibbleE, Char Nibble1 NibbleF, CHR '' '', CHR ''!'',
  Char Nibble2 Nibble2, CHR ''#'', CHR ''$'', CHR ''%'', CHR ''&'',

Char Nibble2 Nibble7, *CHR "("*, *CHR ")"*, *CHR "*"*, *CHR "+"*, *CHR ","*,
CHR "−", *CHR "."*, *CHR "/"*, *CHR "0"*, *CHR "1"*, *CHR "2"*, *CHR "3"*,
CHR "4", *CHR "5"*, *CHR "6"*, *CHR "7"*, *CHR "8"*, *CHR "9"*, *CHR ":"*,
CHR ";", *CHR "<"*, *CHR "="*, *CHR ">"*, *CHR "?"*, *CHR "@"*, *CHR "A"*,
CHR "B", *CHR "C"*, *CHR "D"*, *CHR "E"*, *CHR "F"*, *CHR "G"*, *CHR "H"*,
CHR "I", *CHR "J"*, *CHR "K"*, *CHR "L"*, *CHR "M"*, *CHR "N"*, *CHR "O"*,
CHR "P", *CHR "Q"*, *CHR "R"*, *CHR "S"*, *CHR "T"*, *CHR "U"*, *CHR "V"*,
CHR "W", *CHR "X"*, *CHR "Y"*, *CHR "Z"*, *CHR "["*, *Char Nibble5 NibbleC*,
CHR "]", *CHR "^"*, *CHR "-"*, *Char Nibble6 Nibble0*, *CHR "a"*, *CHR "b"*,
CHR "c", *CHR "d"*, *CHR "e"*, *CHR "f"*, *CHR "g"*, *CHR "h"*, *CHR "i"*,
CHR "j", *CHR "k"*, *CHR "l"*, *CHR "m"*, *CHR "n"*, *CHR "o"*, *CHR "p"*,
CHR "q", *CHR "r"*, *CHR "s"*, *CHR "t"*, *CHR "u"*, *CHR "v"*, *CHR "w"*,
CHR "x", *CHR "y"*, *CHR "z"*, *CHR "{"*, *CHR "|"*, *CHR "}"*, *CHR "~"*,
Char Nibble7 NibbleF, *Char Nibble8 Nibble0*, *Char Nibble8 Nibble1*,
Char Nibble8 Nibble2, *Char Nibble8 Nibble3*, *Char Nibble8 Nibble4*,
Char Nibble8 Nibble5, *Char Nibble8 Nibble6*, *Char Nibble8 Nibble7*,
Char Nibble8 Nibble8, *Char Nibble8 Nibble9*, *Char Nibble8 NibbleA*,
Char Nibble8 NibbleB, *Char Nibble8 NibbleC*, *Char Nibble8 NibbleD*,
Char Nibble8 NibbleE, *Char Nibble8 NibbleF*, *Char Nibble9 Nibble0*,
Char Nibble9 Nibble1, *Char Nibble9 Nibble2*, *Char Nibble9 Nibble3*,
Char Nibble9 Nibble4, *Char Nibble9 Nibble5*, *Char Nibble9 Nibble6*,
Char Nibble9 Nibble7, *Char Nibble9 Nibble8*, *Char Nibble9 Nibble9*,
Char Nibble9 NibbleA, *Char Nibble9 NibbleB*, *Char Nibble9 NibbleC*,
Char Nibble9 NibbleD, *Char Nibble9 NibbleE*, *Char Nibble9 NibbleF*,
Char NibbleA Nibble0, *Char NibbleA Nibble1*, *Char NibbleA Nibble2*,
Char NibbleA Nibble3, *Char NibbleA Nibble4*, *Char NibbleA Nibble5*,
Char NibbleA Nibble6, *Char NibbleA Nibble7*, *Char NibbleA Nibble8*,
Char NibbleA Nibble9, *Char NibbleA NibbleA*, *Char NibbleA NibbleB*,
Char NibbleA NibbleC, *Char NibbleA NibbleD*, *Char NibbleA NibbleE*,
Char NibbleA NibbleF, *Char NibbleB Nibble0*, *Char NibbleB Nibble1*,
Char NibbleB Nibble2, *Char NibbleB Nibble3*, *Char NibbleB Nibble4*,
Char NibbleB Nibble5, *Char NibbleB Nibble6*, *Char NibbleB Nibble7*,
Char NibbleB Nibble8, *Char NibbleB Nibble9*, *Char NibbleB NibbleA*,
Char NibbleB NibbleB, *Char NibbleB NibbleC*, *Char NibbleB NibbleD*,
Char NibbleB NibbleE, *Char NibbleB NibbleF*, *Char NibbleC Nibble0*,
Char NibbleC Nibble1, *Char NibbleC Nibble2*, *Char NibbleC Nibble3*,
Char NibbleC Nibble4, *Char NibbleC Nibble5*, *Char NibbleC Nibble6*,
Char NibbleC Nibble7, *Char NibbleC Nibble8*, *Char NibbleC Nibble9*,
Char NibbleC NibbleA, *Char NibbleC NibbleB*, *Char NibbleC NibbleC*,
Char NibbleC NibbleD, *Char NibbleC NibbleE*, *Char NibbleC NibbleF*,
Char NibbleD Nibble0, *Char NibbleD Nibble1*, *Char NibbleD Nibble2*,
Char NibbleD Nibble3, *Char NibbleD Nibble4*, *Char NibbleD Nibble5*,
Char NibbleD Nibble6, *Char NibbleD Nibble7*, *Char NibbleD Nibble8*,
Char NibbleD Nibble9, *Char NibbleD NibbleA*, *Char NibbleD NibbleB*,
Char NibbleD NibbleC, *Char NibbleD NibbleD*, *Char NibbleD NibbleE*,
Char NibbleD NibbleF, *Char NibbleE Nibble0*, *Char NibbleE Nibble1*,
Char NibbleE Nibble2, *Char NibbleE Nibble3*, *Char NibbleE Nibble4*,
Char NibbleE Nibble5, *Char NibbleE Nibble6*, *Char NibbleE Nibble7*,
Char NibbleE Nibble8, *Char NibbleE Nibble9*, *Char NibbleE NibbleA*,

Char NibbleE NibbleB, *Char NibbleE NibbleC*, *Char NibbleE NibbleD*,
Char NibbleE NibbleE, *Char NibbleE NibbleF*, *Char NibbleF Nibble0*,
Char NibbleF Nibble1, *Char NibbleF Nibble2*, *Char NibbleF Nibble3*,
Char NibbleF Nibble4, *Char NibbleF Nibble5*, *Char NibbleF Nibble6*,
Char NibbleF Nibble7, *Char NibbleF Nibble8*, *Char NibbleF Nibble9*,
Char NibbleF NibbleA, *Char NibbleF NibbleB*, *Char NibbleF NibbleC*,
Char NibbleF NibbleD, *Char NibbleF NibbleE*, *Char NibbleF NibbleF*]

45.3 Strings as dedicated datatype

datatype *literal* = *STR string*

declare *literal.cases* [*code del*] *literal.recs* [*code del*]

lemma [*code*]: *size* (*s*::*literal*) = *0*
by (*cases s*) *simp-all*

lemma [*code*]: *literal-size* (*s*::*literal*) = *0*
by (*cases s*) *simp-all*

45.4 Code generator

use *Tools/string-code.ML*

code-reserved *SML string*
code-reserved *OCaml string*
code-reserved *Scala string*

code-type *literal*
(*SML string*)
(*OCaml string*)
(*Haskell String*)
(*Scala String*)

setup ⟪
fold String-Code.add-literal-string [*SML*, *OCaml*, *Haskell*, *Scala*]
⟫

code-instance *literal* :: *eq*
(*Haskell* −)

code-const *eq-class.eq* :: *literal* ⇒ *literal* ⇒ *bool*
(*SML* !((- : *string*) = -))
(*OCaml* !((- : *string*) = -))
(*Haskell* **infixl** *4* ==)
(*Scala* **infixl** *5* ==)

types-code
char (*string*)
attach (*term-of*) ⟪

```
val term-of-char = HOLogic.mk-char o ord;
⟩⟩
attach (test) ⟨⟨
fun gen-char i =
  let val j = random-range (ord a) (Int.min (ord a + i, ord z))
  in (chr j, fn () => HOLogic.mk-char j) end;
⟩⟩

setup ⟨⟨
let

fun char-codegen thy defs dep thyname b t gr =
  let
    val i = HOLogic.dest-char t;
    val (-, gr') = Codegen.invoke-tycodegen thy defs dep thyname false
      (fastype-of t) gr;
  in SOME (Codegen.str (ML-Syntax.print-string (chr i)), gr')
  end handle TERM - => NONE;

in Codegen.add-codegen char-codegen char-codegen end
⟩⟩

hide-type (open) literal

end
```

46 Typerep: Reflecting Pure types into HOL

```
theory Typerep
imports Plain String
begin

datatype typerep = Typerep String.literal typerep list

class typerep =
  fixes typerep :: 'a itself ⇒ typerep
begin

definition typerep-of :: 'a ⇒ typerep where
  [simp]: typerep-of x = typerep TYPE('a)

end

syntax
  -TYPEREP :: type => logic  ((1TYPEREP/(1'(-'))))

parse-translation ⟨⟨
let
  fun typerep-tr (*-TYPEREP*) [ty] =
```

```
      Syntax.const @{const-syntax typerep} $
        (Syntax.const @{syntax-const -constrain} $ Syntax.const @{const-syntax
TYPE} $
          (Syntax.const @{type-syntax itself} $ ty))
  | typerep-tr (*-TYPEREP*) ts = raise TERM (typerep-tr, ts);
in [(@{syntax-const -TYPEREP}, typerep-tr)] end
⟫

typed-print-translation ⟪
let
  fun typerep-tr' show-sorts (*typerep*)
        (Type (@{type-name fun}, [Type (@{type-name itself}, [T]), -]))
        (Const (@{const-syntax TYPE}, -) :: ts) =
       Term.list-comb
       (Syntax.const @{syntax-const -TYPEREP} $ Syntax.term-of-typ show-sorts
T, ts)
  | typerep-tr' - T ts = raise Match;
in [(@{const-syntax typerep}, typerep-tr')] end
⟫

setup ⟪
let

fun add-typerep tyco thy =
  let
    val sorts = replicate (Sign.arity-number thy tyco) @{sort typerep};
    val vs = Name.names Name.context 'a sorts;
    val ty = Type (tyco, map TFree vs);
   val lhs = Const (@{const-name typerep}, Term.itselfT ty --> @{typ typerep})
     $ Free (T, Term.itselfT ty);
    val rhs = @{term Typerep} $ HOLogic.mk-literal tyco
     $ HOLogic.mk-list @{typ typerep} (map (HOLogic.mk-typerep o TFree) vs);
    val eq = HOLogic.mk-Trueprop (HOLogic.mk-eq (lhs, rhs));
  in
    thy
    |> Theory-Target.instantiation ([tyco], vs, @{sort typerep})
    |> '(fn lthy => Syntax.check-term lthy eq)
    |-> (fn eq => Specification.definition (NONE, (Attrib.empty-binding, eq)))
    |> snd
    |> Class.prove-instantiation-exit (K (Class.intro-classes-tac []))
  end;

fun ensure-typerep tyco thy = if not (can (Sorts.mg-domain (Sign.classes-of thy)
tyco) @{sort typerep})
  andalso can (Sorts.mg-domain (Sign.classes-of thy) tyco) @{sort type}
  then add-typerep tyco thy else thy;

in
```

add-typerep @{*type-name fun*}
#> *Typedef.interpretation ensure-typerep*
#> *Code.datatype-interpretation* (*ensure-typerep o fst*)
#> *Code.abstype-interpretation* (*ensure-typerep o fst*)

end
⟫

lemma [*code*]:
eq-class.eq (*Typerep tyco1 tys1*) (*Typerep tyco2 tys2*) ⟷ *eq-class.eq tyco1 tyco2*
∧ *list-all2 eq-class.eq tys1 tys2*
by (*auto simp add*: *equals-eq* [*symmetric*] *list-all2-eq* [*symmetric*])

code-type *typerep*
(*Eval Term.typ*)

code-const *Typerep*
(*Eval Term.Type*/ (-, -))

code-reserved *Eval Term*

hide-const (**open**) *typerep Typerep*

end

47 Code-Evaluation: Term evaluation using the generic code generator

theory *Code-Evaluation*
imports *Plain Typerep Code-Numeral*
begin

47.1 Term representation

47.1.1 Terms and class *term-of*

datatype *term* = *dummy-term*

definition *Const* :: *String.literal* ⇒ *typerep* ⇒ *term* **where**
Const - - = *dummy-term*

definition *App* :: *term* ⇒ *term* ⇒ *term* **where**
App - - = *dummy-term*

code-datatype *Const App*

class *term-of* = *typerep* +
fixes *term-of* :: *'a* ⇒ *term*

lemma *term-of-anything*: *term-of x* ≡ *t*
by (*rule eq-reflection*) (*cases term-of x*, *cases t*, *simp*)

definition *valapp* :: (*'a* ⇒ *'b*) × (*unit* ⇒ *term*)
⇒ *'a* × (*unit* ⇒ *term*) ⇒ *'b* × (*unit* ⇒ *term*) **where**
valapp f x = (*fst f* (*fst x*), λ*u*. *App* (*snd f* ()) (*snd x* ()))

lemma *valapp-code* [*code*, *code-unfold*]:
valapp (*f*, *tf*) (*x*, *tx*) = (*f x*, λ*u*. *App* (*tf* ()) (*tx* ()))
by (*simp only*: *valapp-def fst-conv snd-conv*)

47.1.2 *term-of* instances

instantiation *fun* :: (*typerep*, *typerep*) *term-of*
begin

definition
term-of (*f* :: *'a* ⇒ *'b*) = *Const* (*STR ''dummy-pattern''*) (*Typerep.Typerep* (*STR ''fun''*)
[*Typerep.typerep TYPE*(*'a*), *Typerep.typerep TYPE*(*'b*)])

instance ..

end

setup ⟪
let
fun add-term-of tyco raw-vs thy =
let
val vs = *map* (*fn* (*v*, -) => (*v*, @{*sort typerep*})) *raw-vs*;
val ty = *Type* (*tyco*, *map TFree vs*);
val lhs = *Const* (@{*const-name term-of*}, *ty* --> @{*typ term*})
$ *Free* (*x*, *ty*);
val rhs = @{*term undefined* :: *term*};
val eq = *HOLogic.mk-Trueprop* (*HOLogic.mk-eq* (*lhs*, *rhs*));
fun triv-name-of t = (*fst o dest-Free o fst o strip-comb o fst*
o HOLogic.dest-eq o HOLogic.dest-Trueprop) *t* ^ *-triv*;
in
thy
|> *Theory-Target.instantiation* ([*tyco*], *vs*, @{*sort term-of*})
|> '(*fn lthy* => *Syntax.check-term lthy eq*)
|-> (*fn eq* => *Specification.definition* (*NONE*, ((*Binding.name* (*triv-name-of eq*), []), *eq*)))
|> *snd*
|> *Class.prove-instantiation-exit* (*K* (*Class.intro-classes-tac* []))
end;
fun ensure-term-of (*tyco*, (*raw-vs*, -)) *thy* =
let

```
    val need-inst = not (can (Sorts.mg-domain (Sign.classes-of thy) tyco) @{sort
term-of})
      andalso can (Sorts.mg-domain (Sign.classes-of thy) tyco) @{sort typerep};
  in if need-inst then add-term-of tyco raw-vs thy else thy end;
in
  Code.datatype-interpretation ensure-term-of
  #> Code.abstype-interpretation ensure-term-of
end
⟩⟩

setup ⟨⟨
let
  fun mk-term-of-eq thy ty vs tyco (c, tys) =
    let
      val t = list-comb (Const (c, tys ---> ty),
        map Free (Name.names Name.context a tys));
      val (arg, rhs) =
      pairself (Thm.cterm-of thy o map-types Logic.unvarifyT-global o Logic.varify-global)
        (t, (map-aterms (fn t as Free (v, ty) => HOLogic.mk-term-of ty t | t =>
t) o HOLogic.reflect-term) t)
      val cty = Thm.ctyp-of thy ty;
    in
      @{thm term-of-anything}
      |> Drule.instantiate' [SOME cty] [SOME arg, SOME rhs]
      |> Thm.varifyT-global
    end;
  fun add-term-of-code tyco raw-vs raw-cs thy =
    let
      val algebra = Sign.classes-of thy;
      val vs = map (fn (v, sort) =>
        (v, curry (Sorts.inter-sort algebra) @{sort typerep} sort)) raw-vs;
      val ty = Type (tyco, map TFree vs);
      val cs = (map o apsnd o map o map-atyps)
        (fn TFree (v, -) => TFree (v, (the o AList.lookup (op =) vs) v)) raw-cs;
      val const = AxClass.param-of-inst thy (@{const-name term-of}, tyco);
      val eqs = map (mk-term-of-eq thy ty vs tyco) cs;
   in
      thy
      |> Code.del-eqns const
      |> fold Code.add-eqn eqs
    end;
  fun ensure-term-of-code (tyco, (raw-vs, cs)) thy =
    let
      val has-inst = can (Sorts.mg-domain (Sign.classes-of thy) tyco) @{sort
term-of};
    in if has-inst then add-term-of-code tyco raw-vs cs thy else thy end;
in
  Code.datatype-interpretation ensure-term-of-code
end
```

⟫

setup ⟪
let
fun mk-term-of-eq thy ty vs tyco abs ty-rep proj =
let
val arg = Var ((x, 0), ty);
val rhs = Abs (y, @{typ term}, HOLogic.reflect-term (Const (abs, ty-rep ---> ty) $ Bound 0)) $
(HOLogic.mk-term-of ty-rep (Const (proj, ty ---> ty-rep) $ arg))
|> Thm.cterm-of thy;
val cty = Thm.ctyp-of thy ty;
in
@{thm term-of-anything}
|> Drule.instantiate' [SOME cty] [SOME (Thm.cterm-of thy arg), SOME rhs]
|> Thm.varifyT-global
end;
fun add-term-of-code tyco raw-vs abs raw-ty-rep proj thy =
let
val algebra = Sign.classes-of thy;
val vs = map (fn (v, sort) =>
(v, curry (Sorts.inter-sort algebra) @{sort typerep} sort)) raw-vs;
val ty = Type (tyco, map TFree vs);
val ty-rep = map-atyps
(fn TFree (v, -) => TFree (v, (the o AList.lookup (op =) vs) v)) raw-ty-rep;
val const = AxClass.param-of-inst thy (@{const-name term-of}, tyco);
val eq = mk-term-of-eq thy ty vs tyco abs ty-rep proj;
in
thy
|> Code.del-eqns const
|> Code.add-eqn eq
end;
fun ensure-term-of-code (tyco, (raw-vs, ((abs, ty), (proj, -)))) thy =
let
val has-inst = can (Sorts.mg-domain (Sign.classes-of thy) tyco) @{sort term-of};
in if has-inst then add-term-of-code tyco raw-vs abs ty proj thy else thy end;
in
Code.abstype-interpretation ensure-term-of-code
end
⟪

47.1.3 Code generator setup

lemmas [*code del*] = *term.recs term.cases term.size*
lemma [*code*, *code del*]: *eq-class.eq (t1::term) t2 ⟷ eq-class.eq t1 t2* **..**

lemma [*code*, *code del*]: (*term-of* :: *typerep* ⇒ *term*) = *term-of* **..**
lemma [*code*, *code del*]: (*term-of* :: *term* ⇒ *term*) = *term-of* **..**

lemma [*code*, *code del*]: (*term-of* :: *String.literal* ⇒ *term*) = *term-of* **..**
lemma [*code*, *code del*]:
(*Code-Evaluation.term-of* :: *'a*::{*type*, *term-of*} *Predicate.pred* ⇒ *Code-Evaluation.term*)
= *Code-Evaluation.term-of* **..**
lemma [*code*, *code del*]:
(*Code-Evaluation.term-of* :: *'a*::{*type*, *term-of*} *Predicate.seq* ⇒ *Code-Evaluation.term*)
= *Code-Evaluation.term-of* **..**

lemma *term-of-char* [*unfolded typerep-fun-def typerep-char-def typerep-nibble-def*,
code]: *Code-Evaluation.term-of c* =
(*let* (*n*, *m*) = *nibble-pair-of-char c*
in Code-Evaluation.App (*Code-Evaluation.App* (*Code-Evaluation.Const* (*STR*
''String.char.Char'') (*TYPEREP*(*nibble* ⇒ *nibble* ⇒ *char*)))
(*Code-Evaluation.term-of n*)) (*Code-Evaluation.term-of m*))
by (*subst term-of-anything*) *rule*

code-type *term*
(*Eval Term.term*)

code-const *Const* **and** *App*
(*Eval Term.Const*/ ((-), (-)) **and** *Term.*$/ ((-), (-)))

code-const *term-of* :: *String.literal* ⇒ *term*
(*Eval HOLogic.mk'-literal*)

code-reserved *Eval HOLogic*

47.1.4 Syntax

definition *termify* :: *'a* ⇒ *term* **where**
[*code del*]: *termify x* = *dummy-term*

abbreviation *valtermify* :: *'a* ⇒ *'a* × (*unit* ⇒ *term*) **where**
valtermify x ≡ (*x*, λ*u*. *termify x*)

setup ⟪
let
fun map-default f xs =
let val ys = *map f xs*
in if exists is-some ys
then SOME (*map2 the-default xs ys*)
else NONE
end;
fun subst-termify-app (*Const* (@{*const-name termify*}, *T*), [*t*]) =
if not (*Term.has-abs t*)
then if fold-aterms (*fn Const - => I | - => K false*) *t true*
then SOME (*HOLogic.reflect-term t*)
else error Cannot termify expression containing variables
else error Cannot termify expression containing abstraction

```
    | subst-termify-app (t, ts) = case map-default subst-termify ts
        of SOME ts' => SOME (list-comb (t, ts'))
         | NONE => NONE
  and subst-termify (Abs (v, T, t)) = (case subst-termify t
        of SOME t' => SOME (Abs (v, T, t'))
         | NONE => NONE)
    | subst-termify t = subst-termify-app (strip-comb t)
  fun check-termify ts ctxt = map-default subst-termify ts
    |> Option.map (rpair ctxt)
in
  Context.theory-map (Syntax.add-term-check 0 termify check-termify)
end;
⟫

locale term-syntax
begin

notation App (infixl <·> 70)
  and valapp (infixl {·} 70)

end

interpretation term-syntax .

no-notation App (infixl <·> 70)
  and valapp (infixl {·} 70)
```

47.2 Numeric types

```
definition term-of-num :: 'a::{semiring-div} ⇒ 'a::{semiring-div} ⇒ term where
  term-of-num two = (λ-. dummy-term)

lemma (in term-syntax) term-of-num-code [code]:
  term-of-num two k = (if k = 0 then termify Int.Pls
    else (if k mod two = 0
      then termify Int.Bit0 <·> term-of-num two (k div two)
      else termify Int.Bit1 <·> term-of-num two (k div two)))
 by (auto simp add: term-of-anything Const-def App-def term-of-num-def Let-def)

lemma (in term-syntax) term-of-nat-code [code]:
  term-of (n::nat) = termify (number-of :: int ⇒ nat) <·> term-of-num (2::nat)
n
  by (simp only: term-of-anything)

lemma (in term-syntax) term-of-int-code [code]:
  term-of (k::int) = (if k = 0 then termify (0 :: int)
    else if k > 0 then termify (number-of :: int ⇒ int) <·> term-of-num (2::int) k
      else termify (uminus :: int ⇒ int) <·> (termify (number-of :: int ⇒ int)
<·> term-of-num (2::int) (- k)))
```

by (*simp only*: *term-of-anything*)

lemma (**in** *term-syntax*) *term-of-code-numeral-code* [*code*]:
term-of (*k*::*code-numeral*) = *termify* (*number-of* :: *int* ⇒ *code-numeral*) <·> *term-of-num* (*2*::*code-numeral*) *k*
by (*simp only*: *term-of-anything*)

47.3 Obfuscate

print-translation ⟪
let
val term = *Const* (<*TERM*>, *dummyT*);
fun tr1' [-, -] = *term*;
fun tr2' [] = *term*;
in
[(@{*const-syntax Const*}, *tr1'*),
(@{*const-syntax App*}, *tr1'*),
(@{*const-syntax dummy-term*}, *tr2'*)]
end
⟫

hide-const *dummy-term App valapp*
hide-const (**open**) *Const termify valtermify term-of term-of-num*

47.4 Tracing of generated and evaluated code

definition *tracing* :: *String.literal* => *'a* => *'a*
where
[*code del*]: *tracing s x* = *x*

ML ⟪
structure Code-Evaluation =
struct

fun tracing s x = (*Output.tracing s*; *x*)

end
⟫

code-const *tracing* :: *String.literal* => *'a* => *'a*
(*Eval Code'-Evaluation.tracing*)

hide-const (**open**) *tracing*
code-reserved *Eval Code-Evaluation*

47.5 Evaluation setup

ML ⟪
signature EVAL =
sig

```
  val eval-ref: (unit -> term) option Unsynchronized.ref
  val eval-term: theory -> term -> term
end;

structure Eval : EVAL =
struct

val eval-ref = Unsynchronized.ref (NONE : (unit -> term) option);

fun eval-term thy t =
  Code-Eval.eval NONE (Eval.eval-ref, eval-ref) I thy (HOLogic.mk-term-of (fastype-of
t) t) [];

end;
⟩⟩

setup ⟨⟨
  Value.add-evaluator (code, Eval.eval-term o ProofContext.theory-of)
⟩⟩

end
```

48 Quickcheck: A simple counterexample generator

```
theory Quickcheck
imports Random Code-Evaluation
uses (Tools/quickcheck-generators.ML)
begin

notation fcomp (infixl o> 60)
notation scomp (infixl o→ 60)
```

48.1 The *random* class

```
class random = typerep +
  fixes random :: code-numeral ⇒ Random.seed ⇒ ('a × (unit ⇒ term)) × Ran-
dom.seed
```

48.2 Fundamental and numeric types

```
instantiation bool :: random
begin

definition
  random i = Random.range 2 o→
   (λk. Pair (if k = 0 then Code-Evaluation.valtermify False else Code-Evaluation.valtermify
True))
```

instance ..

end

instantiation *itself* :: (*typerep*) *random*
begin

definition *random-itself* :: *code-numeral* ⇒ *Random.seed* ⇒ (*'a itself* × (*unit* ⇒ *term*)) × *Random.seed* **where**
random-itself - = *Pair* (*Code-Evaluation.valtermify TYPE('a)*)

instance ..

end

instantiation *char* :: *random*
begin

definition
random - = *Random.select chars o→* (λ*c*. *Pair* (*c*, λ*u*. *Code-Evaluation.term-of c*))

instance ..

end

instantiation *String.literal* :: *random*
begin

definition
random - = *Pair* (*STR ''''*, λ*u*. *Code-Evaluation.term-of* (*STR ''''*))

instance ..

end

instantiation *nat* :: *random*
begin

definition *random-nat* :: *code-numeral* ⇒ *Random.seed* ⇒ (*nat* × (*unit* ⇒ *Code-Evaluation.term*)) × *Random.seed* **where**
random-nat i = *Random.range* (*i* + *1*) *o→* (λ*k*. *Pair* (
let n = *Code-Numeral.nat-of k*
in (*n*, λ-. *Code-Evaluation.term-of n*)))

instance ..

end

instantiation *int* :: *random*
begin

definition
random i = *Random.range* (2 ∗ *i* + 1) *o*→ (λ*k*. *Pair* (
let j = (*if k* ≥ *i then Code-Numeral.int-of* (*k* − *i*) *else* − *Code-Numeral.int-of* (*i* − *k*))
in (*j*, λ-. *Code-Evaluation.term-of j*)))

instance ..

end

48.3 Complex generators

Towards *'a* ⇒ *'b*

axiomatization *random-fun-aux* :: *typerep* ⇒ *typerep* ⇒ (*'a* ⇒ *'a* ⇒ *bool*) ⇒ (*'a* ⇒ *term*)
⇒ (*Random.seed* ⇒ (*'b* × (*unit* ⇒ *term*)) × *Random.seed*) ⇒ (*Random.seed* ⇒ *Random.seed* × *Random.seed*)
⇒ *Random.seed* ⇒ ((*'a* ⇒ *'b*) × (*unit* ⇒ *term*)) × *Random.seed*

definition *random-fun-lift* :: (*Random.seed* ⇒ (*'b* × (*unit* ⇒ *term*)) × *Random.seed*)
⇒ *Random.seed* ⇒ ((*'a*::*term-of* ⇒ *'b*::*typerep*) × (*unit* ⇒ *term*)) × *Random.seed*
where
random-fun-lift f = *random-fun-aux TYPEREP*(*'a*) *TYPEREP*(*'b*) (*op* =) *Code-Evaluation.term-of f Random.split-seed*

instantiation *fun* :: ({*eq*, *term-of*}, *random*) *random*
begin

definition *random-fun* :: *code-numeral* ⇒ *Random.seed* ⇒ ((*'a* ⇒ *'b*) × (*unit* ⇒ *term*)) × *Random.seed* **where**
random i = *random-fun-lift* (*random i*)

instance ..

end

Towards type copies and datatypes

definition *collapse* :: (*'a* ⇒ (*'a* ⇒ *'b* × *'a*) × *'a*) ⇒ *'a* ⇒ *'b* × *'a* **where**
collapse f = (*f o*→ *id*)

definition *beyond* :: *code-numeral* ⇒ *code-numeral* ⇒ *code-numeral* **where**
beyond k l = (*if l* > *k then l else 0*)

lemma *beyond-zero*:

beyond k 0 = 0
by (*simp add*: *beyond-def*)

lemma *random-aux-rec*:
fixes *random-aux* :: *code-numeral* ⇒ *'a*
assumes *random-aux 0 = rhs 0*
and ⋀*k*. *random-aux* (*Suc-code-numeral k*) = *rhs* (*Suc-code-numeral k*)
shows *random-aux k = rhs k*
using *assms* **by** (*rule code-numeral.induct*)

use *Tools/quickcheck-generators.ML*
setup *Quickcheck-Generators.setup*

48.4 Code setup

code-const *random-fun-aux* (*Quickcheck Quickcheck'-Generators.random'-fun*)
— With enough criminal energy this can be abused to derive *False*; for this reason we use a distinguished target *Quickcheck* not spoiling the regular trusted code generation

code-reserved *Quickcheck Quickcheck-Generators*

no-notation *fcomp* (**infixl** *o>* *60*)
no-notation *scomp* (**infixl** *o→* *60*)

48.5 The Random-Predicate Monad

fun *iter'* ::
'a itself => *code-numeral* => *code-numeral* => *code-numeral* ∗ *code-numeral* => (*'a*::*random*) *Predicate.pred*
where
iter' T nrandom sz seed = (*if nrandom = 0 then bot-class.bot else*
let ((*x*, -), *seed'*) = *random sz seed*
in Predicate.Seq (%*u*. *Predicate.Insert x* (*iter' T* (*nrandom* − *1*) *sz seed'*)))

definition *iter* :: *code-numeral* => *code-numeral* => *code-numeral* ∗ *code-numeral* => (*'a*::*random*) *Predicate.pred*
where
iter nrandom sz seed = *iter'* (*TYPE*(*'a*)) *nrandom sz seed*

lemma [*code*]:
iter nrandom sz seed = (*if nrandom = 0 then bot-class.bot else*
let ((*x*, -), *seed'*) = *random sz seed*
in Predicate.Seq (%*u*. *Predicate.Insert x* (*iter* (*nrandom* − *1*) *sz seed'*)))
unfolding *iter-def iter'.simps*[*of* - *nrandom*] **..**

types *'a randompred* = *Random.seed* ⇒ (*'a Predicate.pred* × *Random.seed*)

definition *empty* :: *'a randompred*
where *empty* = *Pair* (*bot-class.bot*)

definition *single* :: *'a => 'a randompred*
where *single x = Pair (Predicate.single x)*

definition *bind* :: *'a randompred ⇒ ('a ⇒ 'b randompred) ⇒ 'b randompred*
where
bind R f = (λs. let
(P, s') = R s;
(s1, s2) = Random.split-seed s'
in (Predicate.bind P (%a. fst (f a s1)), s2))

definition *union* :: *'a randompred ⇒ 'a randompred ⇒ 'a randompred*
where
union R1 R2 = (λs. let
(P1, s') = R1 s; (P2, s'') = R2 s'
in (semilattice-sup-class.sup P1 P2, s''))

definition *if-randompred* :: *bool ⇒ unit randompred*
where
if-randompred b = (if b then single () else empty)

definition *iterate-upto* :: *(code-numeral => 'a) => code-numeral => code-numeral => 'a randompred*
where
iterate-upto f n m = Pair (Code-Numeral.iterate-upto f n m)

definition *not-randompred* :: *unit randompred ⇒ unit randompred*
where
not-randompred P = (λs. let
(P', s') = P s
in if Predicate.eval P' () then (Orderings.bot, s') else (Predicate.single (), s'))

definition *Random* :: *(Random.seed ⇒ ('a × (unit ⇒ term)) × Random.seed) ⇒ 'a randompred*
where *Random g = scomp g (Pair o (Predicate.single o fst))*

definition *map* :: *('a ⇒ 'b) ⇒ ('a randompred ⇒ 'b randompred)*
where *map f P = bind P (single o f)*

hide-fact (**open**) *iter'.simps iter-def empty-def single-def bind-def union-def if-randompred-def iterate-upto-def not-randompred-def Random-def map-def*
hide-type (**open**) *randompred*
hide-const (**open**) *random collapse beyond random-fun-aux random-fun-lift iter' iter empty single bind union if-randompred iterate-upto not-randompred Random map*

end

49 Lazy-Sequence: Lazy sequences

theory *Lazy-Sequence*
imports *List Code-Numeral*
begin

datatype *'a lazy-sequence = Empty | Insert 'a 'a lazy-sequence*

definition *Lazy-Sequence :: (unit => ('a * 'a lazy-sequence) option) => 'a lazy-sequence*
where
Lazy-Sequence f = (case f () of None => Empty | Some (x, xq) => Insert x xq)

code-datatype *Lazy-Sequence*

primrec *yield :: 'a lazy-sequence => ('a * 'a lazy-sequence) option*
where
yield Empty = None
| yield (Insert x xq) = Some (x, xq)

lemma [*simp*]: *yield xq = Some (x, xq') ==> size xq' < size xq*
by (*cases xq*) *auto*

lemma *yield-Seq* [*code*]:
yield (Lazy-Sequence f) = f ()
unfolding *Lazy-Sequence-def* **by** (*cases f ()*) *auto*

lemma *Seq-yield*:
Lazy-Sequence (%u. yield f) = f
unfolding *Lazy-Sequence-def* **by** (*cases f*) *auto*

lemma *lazy-sequence-size-code* [*code*]:
lazy-sequence-size s xq = (case yield xq of None => 0 | Some (x, xq') => s x + lazy-sequence-size s xq' + 1)
by (*cases xq*) *auto*

lemma *size-code* [*code*]:
size xq = (case yield xq of None => 0 | Some (x, xq') => size xq' + 1)
by (*cases xq*) *auto*

lemma [*code*]: *eq-class.eq xq yq = (case (yield xq, yield yq) of*
(None, None) => True | (Some (x, xq'), Some (y, yq')) => (HOL.eq x y) ∧ (eq-class.eq xq yq) | - => False)
apply (*cases xq*) **apply** (*cases yq*) **apply** (*auto simp add*: *eq-class.eq-equals*)
apply (*cases yq*) **apply** (*auto simp add*: *eq-class.eq-equals*) **done**

lemma *seq-case* [*code*]:
lazy-sequence-case f g xq = (case (yield xq) of None => f | Some (x, xq') => g x xq')
by (*cases xq*) *auto*

lemma [*code*]: *lazy-sequence-rec f g xq = (case (yield xq) of None => f | Some (x, xq') => g x xq' (lazy-sequence-rec f g xq'))*
by (*cases xq*) *auto*

definition *empty* :: *'a lazy-sequence*
where
  [*code*]: *empty = Lazy-Sequence (%u. None)*

definition *single* :: *'a => 'a lazy-sequence*
where
  [*code*]: *single x = Lazy-Sequence (%u. Some (x, empty))*

primrec *append* :: *'a lazy-sequence => 'a lazy-sequence => 'a lazy-sequence*
where
  append Empty yq = yq
| *append (Insert x xq) yq = Insert x (append xq yq)*

lemma [*code*]:
  append xq yq = Lazy-Sequence (%u. case yield xq of
    None => yield yq
  | Some (x, xq') => Some (x, append xq' yq))
unfolding *Lazy-Sequence-def*
apply (*cases xq*)
apply *auto*
apply (*cases yq*)
apply *auto*
done

primrec *flat* :: *'a lazy-sequence lazy-sequence => 'a lazy-sequence*
where
  flat Empty = Empty
| *flat (Insert xq xqq) = append xq (flat xqq)*

lemma [*code*]:
  flat xqq = Lazy-Sequence (%u. case yield xqq of
    None => None
  | Some (xq, xqq') => yield (append xq (flat xqq')))
apply (*cases xqq*)
apply (*auto simp add*: *Seq-yield*)
unfolding *Lazy-Sequence-def*
by *auto*

primrec *map* :: *('a => 'b) => 'a lazy-sequence => 'b lazy-sequence*
where
  map f Empty = Empty
| *map f (Insert x xq) = Insert (f x) (map f xq)*

lemma [*code*]:

map f xq = Lazy-Sequence (%u. Option.map (%(x, xq'). (f x, map f xq')) (yield xq))
apply (*cases xq*)
apply (*auto simp add*: *Seq-yield*)
unfolding *Lazy-Sequence-def*
apply *auto*
done

definition *bind* :: *'a lazy-sequence => ('a => 'b lazy-sequence) => 'b lazy-sequence*
where
[code]: *bind xq f = flat (map f xq)*

definition *if-seq* :: *bool => unit lazy-sequence*
where
if-seq b = (if b then single () else empty)

function *iterate-upto* :: *(code-numeral => 'a) => code-numeral => code-numeral => 'a Lazy-Sequence.lazy-sequence*
where
iterate-upto f n m = Lazy-Sequence.Lazy-Sequence (%u. if n > m then None else Some (f n, iterate-upto f (n + 1) m))
by *pat-completeness auto*

termination by (*relation measure (%(f, n, m). Code-Numeral.nat-of (m + 1 − n))*) *auto*

definition *not-seq* :: *unit lazy-sequence => unit lazy-sequence*
where
not-seq xq = (case yield xq of None => single () | Some ((), xq) => empty)

49.1 Code setup

fun *anamorph* :: *('a ⇒ ('b × 'a) option) ⇒ code-numeral ⇒ 'a ⇒ 'b list × 'a*
where
anamorph f k x = (if k = 0 then ([], x)
else case f x of None ⇒ ([], x) | Some (v, y) ⇒
let (vs, z) = anamorph f (k − 1) y
in (v # vs, z))

definition *yieldn* :: *code-numeral ⇒ 'a lazy-sequence ⇒ 'a list × 'a lazy-sequence*
where
yieldn = anamorph yield

code-reflect *Lazy-Sequence*
datatypes *lazy-sequence = Lazy-Sequence*
functions *map yield yieldn*

49.2 With Hit Bound Value

assuming in negative context

types *'a hit-bound-lazy-sequence = 'a option lazy-sequence*

definition *hit-bound :: 'a hit-bound-lazy-sequence*
where
[*code*]: *hit-bound = Lazy-Sequence (%u. Some (None, empty))*

definition *hb-single :: 'a => 'a hit-bound-lazy-sequence*
where
[*code*]: *hb-single x = Lazy-Sequence (%u. Some (Some x, empty))*

primrec *hb-flat :: 'a hit-bound-lazy-sequence hit-bound-lazy-sequence => 'a hit-bound-lazy-sequence*
where
hb-flat Empty = Empty
| *hb-flat (Insert xq xqq) = append (case xq of None => hit-bound | Some xq => xq) (hb-flat xqq)*

lemma [*code*]:
hb-flat xqq = Lazy-Sequence (%u. case yield xqq of
None => None
| Some (xq, xqq') => yield (append (case xq of None => hit-bound | Some xq => xq) (hb-flat xqq')))
apply (*cases xqq*)
apply (*auto simp add*: *Seq-yield*)
unfolding *Lazy-Sequence-def*
by *auto*

primrec *hb-map :: ('a => 'b) => 'a hit-bound-lazy-sequence => 'b hit-bound-lazy-sequence*
where
hb-map f Empty = Empty
| *hb-map f (Insert x xq) = Insert (Option.map f x) (hb-map f xq)*

lemma [*code*]:
hb-map f xq = Lazy-Sequence (%u. Option.map (%(x, xq'). (Option.map f x, hb-map f xq')) (yield xq))
apply (*cases xq*)
apply (*auto simp add*: *Seq-yield*)
unfolding *Lazy-Sequence-def*
apply *auto*
done

definition *hb-bind :: 'a hit-bound-lazy-sequence => ('a => 'b hit-bound-lazy-sequence) => 'b hit-bound-lazy-sequence*
where
[*code*]: *hb-bind xq f = hb-flat (hb-map f xq)*

definition *hb-if-seq* :: *bool* => *unit hit-bound-lazy-sequence*
where
hb-if-seq b = (*if b then hb-single* () *else empty*)

definition *hb-not-seq* :: *unit hit-bound-lazy-sequence* => *unit lazy-sequence*
where
hb-not-seq xq = (*case yield xq of None* => *single* () | *Some* (*x*, *xq*) => *empty*)

hide-type (**open**) *lazy-sequence*
hide-const (**open**) *Empty Insert Lazy-Sequence yield empty single append flat map bind if-seq iterate-upto not-seq*
hide-fact *yield.simps empty-def single-def append.simps flat.simps map.simps bind-def iterate-upto.simps if-seq-def not-seq-def*

end

50 DSequence: Depth-Limited Sequences with failure element

theory *DSequence*
imports *Lazy-Sequence Code-Numeral*
begin

types *'a dseq* = *code-numeral* => *bool* => *'a Lazy-Sequence.lazy-sequence option*

definition *empty* :: *'a dseq*
where
empty = (%*i pol*. *Some Lazy-Sequence.empty*)

definition *single* :: *'a* => *'a dseq*
where
single x = (%*i pol*. *Some* (*Lazy-Sequence.single x*))

fun *eval* :: *'a dseq* => *code-numeral* => *bool* => *'a Lazy-Sequence.lazy-sequence option*
where
eval f i pol = *f i pol*

definition *yield* :: *'a dseq* => *code-numeral* => *bool* => (*'a* * *'a dseq*) *option*
where
yield dseq i pol = (*case eval dseq i pol of*
None => *None*
| *Some s* => *Option.map* (*apsnd* (%*r i pol*. *Some r*)) (*Lazy-Sequence.yield s*))

fun *map-seq* :: (*'a* => *'b dseq*) => *'a Lazy-Sequence.lazy-sequence* => *'b dseq*
where
map-seq f xq i pol = (*case Lazy-Sequence.yield xq of*

```
    None => Some Lazy-Sequence.empty
  | Some (x, xq') => (case eval (f x) i pol of
      None => None
    | Some yq => (case map-seq f xq' i pol of
        None => None
      | Some zq => Some (Lazy-Sequence.append yq zq))))

definition bind :: 'a dseq => ('a => 'b dseq) => 'b dseq
where
  bind x f = (%i pol.
    if i = 0 then
      (if pol then Some Lazy-Sequence.empty else None)
    else
      (case x (i - 1) pol of
        None => None
      | Some xq => map-seq f xq i pol))

definition union :: 'a dseq => 'a dseq => 'a dseq
where
  union x y = (%i pol. case (x i pol, y i pol) of
      (Some xq, Some yq) => Some (Lazy-Sequence.append xq yq)
    | - => None)

definition if-seq :: bool => unit dseq
where
  if-seq b = (if b then single () else empty)

definition not-seq :: unit dseq => unit dseq
where
  not-seq x = (%i pol. case x i (¬pol) of
    None => Some Lazy-Sequence.empty
  | Some xq => (case Lazy-Sequence.yield xq of
      None => Some (Lazy-Sequence.single ())
    | Some - => Some (Lazy-Sequence.empty)))

definition map :: ('a => 'b) => 'a dseq => 'b dseq
where
  map f g = (%i pol. case g i pol of
      None => None
    | Some xq => Some (Lazy-Sequence.map f xq))

ML ⟪
signature DSEQUENCE =
sig
  type 'a dseq = int -> bool -> 'a Lazy-Sequence.lazy-sequence option
  val yield : 'a dseq -> int -> bool -> ('a * 'a dseq) option
  val yieldn : int -> 'a dseq -> int -> bool -> 'a list * 'a dseq
  val map : ('a -> 'b) -> 'a dseq -> 'b dseq
end;
```

structure DSequence : DSEQUENCE =
struct

type 'a dseq = int -> bool -> 'a Lazy-Sequence.lazy-sequence option

val yield = @{code yield}
fun yieldn n s d pol = case s d pol of
NONE => ([], fn d => fn pol => NONE)
| SOME s => let val (xs, s') = Lazy-Sequence.yieldn n s in (xs, fn d => fn pol => SOME s') end

val map = @{code map}

end;
⟩

code-reserved *Eval DSequence*

hide-const (**open**) *empty single eval map-seq bind union if-seq not-seq map*
hide-fact (**open**) *empty-def single-def eval.simps map-seq.simps bind-def union-def if-seq-def not-seq-def map-def*

end

theory *Random-Sequence*
imports *Quickcheck DSequence*
begin

types *'a random-dseq = code-numeral ⇒ code-numeral ⇒ Random.seed ⇒ ('a DSequence.dseq × Random.seed)*

definition *empty :: 'a random-dseq*
where
empty = (%nrandom size. Pair (DSequence.empty))

definition *single :: 'a => 'a random-dseq*
where
single x = (%nrandom size. Pair (DSequence.single x))

definition *bind :: 'a random-dseq => ('a ⇒ 'b random-dseq) ⇒ 'b random-dseq*
where
bind R f = (λnrandom size s. let
(P, s') = R nrandom size s;
(s1, s2) = Random.split-seed s'
in (DSequence.bind P (%a. fst (f a nrandom size s1)), s2))

definition *union* :: *'a random-dseq => 'a random-dseq => 'a random-dseq*
where
union R1 R2 = (λnrandom size s. let
(S1, s') = R1 nrandom size s; (S2, s'') = R2 nrandom size s'
in (DSequence.union S1 S2, s''))

definition *if-random-dseq* :: *bool => unit random-dseq*
where
if-random-dseq b = (if b then single () else empty)

definition *not-random-dseq* :: *unit random-dseq => unit random-dseq*
where
not-random-dseq R = (λnrandom size s. let
(S, s') = R nrandom size s
in (DSequence.not-seq S, s'))

fun *Random* :: *(code-numeral ⇒ Random.seed ⇒ (('a × (unit ⇒ term)) × Random.seed)) ⇒ 'a random-dseq*
where
Random g nrandom = (%size. if nrandom <= 0 then (Pair DSequence.empty) else
(scomp (g size) (%r. scomp (Random g (nrandom − 1) size) (%rs. Pair (DSequence.union (DSequence.single (fst r)) rs)))))

definition *map* :: *('a => 'b) => 'a random-dseq => 'b random-dseq*
where
map f P = bind P (single o f)

hide-const (**open**) *empty single bind union if-random-dseq not-random-dseq Random map*

hide-type *DSequence.dseq random-dseq*
hide-fact (**open**) *empty-def single-def bind-def union-def if-random-dseq-def not-random-dseq-def Random.simps map-def*

end

51 New-DSequence: Depth-Limited Sequences with failure element

theory *New-DSequence*
imports *Random-Sequence*
begin

51.1 Positive Depth-Limited Sequence

types *'a pos-dseq = code-numeral => 'a Lazy-Sequence.lazy-sequence*

definition *pos-empty :: 'a pos-dseq*
where
pos-empty = (%i. Lazy-Sequence.empty)

definition *pos-single :: 'a => 'a pos-dseq*
where
pos-single x = (%i. Lazy-Sequence.single x)

definition *pos-bind :: 'a pos-dseq => ('a => 'b pos-dseq) => 'b pos-dseq*
where
pos-bind x f = (%i.
if i = 0 then
Lazy-Sequence.empty
else
Lazy-Sequence.bind (x (i − 1)) (%a. f a i))

definition *pos-union :: 'a pos-dseq => 'a pos-dseq => 'a pos-dseq*
where
pos-union xq yq = (%i. Lazy-Sequence.append (xq i) (yq i))

definition *pos-if-seq :: bool => unit pos-dseq*
where
pos-if-seq b = (if b then pos-single () else pos-empty)

definition *pos-iterate-upto :: (code-numeral => 'a) => code-numeral => code-numeral => 'a pos-dseq*
where
pos-iterate-upto f n m = (%i. Lazy-Sequence.iterate-upto f n m)

definition *pos-map :: ('a => 'b) => 'a pos-dseq => 'b pos-dseq*
where
pos-map f xq = (%i. Lazy-Sequence.map f (xq i))

51.2 Negative Depth-Limited Sequence

types *'a neg-dseq = code-numeral => 'a Lazy-Sequence.hit-bound-lazy-sequence*

definition *neg-empty :: 'a neg-dseq*
where
neg-empty = (%i. Lazy-Sequence.empty)

definition *neg-single :: 'a => 'a neg-dseq*
where
neg-single x = (%i. Lazy-Sequence.hb-single x)

definition *neg-bind :: 'a neg-dseq => ('a => 'b neg-dseq) => 'b neg-dseq*

where
neg-bind x f = (%i.
if i = 0 then
Lazy-Sequence.hit-bound
else
hb-bind (x (i − 1)) (%a. f a i))

definition *neg-union :: 'a neg-dseq => 'a neg-dseq => 'a neg-dseq*
where
neg-union x y = (%i. Lazy-Sequence.append (x i) (y i))

definition *neg-if-seq :: bool => unit neg-dseq*
where
neg-if-seq b = (if b then neg-single () else neg-empty)

definition *neg-iterate-upto*
where
neg-iterate-upto f n m = (%i. Lazy-Sequence.iterate-upto (%i. Some (f i)) n m)

definition *neg-map :: ('a => 'b) => 'a neg-dseq => 'b neg-dseq*
where
neg-map f xq = (%i. Lazy-Sequence.hb-map f (xq i))

51.3 Negation

definition *pos-not-seq :: unit neg-dseq => unit pos-dseq*
where
pos-not-seq xq = (%i. Lazy-Sequence.hb-not-seq (xq i))

definition *neg-not-seq :: unit pos-dseq => unit neg-dseq*
where
neg-not-seq x = (%i. case Lazy-Sequence.yield (x i) of
None => Lazy-Sequence.hb-single ()
| Some ((), xq) => Lazy-Sequence.empty)

hide-type (**open**) *pos-dseq neg-dseq*

hide-const (**open**)
pos-empty pos-single pos-bind pos-union pos-if-seq pos-iterate-upto pos-not-seq pos-map
neg-empty neg-single neg-bind neg-union neg-if-seq neg-iterate-upto neg-not-seq neg-map
hide-fact (**open**)
pos-empty-def pos-single-def pos-bind-def pos-union-def pos-if-seq-def pos-iterate-upto-def pos-not-seq-def pos-map-def
neg-empty-def neg-single-def neg-bind-def neg-union-def neg-if-seq-def neg-iterate-upto-def neg-not-seq-def neg-map-def

end

theory *New-Random-Sequence*
imports *Quickcheck New-DSequence*
begin

types *'a pos-random-dseq = code-numeral ⇒ code-numeral ⇒ Random.seed ⇒ 'a New-DSequence.pos-dseq*
types *'a neg-random-dseq = code-numeral ⇒ code-numeral ⇒ Random.seed ⇒ 'a New-DSequence.neg-dseq*

definition *pos-empty :: 'a pos-random-dseq*
where
pos-empty = (%nrandom size seed. New-DSequence.pos-empty)

definition *pos-single :: 'a => 'a pos-random-dseq*
where
pos-single x = (%nrandom size seed. New-DSequence.pos-single x)

definition *pos-bind :: 'a pos-random-dseq => ('a ⇒ 'b pos-random-dseq) ⇒ 'b pos-random-dseq*
where
pos-bind R f = (λnrandom size seed. New-DSequence.pos-bind (R nrandom size seed) (%a. f a nrandom size seed))

definition *pos-union :: 'a pos-random-dseq => 'a pos-random-dseq => 'a pos-random-dseq*
where
pos-union R1 R2 = (λnrandom size seed. New-DSequence.pos-union (R1 nrandom size seed) (R2 nrandom size seed))

definition *pos-if-random-dseq :: bool => unit pos-random-dseq*
where
pos-if-random-dseq b = (if b then pos-single () else pos-empty)

definition *pos-iterate-upto :: (code-numeral => 'a) => code-numeral => code-numeral => 'a pos-random-dseq*
where
pos-iterate-upto f n m = (λnrandom size seed. New-DSequence.pos-iterate-upto f n m)

definition *pos-not-random-dseq :: unit neg-random-dseq => unit pos-random-dseq*
where
pos-not-random-dseq R = (λnrandom size seed. New-DSequence.pos-not-seq (R nrandom size seed))

fun *iter :: (code-numeral * code-numeral => ('a * (unit => term)) * code-numeral * code-numeral) => code-numeral => code-numeral * code-numeral => 'a Lazy-Sequence.lazy-sequence*

where
iter random nrandom seed =
(if nrandom = 0 then Lazy-Sequence.empty else Lazy-Sequence.Lazy-Sequence (%u. let ((x, -), seed') = random seed in Some (x, iter random (nrandom − 1) seed')))

definition *Random* :: *(code-numeral ⇒ Random.seed ⇒ (('a × (unit ⇒ term)) × Random.seed)) ⇒ 'a pos-random-dseq*
where
Random g = (%nrandom size seed depth. iter (g size) nrandom seed)

definition *pos-map* :: *('a => 'b) => 'a pos-random-dseq => 'b pos-random-dseq*
where
pos-map f P = pos-bind P (pos-single o f)

definition *neg-empty* :: *'a neg-random-dseq*
where
neg-empty = (%nrandom size seed. New-DSequence.neg-empty)

definition *neg-single* :: *'a => 'a neg-random-dseq*
where
neg-single x = (%nrandom size seed. New-DSequence.neg-single x)

definition *neg-bind* :: *'a neg-random-dseq => ('a ⇒ 'b neg-random-dseq) ⇒ 'b neg-random-dseq*
where
neg-bind R f = (λnrandom size seed. New-DSequence.neg-bind (R nrandom size seed) (%a. f a nrandom size seed))

definition *neg-union* :: *'a neg-random-dseq => 'a neg-random-dseq => 'a neg-random-dseq*
where
neg-union R1 R2 = (λnrandom size seed. New-DSequence.neg-union (R1 nrandom size seed) (R2 nrandom size seed))

definition *neg-if-random-dseq* :: *bool => unit neg-random-dseq*
where
neg-if-random-dseq b = (if b then neg-single () else neg-empty)

definition *neg-iterate-upto* :: *(code-numeral => 'a) => code-numeral => code-numeral => 'a neg-random-dseq*
where
neg-iterate-upto f n m = (λnrandom size seed. New-DSequence.neg-iterate-upto f n m)

definition *neg-not-random-dseq* :: *unit pos-random-dseq => unit neg-random-dseq*
where
neg-not-random-dseq R = (λnrandom size seed. New-DSequence.neg-not-seq (R nrandom size seed))

definition *neg-map* :: ('*a* => '*b*) => '*a neg-random-dseq* => '*b neg-random-dseq*
where
neg-map f P = *neg-bind P* (*neg-single o f*)

hide-const (**open**)
pos-empty pos-single pos-bind pos-union pos-if-random-dseq pos-iterate-upto pos-not-random-dseq iter Random pos-map
neg-empty neg-single neg-bind neg-union neg-if-random-dseq neg-iterate-upto neg-not-random-dseq neg-map
hide-type *New-DSequence.pos-dseq New-DSequence.neg-dseq pos-random-dseq neg-random-dseq*

end

52 Predicate-Compile: A compiler for predicates defined by introduction rules

theory *Predicate-Compile*
imports *New-Random-Sequence*
uses
Tools/Predicate-Compile/predicate-compile-aux.ML
Tools/Predicate-Compile/predicate-compile-compilations.ML
Tools/Predicate-Compile/predicate-compile-core.ML
Tools/Predicate-Compile/predicate-compile-data.ML
Tools/Predicate-Compile/predicate-compile-fun.ML
Tools/Predicate-Compile/predicate-compile-pred.ML
Tools/Predicate-Compile/predicate-compile-specialisation.ML
Tools/Predicate-Compile/predicate-compile.ML
begin

setup *Predicate-Compile.setup*

end

53 Map: Maps

theory *Map*
imports *List*
begin

types ('*a*,'*b*) *map* = '*a* => '*b option* (**infixr** ~=> *0*)
translations (*type*) '*a* ~=> '*b* <= (*type*) '*a* => '*b option*

type-notation (*xsymbols*)

map (**infixr** $\rightharpoonup$ *0*)

abbreviation
empty :: *'a ~=> 'b* **where**
empty == *%x. None*

definition
map-comp :: *('b ~=> 'c) => ('a ~=> 'b) => ('a ~=> 'c)* (**infixl** *o'-m 55*)
where
f o-m g = *(λk. case g k of None ⇒ None | Some v ⇒ f v)*

notation (*xsymbols*)
map-comp (**infixl** $\circ_m$ *55*)

definition
map-add :: *('a ~=> 'b) => ('a ~=> 'b) => ('a ~=> 'b)* (**infixl** *++ 100*)
where
m1 ++ m2 = *(λx. case m2 x of None => m1 x | Some y => Some y)*

definition
restrict-map :: *('a ~=> 'b) => 'a set => ('a ~=> 'b)* (**infixl** *|` 110*) **where**
m|`A = *(λx. if x : A then m x else None)*

notation (*latex* **output**)
restrict-map (*-|- [111,110] 110*)

definition
dom :: *('a ~=> 'b) => 'a set* **where**
dom m = *{a. m a ~= None}*

definition
ran :: *('a ~=> 'b) => 'b set* **where**
ran m = *{b. EX a. m a = Some b}*

definition
map-le :: *('a ~=> 'b) => ('a ~=> 'b) => bool* (**infix** $\subseteq_m$ *50*) **where**
$(m_1 \subseteq_m m_2) = (\forall a \in dom\ m_1.\ m_1\ a = m_2\ a)$

nonterminals
maplets maplet

syntax
-maplet :: *['a, 'a] => maplet* (*- /|->/ -*)
-maplets :: *['a, 'a] => maplet* (*- /[|->]/ -*)
:: *maplet => maplets* (*-*)
-Maplets :: *[maplet, maplets] => maplets* (*-,/ -*)
-MapUpd :: *['a ~=> 'b, maplets] => 'a ~=> 'b* (*-/'(-') [900,0]900*)
-Map :: *maplets => 'a ~=> 'b* ((*1*[-]))

syntax (*xsymbols*)
-maplet :: ['*a*, '*a*] => *maplet* (- /↦/ -)
-maplets :: ['*a*, '*a*] => *maplet* (- /[↦]/ -)

translations
-MapUpd m (*-Maplets xy ms*) == *-MapUpd* (*-MapUpd m xy*) *ms*
-MapUpd m (*-maplet x y*) == *m*(*x* := *CONST Some y*)
-Map ms == *-MapUpd* (*CONST empty*) *ms*
-Map (*-Maplets ms1 ms2*) <= *-MapUpd* (*-Map ms1*) *ms2*
-Maplets ms1 (*-Maplets ms2 ms3*) <= *-Maplets* (*-Maplets ms1 ms2*) *ms3*

primrec
map-of :: ('*a* × '*b*) *list* ⇒ '*a* ⇀ '*b* **where**
map-of [] = *empty*
| *map-of* (*p* # *ps*) = (*map-of ps*)(*fst p* ↦ *snd p*)

definition
map-upds :: ('*a* ⇀ '*b*) ⇒ '*a list* ⇒ '*b list* ⇒ '*a* ⇀ '*b* **where**
map-upds m xs ys = *m* ++ *map-of* (*rev* (*zip xs ys*))

translations
-MapUpd m (*-maplets x y*) == *CONST map-upds m x y*

lemma *map-of-Cons-code* [*code*]:
map-of [] *k* = *None*
map-of ((*l*, *v*) # *ps*) *k* = (*if l* = *k then Some v else map-of ps k*)
by *simp-all*

53.1 *empty*

lemma *empty-upd-none* [*simp*]: *empty*(*x* := *None*) = *empty*
by (*rule ext*) *simp*

53.2 *map-upd*

lemma *map-upd-triv*: *t k* = *Some x* ==> *t*(*k*|->*x*) = *t*
by (*rule ext*) *simp*

lemma *map-upd-nonempty* [*simp*]: *t*(*k*|->*x*) ~= *empty*
proof
assume *t*(*k* ↦ *x*) = *empty*
then have (*t*(*k* ↦ *x*)) *k* = *None* **by** *simp*
then show *False* **by** *simp*
qed

lemma *map-upd-eqD1*:
assumes *m*(*a*↦*x*) = *n*(*a*↦*y*)
shows *x* = *y*
proof −
from *prems* **have** (*m*(*a*↦*x*)) *a* = (*n*(*a*↦*y*)) *a* **by** *simp*

then show *?thesis* **by** *simp*
qed

lemma *map-upd-Some-unfold*:
((*m*(*a*|−>*b*)) *x* = *Some y*) = (*x* = *a* ∧ *b* = *y* ∨ *x* ≠ *a* ∧ *m x* = *Some y*)
by *auto*

lemma *image-map-upd* [*simp*]: *x* ∉ *A* ⟹ *m*(*x* ↦ *y*) ‘ *A* = *m* ‘ *A*
by *auto*

lemma *finite-range-updI*: *finite* (*range f*) ==> *finite* (*range* (*f*(*a*|−>*b*)))
unfolding *image-def*
apply (*simp* (*no-asm-use*) *add*:*full-SetCompr-eq*)
apply (*rule finite-subset*)
prefer *2* **apply** *assumption*
apply (*auto*)
done

53.3 *map-of*

lemma *map-of-eq-None-iff*:
(*map-of xys x* = *None*) = (*x* ∉ *fst* ‘ (*set xys*))
by (*induct xys*) *simp-all*

lemma *map-of-is-SomeD*: *map-of xys x* = *Some y* ⟹ (*x*,*y*) ∈ *set xys*
apply (*induct xys*)
apply *simp*
apply (*clarsimp split*: *if-splits*)
done

lemma *map-of-eq-Some-iff* [*simp*]:
distinct(*map fst xys*) ⟹ (*map-of xys x* = *Some y*) = ((*x*,*y*) ∈ *set xys*)
apply (*induct xys*)
apply *simp*
apply (*auto simp*: *map-of-eq-None-iff* [*symmetric*])
done

lemma *Some-eq-map-of-iff* [*simp*]:
distinct(*map fst xys*) ⟹ (*Some y* = *map-of xys x*) = ((*x*,*y*) ∈ *set xys*)
by (*auto simp del*:*map-of-eq-Some-iff simp add*: *map-of-eq-Some-iff* [*symmetric*])

lemma *map-of-is-SomeI* [*simp*]: ⟦ *distinct*(*map fst xys*); (*x*,*y*) ∈ *set xys* ⟧
⟹ *map-of xys x* = *Some y*
apply (*induct xys*)
apply *simp*
apply *force*
done

lemma *map-of-zip-is-None* [*simp*]:

length xs = length ys ⟹ (map-of (zip xs ys) x = None) = (x ∉ set xs)
by (*induct rule*: *list-induct2*) *simp-all*

lemma *map-of-zip-is-Some*:
assumes *length xs = length ys*
shows *x ∈ set xs ⟷ (∃ y. map-of (zip xs ys) x = Some y)*
using *assms* **by** (*induct rule*: *list-induct2*) *simp-all*

lemma *map-of-zip-upd*:
fixes *x* :: *'a* **and** *xs* :: *'a list* **and** *ys zs* :: *'b list*
assumes *length ys = length xs*
and *length zs = length xs*
and *x ∉ set xs*
and *map-of (zip xs ys)(x ↦ y) = map-of (zip xs zs)(x ↦ z)*
shows *map-of (zip xs ys) = map-of (zip xs zs)*
proof
fix *x'* :: *'a*
show *map-of (zip xs ys) x' = map-of (zip xs zs) x'*
proof (*cases x = x'*)
case *True*
from *assms True map-of-zip-is-None* [*of xs ys x'*]
have *map-of (zip xs ys) x' = None* **by** *simp*
moreover from *assms True map-of-zip-is-None* [*of xs zs x'*]
have *map-of (zip xs zs) x' = None* **by** *simp*
ultimately show *?thesis* **by** *simp*
next
case *False* **from** *assms*
have (*map-of (zip xs ys)(x ↦ y)*) *x' = (map-of (zip xs zs)(x ↦ z)) x'* **by** *auto*
with *False* **show** *?thesis* **by** *simp*
qed
qed

lemma *map-of-zip-inject*:
assumes *length ys = length xs*
and *length zs = length xs*
and *dist*: *distinct xs*
and *map-of*: *map-of (zip xs ys) = map-of (zip xs zs)*
shows *ys = zs*
using *assms(1) assms(2)*[*symmetric*] **using** *dist map-of* **proof** (*induct ys xs zs rule*: *list-induct3*)
case *Nil* **show** *?case* **by** *simp*
next
case (*Cons y ys x xs z zs*)
from ‹*map-of (zip (x#xs) (y#ys)) = map-of (zip (x#xs) (z#zs))*›
have *map-of*: *map-of (zip xs ys)(x ↦ y) = map-of (zip xs zs)(x ↦ z)* **by** *simp*
from *Cons* **have** *length ys = length xs* **and** *length zs = length xs*
and *x ∉ set xs* **by** *simp-all*
then have *map-of (zip xs ys) = map-of (zip xs zs)* **using** *map-of* **by** (*rule*

map-of-zip-upd)
with *Cons.hyps* ‹*distinct* (*x* # *xs*)› **have** *ys* = *zs* **by** *simp*
moreover from *map-of* **have** *y* = *z* **by** (*rule map-upd-eqD1*)
ultimately show *?case* **by** *simp*
qed

lemma *map-of-zip-map*:
map-of (*zip xs* (*map f xs*)) = (λ*x*. *if x* ∈ *set xs then Some* (*f x*) *else None*)
by (*induct xs*) (*simp-all add*: *expand-fun-eq*)

lemma *finite-range-map-of*: *finite* (*range* (*map-of xys*))
apply (*induct xys*)
apply (*simp-all add*: *image-constant*)
apply (*rule finite-subset*)
prefer *2* **apply** *assumption*
apply *auto*
done

lemma *map-of-SomeD*: *map-of xs k* = *Some y* ⟹ (*k*, *y*) ∈ *set xs*
by (*induct xs*) (*simp*, *atomize* (*full*), *auto*)

lemma *map-of-mapk-SomeI*:
inj f ==> *map-of t k* = *Some x* ==>
map-of (*map* (*split* (%*k*. *Pair* (*f k*))) *t*) (*f k*) = *Some x*
by (*induct t*) (*auto simp add*: *inj-eq*)

lemma *weak-map-of-SomeI*: (*k*, *x*) : *set l* ==> ∃*x*. *map-of l k* = *Some x*
by (*induct l*) *auto*

lemma *map-of-filter-in*:
map-of xs k = *Some z* ⟹ *P k z* ⟹ *map-of* (*filter* (*split P*) *xs*) *k* = *Some z*
by (*induct xs*) *auto*

lemma *map-of-map*:
map-of (*map* (λ(*k*, *v*). (*k*, *f v*)) *xs*) = *Option.map f* ∘ *map-of xs*
by (*induct xs*) (*auto simp add*: *expand-fun-eq*)

lemma *dom-option-map*:
dom (λ*k*. *Option.map* (*f k*) (*m k*)) = *dom m*
by (*simp add*: *dom-def*)

53.4 *Option.map* related

lemma *option-map-o-empty* [*simp*]: *Option.map f o empty* = *empty*
by (*rule ext*) *simp*

lemma *option-map-o-map-upd* [*simp*]:
Option.map f o m(*a*|−>*b*) = (*Option.map f o m*)(*a*|−>*f b*)
by (*rule ext*) *simp*

53.5 *map-comp* related

lemma *map-comp-empty* [*simp*]:
m $\circ_m$ *empty* = *empty*
empty $\circ_m$ *m* = *empty*
by (*auto simp add*: *map-comp-def intro*: *ext split*: *option.splits*)

lemma *map-comp-simps* [*simp*]:
m2 k = *None* $\Longrightarrow$ (*m1* $\circ_m$ *m2*) *k* = *None*
m2 k = *Some k'* $\Longrightarrow$ (*m1* $\circ_m$ *m2*) *k* = *m1 k'*
by (*auto simp add*: *map-comp-def*)

lemma *map-comp-Some-iff*:
((*m1* $\circ_m$ *m2*) *k* = *Some v*) = ($\exists$ *k'*. *m2 k* = *Some k'* $\wedge$ *m1 k'* = *Some v*)
by (*auto simp add*: *map-comp-def split*: *option.splits*)

lemma *map-comp-None-iff*:
((*m1* $\circ_m$ *m2*) *k* = *None*) = (*m2 k* = *None* $\vee$ ($\exists$ *k'*. *m2 k* = *Some k'* $\wedge$ *m1 k'* = *None*))
by (*auto simp add*: *map-comp-def split*: *option.splits*)

53.6 ++

lemma *map-add-empty*[*simp*]: *m* ++ *empty* = *m*
by(*simp add*: *map-add-def*)

lemma *empty-map-add*[*simp*]: *empty* ++ *m* = *m*
by (*rule ext*) (*simp add*: *map-add-def split*: *option.split*)

lemma *map-add-assoc*[*simp*]: *m1* ++ (*m2* ++ *m3*) = (*m1* ++ *m2*) ++ *m3*
by (*rule ext*) (*simp add*: *map-add-def split*: *option.split*)

lemma *map-add-Some-iff*:
((*m* ++ *n*) *k* = *Some x*) = (*n k* = *Some x* | *n k* = *None* & *m k* = *Some x*)
by (*simp add*: *map-add-def split*: *option.split*)

lemma *map-add-SomeD* [*dest*!]:
(*m* ++ *n*) *k* = *Some x* $\Longrightarrow$ *n k* = *Some x* $\vee$ *n k* = *None* $\wedge$ *m k* = *Some x*
by (*rule map-add-Some-iff* [*THEN iffD1*])

lemma *map-add-find-right* [*simp*]: !!*xx*. *n k* = *Some xx* ==> (*m* ++ *n*) *k* = *Some xx*
by (*subst map-add-Some-iff*) *fast*

lemma *map-add-None* [*iff*]: ((*m* ++ *n*) *k* = *None*) = (*n k* = *None* & *m k* = *None*)
by (*simp add*: *map-add-def split*: *option.split*)

lemma *map-add-upd*[*simp*]: *f* ++ *g*(*x*|->*y*) = (*f* ++ *g*)(*x*|->*y*)
by (*rule ext*) (*simp add*: *map-add-def*)

lemma *map-add-upds*[*simp*]: *m1 ++ (m2*(*xs*[↦]*ys*)) = (*m1*++*m2*)(*xs*[↦]*ys*)
by (*simp add*: *map-upds-def*)

lemma *map-add-upd-left*: *m*∉*dom e2* ⟹ *e1*(*m* ↦ *u1*) ++ *e2* = (*e1* ++ *e2*)(*m* ↦ *u1*)
by (*rule ext*) (*auto simp*: *map-add-def dom-def split*: *option.split*)

lemma *map-of-append*[*simp*]: *map-of* (*xs* @ *ys*) = *map-of ys* ++ *map-of xs*
unfolding *map-add-def*
apply (*induct xs*)
apply *simp*
apply (*rule ext*)
apply (*simp split add*: *option.split*)
done

lemma *finite-range-map-of-map-add*:
finite (*range f*) ==> *finite* (*range* (*f* ++ *map-of l*))
apply (*induct l*)
apply (*auto simp del*: *fun-upd-apply*)
apply (*erule finite-range-updI*)
done

lemma *inj-on-map-add-dom* [*iff*]:
inj-on (*m* ++ *m'*) (*dom m'*) = *inj-on m'* (*dom m'*)
by (*fastsimp simp*: *map-add-def dom-def inj-on-def split*: *option.splits*)

lemma *map-upds-fold-map-upd*:
m(*ks*[↦]*vs*) = *foldl* (λ*m* (*k*, *v*). *m*(*k* ↦ *v*)) *m* (*zip ks vs*)
unfolding *map-upds-def* **proof** (*rule sym*, *rule zip-obtain-same-length*)
fix *ks* :: *'a list* **and** *vs* :: *'b list*
assume *length ks* = *length vs*
then show *foldl* (λ*m* (*k*, *v*). *m*(*k*↦*v*)) *m* (*zip ks vs*) = *m* ++ *map-of* (*rev* (*zip ks vs*))
by(*induct arbitrary*: *m rule*: *list-induct2*) *simp-all*
qed

lemma *map-add-map-of-foldr*:
m ++ *map-of ps* = *foldr* (λ(*k*, *v*) *m*. *m*(*k* ↦ *v*)) *ps m*
by (*induct ps*) (*auto simp add*: *expand-fun-eq map-add-def*)

53.7 *restrict-map*

lemma *restrict-map-to-empty* [*simp*]: *m*|`{} = *empty*
by (*simp add*: *restrict-map-def*)

lemma *restrict-map-insert*: *f* |` (*insert a A*) = (*f* |` *A*)(*a* := *f a*)
by (*auto simp add*: *restrict-map-def intro*: *ext*)

lemma *restrict-map-empty* [*simp*]: $empty|`D = empty$
by (*simp add*: *restrict-map-def*)

lemma *restrict-in* [*simp*]: $x \in A \Longrightarrow (m|`A)\ x = m\ x$
by (*simp add*: *restrict-map-def*)

lemma *restrict-out* [*simp*]: $x \notin A \Longrightarrow (m|`A)\ x = None$
by (*simp add*: *restrict-map-def*)

lemma *ran-restrictD*: $y \in ran\ (m|`A) \Longrightarrow \exists x{\in}A.\ m\ x = Some\ y$
by (*auto simp*: *restrict-map-def ran-def split*: *split-if-asm*)

lemma *dom-restrict* [*simp*]: $dom\ (m|`A) = dom\ m \cap A$
by (*auto simp*: *restrict-map-def dom-def split*: *split-if-asm*)

lemma *restrict-upd-same* [*simp*]: $m(x{\mapsto}y)|`(-\{x\}) = m|`(-\{x\})$
by (*rule ext*) (*auto simp*: *restrict-map-def*)

lemma *restrict-restrict* [*simp*]: $m|`A|`B = m|`(A{\cap}B)$
by (*rule ext*) (*auto simp*: *restrict-map-def*)

lemma *restrict-fun-upd* [*simp*]:
$m(x := y)|`D = (if\ x \in D\ then\ (m|`(D{-}\{x\}))(x := y)\ else\ m|`D)$
by (*simp add*: *restrict-map-def expand-fun-eq*)

lemma *fun-upd-None-restrict* [*simp*]:
$(m|`D)(x := None) = (if\ x{:}D\ then\ m|`(D - \{x\})\ else\ m|`D)$
by (*simp add*: *restrict-map-def expand-fun-eq*)

lemma *fun-upd-restrict*: $(m|`D)(x := y) = (m|`(D{-}\{x\}))(x := y)$
by (*simp add*: *restrict-map-def expand-fun-eq*)

lemma *fun-upd-restrict-conv* [*simp*]:
$x \in D \Longrightarrow (m|`D)(x := y) = (m|`(D{-}\{x\}))(x := y)$
by (*simp add*: *restrict-map-def expand-fun-eq*)

lemma *map-of-map-restrict*:
map-of (*map* ($\lambda k.\ (k, f\ k)$) *ks*) = (*Some* $\circ$ *f*) |` *set ks*
by (*induct ks*) (*simp-all add*: *expand-fun-eq restrict-map-insert*)

lemma *restrict-complement-singleton-eq*:
f |` (− {*x*}) = *f*(*x* := *None*)
by (*simp add*: *restrict-map-def expand-fun-eq*)

53.8 *map-upds*

lemma *map-upds-Nil1* [*simp*]: $m([]\ [|{->}]\ bs) = m$
by (*simp add*: *map-upds-def*)

lemma *map-upds-Nil2* [*simp*]: $m(as\ [|{->}]\ []) = m$
by (*simp add*:*map-upds-def*)

lemma *map-upds-Cons* [*simp*]: $m(a\#as\ [|{->}]\ b\#bs) = (m(a|{->}b))(as[|{->}]bs)$
by (*simp add*:*map-upds-def*)

lemma *map-upds-append1* [*simp*]: $\bigwedge ys\ m.\ size\ xs < size\ ys \Longrightarrow$
$m(xs@[x]\ [\mapsto]\ ys) = m(xs\ [\mapsto]\ ys)(x \mapsto ys!size\ xs)$
apply(*induct xs*)
apply (*clarsimp simp add*: *neq-Nil-conv*)
apply (*case-tac ys*)
apply *simp*
apply *simp*
done

lemma *map-upds-list-update2-drop* [*simp*]:
$[\![size\ xs \leq i;\ i < size\ ys]\!]$
$\Longrightarrow m(xs[\mapsto]ys[i:=y]) = m(xs[\mapsto]ys)$
apply (*induct xs arbitrary*: *m ys i*)
apply *simp*
apply (*case-tac ys*)
apply *simp*
apply (*simp split*: *nat.split*)
done

lemma *map-upd-upds-conv-if*:
$(f(x|{->}y))(xs\ [|{->}]\ ys) =$
$(if\ x : set(take\ (length\ ys)\ xs)\ then\ f(xs\ [|{->}]\ ys)$
$else\ (f(xs\ [|{->}]\ ys))(x|{->}y))$
apply (*induct xs arbitrary*: *x y ys f*)
apply *simp*
apply (*case-tac ys*)
apply (*auto split*: *split-if simp*: *fun-upd-twist*)
done

lemma *map-upds-twist* [*simp*]:
$a\ {}^{\sim}{:}\ set\ as\ =={>}\ m(a|{->}b)(as[|{->}]bs) = m(as[|{->}]bs)(a|{->}b)$
using *set-take-subset* **by** (*fastsimp simp add*: *map-upd-upds-conv-if*)

lemma *map-upds-apply-nontin* [*simp*]:
$x\ {}^{\sim}{:}\ set\ xs\ =={>}\ (f(xs[|{->}]ys))\ x = f\ x$
apply (*induct xs arbitrary*: *ys*)
apply *simp*
apply (*case-tac ys*)
apply (*auto simp*: *map-upd-upds-conv-if*)
done

lemma *fun-upds-append-drop* [*simp*]:
$size\ xs = size\ ys \Longrightarrow m(xs@zs[\mapsto]ys) = m(xs[\mapsto]ys)$

apply (*induct xs arbitrary*: *m ys*)
apply *simp*
apply (*case-tac ys*)
apply *simp-all*
done

lemma *fun-upds-append2-drop* [*simp*]:
size xs = *size ys* ⟹ *m*(*xs*[↦]*ys*@*zs*) = *m*(*xs*[↦]*ys*)
apply (*induct xs arbitrary*: *m ys*)
apply *simp*
apply (*case-tac ys*)
apply *simp-all*
done

lemma *restrict-map-upds*[*simp*]:
⟦ *length xs* = *length ys*; *set xs* ⊆ *D* ⟧
⟹ *m*(*xs* [↦] *ys*)|`*D* = (*m*|`(*D* − *set xs*))(*xs* [↦] *ys*)
apply (*induct xs arbitrary*: *m ys*)
apply *simp*
apply (*case-tac ys*)
apply *simp*
apply (*simp add*: *Diff-insert* [*symmetric*] *insert-absorb*)
apply (*simp add*: *map-upd-upds-conv-if*)
done

53.9 *dom*

lemma *dom-eq-empty-conv* [*simp*]: *dom f* = {} ⟷ *f* = *empty*
by(*auto intro*!:*ext simp*: *dom-def*)

lemma *domI*: *m a* = *Some b* ==> *a* : *dom m*
by(*simp add*:*dom-def*)

lemma *domD*: *a* : *dom m* ==> ∃ *b*. *m a* = *Some b*
by (*cases m a*) (*auto simp add*: *dom-def*)

lemma *domIff* [*iff*, *simp del*]: (*a* : *dom m*) = (*m a* ~= *None*)
by(*simp add*:*dom-def*)

lemma *dom-empty* [*simp*]: *dom empty* = {}
by(*simp add*:*dom-def*)

lemma *dom-fun-upd* [*simp*]:
dom(*f*(*x* := *y*)) = (*if y*=*None then dom f* − {*x*} *else insert x* (*dom f*))
by(*auto simp add*:*dom-def*)

lemma *dom-if*:

dom (λ*x*. *if P x then f x else g x*) = *dom f* ∩ {*x*. *P x*} ∪ *dom g* ∩ {*x*. ¬ *P x*}
by (*auto split*: *if-splits*)

lemma *dom-map-of-conv-image-fst*:
dom (*map-of xys*) = *fst* ' *set xys*
by (*induct xys*) (*auto simp add*: *dom-if*)

lemma *dom-map-of-zip* [*simp*]: [| *length xs* = *length ys*; *distinct xs* |] ==>
dom(*map-of*(*zip xs ys*)) = *set xs*
by (*induct rule*: *list-induct2*) *simp-all*

lemma *finite-dom-map-of*: *finite* (*dom* (*map-of l*))
by (*induct l*) (*auto simp add*: *dom-def insert-Collect* [*symmetric*])

lemma *dom-map-upds* [*simp*]:
dom(*m*(*xs*[|->]*ys*)) = *set*(*take* (*length ys*) *xs*) *Un dom m*
apply (*induct xs arbitrary*: *m ys*)
apply *simp*
apply (*case-tac ys*)
apply *auto*
done

lemma *dom-map-add* [*simp*]: *dom*(*m*++*n*) = *dom n Un dom m*
by(*auto simp*:*dom-def*)

lemma *dom-override-on* [*simp*]:
dom(*override-on f g A*) =
(*dom f* − {*a*. *a* : *A* − *dom g*}) *Un* {*a*. *a* : *A Int dom g*}
by(*auto simp*: *dom-def override-on-def*)

lemma *map-add-comm*: *dom m1* ∩ *dom m2* = {} ⟹ *m1*++*m2* = *m2*++*m1*
by (*rule ext*) (*force simp*: *map-add-def dom-def split*: *option.split*)

lemma *map-add-dom-app-simps*:
⟦ *m*∈*dom l2* ⟧ ⟹ (*l1*++*l2*) *m* = *l2 m*
⟦ *m*∉*dom l1* ⟧ ⟹ (*l1*++*l2*) *m* = *l2 m*
⟦ *m*∉*dom l2* ⟧ ⟹ (*l1*++*l2*) *m* = *l1 m*
by (*auto simp add*: *map-add-def split*: *option.split-asm*)

lemma *dom-const* [*simp*]:
dom (λ*x*. *Some* (*f x*)) = *UNIV*
by *auto*

lemma *finite-map-freshness*:
finite (*dom* (*f* :: 'a ⇀ 'b)) ⟹ ¬ *finite* (*UNIV* :: 'a *set*) ⟹
∃ *x*. *f x* = *None*
by(*bestsimp dest*:*ex-new-if-finite*)

lemma *dom-minus*:
f x = None ⟹ dom f − insert x A = dom f − A
unfolding *dom-def* **by** *simp*

lemma *insert-dom*:
f x = Some y ⟹ insert x (dom f) = dom f
unfolding *dom-def* **by** *auto*

lemma *map-of-map-keys*:
set xs = dom m ⟹ map-of (map (λk. (k, the (m k))) xs) = m
by (*rule ext*) (*auto simp add*: *map-of-map-restrict restrict-map-def*)

53.10 *ran*

lemma *ranI*: *m a = Some b ==> b : ran m*
by(*auto simp*: *ran-def*)

lemma *ran-empty* [*simp*]: *ran empty = {}*
by(*auto simp*: *ran-def*)

lemma *ran-map-upd* [*simp*]: *m a = None ==> ran(m(a|->b)) = insert b (ran m)*
unfolding *ran-def*
apply *auto*
apply (*subgoal-tac aa ~= a*)
apply *auto*
done

lemma *ran-distinct*:
assumes *dist*: *distinct* (*map fst al*)
shows *ran* (*map-of al*) = *snd ' set al*
using *assms* **proof** (*induct al*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons kv al*)
then have *ran* (*map-of al*) = *snd ' set al* **by** *simp*
moreover from *Cons.prems* **have** *map-of al* (*fst kv*) = *None*
by (*simp add*: *map-of-eq-None-iff*)
ultimately show *?case* **by** (*simp only*: *map-of.simps ran-map-upd*) *simp*
qed

53.11 *map-le*

lemma *map-le-empty* [*simp*]: *empty ⊆m g*
by (*simp add*: *map-le-def*)

lemma *upd-None-map-le* [*simp*]: *f(x := None) ⊆m f*
by (*force simp add*: *map-le-def*)

lemma *map-le-upd*[*simp*]: $f \subseteq_m g \Longrightarrow f(a := b) \subseteq_m g(a := b)$
by (*fastsimp simp add*: *map-le-def*)

lemma *map-le-imp-upd-le* [*simp*]: $m1 \subseteq_m m2 \Longrightarrow m1(x := None) \subseteq_m m2(x \mapsto y)$
by (*force simp add*: *map-le-def*)

lemma *map-le-upds* [*simp*]:
$f \subseteq_m g \Longrightarrow f(as\ [|->]\ bs) \subseteq_m g(as\ [|->]\ bs)$
apply (*induct as arbitrary*: *f g bs*)
apply *simp*
apply (*case-tac bs*)
apply *auto*
done

lemma *map-le-implies-dom-le*: $(f \subseteq_m g) \Longrightarrow (dom\ f \subseteq dom\ g)$
by (*fastsimp simp add*: *map-le-def dom-def*)

lemma *map-le-refl* [*simp*]: $f \subseteq_m f$
by (*simp add*: *map-le-def*)

lemma *map-le-trans*[*trans*]: $[\![m1 \subseteq_m m2; m2 \subseteq_m m3]\!] \Longrightarrow m1 \subseteq_m m3$
by (*auto simp add*: *map-le-def dom-def*)

lemma *map-le-antisym*: $[\![f \subseteq_m g; g \subseteq_m f]\!] \Longrightarrow f = g$
unfolding *map-le-def*
apply (*rule ext*)
apply (*case-tac* $x \in dom\ f$, *simp*)
apply (*case-tac* $x \in dom\ g$, *simp*, *fastsimp*)
done

lemma *map-le-map-add* [*simp*]: $f \subseteq_m (g ++ f)$
by (*fastsimp simp add*: *map-le-def*)

lemma *map-le-iff-map-add-commute*: $(f \subseteq_m f ++ g) = (f++g = g++f)$
by(*fastsimp simp*: *map-add-def map-le-def expand-fun-eq split*: *option.splits*)

lemma *map-add-le-mapE*: $f++g \subseteq_m h \Longrightarrow g \subseteq_m h$
by (*fastsimp simp add*: *map-le-def map-add-def dom-def*)

lemma *map-add-le-mapI*: $[\![f \subseteq_m h; g \subseteq_m h; f \subseteq_m f++g]\!] \Longrightarrow f++g \subseteq_m h$
by (*clarsimp simp add*: *map-le-def map-add-def dom-def split*: *option.splits*)

lemma *dom-eq-singleton-conv*: $dom\ f = \{x\} \longleftrightarrow (\exists\, v.\ f = [x \mapsto v])$
proof(*rule iffI*)
assume $\exists\, v.\ f = [x \mapsto v]$
thus $dom\ f = \{x\}$ **by**(*auto split*: *split-if-asm*)
next
assume $dom\ f = \{x\}$

then obtain v **where** $f\ x = Some\ v$ **by** *auto*
hence $[x \mapsto v] \subseteq_m f$ **by**(*auto simp add*: *map-le-def*)
moreover have $f \subseteq_m [x \mapsto v]$ **using** ‹*dom* $f = \{x\}$› ‹$f\ x = Some\ v$›
by(*auto simp add*: *map-le-def*)
ultimately have $f = [x \mapsto v]$ **by**−(*rule map-le-antisym*)
thus $\exists v.\ f = [x \mapsto v]$ **by** *blast*
qed

53.12 Various

lemma *set-map-of-compr*:
assumes *distinct*: *distinct* (*map fst xs*)
shows *set xs* = {(*k*, *v*). *map-of xs k* = *Some v*}
using *assms* **proof** (*induct xs*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*)
obtain *k v* **where** *x* = (*k*, *v*) **by** (*cases x*) *blast*
with *Cons.prems* **have** *k* ∉ *dom* (*map-of xs*)
by (*simp add*: *dom-map-of-conv-image-fst*)
then have ∗: *insert* (*k*, *v*) {(*k*, *v*). *map-of xs k* = *Some v*} =
{(*k′*, *v′*). (*map-of xs*(*k* ↦ *v*)) *k′* = *Some v′*}
by (*auto split*: *if-splits*)
from *Cons* **have** *set xs* = {(*k*, *v*). *map-of xs k* = *Some v*} **by** *simp*
with ∗ ‹*x* = (*k*, *v*)› **show** *?case* **by** *simp*
qed

lemma *map-of-inject-set*:
assumes *distinct*: *distinct* (*map fst xs*) *distinct* (*map fst ys*)
shows *map-of xs* = *map-of ys* ⟷ *set xs* = *set ys* (**is** *?lhs* ⟷ *?rhs*)
proof
assume *?lhs*
moreover from ‹*distinct* (*map fst xs*)› **have** *set xs* = {(*k*, *v*). *map-of xs k* = *Some v*}
by (*rule set-map-of-compr*)
moreover from ‹*distinct* (*map fst ys*)› **have** *set ys* = {(*k*, *v*). *map-of ys k* = *Some v*}
by (*rule set-map-of-compr*)
ultimately show *?rhs* **by** *simp*
next
assume *?rhs* **show** *?lhs* **proof**
fix *k*
show *map-of xs k* = *map-of ys k* **proof** (*cases map-of xs k*)
case *None*
moreover with ‹*?rhs*› **have** *map-of ys k* = *None*
by (*simp add*: *map-of-eq-None-iff*)
ultimately show *?thesis* **by** *simp*
next
case (*Some v*)

```
    moreover with distinct ‹?rhs› have map-of ys k = Some v
      by simp
    ultimately show ?thesis by simp
  qed
 qed
qed

end
```

54 Quotient: Definition of Quotient Types

theory *Quotient*
imports *Plain Sledgehammer*
uses
(~~/*src*/*HOL*/*Tools*/*Quotient*/*quotient-info.ML*)
(~~/*src*/*HOL*/*Tools*/*Quotient*/*quotient-typ.ML*)
(~~/*src*/*HOL*/*Tools*/*Quotient*/*quotient-def.ML*)
(~~/*src*/*HOL*/*Tools*/*Quotient*/*quotient-term.ML*)
(~~/*src*/*HOL*/*Tools*/*Quotient*/*quotient-tacs.ML*)
begin

Basic definition for equivalence relations that are represented by predicates.

definition
$equivp\ E \equiv \forall x\ y.\ E\ x\ y = (E\ x = E\ y)$

definition
$reflp\ E \equiv \forall x.\ E\ x\ x$

definition
$symp\ E \equiv \forall x\ y.\ E\ x\ y \longrightarrow E\ y\ x$

definition
$transp\ E \equiv \forall x\ y\ z.\ E\ x\ y \wedge E\ y\ z \longrightarrow E\ x\ z$

lemma *equivp-reflp-symp-transp*:
shows $equivp\ E = (reflp\ E \wedge symp\ E \wedge transp\ E)$
unfolding *equivp-def reflp-def symp-def transp-def expand-fun-eq*
by *blast*

lemma *equivp-reflp*:
shows $equivp\ E \Longrightarrow E\ x\ x$
by (*simp only*: *equivp-reflp-symp-transp reflp-def*)

lemma *equivp-symp*:
shows $equivp\ E \Longrightarrow E\ x\ y \Longrightarrow E\ y\ x$
by (*metis equivp-reflp-symp-transp symp-def*)

lemma *equivp-transp*:

shows *equivp E* $\Longrightarrow$ *E x y* $\Longrightarrow$ *E y z* $\Longrightarrow$ *E x z*
by (*metis equivp-reflp-symp-transp transp-def*)

lemma *equivpI*:
assumes *reflp R symp R transp R*
shows *equivp R*
using *assms* **by** (*simp add*: *equivp-reflp-symp-transp*)

lemma *identity-equivp*:
shows *equivp* (*op* =)
unfolding *equivp-def*
by *auto*

Partial equivalences: not yet used anywhere

definition
part-equivp E $\equiv$ ($\exists$ *x*. *E x x*) $\wedge$ ($\forall$ *x y*. *E x y* = (*E x x* $\wedge$ *E y y* $\wedge$ (*E x* = *E y*)))

lemma *equivp-implies-part-equivp*:
assumes *a*: *equivp E*
shows *part-equivp E*
using *a*
unfolding *equivp-def part-equivp-def*
by *auto*

Composition of Relations

abbreviation
rel-conj (**infixr** *OOO 75*)
where
r1 OOO r2 $\equiv$ *r1 OO r2 OO r1*

lemma *eq-comp-r*:
shows ((*op* =) *OOO R*) = *R*
by (*auto simp add*: *expand-fun-eq*)

54.1 Respects predicate

definition
Respects
where
Respects R x $\equiv$ *R x x*

lemma *in-respects*:
shows (*x* $\in$ *Respects R*) = *R x x*
unfolding *mem-def Respects-def*
by *simp*

54.2 Function map and function relation

definition

fun-map (**infixr** $---{>}$ *55*)
where
[*simp*]: *fun-map f g h x* = *g* (*h* (*f x*))

definition
fun-rel (**infixr** $==={>}$ *55*)
where
[*simp*]: *fun-rel E1 E2 f g* = (∀ *x y*. *E1 x y* ⟶ *E2* (*f x*) (*g y*))

lemma *fun-relI* [*intro*]:
assumes ⋀*a b*. *P a b* ⟹ *Q* (*x a*) (*y b*)
shows (*P* $==={>}$ *Q*) *x y*
using *assms* **by** (*simp add*: *fun-rel-def*)

lemma *fun-map-id*:
shows (*id* $---{>}$ *id*) = *id*
by (*simp add*: *expand-fun-eq id-def*)

lemma *fun-rel-eq*:
shows ((*op* =) $==={>}$ (*op* =)) = (*op* =)
by (*simp add*: *expand-fun-eq*)

lemma *fun-rel-id*:
assumes *a*: ⋀*x y*. *R1 x y* ⟹ *R2* (*f x*) (*g y*)
shows (*R1* $==={>}$ *R2*) *f g*
using *a* **by** *simp*

lemma *fun-rel-id-asm*:
assumes *a*: ⋀*x y*. *R1 x y* ⟹ (*A* ⟶ *R2* (*f x*) (*g y*))
shows *A* ⟶ (*R1* $==={>}$ *R2*) *f g*
using *a* **by** *auto*

54.3 Quotient Predicate

definition
Quotient E Abs Rep ≡
(∀ *a*. *Abs* (*Rep a*) = *a*) ∧ (∀ *a*. *E* (*Rep a*) (*Rep a*)) ∧
(∀ *r s*. *E r s* = (*E r r* ∧ *E s s* ∧ (*Abs r* = *Abs s*)))

lemma *Quotient-abs-rep*:
assumes *a*: *Quotient E Abs Rep*
shows *Abs* (*Rep a*) = *a*
using *a*
unfolding *Quotient-def*
by *simp*

lemma *Quotient-rep-reflp*:
assumes *a*: *Quotient E Abs Rep*
shows *E* (*Rep a*) (*Rep a*)

using *a*
unfolding *Quotient-def*
by *blast*

lemma *Quotient-rel*:
assumes *a*: *Quotient E Abs Rep*
shows $E\ r\ s = (E\ r\ r \wedge E\ s\ s \wedge (Abs\ r = Abs\ s))$
using *a*
unfolding *Quotient-def*
by *blast*

lemma *Quotient-rel-rep*:
assumes *a*: *Quotient R Abs Rep*
shows $R\ (Rep\ a)\ (Rep\ b) = (a = b)$
using *a*
unfolding *Quotient-def*
by *metis*

lemma *Quotient-rep-abs*:
assumes *a*: *Quotient R Abs Rep*
shows $R\ r\ r \Longrightarrow R\ (Rep\ (Abs\ r))\ r$
using *a* **unfolding** *Quotient-def*
by *blast*

lemma *Quotient-rel-abs*:
assumes *a*: *Quotient E Abs Rep*
shows $E\ r\ s \Longrightarrow Abs\ r = Abs\ s$
using *a* **unfolding** *Quotient-def*
by *blast*

lemma *Quotient-symp*:
assumes *a*: *Quotient E Abs Rep*
shows *symp E*
using *a* **unfolding** *Quotient-def symp-def*
by *metis*

lemma *Quotient-transp*:
assumes *a*: *Quotient E Abs Rep*
shows *transp E*
using *a* **unfolding** *Quotient-def transp-def*
by *metis*

lemma *identity-quotient*:
shows *Quotient* $(op =)$ *id id*
unfolding *Quotient-def id-def*
by *blast*

lemma *fun-quotient*:
assumes *q1*: *Quotient R1 abs1 rep1*

```
  and     q2: Quotient R2 abs2 rep2
  shows Quotient (R1 ===> R2) (rep1 ---> abs2) (abs1 ---> rep2)
proof -
  have ∀ a. (rep1 ---> abs2) ((abs1 ---> rep2) a) = a
    using q1 q2
    unfolding Quotient-def
    unfolding expand-fun-eq
    by simp
  moreover
  have ∀ a. (R1 ===> R2) ((abs1 ---> rep2) a) ((abs1 ---> rep2) a)
    using q1 q2
    unfolding Quotient-def
    by (simp (no-asm)) (metis)
  moreover
  have ∀ r s. (R1 ===> R2) r s = ((R1 ===> R2) r r ∧ (R1 ===> R2) s s
∧
       (rep1 ---> abs2) r  = (rep1 ---> abs2) s)
    unfolding expand-fun-eq
    apply(auto)
    using q1 q2 unfolding Quotient-def
    apply(metis)
    using q1 q2 unfolding Quotient-def
    apply(metis)
    using q1 q2 unfolding Quotient-def
    apply(metis)
    using q1 q2 unfolding Quotient-def
    apply(metis)
    done
  ultimately
  show Quotient (R1 ===> R2) (rep1 ---> abs2) (abs1 ---> rep2)
    unfolding Quotient-def by blast
qed

lemma abs-o-rep:
  assumes a: Quotient R Abs Rep
  shows Abs o Rep = id
  unfolding expand-fun-eq
  by (simp add: Quotient-abs-rep[OF a])

lemma equals-rsp:
  assumes q: Quotient R Abs Rep
  and     a: R xa xb R ya yb
  shows R xa ya = R xb yb
  using a Quotient-symp[OF q] Quotient-transp[OF q]
  unfolding symp-def transp-def
  by blast

lemma lambda-prs:
  assumes q1: Quotient R1 Abs1 Rep1
```

and *q2*: *Quotient R2 Abs2 Rep2*
shows (*Rep1* −−−> *Abs2*) (λ*x*. *Rep2* (*f* (*Abs1 x*))) = (λ*x*. *f x*)
unfolding *expand-fun-eq*
using *Quotient-abs-rep*[*OF q1*] *Quotient-abs-rep*[*OF q2*]
by *simp*

lemma *lambda-prs1*:
assumes *q1*: *Quotient R1 Abs1 Rep1*
and *q2*: *Quotient R2 Abs2 Rep2*
shows (*Rep1* −−−> *Abs2*) (λ*x*. (*Abs1* −−−> *Rep2*) *f x*) = (λ*x*. *f x*)
unfolding *expand-fun-eq*
using *Quotient-abs-rep*[*OF q1*] *Quotient-abs-rep*[*OF q2*]
by *simp*

lemma *rep-abs-rsp*:
assumes *q*: *Quotient R Abs Rep*
and *a*: *R x1 x2*
shows *R x1* (*Rep* (*Abs x2*))
using *a Quotient-rel*[*OF q*] *Quotient-abs-rep*[*OF q*] *Quotient-rep-reflp*[*OF q*]
by *metis*

lemma *rep-abs-rsp-left*:
assumes *q*: *Quotient R Abs Rep*
and *a*: *R x1 x2*
shows *R* (*Rep* (*Abs x1*)) *x2*
using *a Quotient-rel*[*OF q*] *Quotient-abs-rep*[*OF q*] *Quotient-rep-reflp*[*OF q*]
by *metis*

In the following theorem R1 can be instantiated with anything, but we know some of the types of the Rep and Abs functions; so by solving Quotient assumptions we can get a unique R1 that will be provable; which is why we need to use *apply-rsp* and not the primed version

lemma *apply-rsp*:
fixes *f g*::'*a* ⇒ '*c*
assumes *q*: *Quotient R1 Abs1 Rep1*
and *a*: (*R1* ===> *R2*) *f g R1 x y*
shows *R2* (*f x*) (*g y*)
using *a* **by** *simp*

lemma *apply-rsp'*:
assumes *a*: (*R1* ===> *R2*) *f g R1 x y*
shows *R2* (*f x*) (*g y*)
using *a* **by** *simp*

54.4 lemmas for regularisation of ball and bex

lemma *ball-reg-eqv*:
fixes *P* :: '*a* ⇒ *bool*
assumes *a*: *equivp R*

shows *Ball* (*Respects R*) *P* = (*All P*)
using *a*
unfolding *equivp-def*
by (*auto simp add*: *in-respects*)

lemma *bex-reg-eqv*:
fixes *P* :: *'a* ⇒ *bool*
assumes *a*: *equivp R*
shows *Bex* (*Respects R*) *P* = (*Ex P*)
using *a*
unfolding *equivp-def*
by (*auto simp add*: *in-respects*)

lemma *ball-reg-right*:
assumes *a*: ⋀*x*. *R x* ⟹ *P x* ⟶ *Q x*
shows *All P* ⟶ *Ball R Q*
using *a* **by** (*metis COMBC-def Collect-def Collect-mem-eq*)

lemma *bex-reg-left*:
assumes *a*: ⋀*x*. *R x* ⟹ *Q x* ⟶ *P x*
shows *Bex R Q* ⟶ *Ex P*
using *a* **by** (*metis COMBC-def Collect-def Collect-mem-eq*)

lemma *ball-reg-left*:
assumes *a*: *equivp R*
shows (⋀*x*. (*Q x* ⟶ *P x*)) ⟹ *Ball* (*Respects R*) *Q* ⟶ *All P*
using *a* **by** (*metis equivp-reflp in-respects*)

lemma *bex-reg-right*:
assumes *a*: *equivp R*
shows (⋀*x*. (*Q x* ⟶ *P x*)) ⟹ *Ex Q* ⟶ *Bex* (*Respects R*) *P*
using *a* **by** (*metis equivp-reflp in-respects*)

lemma *ball-reg-eqv-range*:
fixes *P*::*'a* ⇒ *bool*
and *x*::*'a*
assumes *a*: *equivp R2*
shows (*Ball* (*Respects* (*R1* ===> *R2*)) (λ*f*. *P* (*f x*)) = *All* (λ*f*. *P* (*f x*)))
apply(*rule iffI*)
apply(*rule allI*)
apply(*drule-tac x*=λ*y*. *f x* **in** *bspec*)
apply(*simp add*: *in-respects*)
apply(*rule impI*)
using *a equivp-reflp-symp-transp*[*of R2*]
apply(*simp add*: *reflp-def*)
apply(*simp*)
apply(*simp*)
done

lemma *bex-reg-eqv-range*:
assumes *a*: *equivp R2*
shows (*Bex* (*Respects* (*R1* ===> *R2*)) (λ*f*. *P* (*f x*)) = *Ex* (λ*f*. *P* (*f x*)))
apply(*auto*)
apply(*rule-tac x*=λ*y*. *f x* **in** *bexI*)
apply(*simp*)
apply(*simp add*: *Respects-def in-respects*)
apply(*rule impI*)
using *a equivp-reflp-symp-transp*[*of R2*]
apply(*simp add*: *reflp-def*)
done

lemma *all-reg*:
assumes *a*: !*x* :: 'a. (*P x* −−> *Q x*)
and *b*: *All P*
shows *All Q*
using *a b* **by** (*metis*)

lemma *ex-reg*:
assumes *a*: !*x* :: 'a. (*P x* −−> *Q x*)
and *b*: *Ex P*
shows *Ex Q*
using *a b* **by** *metis*

lemma *ball-reg*:
assumes *a*: !*x* :: 'a. (*R x* −−> *P x* −−> *Q x*)
and *b*: *Ball R P*
shows *Ball R Q*
using *a b* **by** (*metis COMBC-def Collect-def Collect-mem-eq*)

lemma *bex-reg*:
assumes *a*: !*x* :: 'a. (*R x* −−> *P x* −−> *Q x*)
and *b*: *Bex R P*
shows *Bex R Q*
using *a b* **by** (*metis COMBC-def Collect-def Collect-mem-eq*)

lemma *ball-all-comm*:
assumes ⋀*y*. (∀ *x*∈*P*. *A x y*) ⟶ (∀ *x*. *B x y*)
shows (∀ *x*∈*P*. ∀ *y*. *A x y*) ⟶ (∀ *x*. ∀ *y*. *B x y*)
using *assms* **by** *auto*

lemma *bex-ex-comm*:
assumes (∃ *y*. ∃ *x*. *A x y*) ⟶ (∃ *y*. ∃ *x*∈*P*. *B x y*)
shows (∃ *x*. ∃ *y*. *A x y*) ⟶ (∃ *x*∈*P*. ∃ *y*. *B x y*)
using *assms* **by** *auto*

54.5 Bounded abstraction

definition
Babs :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$
where
$x \in p \Longrightarrow Babs\ p\ m\ x = m\ x$

lemma *babs-rsp*:
assumes *q*: *Quotient R1 Abs1 Rep1*
and *a*: $(R1 \mathrel{===>} R2)\ f\ g$
shows $(R1 \mathrel{===>} R2)\ (Babs\ (Respects\ R1)\ f)\ (Babs\ (Respects\ R1)\ g)$
apply (*auto simp add*: *Babs-def in-respects*)
apply (*subgoal-tac* $x \in Respects\ R1 \wedge y \in Respects\ R1$)
using *a* **apply** (*simp add*: *Babs-def*)
apply (*simp add*: *in-respects*)
using *Quotient-rel*[*OF q*]
by *metis*

lemma *babs-prs*:
assumes *q1*: *Quotient R1 Abs1 Rep1*
and *q2*: *Quotient R2 Abs2 Rep2*
shows $((Rep1 \mathrel{--->} Abs2)\ (Babs\ (Respects\ R1)\ ((Abs1 \mathrel{--->} Rep2)\ f))) = f$
apply (*rule ext*)
apply (*simp*)
apply (*subgoal-tac Rep1* $x \in Respects\ R1$)
apply (*simp add*: *Babs-def Quotient-abs-rep*[*OF q1*] *Quotient-abs-rep*[*OF q2*])
apply (*simp add*: *in-respects Quotient-rel-rep*[*OF q1*])
done

lemma *babs-simp*:
assumes *q*: *Quotient R1 Abs Rep*
shows $((R1 \mathrel{===>} R2)\ (Babs\ (Respects\ R1)\ f)\ (Babs\ (Respects\ R1)\ g)) = ((R1 \mathrel{===>} R2)\ f\ g)$
apply(*rule iffI*)
apply(*simp-all only*: *babs-rsp*[*OF q*])
apply(*auto simp add*: *Babs-def*)
apply (*subgoal-tac* $x \in Respects\ R1 \wedge y \in Respects\ R1$)
apply(*metis Babs-def*)
apply (*simp add*: *in-respects*)
using *Quotient-rel*[*OF q*]
by *metis*

lemma *babs-reg-eqv*:
shows *equivp R* $\Longrightarrow$ *Babs* (*Respects R*) *P* = *P*
by (*simp add*: *expand-fun-eq Babs-def in-respects equivp-reflp*)

lemma *ball-rsp*:
assumes *a*: $(R ===> (op =))\ f\ g$
shows $Ball\ (Respects\ R)\ f = Ball\ (Respects\ R)\ g$
using *a* **by** (*simp add*: *Ball-def in-respects*)

lemma *bex-rsp*:
assumes *a*: $(R ===> (op =))\ f\ g$
shows $(Bex\ (Respects\ R)\ f = Bex\ (Respects\ R)\ g)$
using *a* **by** (*simp add*: *Bex-def in-respects*)

lemma *bex1-rsp*:
assumes *a*: $(R ===> (op =))\ f\ g$
shows $Ex1\ (\lambda x.\ x \in Respects\ R \wedge f\ x) = Ex1\ (\lambda x.\ x \in Respects\ R \wedge g\ x)$
using *a*
by (*simp add*: *Ex1-def in-respects*) *auto*

lemma *all-prs*:
assumes *a*: *Quotient R absf repf*
shows $Ball\ (Respects\ R)\ ((absf\ ---> id)\ f) = All\ f$
using *a* **unfolding** *Quotient-def Ball-def in-respects fun-map-def id-apply*
by *metis*

lemma *ex-prs*:
assumes *a*: *Quotient R absf repf*
shows $Bex\ (Respects\ R)\ ((absf\ ---> id)\ f) = Ex\ f$
using *a* **unfolding** *Quotient-def Bex-def in-respects fun-map-def id-apply*
by *metis*

54.6 *Bex1-rel* quantifier

definition
$Bex1\text{-}rel :: ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow bool) \Rightarrow bool$
where
$Bex1\text{-}rel\ R\ P \longleftrightarrow (\exists x \in Respects\ R.\ P\ x) \wedge (\forall x \in Respects\ R.\ \forall y \in Respects\ R.\ ((P\ x \wedge P\ y) \longrightarrow (R\ x\ y)))$

lemma *bex1-rel-aux*:
$[\![\forall xa\ ya.\ R\ xa\ ya \longrightarrow x\ xa = y\ ya;\ Bex1\text{-}rel\ R\ x]\!] \Longrightarrow Bex1\text{-}rel\ R\ y$
unfolding *Bex1-rel-def*
apply (*erule conjE*)+
apply (*erule bexE*)
apply *rule*
apply (*rule-tac x*=*xa* **in** *bexI*)
apply *metis*
apply *metis*
apply *rule*+
apply (*erule-tac x*=*xaa* **in** *ballE*)
prefer *2*

apply (*metis*)
apply (*erule-tac x=ya* **in** *ballE*)
prefer *2*
apply (*metis*)
apply (*metis in-respects*)
done

lemma *bex1-rel-aux2*:
⟦∀ *xa ya*. *R xa ya* ⟶ *x xa* = *y ya*; *Bex1-rel R y*⟧ ⟹ *Bex1-rel R x*
unfolding *Bex1-rel-def*
apply (*erule conjE*)+
apply (*erule bexE*)
apply *rule*
apply (*rule-tac x=xa* **in** *bexI*)
apply *metis*
apply *metis*
apply *rule*+
apply (*erule-tac x=xaa* **in** *ballE*)
prefer *2*
apply (*metis*)
apply (*erule-tac x=ya* **in** *ballE*)
prefer *2*
apply (*metis*)
apply (*metis in-respects*)
done

lemma *bex1-rel-rsp*:
assumes *a*: *Quotient R absf repf*
shows ((*R* ===> *op* =) ===> *op* =) (*Bex1-rel R*) (*Bex1-rel R*)
apply *simp*
apply *clarify*
apply *rule*
apply (*simp-all add*: *bex1-rel-aux bex1-rel-aux2*)
apply (*erule bex1-rel-aux2*)
apply *assumption*
done

lemma *ex1-prs*:
assumes *a*: *Quotient R absf repf*
shows ((*absf* ---> *id*) ---> *id*) (*Bex1-rel R*) *f* = *Ex1 f*
apply *simp*
apply (*subst Bex1-rel-def*)
apply (*subst Bex-def*)
apply (*subst Ex1-def*)
apply *simp*
apply *rule*
apply (*erule conjE*)+
apply (*erule-tac exE*)

apply (*erule conjE*)
apply (*subgoal-tac* $\forall y.\ R\ y\ y \longrightarrow f\ (absf\ y) \longrightarrow R\ x\ y$)
apply (*rule-tac x=absf x* **in** *exI*)
apply (*simp*)
apply *rule+*
using *a* **unfolding** *Quotient-def*
apply *metis*
apply *rule+*
apply (*erule-tac x=x* **in** *ballE*)
apply (*erule-tac x=y* **in** *ballE*)
apply *simp*
apply (*simp add*: *in-respects*)
apply (*simp add*: *in-respects*)
apply (*erule-tac exE*)
apply *rule*
apply (*rule-tac x=repf x* **in** *exI*)
apply (*simp only*: *in-respects*)
apply *rule*
apply (*metis Quotient-rel-rep*[*OF a*])
using *a* **unfolding** *Quotient-def* **apply** (*simp*)
apply *rule+*
using *a* **unfolding** *Quotient-def in-respects*
apply *metis*
done

lemma *bex1-bexeq-reg*: $(\exists!x{\in}Respects\ R.\ P\ x) \longrightarrow (Bex1\text{-}rel\ R\ (\lambda x.\ P\ x))$
apply (*simp add*: *Ex1-def Bex1-rel-def in-respects*)
apply *clarify*
apply *auto*
apply (*rule bexI*)
apply *assumption*
apply (*simp add*: *in-respects*)
apply (*simp add*: *in-respects*)
apply *auto*
done

54.7 Various respects and preserve lemmas

lemma *quot-rel-rsp*:
assumes *a*: *Quotient R Abs Rep*
shows (R ===> R ===> *op* =) $R\ R$
apply(*rule fun-rel-id*)+
apply(*rule equals-rsp*[*OF a*])
apply(*assumption*)+
done

lemma *o-prs*:
assumes *q1*: *Quotient R1 Abs1 Rep1*
and *q2*: *Quotient R2 Abs2 Rep2*

and *q3*: *Quotient R3 Abs3 Rep3*
shows ((*Abs2* −−−> *Rep3*) −−−> (*Abs1* −−−> *Rep2*) −−−> (*Rep1* −−−> *Abs3*)) *op* ∘ = *op* ∘
and (*id* −−−> (*Abs1* −−−> *id*) −−−> *Rep1* −−−> *id*) *op* ∘ = *op* ∘
using *Quotient-abs-rep*[*OF q1*] *Quotient-abs-rep*[*OF q2*] *Quotient-abs-rep*[*OF q3*]
unfolding *o-def expand-fun-eq* **by** *simp-all*

lemma *o-rsp*:
((*R2* ===> *R3*) ===> (*R1* ===> *R2*) ===> (*R1* ===> *R3*)) *op* ∘ *op* ∘
(*op* = ===> (*R1* ===> *op* =) ===> *R1* ===> *op* =) *op* ∘ *op* ∘
unfolding *fun-rel-def o-def expand-fun-eq* **by** *auto*

lemma *cond-prs*:
assumes *a*: *Quotient R absf repf*
shows *absf* (*if a then repf b else repf c*) = (*if a then b else c*)
using *a* **unfolding** *Quotient-def* **by** *auto*

lemma *if-prs*:
assumes *q*: *Quotient R Abs Rep*
shows (*id* −−−> *Rep* −−−> *Rep* −−−> *Abs*) *If* = *If*
using *Quotient-abs-rep*[*OF q*]
by (*auto simp add*: *expand-fun-eq*)

lemma *if-rsp*:
assumes *q*: *Quotient R Abs Rep*
shows (*op* = ===> *R* ===> *R* ===> *R*) *If If*
by *auto*

lemma *let-prs*:
assumes *q1*: *Quotient R1 Abs1 Rep1*
and *q2*: *Quotient R2 Abs2 Rep2*
shows (*Rep2* −−−> (*Abs2* −−−> *Rep1*) −−−> *Abs1*) *Let* = *Let*
using *Quotient-abs-rep*[*OF q1*] *Quotient-abs-rep*[*OF q2*]
by (*auto simp add*: *expand-fun-eq*)

lemma *let-rsp*:
shows (*R1* ===> (*R1* ===> *R2*) ===> *R2*) *Let Let*
by *auto*

locale *quot-type* =
fixes *R* :: *'a* ⇒ *'a* ⇒ *bool*
and *Abs* :: (*'a* ⇒ *bool*) ⇒ *'b*
and *Rep* :: *'b* ⇒ (*'a* ⇒ *bool*)
assumes *equivp*: *equivp R*
and *rep-prop*: ⋀*y*. ∃ *x*. *Rep y* = *R x*
and *rep-inverse*: ⋀*x*. *Abs* (*Rep x*) = *x*
and *abs-inverse*: ⋀*x*. (*Rep* (*Abs* (*R x*))) = (*R x*)
and *rep-inject*: ⋀*x y*. (*Rep x* = *Rep y*) = (*x* = *y*)
begin

definition
abs::$'a \Rightarrow 'b$
where
abs x ≡ *Abs* (*R x*)

definition
rep::$'b \Rightarrow 'a$
where
rep a = *Eps* (*Rep a*)

lemma *homeier-lem9*:
shows *R* (*Eps* (*R x*)) = *R x*
proof −
have *a*: *R x x* **using** *equivp* **by** (*simp add*: *equivp-reflp-symp-transp reflp-def*)
then have *R x* (*Eps* (*R x*)) **by** (*rule someI*)
then show *R* (*Eps* (*R x*)) = *R x*
using *equivp* **unfolding** *equivp-def* **by** *simp*
qed

theorem *homeier-thm10*:
shows *abs* (*rep a*) = *a*
unfolding *abs-def rep-def*
proof −
from *rep-prop*
obtain *x* **where** *eq*: *Rep a* = *R x* **by** *auto*
have *Abs* (*R* (*Eps* (*Rep a*))) = *Abs* (*R* (*Eps* (*R x*))) **using** *eq* **by** *simp*
also have ... = *Abs* (*R x*) **using** *homeier-lem9* **by** *simp*
also have ... = *Abs* (*Rep a*) **using** *eq* **by** *simp*
also have ... = *a* **using** *rep-inverse* **by** *simp*
finally
show *Abs* (*R* (*Eps* (*Rep a*))) = *a* **by** *simp*
qed

lemma *homeier-lem7*:
shows (*R x* = *R y*) = (*Abs* (*R x*) = *Abs* (*R y*)) (**is** *?LHS* = *?RHS*)
proof −
have *?RHS* = (*Rep* (*Abs* (*R x*)) = *Rep* (*Abs* (*R y*))) **by** (*simp add*: *rep-inject*)
also have ... = *?LHS* **by** (*simp add*: *abs-inverse*)
finally show *?LHS* = *?RHS* **by** *simp*
qed

theorem *homeier-thm11*:
shows *R r r'* = (*abs r* = *abs r'*)
unfolding *abs-def*
by (*simp only*: *equivp*[*simplified equivp-def*] *homeier-lem7*)

lemma *rep-refl*:
shows *R* (*rep a*) (*rep a*)

unfolding *rep-def*
by (*simp add*: *equivp*[*simplified equivp-def*])

lemma *rep-abs-rsp*:
shows *R f* (*rep* (*abs g*)) = *R f g*
and *R* (*rep* (*abs g*)) *f* = *R g f*
by (*simp-all add*: *homeier-thm10 homeier-thm11*)

lemma *Quotient*:
shows *Quotient R abs rep*
unfolding *Quotient-def*
apply(*simp add*: *homeier-thm10*)
apply(*simp add*: *rep-refl*)
apply(*subst homeier-thm11*[*symmetric*])
apply(*simp add*: *equivp*[*simplified equivp-def*])
done

end

54.8 ML setup

Auxiliary data for the quotient package

use ~~/*src*/*HOL*/*Tools*/*Quotient*/*quotient-info.ML*

declare [[*map fun* = (*fun-map*, *fun-rel*)]]

lemmas [*quot-thm*] = *fun-quotient*
lemmas [*quot-respect*] = *quot-rel-rsp if-rsp o-rsp let-rsp*
lemmas [*quot-preserve*] = *if-prs o-prs let-prs*
lemmas [*quot-equiv*] = *identity-equivp*

Lemmas about simplifying id's.

lemmas [*id-simps*] =
id-def[*symmetric*]
fun-map-id
id-apply
id-o
o-id
eq-comp-r

Translation functions for the lifting process.

use ~~/*src*/*HOL*/*Tools*/*Quotient*/*quotient-term.ML*

Definitions of the quotient types.

use ~~/*src*/*HOL*/*Tools*/*Quotient*/*quotient-typ.ML*

Definitions for quotient constants.

use ~~/*src*/*HOL*/*Tools*/*Quotient*/*quotient-def.ML*

An auxiliary constant for recording some information about the lifted theorem in a tactic.

definition
Quot-True (*x* :: *'a*) ≡ *True*

lemma
shows *QT-all*: *Quot-True* (*All P*) ⟹ *Quot-True P*
and *QT-ex*: *Quot-True* (*Ex P*) ⟹ *Quot-True P*
and *QT-ex1*: *Quot-True* (*Ex1 P*) ⟹ *Quot-True P*
and *QT-lam*: *Quot-True* (λ*x*. *P x*) ⟹ (⋀*x*. *Quot-True* (*P x*))
and *QT-ext*: (⋀*x*. *Quot-True* (*a x*) ⟹ *f x* = *g x*) ⟹ (*Quot-True a* ⟹ *f* = *g*)
by (*simp-all add*: *Quot-True-def ext*)

lemma *QT-imp*: *Quot-True a* ≡ *Quot-True b*
by (*simp add*: *Quot-True-def*)

Tactics for proving the lifted theorems

use ~~/*src*/*HOL*/*Tools*/*Quotient*/*quotient-tacs.ML*

54.9 Methods / Interface

method-setup *lifting* =
⟪ *Attrib.thms* >> (*fn thms* => *fn ctxt* => *SIMPLE-METHOD* (*HEADGOAL* (*Quotient-Tacs.lift-tac ctxt thms*))) ⟫
⟪ *lifts theorems to quotient types* ⟫

method-setup *lifting-setup* =
⟪ *Attrib.thm* >> (*fn thms* => *fn ctxt* => *SIMPLE-METHOD* (*HEADGOAL* (*Quotient-Tacs.procedure-tac ctxt thms*))) ⟫
⟪ *sets up the three goals for the quotient lifting procedure* ⟫

method-setup *regularize* =
⟪ *Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD* (*HEADGOAL* (*Quotient-Tacs.regularize-tac ctxt*))) ⟫
⟪ *proves the regularization goals from the quotient lifting procedure* ⟫

method-setup *injection* =
⟪ *Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD* (*HEADGOAL* (*Quotient-Tacs.all-injection-tac ctxt*))) ⟫
⟪ *proves the rep/abs injection goals from the quotient lifting procedure* ⟫

method-setup *cleaning* =
⟪ *Scan.succeed* (*fn ctxt* => *SIMPLE-METHOD* (*HEADGOAL* (*Quotient-Tacs.clean-tac ctxt*))) ⟫
⟪ *proves the cleaning goals from the quotient lifting procedure* ⟫

attribute-setup *quot-lifted* =
  ⟪ *Scan.succeed Quotient-Tacs.lifted-attrib* ⟫
  ⟪ *lifts theorems to quotient types* ⟫

no-notation
  rel-conj (**infixr** *OOO 75*) **and**
  fun-map (**infixr** *−−−> 55*) **and**
  fun-rel (**infixr** *===> 55*)

end

55 Refute: Refute

theory *Refute*
imports *Hilbert-Choice List*
uses
  Tools/refute.ML
  Tools/refute-isar.ML
begin

setup *Refute.setup*

```
(* ------------------------------------------------------------------------- *)
(* REFUTE                                                                    *)
(*                                                                           *)
(* We use a SAT solver to search for a (finite) model that refutes a given   *)
(* HOL formula.                                                              *)
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
(* NOTE                                                                      *)
(*                                                                           *)
(* I strongly recommend that you install a stand-alone SAT solver if you     *)
(* want to use 'refute'.  For details see 'HOL/Tools/sat_solver.ML'.  If you *)
(* have installed (a supported version of) zChaff, simply set 'ZCHAFF_HOME'  *)
(* in 'etc/settings'.                                                        *)
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
(* USAGE                                                                     *)
(*                                                                           *)
(* See the file 'HOL/ex/Refute_Examples.thy' for examples.  The supported    *)
(* parameters are explained below.                                           *)
(* ------------------------------------------------------------------------- *)

(* ------------------------------------------------------------------------- *)
```

```
(* CURRENT LIMITATIONS                                                    *)
(*                                                                        *)
(* 'refute' currently accepts formulas of higher-order predicate logic (with *)
(* equality), including free/bound/schematic variables, lambda abstractions, *)
(* sets and set membership, "arbitrary", "The", "Eps", records and        *)
(* inductively defined sets.  Constants are unfolded automatically, and sort *)
(* axioms are added as well.  Other, user-asserted axioms however are     *)
(* ignored.  Inductive datatypes and recursive functions are supported, but  *)
(* may lead to spurious countermodels.                                    *)
(*                                                                        *)
(* The (space) complexity of the algorithm is non-elementary.             *)
(*                                                                        *)
(* Schematic type variables are not supported.                            *)
(* ---------------------------------------------------------------------- *)

(* ---------------------------------------------------------------------- *)
(* PARAMETERS                                                             *)
(*                                                                        *)
(* The following global parameters are currently supported (and required,    *)
(* except for "expect"):                                                  *)
(*                                                                        *)
(* Name          Type    Description                                      *)
(*                                                                        *)
(* "minsize"     int     Only search for models with size at least        *)
(*                       'minsize'.                                       *)
(* "maxsize"     int     If >0, only search for models with size at most  *)
(*                       'maxsize'.                                       *)
(* "maxvars"     int     If >0, use at most 'maxvars' boolean variables   *)
(*                       when transforming the term into a propositional  *)
(*                       formula.                                         *)
(* "maxtime"     int     If >0, terminate after at most 'maxtime' seconds.   *)
(*                       This value is ignored under some ML compilers.   *)
(* "satsolver"   string  Name of the SAT solver to be used.               *)
(* "no_assms"    bool    If "true", assumptions in structured proofs are  *)
(*                       not considered.                                  *)
(* "expect"      string  Expected result ("genuine", "potential", "none", or *)
(*                       "unknown").                                      *)
(*                                                                        *)
(* See 'HOL/SAT.thy' for default values.                                  *)
(*                                                                        *)
(* The size of particular types can be specified in the form type=size    *)
(* (where 'type' is a string, and 'size' is an int).  Examples:           *)
(* "'a"=1                                                                 *)
(* "List.list"=2                                                          *)
(* ---------------------------------------------------------------------- *)

(* ---------------------------------------------------------------------- *)
(* FILES                                                                  *)
(*                                                                        *)
```

```
(* HOL/Tools/prop_logic.ML    Propositional logic                          *)
(* HOL/Tools/sat_solver.ML    SAT solvers                                  *)
(* HOL/Tools/refute.ML        Translation HOL -> propositional logic and   *)
(*                            Boolean assignment -> HOL model              *)
(* HOL/Tools/refute_isar.ML   Adds 'refute'/'refute_params' to Isabelle's  *)
(*                            syntax                                       *)
(* HOL/Refute.thy             This file: loads the ML files, basic setup,  *)
(*                            documentation                                *)
(* HOL/SAT.thy                Sets default parameters                      *)
(* HOL/ex/RefuteExamples.thy  Examples                                     *)
(* ------------------------------------------------------------------------ *)
```

end

56 SAT: Reconstructing external resolution proofs for propositional logic

theory *SAT*
imports *Refute*
uses
Tools/cnf-funcs.ML
Tools/sat-funcs.ML
begin

Late package setup: default values for refute, see also theory *Refute*.

refute-params
[*itself=1*,
minsize=1,
maxsize=8,
maxvars=10000,
maxtime=60,
satsolver=auto,
no-assms=false]

ML ⟨⟨ *structure sat = SATFunc(cnf)* ⟩⟩

method-setup *sat* = ⟨⟨ *Scan.succeed (SIMPLE-METHOD' o sat.sat-tac)* ⟩⟩
SAT solver

method-setup *satx* = ⟨⟨ *Scan.succeed (SIMPLE-METHOD' o sat.satx-tac)* ⟩⟩
SAT solver (with definitional CNF)

end

57 Nitpick: Nitpick: Yet Another Counterexample Generator for Isabelle/HOL

theory *Nitpick*
imports *Map Quotient SAT*
uses (*Tools/Nitpick/kodkod.ML*)
(*Tools/Nitpick/kodkod-sat.ML*)
(*Tools/Nitpick/nitpick-util.ML*)
(*Tools/Nitpick/nitpick-hol.ML*)
(*Tools/Nitpick/nitpick-preproc.ML*)
(*Tools/Nitpick/nitpick-mono.ML*)
(*Tools/Nitpick/nitpick-scope.ML*)
(*Tools/Nitpick/nitpick-peephole.ML*)
(*Tools/Nitpick/nitpick-rep.ML*)
(*Tools/Nitpick/nitpick-nut.ML*)
(*Tools/Nitpick/nitpick-kodkod.ML*)
(*Tools/Nitpick/nitpick-model.ML*)
(*Tools/Nitpick/nitpick.ML*)
(*Tools/Nitpick/nitpick-isar.ML*)
(*Tools/Nitpick/nitpick-tests.ML*)
(*Tools/Nitpick/minipick.ML*)
begin

typedecl *bisim-iterator*

axiomatization *unknown* :: *'a*
and *is-unknown* :: *'a* ⇒ *bool*
and *undefined-fast-The* :: *'a*
and *undefined-fast-Eps* :: *'a*
and *bisim* :: *bisim-iterator* ⇒ *'a* ⇒ *'a* ⇒ *bool*
and *bisim-iterator-max* :: *bisim-iterator*
and *Quot* :: *'a* ⇒ *'b*
and *safe-The* :: (*'a* ⇒ *bool*) ⇒ *'a*
and *safe-Eps* :: (*'a* ⇒ *bool*) ⇒ *'a*

datatype (*'a*, *'b*) *fin-fun* = *FinFun* (*'a* ⇒ *'b*)
datatype (*'a*, *'b*) *fun-box* = *FunBox* (*'a* ⇒ *'b*)
datatype (*'a*, *'b*) *pair-box* = *PairBox* *'a* *'b*

typedecl *unsigned-bit*
typedecl *signed-bit*

datatype *'a word* = *Word* (*'a set*)

Alternative definitions.

lemma *If-def* [*nitpick-def*, *no-atp*]:
(*if P then Q else R*) ≡ (*P* ⟶ *Q*) ∧ (¬ *P* ⟶ *R*)
by (*rule eq-reflection*) (*rule if-bool-eq-conj*)

lemma *Ex1-def* [*nitpick-def*, *no-atp*]:
Ex1 P ≡ ∃ *x*. *P* = {*x*}
apply (*rule eq-reflection*)
apply (*simp add*: *Ex1-def expand-set-eq*)
apply (*rule iffI*)
apply (*erule exE*)
apply (*erule conjE*)
apply (*rule-tac x* = *x* **in** *exI*)
apply (*rule allI*)
apply (*rename-tac y*)
apply (*erule-tac x* = *y* **in** *allE*)
by (*auto simp*: *mem-def*)

lemma *split-def* [*nitpick-def*]: *split f* = (λ*p*. *f* (*fst p*) (*snd p*))
by (*simp add*: *prod-case-unfold split-def*)

lemma *rtrancl-def* [*nitpick-def*, *no-atp*]: *r** ≡ (*r*$^{+}$)$^{=}$
by *simp*

lemma *rtranclp-def* [*nitpick-def*, *no-atp*]:
rtranclp r a b ≡ (*a* = *b* ∨ *tranclp r a b*)
by (*rule eq-reflection*) (*auto dest*: *rtranclpD*)

lemma *tranclp-def* [*nitpick-def*, *no-atp*]:
tranclp r a b ≡ *trancl* (*split r*) (*a*, *b*)
by (*simp add*: *trancl-def Collect-def mem-def*)

definition *refl'* :: ('*a* × '*a* ⇒ *bool*) ⇒ *bool* **where**
refl' r ≡ ∀ *x*. (*x*, *x*) ∈ *r*

definition *wf'* :: ('*a* × '*a* ⇒ *bool*) ⇒ *bool* **where**
wf' r ≡ *acyclic r* ∧ (*finite r* ∨ *unknown*)

axiomatization *wf-wfrec* :: ('*a* × '*a* ⇒ *bool*) ⇒ (('*a* ⇒ '*b*) ⇒ '*a* ⇒ '*b*) ⇒ '*a* ⇒ '*b*

definition *wf-wfrec'* :: ('*a* × '*a* ⇒ *bool*) ⇒ (('*a* ⇒ '*b*) ⇒ '*a* ⇒ '*b*) ⇒ '*a* ⇒ '*b* **where**
[*nitpick-simp*]: *wf-wfrec' R F x* = *F* (*Recdef.cut* (*wf-wfrec R F*) *R x*) *x*

definition *wfrec'* :: ('*a* × '*a* ⇒ *bool*) ⇒ (('*a* ⇒ '*b*) ⇒ '*a* ⇒ '*b*) ⇒ '*a* ⇒ '*b* **where**
wfrec' R F x ≡ *if wf R then wf-wfrec' R F x*
else THE y. *wfrec-rel R* (%*f x*. *F* (*Recdef.cut f R x*) *x*) *x y*

definition *card'* :: ('*a* ⇒ *bool*) ⇒ *nat* **where**
card' A ≡ *if finite A then length* (*safe-Eps* (λ*xs*. *set xs* = *A* ∧ *distinct xs*)) *else 0*

definition *setsum'* :: ('*a* ⇒ '*b*::*comm-monoid-add*) ⇒ ('*a* ⇒ *bool*) ⇒ '*b* **where**
setsum' f A ≡ *if finite A then listsum* (*map f* (*safe-Eps* (λ*xs*. *set xs* = *A* ∧ *distinct*

xs))) *else 0*

inductive *fold-graph'* :: ('*a* ⇒ '*b* ⇒ '*b*) ⇒ '*b* ⇒ ('*a* ⇒ *bool*) ⇒ '*b* ⇒ *bool* **where**
fold-graph' f z {} *z* |
⟦*x* ∈ *A*; *fold-graph' f z* (*A* − {*x*}) *y*⟧ ⟹ *fold-graph' f z A* (*f x y*)

The following lemmas are not strictly necessary but they help the *special_level* optimization.

lemma *The-psimp* [*nitpick-psimp*, *no-atp*]:
P = {*x*} ⟹ *The P* = *x*
by (*subgoal-tac* {*x*} = (λ*y*. *y* = *x*)) (*auto simp*: *mem-def*)

lemma *Eps-psimp* [*nitpick-psimp*, *no-atp*]:
⟦*P x*; ¬ *P y*; *Eps P* = *y*⟧ ⟹ *Eps P* = *x*
apply (*case-tac P* (*Eps P*))
apply *auto*
apply (*erule contrapos-np*)
by (*rule someI*)

lemma *unit-case-def* [*nitpick-def*, *no-atp*]:
unit-case x u ≡ *x*
apply (*subgoal-tac u* = ())
apply (*simp only*: *unit.cases*)
by *simp*

declare *unit.cases* [*nitpick-simp del*]

lemma *nat-case-def* [*nitpick-def*, *no-atp*]:
nat-case x f n ≡ *if n* = *0 then x else f* (*n* − *1*)
apply (*rule eq-reflection*)
by (*case-tac n*) *auto*

declare *nat.cases* [*nitpick-simp del*]

lemma *list-size-simp* [*nitpick-simp*, *no-atp*]:
list-size f xs = (*if xs* = [] *then 0*
else Suc (*f* (*hd xs*) + *list-size f* (*tl xs*)))
size xs = (*if xs* = [] *then 0 else Suc* (*size* (*tl xs*)))
by (*case-tac xs*) *auto*

Auxiliary definitions used to provide an alternative representation for *rat* and *real*.

function *nat-gcd* :: *nat* ⇒ *nat* ⇒ *nat* **where**
[*simp del*]: *nat-gcd x y* = (*if y* = *0 then x else nat-gcd y* (*x mod y*))
by *auto*
termination
apply (*relation measure* (λ(*x*, *y*). *x* + *y* + (*if y* > *x then 1 else 0*)))
apply *auto*
apply (*metis mod-less-divisor xt1*(*9*))

by (*metis mod-mod-trivial mod-self nat-neq-iff xt1*(*10*))

definition *nat-lcm* :: *nat* ⇒ *nat* ⇒ *nat* **where**
nat-lcm x y = *x* ∗ *y div* (*nat-gcd x y*)

definition *int-gcd* :: *int* ⇒ *int* ⇒ *int* **where**
int-gcd x y = *int* (*nat-gcd* (*nat* (*abs x*)) (*nat* (*abs y*)))

definition *int-lcm* :: *int* ⇒ *int* ⇒ *int* **where**
int-lcm x y = *int* (*nat-lcm* (*nat* (*abs x*)) (*nat* (*abs y*)))

definition *Frac* :: *int* × *int* ⇒ *bool* **where**
Frac ≡ λ(*a*, *b*). *b* > *0* ∧ *int-gcd a b* = *1*

axiomatization *Abs-Frac* :: *int* × *int* ⇒ *'a*
and *Rep-Frac* :: *'a* ⇒ *int* × *int*

definition *zero-frac* :: *'a* **where**
zero-frac ≡ *Abs-Frac* (*0*, *1*)

definition *one-frac* :: *'a* **where**
one-frac ≡ *Abs-Frac* (*1*, *1*)

definition *num* :: *'a* ⇒ *int* **where**
num ≡ *fst o Rep-Frac*

definition *denom* :: *'a* ⇒ *int* **where**
denom ≡ *snd o Rep-Frac*

function *norm-frac* :: *int* ⇒ *int* ⇒ *int* × *int* **where**
[*simp del*]: *norm-frac a b* = (*if b* < *0 then norm-frac* (− *a*) (− *b*)
else if a = *0* ∨ *b* = *0 then* (*0*, *1*)
else let c = *int-gcd a b in* (*a div c*, *b div c*))
by *pat-completeness auto*
termination by (*relation measure* (λ(-, *b*). *if b* < *0 then 1 else 0*)) *auto*

definition *frac* :: *int* ⇒ *int* ⇒ *'a* **where**
frac a b ≡ *Abs-Frac* (*norm-frac a b*)

definition *plus-frac* :: *'a* ⇒ *'a* ⇒ *'a* **where**
[*nitpick-simp*]:
plus-frac q r = (*let d* = *int-lcm* (*denom q*) (*denom r*) *in*
frac (*num q* ∗ (*d div denom q*) + *num r* ∗ (*d div denom r*)) *d*)

definition *times-frac* :: *'a* ⇒ *'a* ⇒ *'a* **where**
[*nitpick-simp*]:
times-frac q r = *frac* (*num q* ∗ *num r*) (*denom q* ∗ *denom r*)

definition *uminus-frac* :: *'a* ⇒ *'a* **where**

uminus-frac q ≡ *Abs-Frac* (− *num q*, *denom q*)

definition *number-of-frac* :: *int* ⇒ *'a* **where**
number-of-frac n ≡ *Abs-Frac* (*n*, *1*)

definition *inverse-frac* :: *'a* ⇒ *'a* **where**
inverse-frac q ≡ *frac* (*denom q*) (*num q*)

definition *less-eq-frac* :: *'a* ⇒ *'a* ⇒ *bool* **where**
[*nitpick-simp*]:
less-eq-frac q r ⟷ *num* (*plus-frac q* (*uminus-frac r*)) ≤ *0*

definition *of-frac* :: *'a* ⇒ *'b*::{*inverse*,*ring-1*} **where**
of-frac q ≡ *of-int* (*num q*) / *of-int* (*denom q*)

use *Tools/Nitpick/kodkod.ML*
use *Tools/Nitpick/kodkod-sat.ML*
use *Tools/Nitpick/nitpick-util.ML*
use *Tools/Nitpick/nitpick-hol.ML*
use *Tools/Nitpick/nitpick-mono.ML*
use *Tools/Nitpick/nitpick-preproc.ML*
use *Tools/Nitpick/nitpick-scope.ML*
use *Tools/Nitpick/nitpick-peephole.ML*
use *Tools/Nitpick/nitpick-rep.ML*
use *Tools/Nitpick/nitpick-nut.ML*
use *Tools/Nitpick/nitpick-kodkod.ML*
use *Tools/Nitpick/nitpick-model.ML*
use *Tools/Nitpick/nitpick.ML*
use *Tools/Nitpick/nitpick-isar.ML*
use *Tools/Nitpick/nitpick-tests.ML*
use *Tools/Nitpick/minipick.ML*

setup ⟪ *Nitpick-Isar.setup* ⟫

hide-const (**open**) *unknown is-unknown undefined-fast-The undefined-fast-Eps bisim*

bisim-iterator-max Quot safe-The safe-Eps FinFun FunBox PairBox Word refl' wf' wf-wfrec wf-wfrec' wfrec' card' setsum' fold-graph' nat-gcd nat-lcm int-gcd int-lcm Frac Abs-Frac Rep-Frac zero-frac one-frac num denom norm-frac frac plus-frac times-frac uminus-frac number-of-frac inverse-frac less-eq-frac of-frac

hide-type (**open**) *bisim-iterator fin-fun fun-box pair-box unsigned-bit signed-bit word*

hide-fact (**open**) *If-def Ex1-def split-def rtrancl-def rtranclp-def tranclp-def refl'-def wf'-def wf-wfrec'-def wfrec'-def card'-def setsum'-def fold-graph'-def The-psimp Eps-psimp unit-case-def nat-case-def list-size-simp nat-gcd-def nat-lcm-def int-gcd-def int-lcm-def Frac-def zero-frac-def one-frac-def num-def denom-def norm-frac-def frac-def plus-frac-def times-frac-def uminus-frac-def number-of-frac-def*

inverse-frac-def less-eq-frac-def of-frac-def

end

58 SMT: Bindings to Satisfiability Modulo Theories (SMT) solvers

theory *SMT*
imports *List*
uses
"~~/src/Tools/cache-io.ML"
("Tools/SMT/smt-monomorph.ML")
("Tools/SMT/smt-normalize.ML")
("Tools/SMT/smt-translate.ML")
("Tools/SMT/smt-solver.ML")
("Tools/SMT/smtlib-interface.ML")
("Tools/SMT/z3-proof-parser.ML")
("Tools/SMT/z3-proof-tools.ML")
("Tools/SMT/z3-proof-literals.ML")
("Tools/SMT/z3-proof-reconstruction.ML")
("Tools/SMT/z3-model.ML")
("Tools/SMT/z3-interface.ML")
("Tools/SMT/z3-solver.ML")
("Tools/SMT/cvc3-solver.ML")
("Tools/SMT/yices-solver.ML")
begin

58.1 Triggers for quantifier instantiation

Some SMT solvers support triggers for quantifier instantiation. Each trigger consists of one ore more patterns. A pattern may either be a list of positive subterms (each being tagged by "pat"), or a list of negative subterms (each being tagged by "nopat").

When an SMT solver finds a term matching a positive pattern (a pattern with positive subterms only), it instantiates the corresponding quantifier accordingly. Negative patterns inhibit quantifier instantiations. Each pattern should mention all preceding bound variables.

datatype *pattern* = *Pattern*

definition *pat* :: *'a* ⇒ *pattern* **where** *pat* - = *Pattern*
definition *nopat* :: *'a* ⇒ *pattern* **where** *nopat* - = *Pattern*

definition *trigger* :: *pattern list list* ⇒ *bool* ⇒ *bool*
where *trigger* - *P* = *P*

58.2 Higher-order encoding

Application is made explicit for constants occurring with varying numbers of arguments. This is achieved by the introduction of the following constant.

definition *fun-app* **where** *fun-app f x = f x*

Some solvers support a theory of arrays which can be used to encode higher-order functions. The following set of lemmas specifies the properties of such (extensional) arrays.

lemmas *array-rules = ext fun-upd-apply fun-upd-same fun-upd-other fun-upd-upd fun-app-def*

58.3 First-order logic

Some SMT solvers require a strict separation between formulas and terms. When translating higher-order into first-order problems, all uninterpreted constants (those not builtin in the target solver) are treated as function symbols in the first-order sense. Their occurrences as head symbols in atoms (i.e., as predicate symbols) is turned into terms by equating such atoms with *True* using the following term-level equation symbol.

definition *term-eq* :: *bool* ⇒ *bool* ⇒ *bool* **where** *term-eq x y* = (*x* = *y*)

58.4 Integer division and modulo for Z3

definition *z3div* :: *int* ⇒ *int* ⇒ *int* **where**
z3div k l = (*if* 0 ≤ *l then k div l else* −(*k div* (−*l*)))

definition *z3mod* :: *int* ⇒ *int* ⇒ *int* **where**
z3mod k l = (*if* 0 ≤ *l then k mod l else k mod* (−*l*))

lemma *div-by-z3div*: *k div l* = (
  if k = 0 ∨ *l* = 0 *then* 0
  else if (0 < *k* ∧ 0 < *l*) ∨ (*k* < 0 ∧ 0 < *l*) *then z3div k l*
  else z3div (−*k*) (−*l*))
by (*auto simp add*: *z3div-def*)

lemma *mod-by-z3mod*: *k mod l* = (
  if l = 0 *then k*
  else if k = 0 *then* 0
  else if (0 < *k* ∧ 0 < *l*) ∨ (*k* < 0 ∧ 0 < *l*) *then z3mod k l*
  else − *z3mod* (−*k*) (−*l*))
by (*auto simp add*: *z3mod-def*)

58.5 Setup

use *Tools/SMT/smt-monomorph.ML*
use *Tools/SMT/smt-normalize.ML*

use *Tools/SMT/smt-translate.ML*
use *Tools/SMT/smt-solver.ML*
use *Tools/SMT/smtlib-interface.ML*
use *Tools/SMT/z3-interface.ML*
use *Tools/SMT/z3-proof-parser.ML*
use *Tools/SMT/z3-proof-tools.ML*
use *Tools/SMT/z3-proof-literals.ML*
use *Tools/SMT/z3-proof-reconstruction.ML*
use *Tools/SMT/z3-model.ML*
use *Tools/SMT/z3-solver.ML*
use *Tools/SMT/cvc3-solver.ML*
use *Tools/SMT/yices-solver.ML*

setup ⟪
SMT-Solver.setup #>
Z3-Proof-Reconstruction.setup #>
Z3-Solver.setup #>
CVC3-Solver.setup #>
Yices-Solver.setup
⟫

58.6 Configuration

The current configuration can be printed by the command *smt-status*, which shows the values of most options.

58.7 General configuration options

The option *smt-solver* can be used to change the target SMT solver. The possible values are *cvc3*, *yices*, and *z3*. It is advisable to locally install the selected solver, although this is not necessary for *cvc3* and *z3*, which can also be used over an Internet-based service.

When using local SMT solvers, the path to their binaries should be declared by setting the following environment variables: *CVC3-SOLVER*, *YICES-SOLVER*, and *Z3-SOLVER*.

declare [[*smt-solver* = *z3*]]

Since SMT solvers are potentially non-terminating, there is a timeout (given in seconds) to restrict their runtime. A value greater than 120 (seconds) is in most cases not advisable.

declare [[*smt-timeout* = *20*]]

58.8 Certificates

By setting the option *smt-certificates* to the name of a file, all following applications of an SMT solver a cached in that file. Any further application

of the same SMT solver (using the very same configuration) re-uses the cached certificate instead of invoking the solver. An empty string disables caching certificates.
The filename should be given as an explicit path. It is good practice to use the name of the current theory (with ending *.certs* instead of *.thy*) as the certificates file.

declare [[*smt-certificates* =]]

The option *smt-fixed* controls whether only stored certificates are should be used or invocation of an SMT solver is allowed. When set to *true*, no SMT solver will ever be invoked and only the existing certificates found in the configured cache are used; when set to *false* and there is no cached certificate for some proposition, then the configured SMT solver is invoked.

declare [[*smt-fixed* = *false*]]

58.9 Tracing

For tracing the generated problem file given to the SMT solver as well as the returned result of the solver, the option *smt-trace* should be set to *true*.

declare [[*smt-trace* = *false*]]

58.10 Z3-specific options

Z3 is the only SMT solver whose proofs are checked (or reconstructed) in Isabelle (all other solvers are implemented as oracles). Enabling or disabling proof reconstruction for Z3 is controlled by the option *z3-proofs*.

declare [[*z3-proofs* = *true*]]

From the set of assumptions given to Z3, those assumptions used in the proof are traced when the option *z3-trace-assms* is set to *true*.

declare [[*z3-trace-assms* = *false*]]

Z3 provides several commandline options to tweak its behaviour. They can be configured by writing them literally as value for the option *z3-options*.

declare [[*z3-options* =]]

58.11 Schematic rules for Z3 proof reconstruction

Several prof rules of Z3 are not very well documented. There are two lemma groups which can turn failing Z3 proof reconstruction attempts into succeeding ones: the facts in *z3-rule* are tried prior to any implemented reconstruction procedure for all uncertain Z3 proof rules; the facts in *z3-simp* are only fed to invocations of the simplifier when reconstructing theory-specific proof steps.

lemmas [*z3-rule*] =
refl eq-commute conj-commute disj-commute simp-thms nnf-simps
ring-distribs field-simps times-divide-eq-right times-divide-eq-left
if-True if-False not-not

lemma [*z3-rule*]:
$(P \longrightarrow Q) = (Q \vee \neg P)$
$(\neg P \longrightarrow Q) = (P \vee Q)$
$(\neg P \longrightarrow Q) = (Q \vee P)$
by *auto*

lemma [*z3-rule*]:
$((P = Q) \longrightarrow R) = (R \mid (Q = (\neg P)))$
by *auto*

lemma [*z3-rule*]:
$((\neg P) = P) = False$
$(P = (\neg P)) = False$
$(P \neq Q) = (Q = (\neg P))$
$(P = Q) = ((\neg P \vee Q) \wedge (P \vee \neg Q))$
$(P \neq Q) = ((\neg P \vee \neg Q) \wedge (P \vee Q))$
by *auto*

lemma [*z3-rule*]:
$(if\ P\ then\ P\ else\ \neg P) = True$
$(if\ \neg P\ then\ \neg P\ else\ P) = True$
$(if\ P\ then\ True\ else\ False) = P$
$(if\ P\ then\ False\ else\ True) = (\neg P)$
$(if\ \neg P\ then\ x\ else\ y) = (if\ P\ then\ y\ else\ x)$
by *auto*

lemma [*z3-rule*]:
$P = Q \vee P \vee Q$
$P = Q \vee \neg P \vee \neg Q$
$(\neg P) = Q \vee \neg P \vee Q$
$(\neg P) = Q \vee P \vee \neg Q$
$P = (\neg Q) \vee \neg P \vee Q$
$P = (\neg Q) \vee P \vee \neg Q$
$P \neq Q \vee P \vee \neg Q$
$P \neq Q \vee \neg P \vee Q$
$P \neq (\neg Q) \vee P \vee Q$
$(\neg P) \neq Q \vee P \vee Q$
$P \vee Q \vee P \neq (\neg Q)$
$P \vee Q \vee (\neg P) \neq Q$
$P \vee \neg Q \vee P \neq Q$
$\neg P \vee Q \vee P \neq Q$
by *auto*

lemma [*z3-rule*]:

$0 + (x::int) = x$
$x + 0 = x$
$0 * x = 0$
$1 * x = x$
$x + y = y + x$
by *auto*

hide-type (**open**) *pattern*
hide-const *Pattern term-eq*
hide-const (**open**) *trigger pat nopat fun-app z3div z3mod*

end

59 Main: Main HOL

theory *Main*
imports *Plain Predicate-Compile Nitpick SMT*
begin

Classical Higher-order Logic – only "Main", excluding real and complex numbers etc.

See further [1]

end

References

[1] T. Nipkow, L. C. Paulson, and M. Wenzel. *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*, volume 2283 of *LNCS*. Springer-Verlag, 2002.